

Álgebra Linear - 2019.1

Lista 7 - Determinantes

- 1) Encontre o determinante de cada uma das seguintes matrizes

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 8 & 4 & 0 \\ 2 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \text{ e } C = \begin{pmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{pmatrix}.$$

Solução

Vamos calcular o determinante da matriz A aplicando o desenvolvimento de Laplace na terceira coluna, isto é,

$$\det A = \det \begin{pmatrix} 1 & 3 & 2 \\ 8 & 4 & 0 \\ 2 & 1 & 0 \end{pmatrix} = (-1)^{1+3} 2 \det \begin{pmatrix} 8 & 4 \\ 2 & 1 \end{pmatrix} = 2(8 - 8) = 2(0) = 0.$$

Para a matriz B , note que

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \xrightarrow{L_3 + L_2 \rightarrow L_3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Logo, aplicando o desenvolvimento de Laplace na terceira linha, segue que

$$\det B = \det \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = (-1)^{3+1} 2 \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2(-1 - 1) = 2(-2) = -4.$$

Já para a matriz C , por sabermos que o determinante não se altera com operações elementares sobre colunas, procedemos da seguinte forma:

$$\begin{pmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{pmatrix} \xrightarrow{\substack{C_2 - aC_1 \rightarrow C_2 \\ C_3 - a^2C_1 \rightarrow C_3}} \begin{pmatrix} a & 0 & 0 \\ b & b^2 - ab & b^3 - a^2b \\ c & c^2 - ac & c^3 - a^2c \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ b & b(b-a) & b(b-a)(b+a) \\ c & c(c-a) & c(c-a)(c+a) \end{pmatrix}.$$

Assim,

$$\det C = \det \begin{pmatrix} a & 0 & 0 \\ b & b(b-a) & b(b-a)(b+a) \\ c & c(c-a) & c(c-a)(c+a) \end{pmatrix} = a \det \begin{pmatrix} b(b-a) & b(b-a)(b+a) \\ c(c-a) & c(c-a)(c+a) \end{pmatrix}.$$

Note que, na última igualdade, foi aplicado o desenvolvimento de Laplace na primeira linha. Agora, usando a linearidade do determinante em cada linha, concluímos que

$$a \det \begin{pmatrix} b(b-a) & b(b-a)(b+a) \\ c(c-a) & c(c-a)(c+a) \end{pmatrix} = ab(b-a)c(c-a) \det \begin{pmatrix} 1 & (b+a) \\ 1 & (c+a) \end{pmatrix} = abc(b-a)(c-a)(c-b).$$

Portanto, $\det C = abc(b-a)(c-a)(c-b)$.

- 2) Encontre o determinante da matriz

$$B = \begin{pmatrix} 3 & 4 & 5 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 3 & 6 & 3 \\ 7 & 2 & 9 & 4 \end{pmatrix}$$

Solução

Para calcularmos o determinante de B vamos aplicar o desenvolvimento de Laplace na segunda linha, isto é,

$$\det B = (-1)^{2+1} 1 \det \begin{pmatrix} 4 & 5 & 2 \\ 3 & 6 & 3 \\ 2 & 9 & 4 \end{pmatrix} + (-1)^{2+3} 1 \det \begin{pmatrix} 3 & 4 & 2 \\ 2 & 3 & 3 \\ 7 & 2 & 4 \end{pmatrix} = (-1)(-12) + (-1)36 = -24.$$

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3) Mostre que

$$\det \begin{pmatrix} 2 & 3 & 7 & 1 & 3 \\ 2 & 3 & 7 & 1 & 5 \\ 2 & 3 & 6 & 1 & 9 \\ 4 & 6 & 2 & 3 & 4 \\ 5 & 8 & 7 & 4 & 5 \end{pmatrix} = 2$$

Solução

Note que

$$\begin{pmatrix} 2 & 3 & 7 & 1 & 3 \\ 2 & 3 & 7 & 1 & 5 \\ 2 & 3 & 6 & 1 & 9 \\ 4 & 6 & 2 & 3 & 4 \\ 5 & 8 & 7 & 4 & 5 \end{pmatrix} \xrightarrow{L_2 - L_1 \rightarrow L_2} \begin{pmatrix} 2 & 3 & 7 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 \\ 2 & 3 & 6 & 1 & 9 \\ 4 & 6 & 2 & 3 & 4 \\ 5 & 8 & 7 & 4 & 5 \end{pmatrix}.$$

Logo,

$$\det \begin{pmatrix} 2 & 3 & 7 & 1 & 3 \\ 2 & 3 & 7 & 1 & 5 \\ 2 & 3 & 6 & 1 & 9 \\ 4 & 6 & 2 & 3 & 4 \\ 5 & 8 & 7 & 4 & 5 \end{pmatrix} = \det \begin{pmatrix} 2 & 3 & 7 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 \\ 2 & 3 & 6 & 1 & 9 \\ 4 & 6 & 2 & 3 & 4 \\ 5 & 8 & 7 & 4 & 5 \end{pmatrix} = (-1)^{2+5} 2 \det \begin{pmatrix} 2 & 3 & 7 & 1 \\ 2 & 3 & 6 & 1 \\ 4 & 6 & 2 & 3 \\ 5 & 8 & 7 & 4 \end{pmatrix}.$$

Note que, na última igualdade, foi aplicado o desenvolvimento de Laplace na segunda linha. Contudo,

$$\begin{pmatrix} 2 & 3 & 7 & 1 \\ 2 & 3 & 6 & 1 \\ 4 & 6 & 2 & 3 \\ 5 & 8 & 7 & 4 \end{pmatrix} \xrightarrow{L_2 - L_1 \rightarrow L_2} \begin{pmatrix} 2 & 3 & 7 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 6 & 2 & 3 \\ 5 & 8 & 7 & 4 \end{pmatrix}.$$

Assim,

$$(-2) \det \begin{pmatrix} 2 & 3 & 7 & 1 \\ 2 & 3 & 6 & 1 \\ 4 & 6 & 2 & 3 \\ 5 & 8 & 7 & 4 \end{pmatrix} = (-2) \det \begin{pmatrix} 2 & 3 & 7 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 6 & 2 & 3 \\ 5 & 8 & 7 & 4 \end{pmatrix} = (-2)(-1)^{2+3} (-1) \det \begin{pmatrix} 2 & 3 & 1 \\ 4 & 6 & 3 \\ 5 & 8 & 4 \end{pmatrix} = 2.$$

Note que novamente, na penúltima igualdade, foi aplicado o desenvolvimento de Laplace na segunda linha.

Portanto,

$$\det \begin{pmatrix} 2 & 3 & 7 & 1 & 3 \\ 2 & 3 & 7 & 1 & 5 \\ 2 & 3 & 6 & 1 & 9 \\ 4 & 6 & 2 & 3 & 4 \\ 5 & 8 & 7 & 4 & 5 \end{pmatrix} = 2.$$

4) Mostre que

$$\det \begin{pmatrix} 2 & 1 & 5 & 1 & 3 \\ 2 & 1 & 5 & 1 & 2 \\ 4 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 0 & 1 \\ 2 & 1 & 6 & \pi & 7 \end{pmatrix} = 2$$

Solução

Temos que

$$\begin{pmatrix} 2 & 1 & 5 & 1 & 3 \\ 2 & 1 & 5 & 1 & 2 \\ 4 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 0 & 1 \\ 2 & 1 & 6 & \pi & 7 \end{pmatrix} \xrightarrow{L_2 - L_1 \rightarrow L_2} \begin{pmatrix} 2 & 1 & 5 & 1 & 3 \\ 0 & 0 & 0 & 0 & -1 \\ 4 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 0 & 1 \\ 2 & 1 & 6 & \pi & 7 \end{pmatrix}.$$

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Consequentemente,

$$\det \begin{pmatrix} 2 & 1 & 5 & 1 & 3 \\ 2 & 1 & 5 & 1 & 2 \\ 4 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 0 & 1 \\ 2 & 1 & 6 & \pi & 7 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 & 5 & 1 & 3 \\ 0 & 0 & 0 & 0 & -1 \\ 4 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 0 & 1 \\ 2 & 1 & 6 & \pi & 7 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 & 5 & 1 \\ 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 0 \\ 2 & 1 & 6 & \pi \end{pmatrix}.$$

Note que, na última igualdade, foi aplicado o desenvolvimento de Laplace na segunda linha. Entretanto,

$$\begin{pmatrix} 2 & 1 & 5 & 1 \\ 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 0 \\ 2 & 1 & 6 & \pi \end{pmatrix} \xrightarrow{L_3 - L_2 \rightarrow L_3} \begin{pmatrix} 2 & 1 & 5 & 1 \\ 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & -1 \\ 2 & 1 & 6 & \pi \end{pmatrix}.$$

Em vista disso,

$$\det \begin{pmatrix} 2 & 1 & 5 & 1 \\ 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 0 \\ 2 & 1 & 6 & \pi \end{pmatrix} = \det \begin{pmatrix} 2 & 1 & 5 & 1 \\ 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & -1 \\ 2 & 1 & 6 & \pi \end{pmatrix} = \det \begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & 2 \\ 2 & 1 & 6 \end{pmatrix} = 2.$$

Note que novamente, na penúltima igualdade, foi aplicado o desenvolvimento de Laplace na terceira linha.

Por conseguinte,

$$\det \begin{pmatrix} 2 & 1 & 5 & 1 & 3 \\ 2 & 1 & 5 & 1 & 2 \\ 4 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 0 & 1 \\ 2 & 1 & 6 & \pi & 7 \end{pmatrix} = 2.$$

5) Use eliminação Gaussiana (escalonamento) para calcular os determinantes das seguintes matrizes:

$$A = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 1 & 5 & -7 & 2 \\ 3 & 1 & -5 & 3 \\ 2 & 3 & -6 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 1 & -1 & 4 & 3 \\ 2 & 2 & -1 & 1 \end{pmatrix}$$

Solução

Escalonando a matriz A , obtemos

$$\begin{pmatrix} 1 & -2 & 1 & -1 \\ 1 & 5 & -7 & 2 \\ 3 & 1 & -5 & 3 \\ 2 & 3 & -6 & 0 \end{pmatrix} \xrightarrow[L_3 - 3L_1 \rightarrow L_3]{L_4 - 2L_1 \rightarrow L_4} \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 7 & -8 & 3 \\ 0 & 7 & -8 & 6 \\ 0 & 7 & -8 & 2 \end{pmatrix} \xrightarrow[L_3 - L_2 \rightarrow L_3]{L_4 - L_2 \rightarrow L_4} \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 7 & -8 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Portanto,

$$\det \begin{pmatrix} 1 & -2 & 1 & -1 \\ 1 & 5 & -7 & 2 \\ 3 & 1 & -5 & 3 \\ 2 & 3 & -6 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 7 & -8 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} = 0.$$

Já para a matriz B , temos

$$\begin{pmatrix} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 1 & -1 & 4 & 3 \\ 2 & 2 & -1 & 1 \end{pmatrix} \xrightarrow[L_3 - \frac{1}{2}L_1 \rightarrow L_3]{L_4 - L_1 \rightarrow L_4} \begin{pmatrix} 2 & 1 & 3 & 2 \\ 0 & -\frac{3}{2} & -\frac{7}{2} & -5 \\ 0 & -\frac{3}{2} & \frac{5}{2} & 2 \\ 0 & 1 & -4 & -1 \end{pmatrix} \xrightarrow[L_3 - L_2 \rightarrow L_3]{L_4 + \frac{2}{3}L_2 \rightarrow L_4} \begin{pmatrix} 2 & 1 & 3 & 2 \\ 0 & -\frac{3}{2} & -\frac{7}{2} & -5 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & \frac{55}{18} \end{pmatrix}$$

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Portanto,

$$\det \begin{pmatrix} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 1 & -1 & 4 & 3 \\ 2 & 2 & -1 & 1 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 & 3 & 2 \\ 0 & -\frac{3}{2} & -\frac{7}{2} & -5 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & \frac{55}{18} \end{pmatrix} = -55.$$

- 6) Se A e B são matrizes quadradas das mesmas dimensões e $\det(A) = 2$ e $\det(B) = 3$, encontre $\det(A^2B^{-1})$.

Solução

Sabemos que

$$BB^{-1} = I \Leftrightarrow \det(BB^{-1}) = \det I \Leftrightarrow \det B \det B^{-1} = 1 \Leftrightarrow \det B^{-1} = \frac{1}{\det B}.$$

Além disso,

$$\det(A^2) = \det(AA) = \det A \det A = (\det A)^2.$$

Portanto,

$$\det(A^2B^{-1}) = \det A^2 \det B^{-1} = (\det A)^2 \frac{1}{\det B} = (2)^2 \frac{1}{3} = \frac{4}{3}.$$

- 7) Para cada uma das matrizes abaixo calcule sua adjunta e utilize estes cálculos para calcular a matriz inversa:

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 3 & 5 & 4 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Solução

A adjunta da matriz A é à transposta da matriz dos cofatores de A . O cofator Δ_{ij} do elemento a_{ij} da matriz é $(-1)^{i+j} \det A_{ij}$, em que A_{ij} é a submatriz de A , obtida extraíndo-se a i -ésima linha e j -ésima coluna. Isto posto, calculemos

$$\begin{aligned} \Delta_{11} &= (-1)^{1+1} \det \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} = -1, & \Delta_{12} &= (-1)^{1+2} \det \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = 0, \\ \Delta_{13} &= (-1)^{1+3} \det \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} = 2, & \Delta_{21} &= (-1)^{2+1} \det \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} = 3, \\ \Delta_{22} &= (-1)^{2+2} \det \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} = 0, & \Delta_{23} &= (-1)^{2+3} \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = -1, \\ \Delta_{31} &= (-1)^{3+1} \det \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} = 7, & \Delta_{32} &= (-1)^{3+2} \det \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = 5, \\ \Delta_{33} &= (-1)^{3+3} \det \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} = -4. \end{aligned}$$

Por conseguinte,

$$\text{adj } A = \begin{pmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} \end{pmatrix} = \begin{pmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{pmatrix}.$$

Agora, para a matriz B , calculemos

$$\begin{aligned} \Delta_{11} &= (-1)^{1+1} \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1, & \Delta_{12} &= (-1)^{1+2} \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = -1, \\ \Delta_{13} &= (-1)^{1+3} \det \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = -1, & \Delta_{21} &= (-1)^{2+1} \det \begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix} = -5, \\ \Delta_{22} &= (-1)^{2+2} \det \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix} = -1, & \Delta_{23} &= (-1)^{2+3} \det \begin{pmatrix} 3 & 5 \\ 1 & 0 \end{pmatrix} = 5, \\ \Delta_{31} &= (-1)^{3+1} \det \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix} = 1, & \Delta_{32} &= (-1)^{3+2} \det \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} = 5, \\ \Delta_{33} &= (-1)^{3+3} \det \begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} = -7. \end{aligned}$$

Portanto,

$$\text{adj } B = \begin{pmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & -5 & 1 \\ -1 & -1 & 5 \\ -1 & 5 & -7 \end{pmatrix}.$$

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Uma vez que

$$\det A = \det \begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = 5 \text{ e } \det B = \det \begin{pmatrix} 3 & 5 & 4 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = -6,$$

segue que

$$A^{-1} = \frac{1}{\det A} (\text{adj } A) = \frac{1}{5} \begin{pmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 1 \\ \frac{2}{5} & -\frac{1}{5} & -\frac{4}{5} \end{pmatrix}$$

e

$$B^{-1} = \frac{1}{\det B} (\text{adj } B) = -\frac{1}{6} \begin{pmatrix} 1 & -5 & 1 \\ -1 & -1 & 5 \\ -1 & 5 & -7 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \end{pmatrix}.$$

8) Calcule os seguintes determinantes

$$A = \begin{pmatrix} \sqrt{\pi} & 10 & 100 \\ 0 & 1 & e \\ 0 & 0 & \sqrt{\pi} \end{pmatrix}, B = \begin{pmatrix} \sqrt{\pi} & 0 & 0 \\ 10 & 1 & 0 \\ 100 & e & \sqrt{\pi} \end{pmatrix}$$

Solução

A matriz A é triangular superior e a matriz B é triangular inferior. Em vista disso, os respectivos determinantes são o produto dos elementos da diagonal principal, isto é,

$$\det A = \det \begin{pmatrix} \sqrt{\pi} & 10 & 100 \\ 0 & 1 & e \\ 0 & 0 & \sqrt{\pi} \end{pmatrix} = \pi \text{ e } \det B = \det \begin{pmatrix} \sqrt{\pi} & 0 & 0 \\ 10 & 1 & 0 \\ 100 & e & \sqrt{\pi} \end{pmatrix} = \pi.$$

9) Determine a matriz de cofatores e a matriz adjunta das seguintes matrizes:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & -2 & 0 \end{pmatrix}.$$

Solução

Os cofatores da matriz A são:

$$\begin{aligned} \Delta_{11} &= (-1)^{1+1} \det(4) = 4, & \Delta_{12} &= (-1)^{1+2} \det(3) = -3, \\ \Delta_{13} &= (-1)^{2+1} \det(2) = -2, & \Delta_{21} &= (-1)^{2+2} \det(1) = 1. \end{aligned}$$

Portanto,

$$\bar{A} = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

e

$$\text{adj } A = \begin{pmatrix} \Delta_{11} & \Delta_{21} \\ \Delta_{12} & \Delta_{22} \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix},$$

em que \bar{A} é a matriz dos cofatores de A .

Os cofatores da matriz B são:

$$\begin{aligned} \Delta_{11} &= (-1)^{1+1} \det \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} = 2, & \Delta_{12} &= (-1)^{1+2} \det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -1, \\ \Delta_{13} &= (-1)^{1+3} \det \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} = 1, & \Delta_{21} &= (-1)^{2+1} \det \begin{pmatrix} -1 & 3 \\ -2 & 0 \end{pmatrix} = -6, \\ \Delta_{22} &= (-1)^{2+2} \det \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} = 3, & \Delta_{23} &= (-1)^{2+3} \det \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} = 5, \\ \Delta_{31} &= (-1)^{3+1} \det \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} = -4, & \Delta_{32} &= (-1)^{3+2} \det \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = -2, \\ \Delta_{33} &= (-1)^{3+3} \det \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = 2. \end{aligned}$$

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Por conseguinte,

$$\bar{B} = \begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ -6 & 3 & 5 \\ -4 & -2 & 2 \end{pmatrix}$$

e

$$adj B = \begin{pmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} \end{pmatrix} = \begin{pmatrix} 2 & -6 & -4 \\ -1 & 3 & -2 \\ 1 & 5 & 2 \end{pmatrix},$$

em que \bar{B} é a matriz dos cofatores de B .

- 10) Calcule as inversas das matrizes do exercício anterior usando adjunta e escalonamento.

Solução

Primeiramente vamos calcular as inversas das matrizes A e B usando as respectivas adjuntas. Haja vista que

$$det A = det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = -2 \text{ e } det B = det \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & -2 & 0 \end{pmatrix} = 8,$$

segue que

$$A^{-1} = \frac{1}{det A} (adj A) = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

e

$$B^{-1} = \frac{1}{det B} (adj B) = \frac{1}{8} \begin{pmatrix} 2 & -6 & -4 \\ -1 & 3 & -2 \\ 1 & 5 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{8} & \frac{3}{8} & -\frac{1}{4} \\ \frac{1}{8} & \frac{5}{8} & \frac{1}{4} \end{pmatrix}.$$

Agora vamos calcular A^{-1} e B^{-1} usando a eliminação de Gauss-Jordan. Então,

$$\begin{array}{c} \left(\begin{array}{cc|cc} 1 & 2 & \vdots & 1 & 0 \\ 3 & 4 & \vdots & 0 & 1 \end{array} \right) \xrightarrow{L_2 - 3L_1 \rightarrow L_2} \left(\begin{array}{cc|cc} 1 & 2 & \vdots & 1 & 0 \\ 0 & -2 & \vdots & -3 & 1 \end{array} \right) \xrightarrow{-\frac{1}{2}L_2 \rightarrow L_2} \\ \left(\begin{array}{cc|cc} 1 & 2 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & \frac{3}{2} & -\frac{1}{2} \end{array} \right) \xrightarrow{L_1 - 2L_2 \rightarrow L_1} \left(\begin{array}{cc|cc} 1 & 0 & \vdots & -2 & 1 \\ 0 & 1 & \vdots & \frac{3}{2} & -\frac{1}{2} \end{array} \right). \end{array}$$

Portanto,

$$A^{-1} = \begin{pmatrix} -2 & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

Já para B^{-1} , segue

$$\begin{array}{c}
 \left(\begin{array}{cccccc} 2 & -1 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & 1 & : & 0 & 1 & 0 \\ -1 & -2 & 0 & : & 0 & 0 & 1 \end{array} \right) \xrightarrow{\frac{1}{2}L_1 \rightarrow L_1} \left(\begin{array}{cccccc} 1 & -\frac{1}{2} & \frac{3}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & : & 0 & 1 & 0 \\ -1 & -2 & 0 & : & 0 & 0 & 1 \end{array} \right) \xrightarrow{L_3 + L_1 \rightarrow L_3} \\
 \left(\begin{array}{cccccc} 1 & -\frac{1}{2} & \frac{3}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & -\frac{5}{2} & \frac{3}{2} & : & \frac{1}{2} & 0 & 1 \end{array} \right) \xrightarrow{L_3 + \frac{5}{2}L_2 \rightarrow L_3} \left(\begin{array}{cccccc} 1 & -\frac{1}{2} & \frac{3}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & 0 & 4 & : & \frac{1}{2} & \frac{5}{2} & 1 \end{array} \right) \xrightarrow{\frac{1}{4}L_3 \rightarrow L_3} \\
 \left(\begin{array}{cccccc} 1 & -\frac{1}{2} & \frac{3}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & : & 0 & 1 & 0 \\ 0 & 0 & 1 & : & \frac{1}{8} & \frac{5}{8} & \frac{1}{4} \end{array} \right) \xrightarrow{L_2 - L_3 \rightarrow L_2} \left(\begin{array}{cccccc} 1 & -\frac{1}{2} & \frac{3}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & : & -\frac{1}{8} & \frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & : & \frac{1}{8} & \frac{5}{8} & \frac{1}{4} \end{array} \right) \xrightarrow{L_1 - \frac{3}{2}L_3 \rightarrow L_1} \\
 \left(\begin{array}{cccccc} 1 & -\frac{1}{2} & 0 & : & \frac{5}{16} & -\frac{15}{16} & -\frac{3}{8} \\ 0 & 1 & 0 & : & -\frac{1}{8} & \frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & : & \frac{1}{8} & \frac{5}{8} & \frac{1}{4} \end{array} \right) \xrightarrow{L_1 + \frac{1}{2}L_2 \rightarrow L_1} \left(\begin{array}{cccccc} 1 & -\frac{1}{2} & 0 & : & \frac{1}{4} & -\frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & : & -\frac{1}{8} & \frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & : & \frac{1}{8} & \frac{5}{8} & \frac{1}{4} \end{array} \right).
 \end{array}$$

Portanto,

$$B^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{8} & \frac{3}{8} & -\frac{1}{4} \\ \frac{1}{8} & \frac{5}{8} & \frac{1}{4} \end{pmatrix}.$$

11) (a) Use eliminação Gaussiana (escalonamento) para determinar se existem as inversas das seguintes matrizes:

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix}.$$

Solução

Vamos calcular a forma escalonada da matriz A , isto é,

$$\begin{array}{c}
 \left(\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 3 \end{array} \right) \xrightarrow{L_2 - \frac{1}{2}L_1 \rightarrow L_2} \left(\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 3 \end{array} \right) \xrightarrow{L_4 + \frac{1}{2}L_1 \rightarrow L_4} \\
 \left(\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 & 3 \end{array} \right) \xrightarrow{L_3 \leftrightarrow L_2} \left(\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -\frac{1}{2} & -1 & 1 \\ 0 & \frac{1}{2} & 0 & 3 \end{array} \right) \xrightarrow{L_3 + \frac{1}{2}L_2 \rightarrow L_3} \\
 \left(\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & 0 & 3 \end{array} \right) \xrightarrow{L_4 - \frac{1}{2}L_2 \rightarrow L_4} \left(\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{5}{2} \end{array} \right) \xrightarrow{L_4 - L_3 \rightarrow L_4} \\
 \left(\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{array} \right).
 \end{array}$$

Uma vez que durante o escalonamento houve uma permutação de linhas, segue que

$$\det \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 3 \end{pmatrix} = -\det \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1.$$

Portanto existe A^{-1} , pois $\det A = 1 \neq 0$.

Já para a matriz B , temos

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{L_2-L_1 \rightarrow L_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{L_3-L_2 \rightarrow L_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Assim,

$$\det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Portanto, B^{-1} não existe, pois $\det B = 0$.

(b) Calcular a inversa quando existe.

Solução

Vamos calcular A^{-1} utilizando a eliminação de Gauss-Jordam, isto é,

$$\begin{array}{c} \left(\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & : & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & : & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 & : & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{\frac{1}{2}L_1 \rightarrow L_1} \left(\begin{array}{cccc|cccc} 1 & \frac{1}{2} & 0 & 0 & : & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & : & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & : & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 & : & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{L_2-L_1 \rightarrow L_2} \\ \left(\begin{array}{cccc|cccc} 1 & \frac{1}{2} & 0 & 0 & : & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & 1 & : & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & : & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 & : & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{L_4+L_1 \rightarrow L_4} \left(\begin{array}{cccc|cccc} 1 & \frac{1}{2} & 0 & 0 & : & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & 1 & : & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & : & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 3 & : & \frac{1}{2} & 0 & 0 & 1 \end{array} \right) \xrightarrow{-2L_2 \rightarrow L_2} \end{array}$$

$$\begin{array}{c}
 \left(\begin{array}{ccccccc|ccccc} 1 & \frac{1}{2} & 0 & 0 & \vdots & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 & \vdots & 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 1 & \vdots & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 3 & \vdots & \frac{1}{2} & 0 & 0 & 1 \end{array} \right) \xrightarrow{L_3 - L_2 \rightarrow L_3} \left(\begin{array}{ccccccc|ccccc} 1 & \frac{1}{2} & 0 & 0 & \vdots & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 & \vdots & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 3 & \vdots & -1 & 2 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 3 & \vdots & \frac{1}{2} & 0 & 0 & 1 \end{array} \right) \xrightarrow{L_4 - \frac{1}{2}L_2 \rightarrow L_4} \\
 \left(\begin{array}{ccccccc|ccccc} 1 & \frac{1}{2} & 0 & 0 & \vdots & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 & \vdots & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 3 & \vdots & -1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 4 & \vdots & 0 & 1 & 0 & 1 \end{array} \right) \xrightarrow{(-1)L_3 \rightarrow L_3} \left(\begin{array}{ccccccc|ccccc} 1 & \frac{1}{2} & 0 & 0 & \vdots & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 & \vdots & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -3 & \vdots & 1 & -2 & -1 & 0 \\ 0 & 0 & -1 & 4 & \vdots & 0 & 1 & 0 & 1 \end{array} \right) \xrightarrow{L_4 - L_3 \rightarrow L_4} \\
 \left(\begin{array}{ccccccc|ccccc} 1 & \frac{1}{2} & 0 & 0 & \vdots & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 & \vdots & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -3 & \vdots & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 1 & -1 & -1 & 1 \end{array} \right) \xrightarrow{L_3 + 3L_4 \rightarrow L_3} \left(\begin{array}{ccccccc|ccccc} 1 & \frac{1}{2} & 0 & 0 & \vdots & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 & \vdots & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 4 & -5 & -4 & 3 \\ 0 & 0 & 0 & 1 & \vdots & 1 & -1 & -1 & 1 \end{array} \right) \xrightarrow{L_2 + 2L_4 \rightarrow L_2} \\
 \left(\begin{array}{ccccccc|ccccc} 1 & \frac{1}{2} & 0 & 0 & \vdots & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & \vdots & 3 & -4 & -2 & 2 \\ 0 & 0 & 1 & 0 & \vdots & 4 & -5 & -4 & 3 \\ 0 & 0 & 0 & 1 & \vdots & 1 & -1 & -1 & 1 \end{array} \right) \xrightarrow{L_2 - 2L_3 \rightarrow L_2} \left(\begin{array}{ccccccc|ccccc} 1 & \frac{1}{2} & 0 & 0 & \vdots & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & -5 & 6 & 6 & -4 \\ 0 & 0 & 1 & 0 & \vdots & 4 & -5 & -4 & 3 \\ 0 & 0 & 0 & 1 & \vdots & 1 & -1 & -1 & 1 \end{array} \right) \xrightarrow{L_1 - \frac{1}{2}L_2 \rightarrow L_1} \\
 \left(\begin{array}{ccccccc|ccccc} 1 & 0 & 0 & 0 & \vdots & 3 & -3 & -3 & 2 \\ 0 & 1 & 0 & 0 & \vdots & -5 & 6 & 6 & -4 \\ 0 & 0 & 1 & 0 & \vdots & 4 & -5 & -4 & 3 \\ 0 & 0 & 0 & 1 & \vdots & 1 & -1 & -1 & 1 \end{array} \right)
 \end{array}$$

Portanto,

$$A^{-1} = \begin{pmatrix} 3 & -3 & -3 & 2 \\ -5 & 6 & 6 & -4 \\ 4 & -5 & -4 & 3 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

- 12) Resolva os seguintes sistemas, com a, b, c constantes, por dois métodos: (i) escalonamento e (ii) escrevendo o sistema na forma matricial $Ax = \mathbf{b}$ e encontrando a inversa da matriz A .

(a)

$$\begin{aligned} 2x + y &= a \\ 3x + 6y &= b \end{aligned} \tag{1}$$

Solução

Vamos transformar a matriz ampliada do sistema (1) em uma matriz na forma escalonada, isto é,

$$\left(\begin{array}{cc|c} 2 & 1 & a \\ 3 & 6 & b \end{array} \right) \xrightarrow{L_2 - \frac{3}{2}L_1 \rightarrow L_2} \left(\begin{array}{cc|c} 2 & 1 & a \\ 0 & \frac{9}{2} & \frac{2b-3a}{2} \end{array} \right).$$

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Logo, o sistema de equações correspondente à forma escalonada da matriz ampliada do sistema (1) é:

$$\begin{aligned} 2x + y &= a \\ \frac{9}{2}y &= \frac{2b-3a}{2}. \end{aligned} \quad (2)$$

Da segunda linha de (2) obtemos $y = \frac{2b-3a}{9}$, que substituindo na primeira linha de (2) nos possibilita concluir que $x = \frac{6a-b}{9}$.

Portanto, a solução do sistema (1) é:

$$S_1 = \begin{pmatrix} \frac{6a-b}{9} \\ \frac{2b-3a}{9} \end{pmatrix}.$$

Agora, reescrevendo o sistema (1) na forma matricial $A\mathbf{x} = \mathbf{b}$, obtemos

$$\begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Temos que

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix}$$

é a matriz dos coeficientes. Então,

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{9} \\ -\frac{1}{3} & \frac{2}{9} \end{pmatrix}.$$

Logo, a solução do sistema (1) é $\mathbf{x} = A^{-1}\mathbf{b}$, ou seja,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{9} \\ -\frac{1}{3} & \frac{2}{9} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{6a-b}{9} \\ \frac{2b-3a}{9} \end{pmatrix}.$$

(b)

$$\begin{aligned} x + y + z &= a \\ 2x + 2z &= b \\ 3y + 3z &= c \end{aligned} \quad (3)$$

Solução

Vamos transformar a matriz ampliada do sistema (3) em uma matriz na forma escalonada, isto é,

$$\begin{aligned} \left(\begin{array}{ccc|c} 1 & 1 & 1 & : & a \\ 2 & 0 & 2 & : & b \\ 0 & 3 & 3 & : & c \end{array} \right) &\xrightarrow{L_2-2L_1 \rightarrow L_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & : & a \\ 0 & -2 & 0 & : & b-2a \\ 0 & 3 & 3 & : & c \end{array} \right) \xrightarrow{L_3+\frac{3}{2}L_2 \rightarrow L_3} \\ &\left(\begin{array}{ccc|c} 1 & 1 & 1 & : & a \\ 0 & -2 & 0 & : & b-2a \\ 0 & 0 & 3 & : & \frac{3b+2c-6a}{2} \end{array} \right). \end{aligned}$$

Logo, o sistema de equações correspondente à forma escalonada da matriz ampliada do sistema (3) é:

$$\begin{aligned} x + y + z &= a \\ -2y &= -2a + b \\ 3z &= \frac{3b+2c-6a}{2} \end{aligned} \quad (4)$$

Da terceira linha de (4), obtemos $z = \frac{3b+2c-6a}{6}$. Já, na segunda linha de (4) determinamos que $y = \frac{2a-b}{2}$. Por fim, substituindo os resultados de y e z na primeira linha de (4), concluímos que $x = \frac{3a-c}{3}$.

Portanto, a solução do sistema (3) é:

$$S_2 = \begin{pmatrix} \frac{3a-c}{3} \\ \frac{2a-b}{2} \\ \frac{3b+2c-6a}{6} \end{pmatrix}.$$

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Agora, reescrevendo o sistema (3) na forma matricial $A\mathbf{x} = \mathbf{b}$, obtemos

$$\begin{pmatrix} 1 & 1 & 1 & \vdots & a \\ 2 & 0 & 2 & \vdots & b \\ 0 & 3 & 3 & \vdots & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Temos que

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 3 & 3 \end{pmatrix}$$

é a matriz dos coeficientes. Então,

$$A^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 1 & -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

Logo, a solução do sistema (3) é $\mathbf{x} = A^{-1}\mathbf{b}$, ou seja,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 1 & -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{3a-c}{3} \\ \frac{2a-b}{3} \\ \frac{3b+2c-6a}{6} \end{pmatrix}.$$

13) Mostrar para a seguinte matriz de $M_{n \times n}$ a igualdade:

$$\left| \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix} \right| = n + 1$$

Solução 1

A demonstração será feita de maneira induktiva sobre a ordem da matriz $n \geq 1$. Se $n = 1$, então a matriz $A = (2)$ e, neste caso, $\det A = \det(2) = 2 = 1 + 1 = n + 1$.

Agora, como hipótese de indução, assumimos a validade da afirmação para todo $n \leq k$ e vamos provar que vale para $n = k + 1$.

De fato,

$$\begin{aligned}
 \det \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix}_{k+1 \times k+1} &= (-1)^{1+1} 2 \det \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix}_{k \times k} \\
 &\quad + (-1)^{1+2} 1 \det \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix}_{k \times k} \\
 &\stackrel{\text{hip}}{=} 2(k+1) \\
 &\quad - (-1)^{1+1} 1 \det \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix}_{k-1 \times k-1} \\
 &\stackrel{\text{hip}}{=} 2(k+1) - k \\
 &= (k+1) + 1,
 \end{aligned}$$

o que termina a prova por indução.

Solução 2

Nesta solução é apresentado o procedimento para o cálculo do determinante por escalonamento, isto é,

$$\begin{aligned}
 \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix} &\xrightarrow{L_2 - \frac{1}{2} L_1 \rightarrow L_2} \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix} \\
 &\xrightarrow{L_3 - \frac{2}{3} L_2 \rightarrow L_3} \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix} \dots \xrightarrow{L_{n-1} - \frac{n-2}{n-1} L_{n-2} \rightarrow L_{n-1}} \\
 &\quad \vdots \\
 \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{n-1}{n} & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix} &\xrightarrow{L_n - \frac{n-1}{n} L_{n-1} \rightarrow L_n} \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{n-1}{n} & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{n+1}{n} \end{pmatrix}.
 \end{aligned}$$

Logo,

$$\begin{aligned}
 & \det \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{n}{n-1} & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{n+1}{n} \end{pmatrix} \\
 & = 2 \frac{3}{2} \frac{4}{3} \cdots \frac{n}{n-1} \frac{n+1}{n} = \prod_{i=1}^n \frac{i+1}{i} = n+1.
 \end{aligned}$$