Advanced Digital Signal Processing (ADSP)

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Summary of last lecture

DoA estimation

- Maximize steered response power
- Using high resolution spectral estimation concepts
- Using time difference of arrival

Part C: Array signal processing

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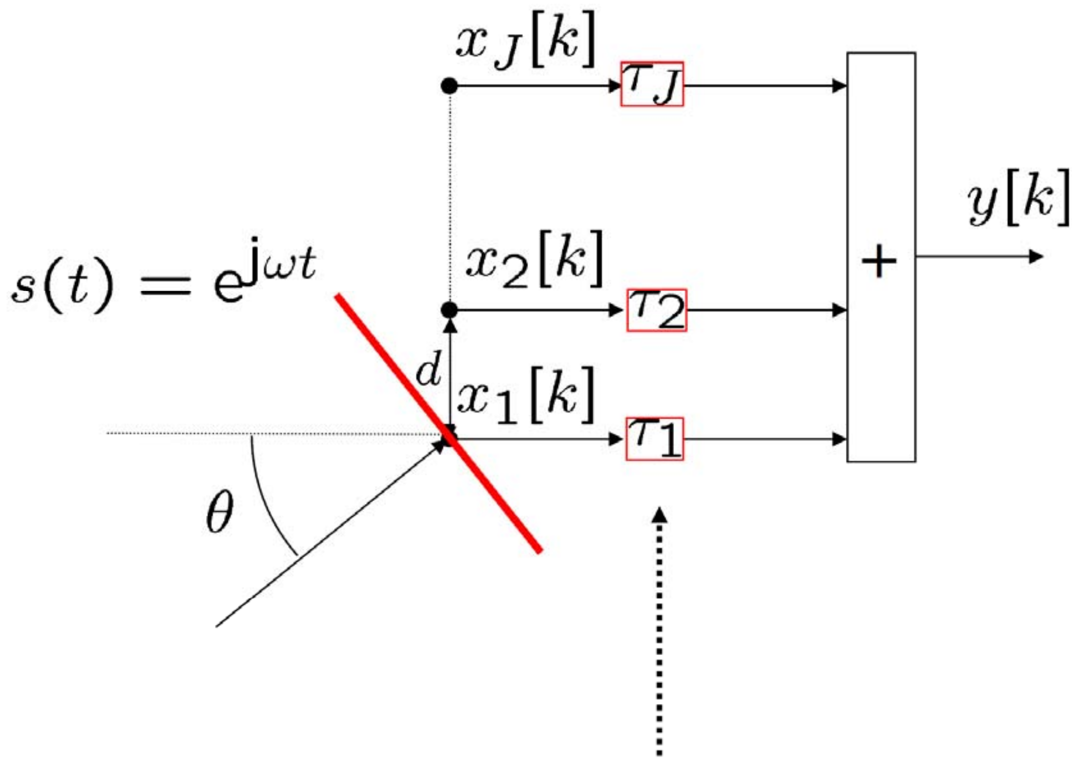

For ULA choose: $\tau_i = (i-1)\tau \leftrightarrow w_i^* = e^{j(i-1)\omega\tau}$

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Maximizing steered response power

- Steer the beamformer to a number of candidate directions θ.

- Output power:
  $P_y(\theta) = E\{|y[k]|^2\} = \underline{\mathbf{w}}^h \cdot \mathbf{R}_x \cdot \underline{\mathbf{w}}$

- Spatial spectrum: $P(\theta) = \frac{P_y(\theta)}{|\underline{\mathbf{w}}|^2} = \frac{\underline{\mathbf{a}}^h(\theta)\mathbf{R}_x\underline{\mathbf{a}}(\theta)}{J}$

- Clearly $P(\theta)$ attains its maximum when $\theta = \theta^*$. Thus, the peak in $P(\theta)$ is the DoA estimate

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Maximizing steered response power

- In practice, the search space is discretized depending on the desired accuracy
- Efficient search strategies may be implemented
- The autocorrelation matrix may be computed as $\hat{\mathbf{R}}_x = \frac{1}{T}\sum_{t=1}^{T} \underline{\mathbf{x}}[t] \cdot \underline{\mathbf{x}}^h[t]$

Multiple Signal Classification: MUSIC

- Consider J sensors and P sources, $P < J$
- $\underline{\mathbf{x}}[k] = \mathbf{A}\underline{\mathbf{s}}[k] + \underline{\mathbf{n}}[k]$
- $\mathbf{R}_x = E\{\underline{\mathbf{x}\mathbf{x}}^h\} = \mathbf{A}\mathbf{R}_s\mathbf{A}^h + \mathbf{R}_n$
- $\mathbf{A}\mathbf{R}_s\mathbf{A}^h$ is a $J \times J$ matrix of rank P, and has $J - P$ zero eigenvalues
- $\Rightarrow \mathbf{A}^h\underline{\mathbf{u}}_i = \underline{0}_{P\times 1}$
- Define $P_{SM}(\theta) = \frac{1}{\sum_{i=1}^{J-P} |\underline{\mathbf{a}}^h(\theta)\underline{\mathbf{u}}_i|^2} = \frac{1}{\underline{\mathbf{a}}^h(\theta)\mathbf{U}_n\mathbf{U}_n^h\underline{\mathbf{a}}(\theta)}$
- So the eigen vectors of $\mathbf{R}_x$ are also the eigen vectors of $\mathbf{A}\mathbf{R}_s\mathbf{A}^h$! (note that we assumed the noise to be spatially white)

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Multiple Signal Classification: MUSIC

- Compute/Estimate $\mathbf{R}_x$
- Perform EVD of $\mathbf{R}_x$; determine $\mathbf{U}_n$ as the matrix containing the eigenvectors corresponding to the $J-P$ smallest eigenvalues
- Evaluate pseudo-spectrum: $P_{SM}(\theta) = \frac{1}{\underline{\mathbf{a}}^h(\theta)\mathbf{U}_n\mathbf{U}_n^h\underline{\mathbf{a}}(\theta)}$
- Locate P sharpest peaks in $P_{SM}(\theta)$

Data dependent beamforming

- Optimum: Assume knowledge of array data statistics. Optimal with respect to a certain optimization criterion (e.g., min MSE, max SINR)

- Adaptive: Estimate the required statistics as data becomes available (when statistics are unknown or time-varying). The derivation is done assuming known statistics. These statistics are estimated on-line.

Optimum (data-dependent) beamforming

- Minimum mean squared error (MMSE)
- Multiple sidelobe canceller (MSC)
- Max. Signal-to-interference-plus-noise ratio (Max. SINR)
- Minimum variance distortionless response (MVDR)
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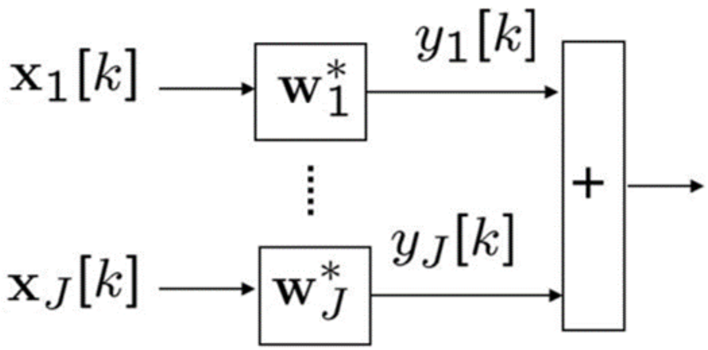
MMSE



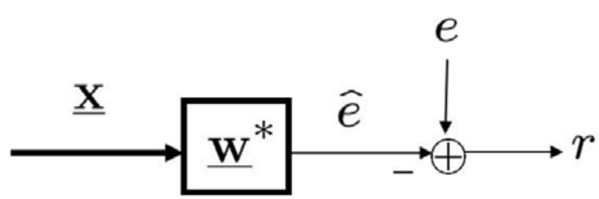
$$y[k] = \sum_{i=1}^{J} \mathbf{w}_i^h \mathbf{x}_i[k]$$

Short notation: $y[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{x}}[k]$

$$\begin{aligned} \underline{\mathbf{x}}[k] &= (\mathbf{x}_1[k], \cdots \mathbf{x}_J[k])^t \\ \underline{\mathbf{w}} &= (\mathbf{w}_1, \cdots, \mathbf{w}_J)^t \end{aligned}$$

$$\underline{\mathbf{x}}[k] \longrightarrow \boxed{\underline{\mathbf{w}}^*} \longrightarrow y[k]$$

MMSE



- Cost function: $\xi = E\{|r|^2\}$
- Solution: $\underline{\mathbf{w}}_{\mathrm{mse}} = \arg\min_{\underline{\mathbf{w}}} \xi = \mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{xe^*}$

$$\mathbf{R}_x = E\{\underline{\mathbf{x}} \cdot \underline{\mathbf{x}}^h\} \qquad \underline{\mathbf{r}}_{xe^*} = E\{\underline{\mathbf{x}} \cdot e^*\}$$

- Need to know $\mathbf{R}_x$ and $\underline{\mathbf{r}}_{xe^*}$ (from measurements)

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MMSE

Example: ULA, one source, narrowband, farfield

With $\underline{\mathbf{x}} = \underline{\mathbf{a}} \cdot s + \underline{\mathbf{n}}$ and $e = s$

$\Rightarrow$

$\mathbf{R}_x = E\{\underline{\mathbf{x}} \cdot \underline{\mathbf{x}}^h\} = (\underline{\mathbf{a}} \cdot \underline{\mathbf{a}}^h) \cdot \sigma_s^2 + \sigma_n^2 \cdot \mathbf{I}$ and

$\underline{\mathbf{r}}_{xe^*} = E\{\underline{\mathbf{x}} \cdot e^*\} = \underline{\mathbf{a}} \cdot \sigma_s^2$

$$\underline{\mathbf{w}}_{mse} = \mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{xe^*} = \left(\underline{\mathbf{a}} \cdot \underline{\mathbf{a}}^h \cdot \sigma_s^2 + \sigma_n^2 \cdot \mathbf{I}\right)^{-1} \cdot \underline{\mathbf{a}} \cdot \sigma_s^2$$

特征值之和
等于对角线之和

匹配时 $\beta=1$

$$= \left(\frac{(\sigma_s^2/\sigma_n^2)}{1 + J \cdot (\sigma_s^2/\sigma_n^2)}\right) \cdot \underline{\mathbf{a}} = \beta \cdot \underline{\mathbf{a}}$$

记：$R_x^{-1} \cdot \underline{r}_{xe} = \dfrac{\sigma_s^2}{1+J\frac{\sigma_s^2}{\sigma_n^2}} \underline{r}_{xe}$ 即记 R_x^{-1} 的特征值为 $\dfrac{\sigma_s^2}{1+J\frac{\sigma_s^2}{\sigma_n^2}}$

Moving forward…

$$\underline{\mathbf{w}}_{mse} = \mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{xe*} = \left(\underline{\mathbf{a}} \cdot \underline{\mathbf{a}}^h \cdot \sigma_s^2 + \sigma_n^2 \cdot \mathbf{I}\right)^{-1} \cdot \underline{\mathbf{a}} \cdot \sigma_s^2$$

$$= \left(\frac{(\sigma_s^2/\sigma_n^2)}{1 + J \cdot (\sigma_s^2/\sigma_n^2)}\right) \cdot \underline{\mathbf{a}} = \beta \cdot \underline{\mathbf{a}}$$

$\beta \simeq \frac{1}{J}$

σ_s^2

$$J_{min} = E\{|e|^2\} - \underline{\mathbf{r}}_{e*x}^h \mathbf{R}_x^{-1} \underline{\mathbf{r}}_{e*x} = \left(\frac{(\sigma_s^2/\sigma_n^2)}{1 + J \cdot (\sigma_s^2/\sigma_n^2)}\right) \cdot \sigma_n^2$$

$$= \beta \cdot \sigma_n^2$$

降低噪声功率（$\frac{1}{J}$）

$= \sigma_s^2 - \underline{a}^h \sigma_s^2 \beta \underline{a} = \sigma_s^2 (1 - \underline{a}^h \beta \underline{a})$

$= \sigma_s^2 (1 - \beta \underline{a}^h \underline{a}) = \sigma_s^2 (1 - \beta J)$

MMSE

Thus for ULA, one source, narrowband, farfield:

- $\underline{\mathbf{w}}_{mse} = \beta \cdot \underline{\mathbf{a}}$, which is equivalent to matched filter result, which maximizes SNR
- $J_{min} \approx \frac{1}{J} \cdot \sigma_n^2 \Rightarrow$ SNR impoved approx. by factor J

Conclusion MMSE

缺点：不知道 σ_s^2

+ Simple
+ Direction of desired signal may be unknown
- Must generate reference signal

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Optimum (data-dependent) beamforming

- Minimum mean squared error (MMSE)
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- Max. Signal-to-interference-plus-noise ratio (Max. SINR)
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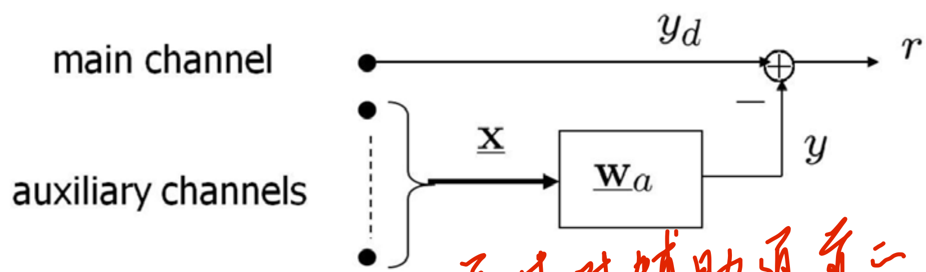
Multiple sidelobe canceller (MSC)



出噪声

通过对辅助通道二滤波，在主通道中减去

- Variation on MMSE: known reference
- Interference assumed to be present in both main and auxiliary channels.
- Desired signal present in main channel, but below noise level in auxiliary channels
- Use auxiliary channels to cancel interference in main channel

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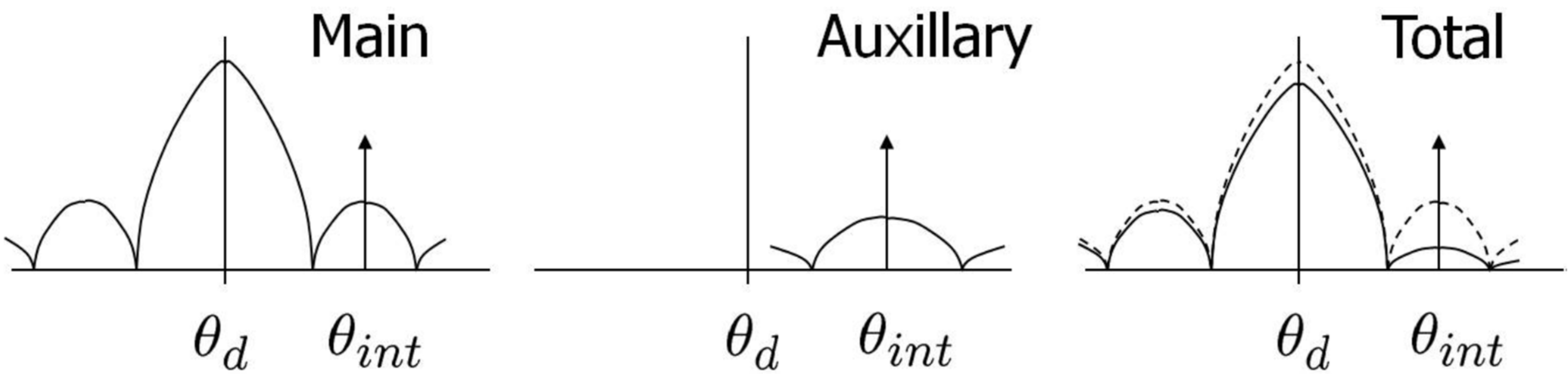
Multiple sidelobe canceller (MSC)



估计的噪声

$$\underline{\mathbf{w}}_o = arg \min_{\underline{\mathbf{w}}_a} \{ |y_d - \underline{\mathbf{w}}_a^h \cdot \underline{\mathbf{x}}|^2 \} \Rightarrow \boxed{\underline{\mathbf{w}}_o = \mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{xy_d^*}}$$

$$\boxed{P_{out} = \sigma_{y_d}^2 - \underline{\mathbf{r}}_{xy_d} \mathbf{R}_x^{-1} \underline{\mathbf{r}}_{xy_d}}$$

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Max. SINR

$a_u \; s_u$

$$\begin{aligned}\underline{\mathbf{x}} &= \underline{\mathbf{a}}s + (\underline{\mathbf{i}} + \underline{\mathbf{n}}) \\ &= \underline{\mathbf{x}}_d + \underline{\mathbf{x}}_u\end{aligned}$$

$$\mathbf{R}_x = \mathbf{R}_{x_d} + \mathbf{R}_{x_u} = \sigma_s^2 \underline{\mathbf{a}}\underline{\mathbf{a}}^h + (\mathbf{R}_i + \sigma_n^2 \mathbf{I})$$

BF output: $y = \underline{\mathbf{w}}^h \underline{\mathbf{x}} = \underbrace{\underline{\mathbf{w}}^h \underline{\mathbf{x}}_d}_{y_d} + \underbrace{\underline{\mathbf{w}}^h \underline{\mathbf{x}}_u}_{y_u}$

beamforing

$$\mathrm{SINR} \underset{\eta}{\doteq} \lambda = \frac{E|y_d|^2}{E|y_u|^2} = \frac{\underline{\mathbf{w}}^h \mathbf{R}_{x_d} \underline{\mathbf{w}}}{\underline{\mathbf{w}}^h \mathbf{R}_{x_u} \underline{\mathbf{w}}} \qquad \text{(SINR-1)}$$

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ADSP Max. SINR

$$\text{SINR} = \eta = \frac{|y_d|^2}{|y_u|^2} = \frac{\underline{\mathbf{w}}^h \mathbf{R}_{x_d} \underline{\mathbf{w}}}{\underline{\mathbf{w}}^h \mathbf{R}_{x_u} \underline{\mathbf{w}}} \quad \Rightarrow \quad \underline{\mathbf{w}}_{opt} = arg \max_{\underline{\mathbf{w}}} \{\eta\}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}\underline{\mathbf{w}}} = \frac{\mathbf{R}_{x_d}\underline{\mathbf{w}}(\underline{\mathbf{w}}^h\mathbf{R}_{x_u}\underline{\mathbf{w}}) - \mathbf{R}_{x_u}\underline{\mathbf{w}}(\underline{\mathbf{w}}^h\mathbf{R}_{x_d}\underline{\mathbf{w}})}{(\underline{\mathbf{w}}^h\mathbf{R}_{x_u}\underline{\mathbf{w}})^2} = 0$$

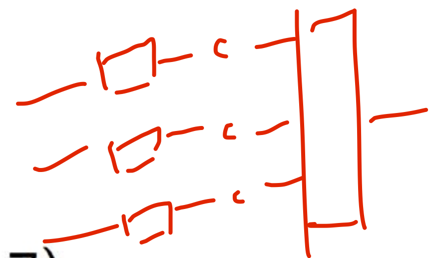
$$\Rightarrow \mathbf{R}_{x_d} \cdot \underline{\mathbf{w}} = \left(\frac{\underline{\mathbf{w}}^h \mathbf{R}_{x_d} \underline{\mathbf{w}}}{\underline{\mathbf{w}}^h \mathbf{R}_{x_u} \underline{\mathbf{w}}} \right) \mathbf{R}_{x_u} \cdot \underline{\mathbf{w}}$$

$\sigma_s^2 \underline{a}\,\underline{a}^h \underline{w}$

瑞利公式

ADSP Max. SINR

$$\underline{\mathbf{a}} = \underbrace{\frac{\underline{\mathbf{w}}_{opt}^{h}\underline{\mathbf{a}}}{\underline{\mathbf{w}}_{opt}^{h}\mathbf{R}_{x_u}\underline{\mathbf{w}}_{opt}}}_{1/c}\mathbf{R}_{x_u}\underline{\mathbf{w}}_{opt}, \quad \text{so that}$$



$$\underline{\mathbf{w}}_{opt} = c\mathbf{R}_{x_u}^{-1}\underline{\mathbf{a}} \qquad \text{(SINR-7)}$$

To obtain c, impose a constraint on $\underline{\mathbf{w}}$, e.g.,

$y = \underline{w}_{opt}^{h}\underline{a} \cdot s$

- Unit gain in look direction: $\underline{\mathbf{w}}_{opt}^{h}\underline{\mathbf{a}} = 1$, which using (SINR-7) yields $c = \frac{1}{\underline{\mathbf{a}}^{h}\mathbf{R}_{x_u}^{-1}\underline{\mathbf{a}}}$

如果仅满足SINR. c可任意

- Unit norm: $\underline{\mathbf{w}}_{opt}^{h}\underline{\mathbf{w}}_{opt} = 1$. Work out the corresp. solution.

$$\underline{\mathbf{w}}_{opt} = \left(\frac{\mathbf{R}_{x_u}^{-1}}{(\underline{\mathbf{a}}^{h}\mathbf{R}^{-2}\underline{\mathbf{a}})^{1/2}}\right)\cdot\underline{\mathbf{a}}$$

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Optimum (data-dependent) beamforming

- Minimum mean squared error (MMSE)
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噪声的方差

$y = s$, $y = \underbrace{w^H a}_{\to 1} s$

- Minimum variance distortionless response (MVDR)

信号无失真

- Linearly constrained minimum variance (LCMV)
- Generalized sidelobe canceller (GSC)

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MVDR derivation

只有一个源

- Signal model: $\underline{\mathbf{x}} = \underline{\mathbf{a}}s + \underline{\mathbf{n}}$
- Beamformer output: $y = \underline{\mathbf{w}}^h\underline{\mathbf{x}} = y_s + y_n$
- MVDR problem statement: Minimize $E|y_n|^2$ subject to $\underline{\mathbf{w}}^h\underline{\mathbf{a}} = 1$

$y_s = s$

- Noise cov: $E|y_n|^2 = E\{\underline{\mathbf{w}}^h\underline{\mathbf{n}}\underline{\mathbf{n}}^h\underline{\mathbf{w}}\} = \underline{\mathbf{w}}_h\mathbf{R}_n\underline{\mathbf{w}}$
- Apply the Lagrange multiplier method to solve the constrained optimization problem

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MVDR derivation

- Cost function: $\xi = \underline{\mathbf{w}}^h \mathbf{R}_n \underline{\mathbf{w}} + \lambda(\underline{\mathbf{w}}^h \underline{\mathbf{a}} - 1)$

$\frac{1}{2}$ 被忽略了

- $0 = \frac{d\xi}{d\underline{\mathbf{w}}} = \mathbf{R}_n \underline{\mathbf{w}} + \lambda \underline{a} \Rightarrow \underline{\mathbf{w}}_{MVDR} = -\lambda \mathbf{R}_n^{-1} \underline{\mathbf{a}}$

- Constraint
  $\underline{\mathbf{w}}^h \underline{\mathbf{a}} = 1 \Rightarrow (-\lambda \underline{\mathbf{a}}^h \mathbf{R}_n^{-1}) \underline{\mathbf{a}} = 1 \Rightarrow \lambda = -\frac{1}{\underline{\mathbf{a}}^h \mathbf{R}_n^{-1} \underline{\mathbf{a}}}$

要求 $\underline{a}$ 准确

$$\boxed{\underline{\mathbf{w}}_{MVDR} = \frac{\mathbf{R}_n^{-1} \underline{\mathbf{a}}}{\underline{\mathbf{a}}^h \mathbf{R}_n^{-1} \underline{\mathbf{a}}}}$$

信号无失真，则噪声的量最小时，相当于 x 的量最小

- Alternative: Min. $E|y|^2$ instead of $E|y_n|^2$ subject to same constraint $\Rightarrow \underline{\mathbf{w}}_{MVDR} = \frac{\mathbf{R}_x^{-1} \underline{\mathbf{a}}}{\underline{\mathbf{a}}^h \mathbf{R}_x^{-1} \underline{\mathbf{a}}}$

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LCMV

- MVDR sensitive to array perturbation and errors in DoA estimation
- LCMV: introduce additional constraints for more robustness

Principle behind LCMV

Design weight vector by minimizing average output power subject to M constraints on the filter response.

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LCMV

- Distortionless response constraint
  $\underline{\mathbf{w}}^h \underline{\mathbf{a}}(\theta_s) = 1$

- Directional constraints for robustness against steering errors
  $\underline{\mathbf{w}}^h \underline{\mathbf{a}}(\theta_s + \Delta\theta) = 1$
  $\underline{\mathbf{w}}^h \underline{\mathbf{a}}(\theta_s - \Delta\theta) = 1$

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LCMV

- Relevant if there are interfering signal(s) from known direction(s)
  $\underline{\mathbf{w}}^h \underline{\mathbf{a}}(\theta_i) = 0, \quad i = 1 \cdots M$

- For robustness against errors in interferer DoA estimation
  $\underline{\mathbf{w}}^h \underline{\mathbf{a}}(\theta_i + \Delta\theta) = 0$
  $\underline{\mathbf{w}}^h \underline{\mathbf{a}}(\theta_i - \Delta\theta) = 0$

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Linearly constrained minimum variance

- Same signal model as MVDR, $E|y|^2 = \underline{\mathbf{w}}^h \mathbf{R}_x \underline{\mathbf{w}}$
- M linearly independent constraints $\mathbf{C}^h \underline{\mathbf{w}} = \underline{\mathbf{r}}_d$, where C is a $J \times M$ constraint matrix, and $M < J$
- Constrained opt. problem: $\min_{\underline{\mathbf{w}}} \underline{\mathbf{w}}^h \mathbf{R}_x \underline{\mathbf{w}}$ subject to $\mathbf{C}^h \underline{\mathbf{w}} = \underline{\mathbf{r}}_d$
- As before apply Lagrange multiplier method

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LCMV

- Cost function: $\xi = \underline{\mathbf{w}}^h \mathbf{R}_x \underline{\mathbf{w}} + \underline{\lambda}(\mathbf{C}^h \underline{\mathbf{w}} - \underline{\mathbf{r}}_d)$. Note we now have a vector of Lagrange multipliers, one for each constraint
- As with the MVDR, set the derivative w.r.t $\underline{\mathbf{w}}$ to zero to obtain $\underline{\mathbf{w}}_{LCMV} = -\mathbf{R}_x^{-1}\mathbf{C}\underline{\lambda}$
- Use the constraint to solve for $\underline{\lambda}$. We get

$$\boxed{\underline{\mathbf{w}}_{LCMV} = \mathbf{R}_x^{-1}\mathbf{C}(\mathbf{C}^h\mathbf{R}_x^{-1}\mathbf{C})^{-1}\underline{\mathbf{r}}_d}$$