Advanced Digital Signal Processing (ADSP)

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Part A: Stochastic signal processing

Contents

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Random Vectors

Definition: $x(\xi) = [x_1(\xi), x_2(\xi), \dots, x_M(\xi)]^T$, *T denotes the transpose.*

A mapping from an abstract probability space to a vector-valued real space $\mathbb{R}^M$

Distribution function:

$$F_{\mathrm{x}}(x_1, x_2, \dots, x_M) \triangleq \Pr\{x_1(\xi) \le x_1, x_2(\xi) \le x_2, \dots, x_M(\xi) \le x_M\}.$$

vector

$$F_{\mathrm{x}}(\mathrm{x}) \triangleq \Pr\{\mathrm{x}(\xi) \le \mathrm{x}\}.$$

Joint probability density function:

$$f_{\mathrm{x}}(\mathrm{x}) = \lim_{\substack{\Delta x_1 \to 0 \\ \vdots \\ \Delta x_M \to 0}} \frac{\Pr\{x_1 < x_1(\xi) \le x_1 + \Delta x_1, x_M < x_M(\xi) \le x_M + \Delta x_M\}}{\Delta x_1 \dots \Delta x_M}$$

$$\triangleq \frac{\partial}{\partial x_1} \dots \frac{\partial}{\partial x_M} F_{\mathrm{x}}(\mathrm{x}).$$

ADSP Random Vectors

Marginal density function

$$f_{x_j}(x_j) = \underbrace{\int \dots \int}_{M-1} f_{\mathrm{x}}(\mathrm{x})\mathrm{d}x_1 \dots \mathrm{d}x_{j-1}\mathrm{d}x_{j+1} \dots \mathrm{d}x_M$$

$$F_{\mathrm{x}}(\mathrm{x}) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_M} f_{\mathrm{x}}(\mathbf{v})\,\mathrm{d}v_1 \dots \mathrm{d}v_M = \int_{-\infty}^{\mathrm{x}} f_{\mathrm{x}}(\mathbf{v})\mathrm{d}\mathbf{v}$$

Independent

事件

$x_1(\xi)$ and $x_2(\xi)$ independent: $\{x_1(\xi) \leq x_1\}$ and $\{x_2(\xi) \leq x_2\}$ jointly independent.

$$\Pr\{x_1(\xi) \leq x_1, x_2(\xi) \leq x_2\} = \Pr\{x_1(\xi) \leq x_1\} \Pr\{x_2(\xi) \leq x_2\}$$

$$F_{x_1,x_2}(x_1, x_2) = F_{x_1}(x_1)\, F_{x_2}(x_2)$$

$$f_{x_1,x_2}(x_1, x_2) = f_{x_1}(x_1)\, f_{x_2}(x_2)$$

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Random Vectors: statistics

Mean Vector 均值向量

$$\mu_{\mathrm{x}} = \mathrm{E}\{\mathrm{x}(\xi)\} = \begin{bmatrix} \mathrm{E}\{x_1(\xi)\} \\ \vdots \\ \mathrm{E}\{x_M(\xi)\} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_M \end{bmatrix}$$

Correlation matrix 相关矩阵 (原点矩)

$$\mathrm{R}_{\mathrm{x}} \triangleq \mathrm{E}\{\mathrm{x}(\xi)\mathrm{x}^H(\xi)\} = \begin{bmatrix} r_{11} & \cdots & r_{1M} \\ \vdots & \ddots & \vdots \\ r_{M1} & \cdots & r_{MM} \end{bmatrix}$$

H denotes the conjugate transpose

Diagonal terms: $r_{ii} \triangleq \mathrm{E}\{|x_i(\xi)|^2\}, \quad i = 1, \ldots, M$ $\quad r_{x_i}^{(2)}$

Off-diagonal terms: $r_{ij} \triangleq \mathrm{E}\{x_i(\xi)x_j^*(\xi)\} = r_{ji}^*, \quad i \neq j$

r_{ij}: the statistical similarity between $x_i(\xi)$ and $x_j(\xi)$

R_{x} is conjugate symmetric $\quad \mathrm{R}_{\mathrm{x}} = \mathrm{R}_{\mathrm{x}}^H \quad$ Hermitian Matrix

ADSP Random Vectors: statistics

Auto-covariance 协方差矩阵（中心矩）

$$\Gamma_{\mathrm{x}} \triangleq \mathrm{E}\{[\mathrm{x}(\xi)-\mu_{\mathrm{x}}][\mathrm{x}(\xi)-\mu_{\mathrm{x}}]^H\} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1M} \\ \vdots & \ddots & \vdots \\ \gamma_{M1} & \cdots & \gamma_{MM} \end{bmatrix}$$

Diagonal terms:

$$\gamma_{ii} \triangleq \mathrm{E}\{|x_i(\xi)-\mu_i|^2\} = \mathrm{E}\{|x_i(\xi)|^2\} - \mu_i^2 \quad i = 1, \dots, M \qquad \sigma_{x_i}^{(2)} \text{ 方差}$$

Off-diagonal terms:

$$\gamma_{ij} \triangleq \mathrm{E}\{[x_i(\xi)-\mu_i][x_j(\xi)-\mu_j]^*\} = \mathrm{E}\{x_i(\xi)x_j^*(\xi)\} - \mu_i\mu_j^* = \gamma_{ji}^*, \quad i \neq j$$

γ_{ij}: the covariance between $x_i(\xi)$ and $x_j(\xi)$

Γ_{x} is also a Hermitian matrix

原点矩

中心矩 $\gamma_{ij} = \mathrm{E}\{x_i(\xi)x_j^*(\xi)\} - \mu_i\mu_j^* = r_{ij} - \mu_i\mu_j^*$

协方差矩阵与自相关矩阵间的关系

$$\Gamma_{\mathrm{x}} \triangleq \mathrm{E}\{[\mathrm{x}(\xi)-\mu_{\mathrm{x}}][\mathrm{x}(\xi)-\mu_{\mathrm{x}}]^H\} = \mathrm{R}_{\mathrm{x}} - \mu_{\mathrm{x}}\mu_{\mathrm{x}}^H$$

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Random Vectors: statistics

Proof: $|\gamma_{ij}| \le |\sigma_{x_i}||\sigma_{x_j}|$

to proof: $\gamma_{ij}^2 \le \sigma_{x_i}^2 \sigma_{x_j}^2$

$E^2[\underbrace{(x_i - u_i)}_{X}\underbrace{(x_j - u_j)}_{Y}] \le E[(x_i - u_i)^2]\,E[(x_j - u_j)^2]$

$E^2[XY] \le E[X^2]\,E[Y^2]$ Cauchy Schwartz Inequality

相关系数

Correlation coefficient between $x_i(\xi)$ and $x_j(\xi)$

we have $E[(aX+Y)^2] \ge 0$ for all $a \ne 0$;

$\Rightarrow E[(a^2X^2 + 2aXY + Y^2)] \ge 0$

$\Rightarrow a^2E[X^2] + 2aE[XY] + E[Y^2] \ge 0$

then $\Delta = 4E^2[XY] - 4E[X^2]E[Y^2] \le 0$

$\Rightarrow E^2[XY] \le E[X^2]E[Y^2]$

$$\rho_{ij} \triangleq \frac{\gamma_{ij}}{\sigma_{x_i}\sigma_{x_j}}$$

作归一化

ρ_{ij} measures the statistical similarity between $x_i(\xi)$ and $x_j(\xi)$

$$|\rho_{ij}| \le 1$$

Cauchy Schwartz Inequality

- $\rho_{ij} = 1, x_i(\xi)$ and $x_j(\xi)$ are perfectly correlated;
- $\rho_{ij} = 0$, uncorrelated;
- $\rho_{ij} = -1$, negatively correlated.

Linear correlation

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Random Vectors: statistics

Uncorrelated vs. independent

Uncorrelated $$\rho_{ij} \triangleq \frac{\gamma_{ij}}{\sigma_{x_i}\sigma_{x_j}} = \frac{\mathrm{E}\{x_i(\xi)x_j^*(\xi)\} - \mu_i\mu_j^*}{\sigma_{x_i}\sigma_{x_j}} = 0$$

Independent $$f_{x_1,x_2}(x_1, x_2) = f_{x_1}(x_1)\, f_{x_2}(x_2)$$

$$\Downarrow$$

$$\mathrm{E}\{x_i(\xi)x_j^*(\xi)\} = \mathrm{E}\{x_i(\xi)\}\,\mathrm{E}\{x_j^*(\xi)\} = \mu_i\mu_j^*$$

Independent $\Rightarrow$ Uncorrelated

$$E\{x_i(\xi)x_j^*(\xi)\} = \iint_{-\infty}^{\infty} x_i \cdot x_j\, f_{ij}(x_i, x_j)\, dx_i\, dx_j = \int_{-\infty}^{+\infty} x_i f(x_i)\, dx_i \int_{-\infty}^{+\infty} x_j f(x_j)\, dx_j$$

Uncorrelated $\not\Rightarrow$ Independent

有线性含义

Uncorrelated + Gaussian $\Rightarrow$ Independent

ADSP Random Vectors: statistics

① 不相关 ⇔ 中心矩为0
② 正交 ⇔ 原点矩为0

Uncorrelated vs. orthogonal

Uncorrelated $\rho_{ij} \triangleq \frac{\gamma_{ij}}{\sigma_{x_i}\sigma_{x_j}} = \frac{\mathrm{E}\{[x_i(\xi)-\mu_i][x_j(\xi)-\mu_j]^*\}}{\sigma_{x_i}\sigma_{x_j}} = 0$

Orthogonal $\mathrm{E}\{x_i(\xi)x_j^*(\xi)\} = 0$ 原点矩

{Uncorrelated} + $\{\mu_i = 0$ or $\mu_j = 0\}$ ⟹ Orthogonal

{ Orthogonal } + $\{\mu_i = 0$ or $\mu_j = 0\}$ ⟹ Uncorrelated

Example: $x_i = w-1$, $x_j = w+1$, $w \sim N(u, \delta^2)$

$\gamma_{ij} = E[x_i x_j] - u_i u_j = E(w^2-1) - u^2 = E[w^2] - u^2 - 1 = \delta^2 + u^2 - u^2 - 1 = \delta^2 - 1$

$E[x_i x_j] = E[w^2-1] = E[w^2] - 1 = \delta^2 + u^2 - 1$

① if uncorrelated: $\gamma_{ij} = 0 \Rightarrow \delta^2 = 1$ and if $u_i = u_j = u = 0$, we get $E(x_i x_j) = 0$ $\Rightarrow$ Orthogonal

② if Orthogonal : $E[x_i x_j] = \delta^2 + u^2 - 1 = 0$ and if $u_i = u_j = u = 0$, we get $\delta^2 - 1 = 0 \Rightarrow \gamma_{ij} = 0$

$\Rightarrow$ Uncorrelated

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Part A: Stochastic signal processing

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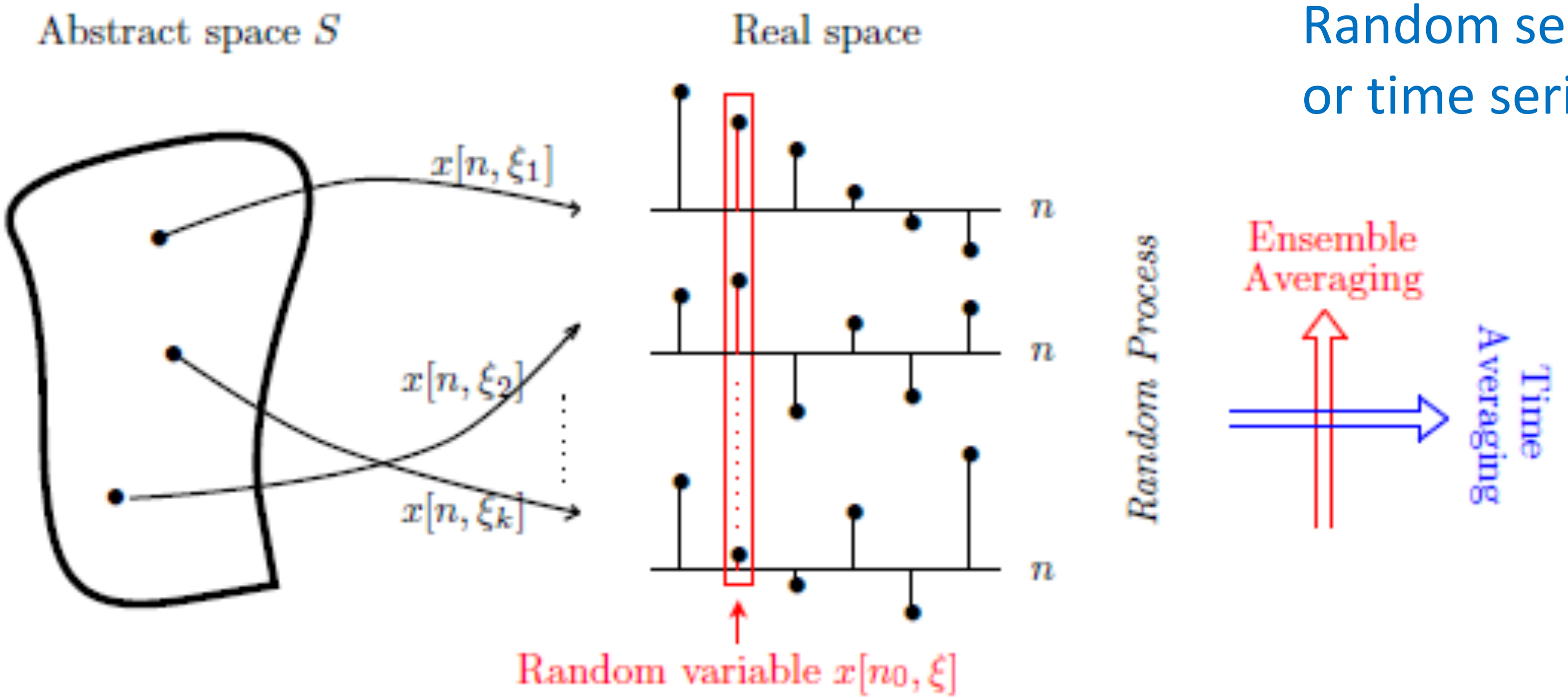
Discrete-time stochastic process

Random sequence or time series



Ensemble: set of all possible sequences $\{x(n,\xi)\}$

- $x(n,\xi)$: a random variable if n is $fixed$ and ξ is a variable.
- $x(n,\xi)$: a sample sequence if ξ is $fixed$ and n is a variable.
- $x(n,\xi)$: a number if both n and ξ are $fixed$.
- $x(n,\xi)$: a stochastic process if both n and ξ are variables.

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Discrete-time stochastic process

k^{th}-order distribution function

$$F_x(x_1, \ldots, x_k; n_1, \ldots, n_k) = \Pr\{x(n_1) \le x_1, \ldots, x(n_k) \le x_k\}$$

k^{th}-order pdf

$x\,(n)$ is assumed to be real-valued

$$f_x(x_1, \ldots, x_k; n_1, \ldots, n_k) \triangleq \frac{\partial F_x(x_1, \ldots, x_k; n_1, \ldots, n_k)}{\partial x_1 \ldots \partial x_{\cancel{M}\,k}} \qquad k \ge 1$$

Notes:

- Probabilistic description requires a lot of information
- Use statistical description in practice
- Skip ξ, thus x[n] for random process or single realization

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Part A: Stochastic signal processing

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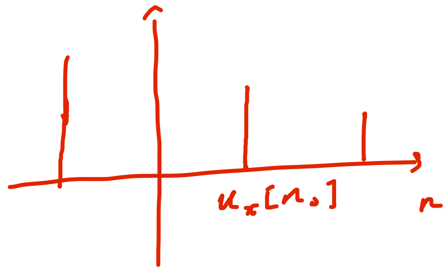
Stochastic process: 2nd order statistics



Statistical properties of stochastic process $x[n]$ **at time n**

Mean : $\mu_x[n] = E\{x[n]\}$

Variance : $\sigma_x^2[n] = E\{|x[n] - \mu_x[n]|^2\} = E\{|x[n]|^2\} - E\{|\mu_x[n]|^2\}$

Autocorrelation : $r_x[n_1, n_2] = E\{x[n_1] \cdot x^*[n_2]\}$

Autocovariance : $\gamma_x[n_1, n_2] = E\{(x[n_1] - \mu_x[n_1]) \cdot (x[n_2] - \mu_x[n_2])^*\}$

自协方差（来自同一个随机过程） $= r_x[n_1, n_2] - \mu_x[n_1] \cdot \mu_x^*[n_2]$

Random variable $x[\xi]$

Mean : $\mu_x \triangleq E\{x(\xi)\} = \int_{-\infty}^{\infty} x\, f_x(x) \mathrm{d}x$

Variance : ${\sigma_x}^2 \triangleq {\gamma_x}^{(2)} = E\{[x(\xi) - \mu_x]^2\}$

Correlation : $r_{ij} \triangleq \mathrm{E}\{x_i(\xi) x_j^*(\xi)\} = r_{ji}^*, \quad i \neq j$

Covariance : $\gamma_{ij} \triangleq \mathrm{E}\{[x_i(\xi) - \mu_i][x_j(\xi) - \mu_j]^*$

$= \mathrm{E}\{x_i(\xi) x_j^*(\xi)\} - \mu_i \mu_j^*, \quad i \neq j$

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Stochastic process: 2^{nd} order statistics

Statistical relation between two of stochastic process $x[n]$ and $y[n]$

互相关

Cross-correlation : $r_{xy}[n_1, n_2] = E\{x[n_1] \cdot y^*[n_2]\}$

Cross-covariance : $\gamma_{xy}[n_1, n_2] = E\{(x[n_1] - \mu_x[n_1]) \cdot (y[n_2] - \mu_y[n_2])^*\}$

互协方差

$$= r_{xy}[n_1, n_2] - \mu_x[n_1] \cdot \mu_y^*[n_2]$$

（来自不同的随机过程）

Normalized γ_{xy} : $\rho_{xy}[n_1, n_2] = \dfrac{\gamma_{xy}[n_1, n_2]}{\sigma_x[n_1] \cdot \sigma_y[n_2]}$

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Stochastic process: some definitions

Independent

If $f_x[x_1, \dots, x_k; n_1, \dots, n_k] = f_1[x_1; n_1] \dots f_k[x_k; n_k] \quad \forall k, n_i, i = 1, \dots, k$

$x[n]$ is a sequence of independent random variables.

随机过程的独立 指 任一个随机变量独立

IID (Independent and Identically Distributed) 独立同分布

If $f_1[x_1; n_1] = f_2\,[x_2; n_2] = \cdots = f_k[x_k; n_k] \quad \forall k, n_i, i = 1, \dots, k$

$x[n]$ is a IID random sequence.

$\gamma_x[n_1, n_2] = E[(x[n_1] - u_1)(x[n_2] - u_2)] = E[x[n_1]x[n_2]] - u_1 u_2$

ADSP Stochastic process: some definitions

Uncorrelated

任意二 n_1, n_2

If $\gamma_x[n_1, n_2] = \begin{cases} \sigma_x^2[n_1] & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases} = \sigma_x^2[n_1]\delta(n_1 - n_2)$

$x[n]$ is a sequence of uncorrelated random variables.

$$\gamma_x[n_1, n_2] = r_x[n_1, n_2] - \mu_x[n_1] \cdot \mu_x^*[n_2]$$

$$r_x[n_1, n_2] = \begin{cases} \sigma_x^2[n_1] + |\mu_x[n_1]|^2 & n_1 = n_2 \\ \mu_x[n_1] \cdot \mu_x^*[n_2] & n_1 \neq n_2 \end{cases}$$

$\gamma_x[n_1, n_2] = 0$

Orthogonal

$$r_x[n_1, n_2] = \begin{cases} \sigma_x^2[n_1] + |\mu_x[n_1]|^2 & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases} = \mathrm{E}\{|x[n_1]|^2\}\,\delta(n_1 - n_2)$$

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Stochastic process: stationary

Stationary of order N N阶平稳

$$f_x[x_1, \dots, x_N; n_1, \dots, n_N] = f_x[x_1, \dots, x_N; n_1 + k, \dots, n_N + k] \quad \forall k$$

Strict-sense stationary (SSS) 严平稳

$x[n]$ is stationary for all orders N=*1, 2,* …. $\forall k$

An IID sequence is SSS.

Wide-sense stationary (WSS): stationary up to order 2 宽平稳（可用统计量来判断）

二阶统计量与n无关

- Mean is a constant independent of n: $\mathrm{E}\{x(n)\} = \mu_x$
- Variance is a constant independent of n: $\mathrm{var}\{x(n)\} = \sigma_x^2$
- Autocorrelation depends only on l ($l = n_1 - n_2$) 和时间起点无关（和时间间隔有关）

$$r_x(n_1, n_2) = r_x(n_1 - n_2) = r_x\ (l)$$

同一随机过程的不同随机变量

ADSP Stochastic process: stationary

Wide-sense stationary (WSS)

对宽平稳而言. 二阶统计量只与 l 有关

Mean : $\mu_x = E\{x[n]\}$

Variance : $\sigma_x^2 = E\{(x[n]-\mu_x)^2\} = E\{x^2[n]\} - \mu_x^2$

Autocorrelation : $r_x[l] = E\{x[n]\cdot x[n-l]\} = E\{x[n+l]x[n]\}$

Notes : Power $E\{x^2[n]\} = r_x[0] \geq 0$ and $r_x[0] \geq r_x[l] \quad \forall l$

$Symmetry r_x[l] = r_x[-l]$

Autocovariance : $\gamma_x[l] = E\{(x[n]-\mu_x)\cdot(x[n-l]-\mu_x)\}$

$$= E\{(x[n+l]-\mu_x)\cdot(x[n]-\mu_x)\} = r_x[l] - \mu_x^2$$

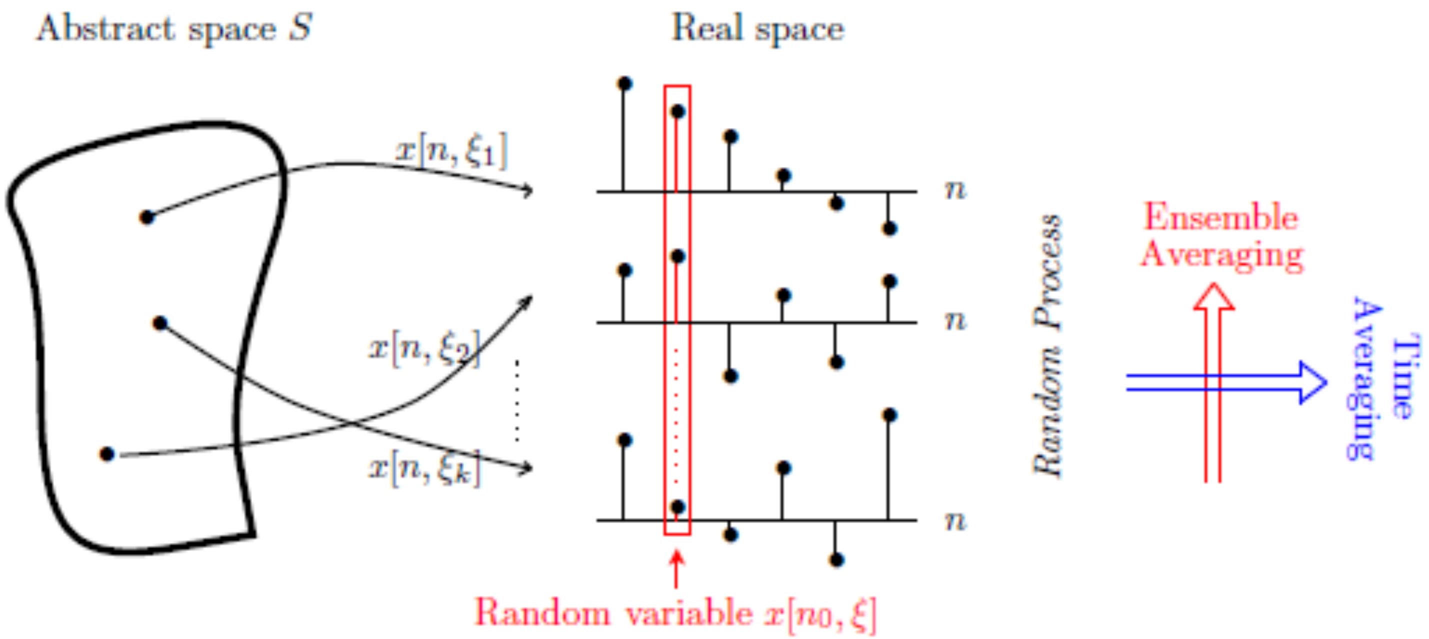
Cross-correlation : $r_{xy}[l] = E\{x[n]\cdot y[n-l]\} \neq E\{x[n+l]\cdot y[n]\}$

Cross-covariance : $\gamma_{xy}[l] = E\{(x[n]-\mu_x)\cdot(y[n-l]-\mu_y)\}$

$$= r_{xy}[l] - \mu_x \cdot \mu_y$$

Normalized γ_{xy} : $\rho_{xy}[l] = \dfrac{\gamma_{xy}[l]}{\sigma_x \cdot \sigma_y}$

ADSP Stochastic process: ergodicity 各态历经



- Ensemble averages: $\mathrm{E}\{\cdot\}$
- Time averages: $\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}(\cdot)$
- Ergodic: $\mathrm{E}\{\cdot\}=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}(\cdot)$
- In practice: $\mathrm{E}\{\cdot\}=\frac{1}{2N+1}\sum_{n=-N}^{N}(\cdot)$

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Stochastic process: ergodicity

Mean : $\hat{\mu}_x = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$

Variance : $\hat{\sigma}_x^2 = \frac{1}{N}\sum_{n=0}^{N-1}(x[n]-\hat{\mu}_x)^2 = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n] - \hat{\mu}_x^2$

Autocorrelation : $\hat{r}_x[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|} x[n]x[n+l]$ for $|l| \leq L-1$

Autocovariance : $\hat{\gamma}_x[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|}(x[n]-\hat{\mu}_x)(x[n+l]-\hat{\mu}_x)$

$$= \hat{r}_x[l] - \left(\frac{N-|l|}{N}\right)\hat{\mu}_x^2$$

- Ergodic is also WSS
- Only WSS can be ergodic

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Stochastic process: ergodicity

Cross-correlation : $\hat{r}_{xy}[l] = \frac{1}{N} \sum_{n=0}^{N-1-|l|} x[n]y[n+l]$

Cross-covariance : $\hat{\gamma}_{xy}[l] = \frac{1}{N} \sum_{n=0}^{N-1-|l|} (x[n] - \hat{\mu}_x)(y[n+l] - \hat{\mu}_y)$

$$= \hat{r}_{xy}[l] - \hat{\mu}_x \cdot \hat{\mu}_y$$

Normalized $\hat{\gamma}_{xy}$: $\hat{\rho}_{xy}[l] = \frac{\hat{\gamma}_{xy}[l]}{\hat{\sigma}_x \cdot \hat{\sigma}_y}$

Note: Equations $\hat{r}[l], \cdots, \hat{\rho}_{xy}[l]$ are biased

THANKS!

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