ADSP

Stochastic process: stationary

Stationary of order N

$$f_x[x_1, \dots, x_N; n_1, \dots, n_N] = f_x[x_1, \dots, x_N; n_1 + k, \dots, n_N + k] \quad \forall k$$

Strict-sense stationary (SSS)

$x[n]$ is stationary for all orders $N=1, 2, \dots$.

An IID sequence is SSS.

Wide-sense stationary (WSS): stationary up to order 2

- Mean is a constant independent of n: $\mathrm{E}\{x[n]\} = \mu_x$
- Variance is a constant independent of n: $\mathrm{var}\{x[n]\} = \sigma_x^2$
- Autocorrelation depends only on l ($l = n_1 - n_2$)

$$r_x[n_1, n_2] = r_x[n_1 - n_2] = r_x\,[l]$$

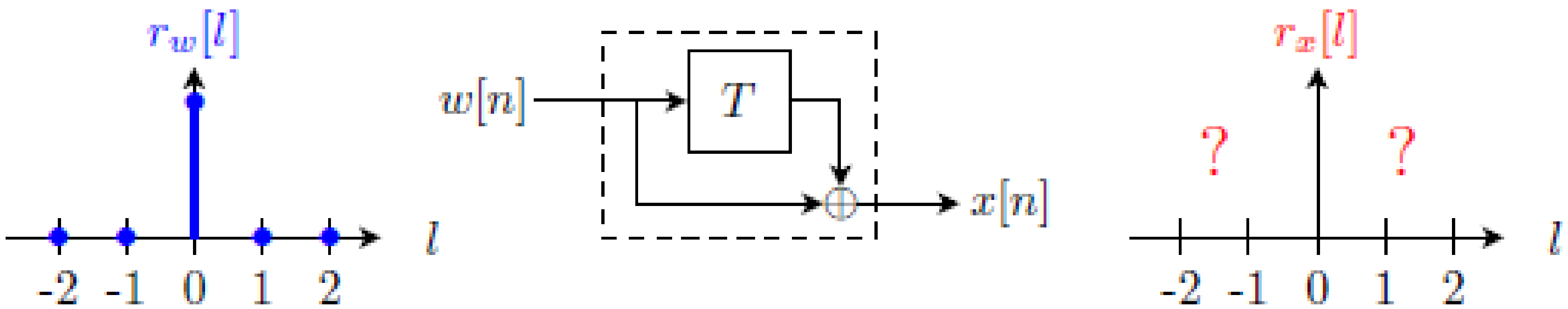
ADSP Stochastic process: stationary

Example:

Let $w[n]$ be a zero-mean, uncorrelated Gaussian random sequence with variance $\sigma^2[n] = 1$.

a. Characterize the random sequence $w[n]$.

b. Define $x[\mathrm{n}] = w[n] + w[n-1]$. Determine the mean and autocorrelation of $x[n]$. Also characterize $x[n]$.



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Stochastic process: stationary

Wide-sense stationary (WSS)

Mean : $\mu_x = E\{x[n]\}$

Variance : $\sigma_x^2 = E\{(x[n] - \mu_x)^2\} = E\{x^2[n]\} - \mu_x^2$

Autocorrelation : $r_x[l] = E\{x[n] \cdot x[n-l]\} = E\{x[n+l]x[n]\}$

Autocovariance : $\gamma_x[l] = E\{(x[n] - \mu_x) \cdot (x[n-l] - \mu_x)\}$
$= E\{(x[n+l] - \mu_x) \cdot (x[n] - \mu_x)\} = r_x[l] - \mu_x^2$

Properties $r_x[l]$

$$r_x[0] = \mathrm{E}\{x^2[n]\} = \sigma_x^2 + \mu_x^2 \geq 0$$

$$r_x[0] \geq r_x[l]$$

$\mathrm{E}\{x^2[n]\}$: Power of $x[n]$

$$r_x[l] = r_x[-l]$$

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Stochastic process: stationary

Joint wide-sense stationary (WSS)

$x[n]$ is WSS, $y[n]$ is WSS, and

Cross-correlation : $r_{xy}[l] = E\{x[n] \cdot y[n-l]\}$

Cross-covariance : $\gamma_{xy}[l] = E\{(x[n] - \mu_x) \cdot (y[n-l] - \mu_y)\}$
$= r_{xy}[l] - \mu_x \cdot \mu_y$

Normalized γ_{xy} : $\rho_{xy}[l] = \dfrac{\gamma_{xy}[l]}{\sigma_x \cdot \sigma_y}$

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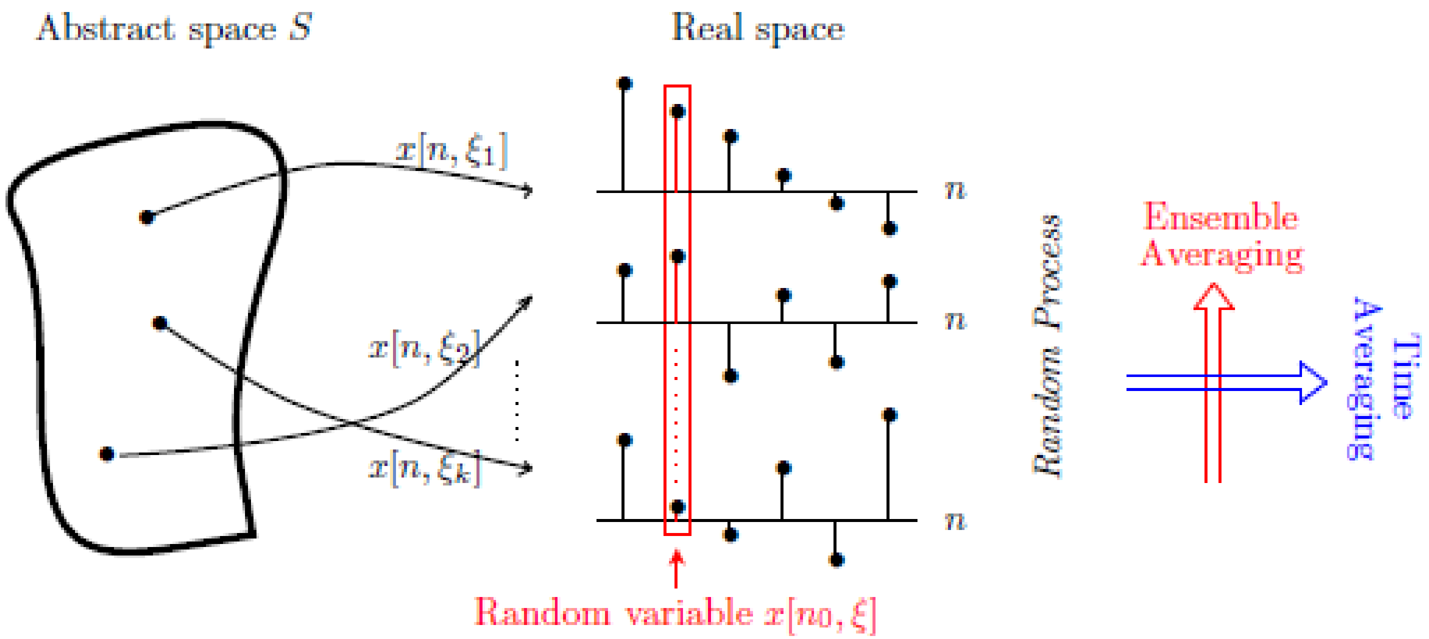
Stochastic process: ergodicity



- Ensemble averages: $\mathrm{E}\{\cdot\}$
- Time averages: $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (\cdot)$
- Ergodic: $\mathrm{E}\{\cdot\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (\cdot)$
- In practice: $\mathrm{E}\{\cdot\} = \frac{1}{2N+1} \sum_{n=-N}^{N} (\cdot)$

Stochastic process: ergodicity

Mean : $\hat{\mu}_x = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$

Variance : $\hat{\sigma}_x^2 = \frac{1}{N}\sum_{n=0}^{N-1}(x[n]-\hat{\mu}_x)^2 = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n] - \hat{\mu}_x^2$

Autocorrelation : $\hat{r}_x[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|} x[n]x[n+l]$ for $|l| \leq L-1$

Autocovariance : $\hat{\gamma}_x[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|} (x[n]-\hat{\mu}_x)(x[n+l]-\hat{\mu}_x)$

$$= \hat{r}_x[l] - \left(\frac{N-|l|}{N}\right)\hat{\mu}_x^2$$

- Ergodic is also WSS
- Only WSS can be ergodic
- WSS does not imply ergodicity of any kind

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Stochastic process: ergodicity

Joint ergodicity

"加号"

Cross-correlation : $\hat{r}_{xy}[l] = \frac{1}{N} \sum_{n=0}^{N-1-|l|} x[n]y[n+l]$

Cross-covariance : $\hat{\gamma}_{xy}[l] = \frac{1}{N} \sum_{n=0}^{N-1-|l|} (x[n]-\hat{\mu}_x)(y[n+l]-\hat{\mu}_y)$

$= \hat{r}_{xy}[l] - \hat{\mu}_x \cdot \hat{\mu}_y$

Normalized $\hat{\gamma}_{xy}$: $\hat{\rho}_{xy}[l] = \frac{\hat{\gamma}_{xy}[l]}{\hat{\sigma}_x \cdot \hat{\sigma}_y}$

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Part A: Stochastic signal processing

Contents

- Random variable
- Random vector
- Stochastic process
- Second order statistics
- Power spectrum estimation

ADSP

Power spectral density (PSD)

PSD and $r_x(l)$ DTFT pair

$x[n]$ stationary, $\mu_x = 0$

$$P_x(e^{j\omega}) = \sum_{l=-\infty}^{\infty} r_x[l] e^{-j\omega l} \quad \circ\!\!-\!\!\circ \quad r_x[l] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_x(e^{j\omega})\, e^{j\omega l} \mathrm{d}\omega$$

Example: $r_x[l] = a^{|l|}$ with $|a| < 1$. Calculate $P_x(e^{j\omega})$

Properties PSD :

- $P_x(e^{j\omega})$ real-valued periodic function of frequency (period 2π)
- $x[n]$ real, PSD even: $P_x(e^{j\omega}) = P_x(e^{-j\omega})$
- PSD nonnegative $P_x(e^{j\omega}) \geq 0$
- Average power: $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(e^{j\omega})\mathrm{d}\omega = r_x(0) = \mathrm{E}\{|x[n]|^2\} \geq 0$

ADSP

Power spectral density (PSD)

Periodicity : $X(e^{j\theta}) = X(e^{j\theta + l \cdot 2\pi}) \quad l \in \mathbb{N}$

Symmetry :

$x[n]$	$X(e^{j\theta})$
Real, even	Real, even
Real, odd	Imaginary, odd
Imaginary, even	Imaginary, even
Imaginary, odd	Real, odd

Properties PSD :

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ADSP

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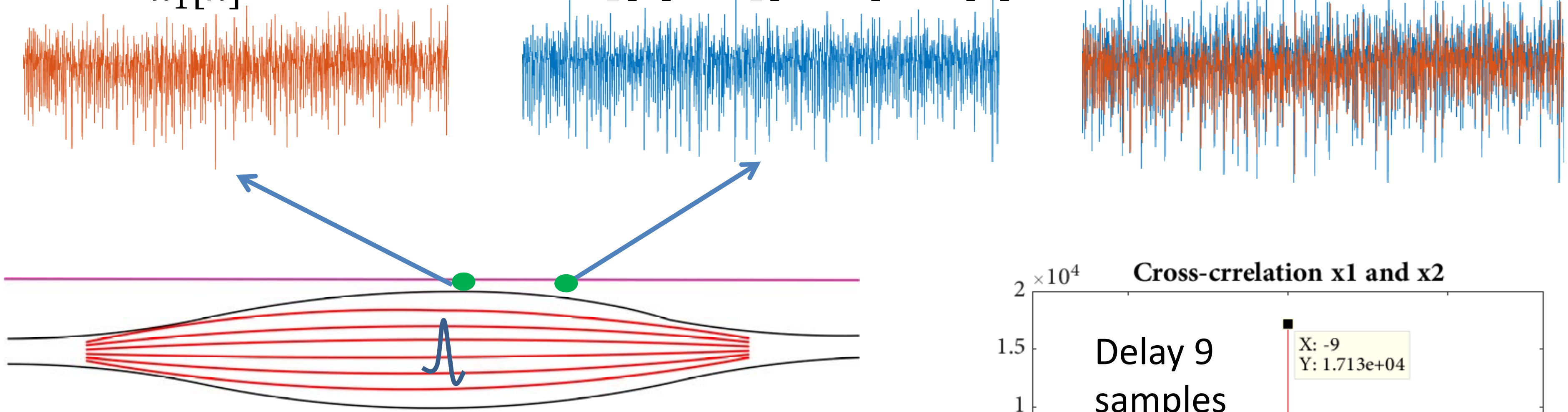
Estimate PSD:

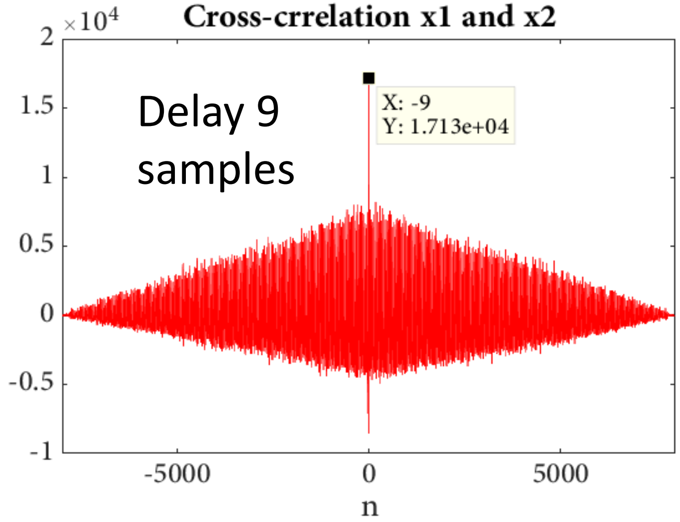
$\{\hat{r}_x[l]\}|_{l=-(L-1)}^{L-1}$ $\hat{P}_x(e^{j\omega}) = \mathrm{fft}(\hat{r}_x[l], M)$ with $M \geq 2L-1$

ADSP Course Introduction: motivation

Stochastic signal processing: EMG conduction velocity estimation

$x_1[n]$ $\quad$ $x_2[n] = x_1[n-\tau] + \omega[n]$





$$R_{x_1x_2}[n] = \begin{cases} \sum_{m=0}^{M-n-1} x_1[m+n]x_2[m], & n \geq 0 \\ R_{x_2x_1}[n], & n < 0 \end{cases}$$

ADSP

Power spectral density (PSD)

White noise: $w[n] \sim \mathrm{WN}(\mu_w, \sigma_w^2)$

$$\mathrm{E}\{w[n]\} = \mu_w \quad \text{and} \quad r_w[l] = \mathrm{E}(w[n]w[n-l]) = \sigma_w^2\, \delta(l)$$

$$P_w(e^{j\omega}) = \sigma_w^2 \qquad \forall\omega$$

White Gaussian noise:

- $w[n]$ is white and Gaussian, $w[n] \sim \mathrm{WGN}(\mu_w, \sigma_w^2)$

IID:

- $w[n]$ are independently and identically distributed with mean μ_w and variance σ_w^2, $w[n] \sim \mathrm{IID}(\mu_w, \sigma_w^2)$

ADSP

Power spectral density (PSD)

Harmonic process: $x[n] = \sum_{k=1}^{M} A_k \cos(\omega_k n + \phi_k)$

$\{\phi_k\}$ pairwise independent random variables uniform in $[0, 2\pi]$.

$$E\{x[n]\} = 0 \quad \forall n \qquad r_x[l] = \frac{1}{2}\sum_{k=1}^{M} A_k^2 \cos(\omega_k l) \quad -\infty < l < \infty$$

$$P_x[e^{j\omega}] = \sum_{k=-M}^{M} 2\pi\left(\frac{A_k^2}{4}\right) \delta(\omega - \omega_k) = \sum_{k=-M}^{M} \frac{\pi}{2} A_k^2 \delta(\omega - \omega_k) \quad -\pi < \omega < \pi$$

$\omega_k/(2\pi)$ rational number, spectral lines equidistant (harmonically related)

Example: $x[n] = \cos(0.1\pi n + \phi_1) + 2\sin(1.5n + \phi_2)$

ϕ_1 and ϕ_2 IID, uniform in $[0, 2\pi]$