Part A: Stochastic signal processing

上海科技大学 ShanghaiTech University

ADSP

Power spectral density (PSD)

确定

Harmonic process: $x[n] = \sum_{k=1}^{M} A_k \cos(\omega_k n + \phi_k)$

两两独立　　　　　　保证E和n无关，满足wss

$\{\phi_k\}$ pairwise independent random variables uniform in $[0, 2\pi]$.

$$E\{x[n]\} = 0 \quad \forall n \qquad\qquad r_x[l] = \frac{1}{2}\sum_{k=1}^{M} A_k^2 \cos(\omega_k l) \quad -\infty < l < \infty$$

$$P_x[e^{j\omega}] = \sum_{k=-M}^{M} 2\pi\left(\frac{A_k^2}{4}\right)\delta(\omega - \omega_k) = \sum_{k=-M}^{M} \frac{\pi}{2} A_k^2\, \delta(\omega - \omega_k) \quad -\pi < \omega < \pi$$

单边谱

$\omega_k/(2\pi)$ rational number, spectral lines equidistant (harmonically related)

Example: $x[n] = \cos(0.1\pi n + \phi_1) + 2\sin(1.5n + \phi_2)$ 非谐波过程

ϕ_1 and ϕ_2 IID, uniform in $[0, 2\pi]$

① 计算E. ② 计算 $r_x[n_1, n_2] = \frac{1}{2}\cos(0.1\pi l) + 2\cos(1.5l)$

$P_x[e^{jw}] = \frac{1}{2}\pi[\delta(w-0.1\pi) + \delta(w+0.1\pi)] + 2\pi[\delta(w-1.5) + \delta(w+1.5)]$

ADSP

Power spectral density (PSD)

Cross-Power spectral density Zero-mean, joint stationary

$R_{xy}(e^{j\omega})$

$$\underbrace{P_{xy}(e^{j\omega})}_{\text{complex value}} = \sum_{l=-\infty}^{\infty} r_{xy}[l]e^{-j\omega l} \quad \circ\!-\!\circ \quad r_{xy}[l] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xy}(e^{j\omega})\, e^{j\omega l} d\omega$$

$$r_{xy}[l] = r_{yx}^{*}[-l] \qquad P_{xy}(e^{j\omega}) = P_{yx}^{*}(e^{j\omega})$$ 模相同，相位相反

Coherence function (normalized cross-PSD):

相干函数（频域）

相关系数（时域）

$\rho_{xy} = \frac{\gamma_{xy}}{\sigma_x \sigma_y}$, zero-mean $\Rightarrow \gamma_{xy} = r_{xy}$

$$C_{xy}(e^{j\omega}) \triangleq \frac{P_{xy}(e^{j\omega})}{\sqrt{P_x(e^{j\omega})}\sqrt{P_y(e^{j\omega})}}$$

$r_{xy}[l] = a + jb$

$r_{yx}[-l] = a - jb$

to proof: $P_{xy}(e^{j\omega}) = P_{yx}^{*}(e^{j\omega})$

Properties coherence function:

- $0 \le |C_{xy}(e^{j\omega})| \le 1$
- $C_{xy}(e^{j\omega}) = 1$: $x[n] = y[n]$
- $C_{xy}(e^{j\omega}) = 0$: $x[n]$ and $y[n]$ not correlated

$P_{xy} = \sum_{l=-}^{+} r_{xy}[l]\, e^{-j\omega l}$

$P_{yx} = \sum_{l=-\infty}^{+\infty} r_{yx}[l]\, e^{-j\omega l}$,

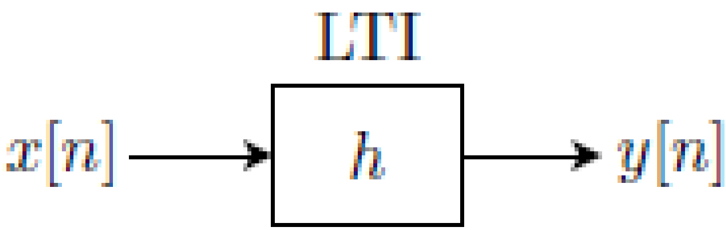
$P_{yx} = \sum_{s=+\infty}^{-\infty} \underbrace{r_{yx}[-s]}_{a-jb}\, e^{+j\omega s}$

$P_{yx}^{*} = \sum_{s=+}^{-} (a+jb)\, e^{j\omega s}$

$= \sum_{l=-\infty}^{+\infty} (a+jb) e^{-j\omega l}$

ADSP Part A: Stochastic signal processing

Linear system with stationary random input



$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

- $x[n]$: stationary (WSS)
- $h[n]$: BIBO stable

Bounded Input Bounded Output

⇨ $y[n]$: stationary

ADSP Linear system with stationary random input

$$x[n] \longrightarrow \overset{\text{LTI}}{\boxed{h}} \longrightarrow y[n]$$

$H(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} h[k]\, e^{-j\omega k}$

Time domain analysis

Output mean value

WSS

$$\mu_y = \sum_{k=-\infty}^{\infty} h[k] \underbrace{E\{x[n-k]\}}_{} = \mu_x \sum_{k=-\infty}^{\infty} h[k] = \mu_x \cdot \underbrace{H(e^{j0})}_{\text{直流增益}} \text{ = constant}$$

$H(e^{j0})$ is the DC gain of the system

Input output cross-correlation

$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$

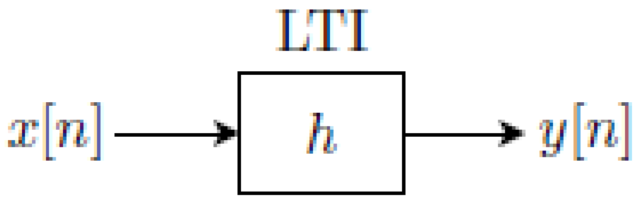
$$r_{xy}[l] = \mathrm{E}\{x[n+l]y^*[n]\} = \sum_{k=-\infty}^{\infty} h^*[k]\mathrm{E}\{x[n+l]x^*[n-k]\}$$

$$= \sum_{k=-\infty}^{\infty} h^*[k] r_x[l+k] = \sum_{m=-\infty}^{\infty} h^*[-m] r_x[l-m]$$

$$r_{xy}[l] = h^*[-l] \star r_x[l] \qquad\qquad r_{yx}[l] = h[l] \star r_x[l]$$

$r_{yx}^*[-l] = h^*[-l] * r_x^*[-l] = h^*[-l] * r_x[l] = r_{xy}[l]$

ADSP Linear system with stationary random input



Output auto-correlation

$$r_y[l] = \mathrm{E}\{y[n]y^*[n-l]\} = \sum_{k=-\infty}^{\infty} h[k]\mathrm{E}\{x[n-k]y^*[n-l]\}$$

$$= \sum_{k=-\infty}^{\infty} h[k]r_{xy}[l-k] = h[l] \star r_{xy}[l]$$

$$r_{xy}[l] = h^*[-l] \star r_x[l]$$

$$r_y[l] = h[l] \star h^*[-l] \star r_x[l] = r_h[l] \star r_x[l]$$

相当于系统的自相关

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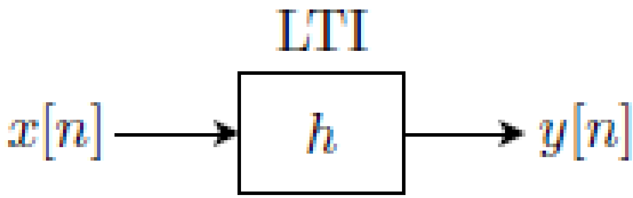
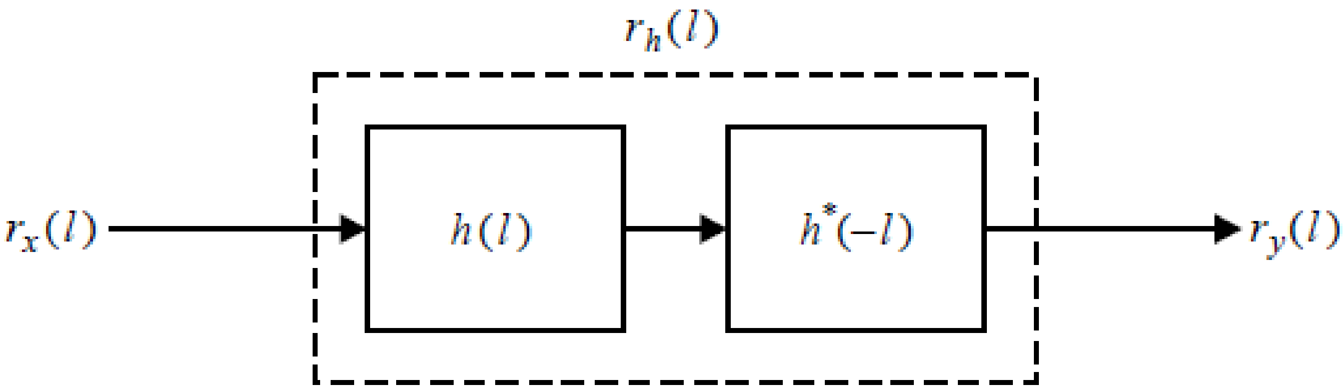
Linear system with stationary random input



Output auto-correlation

$$r_h[l] = h[l] \star h^*[-l] = \sum_{n=-\infty}^{\infty} h[n]h^*[\boldsymbol{n}+\boldsymbol{l}]$$



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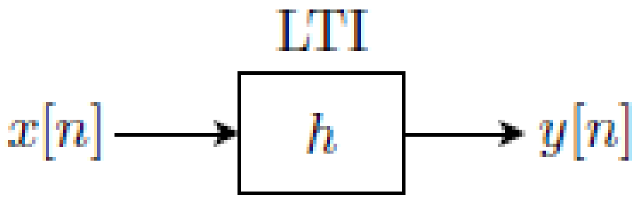
Linear system with stationary random input

Output power



平均功率

$$P_y = r_y[0] = r_h[l] \star r_x[l]_{l=0} = \sum_{k=-\infty}^{\infty} r_h[k] r_x[-k] = \sum_{k=-\infty}^{\infty} r_h[k] r_x[k]$$

$$\sum_{k=-\infty}^{\infty} r_h[k]\, r_x^*[k]$$

$$r_y[l] = r_h[l] \star r_x[l]$$

If $x[n]$ is real

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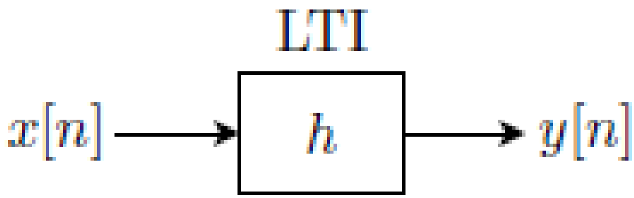
Linear system with stationary random input



Frequency domain

$$Z\{h[n]\} = H(z) \qquad Z\{h^*[-n]\} = H^*\left(\frac{1}{z^*}\right)$$

cross-power spectral density (cross - spectrum)

$$r_{xy}[l] = h^*[-l] \star r_x[l] \qquad R_{xy}(z) = H^*\left(\frac{1}{z^*}\right) R_x(z)$$

$$r_{yx}[l] = h[l] \star r_x[l] \qquad R_{yx}(z) = H(z) R_x(z)$$

$$r_y[l] = h[l] \star h^*[-l] \star r_x[l] \qquad R_y(z) = H(z) H^*\left(\frac{1}{z^*}\right) R_x(z)$$

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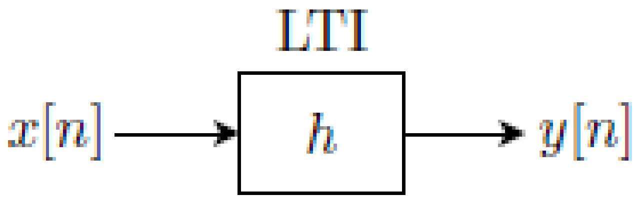
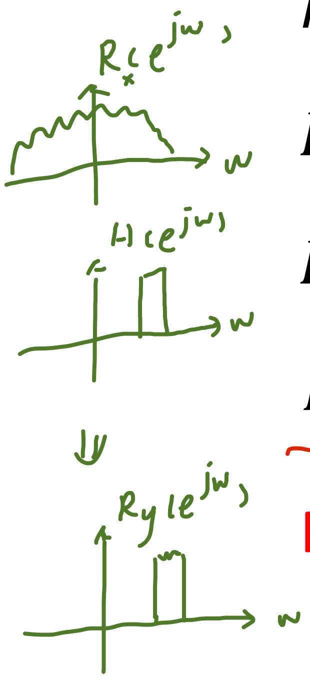
Linear system with stationary random input



Frequency domain



$h[n]$ stable, $z = e^{j\omega}$ within the ROC of $\mathrm{H}[z]$ and $\mathrm{H}[z^{-1}]$

$$R_{xy}(\mathrm{e}^{j\omega}) = H^*(e^{j\omega})R_x(e^{j\omega})$$

$H^*(\frac{1}{z^*})\Big|_{z=e^{jw}} = H^*(\frac{1}{e^{-jw}}) = H^*(e^{jw})$

$$R_{yx}(e^{j\omega}) = H(e^{j\omega})R_x(e^{j\omega})$$

from input PSD to output PSD.

$$R_y(z) = H(e^{j\omega})H^*(e^{j\omega})R_x\left(e^{j\omega}\right) = |H(e^{j\omega})|^2 R_x(e^{j\omega})$$

积分后变成功率（不是密度了）

Power: $P_y = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 P_x\left(e^{j\omega}\right)\mathrm{d}\omega$

类似帕斯瓦尔定理

$$= r_y(0) = \sum_{l=-\infty}^{\infty} r_h[l]\, r_x\,[l] = E\{|x[n]|^2\}$$

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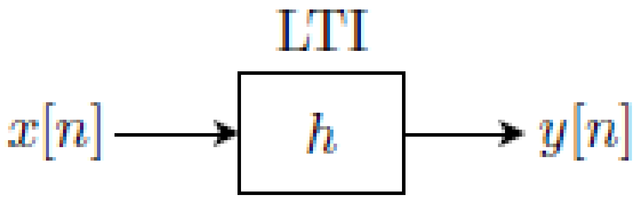
Linear system with stationary random input



Frequency domain

$$H(e^{j\omega}) = \begin{cases} 1 & \omega_c - \dfrac{\Delta\omega}{2} \le \omega \le \omega_c + \dfrac{\Delta\omega}{2} \\ 0 & \text{elsewhere} \end{cases}$$

$$P_y = \mathrm{E}\{|x[n]|^2\} = \frac{1}{2\pi}\int_{\omega_c-\frac{\Delta\omega}{2}}^{\omega_c+\frac{\Delta\omega}{2}} |H(e^{j\omega})|^2 P_x(e^{j\omega})\mathrm{d}\omega = \frac{\Delta\omega}{2\pi} P_x(e^{j\omega})|_{\omega=\omega_c}$$