Beampattern of ULA

main properties

ULA Beampattern

Assumptions:

- Single source $s(t) = \mathrm{e}^{\mathrm{j}\omega t}$
- Frequency relations: $\omega = 2\pi \cdot f = 2\pi \cdot c/\lambda$
- Wavenumber λ
- Speed of propagation: c ($\approx$ 343 [m/sec])
- Directional Of Arrival (DOA): θ
- Far field $\Rightarrow$ Plane wave
- ULA with distance d between sensors
- J omnidirectional sensors
- Array aperture size: $L = J \cdot d$
- No noise, no interferences: $\underline{\mathbf{x}}[k] = \underline{\mathbf{a}}(\theta) \cdot s[k]$
- ASP unit: Single complex weight for each sensor

因为延时的原因（时延等于相移）

[No Title]

ADSP

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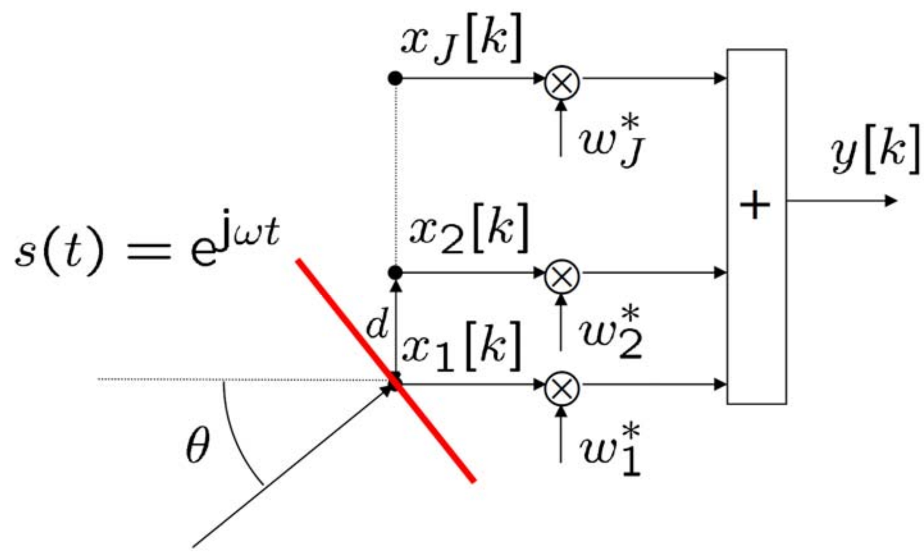
空间



one source

$$\Rightarrow \; y[k] = \sum_{i=1}^{J} w_i^* x_i[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{x}}[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta) \cdot s[k]$$

$$\text{with: } (\underline{\mathbf{a}}(\theta))_i = a_i(\theta) = e^{-j2\pi(i-1)\frac{d \sin(\theta)}{\lambda}}$$

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Thus: $y[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{x}}[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta) \cdot s[k] = r(\theta) \cdot s[k]$

Array response: $r(\theta) = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta)$ 空间

Other names: *angular repsonse* or *directivity pattern*

Array beam pattern: $B(\theta) = \frac{1}{J^2} \cdot |r(\theta)|^2$

$\underline{a}(\theta) = e^{-j2\pi \cdot (i-1) \frac{d \sin\theta}{\lambda}}$

$= e^{-j\omega(i-1) \frac{d \sin\theta}{c}}$

$=$ e

Comparison with FIR: 做傅利叶

Frequency response:

$$W(\omega) = \sum_{i=1}^{J} w_i e^{-j\omega(i-1)T} = \underline{\mathbf{w}}^t \cdot \underline{\mathbf{a}}(\omega)$$

角数

$\omega = \frac{2\pi}{\lambda} \cdot c = 2\pi f$

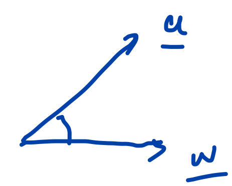
with $a_i(\omega) = (\underline{\mathbf{a}}(\omega))_i = e^{-j\omega \cdot (i-1) \cdot T}$

$\Rightarrow$ Depending on ω, **not** on θ !

ULA Beampattern

Notes:

- Array response vector: (noise free) response to unit-amplitude plane wave from direction θ
- Nonideal sensor characteristics can be incorporated
- Weights effect both temporal and spatial response
- Vector space interpretation:
  Angle between $\underline{\mathbf{w}}$ and $\underline{\mathbf{a}}$ determine response



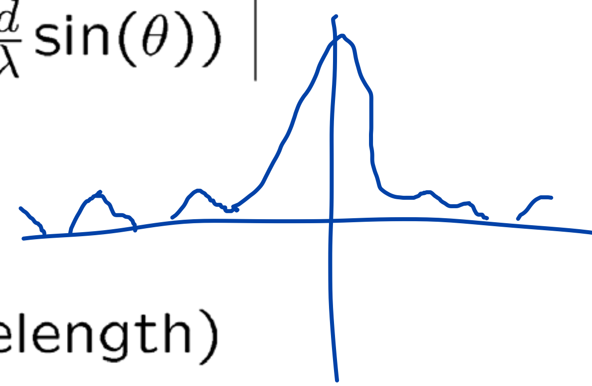
- To evaluate beampattern: choose all weights equal

$$\underline{\mathbf{w}} = (1, \cdots, 1)^t$$

ULA Beampattern

$$B(\theta) = \frac{1}{J^2}|\underline{\mathbf{1}}^t \cdot \underline{\mathbf{a}}(\theta)|^2 = \frac{1}{J^2}\left|\sum_{i=1}^{J} \mathrm{e}^{-\mathrm{j}2\pi(i-1)\frac{d}{\lambda}\sin(\theta)}\right|^2$$

$$= \frac{1}{J^2}\left|\frac{1-\mathrm{e}^{-\mathrm{j}J2\pi\frac{d}{\lambda}\sin(\theta)}}{1-\mathrm{e}^{-\mathrm{j}2\pi\frac{d}{\lambda}\sin(\theta)}}\right|^2 = \frac{1}{J^2}\left|\frac{\sin(J\pi\frac{d}{\lambda}\sin(\theta))}{\sin(\pi\frac{d}{\lambda}\sin(\theta))}\right|^2$$



Important parameters:

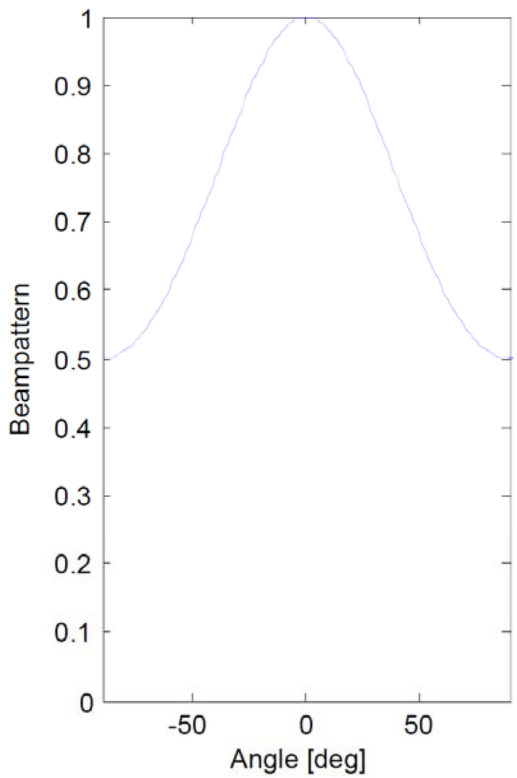
- DOA θ
- Ratio $\frac{d}{\lambda}$ (everything scales with wavelength)
- Number of sensors J
- Element spacing d
- Array aperture $L = J \cdot d$

ULA Beampattern

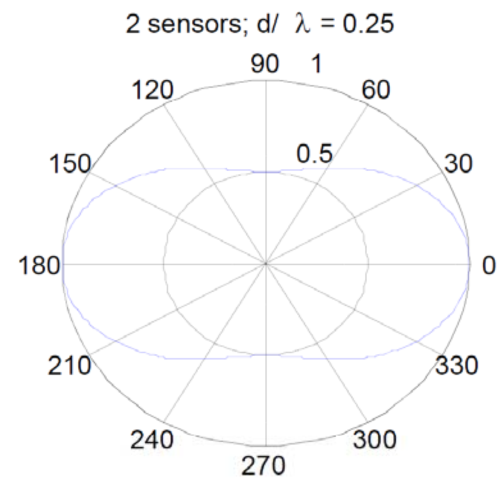
Example: $J = 2$ and $\frac{d}{\lambda} = \frac{1}{4}$ `B = abs(a'*w*w'*a)`

Linear

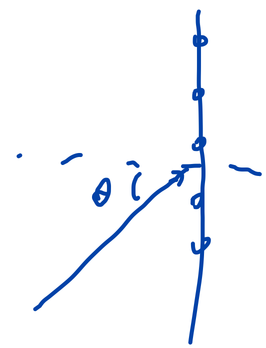
scale!



`plot(theta*180/pi,B)`



`polar(theta,B)`



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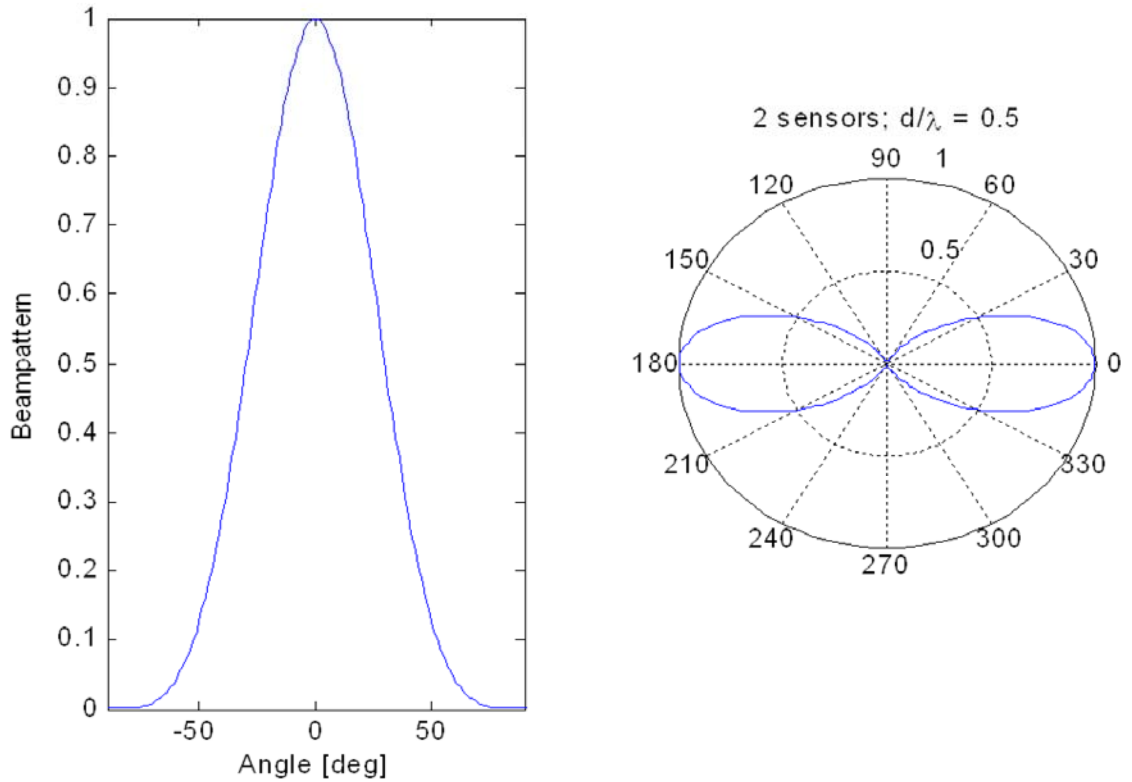
$J = 2$ and $\boxed{\frac{d}{\lambda} = \frac{1}{2}}$ $d \uparrow$



ULA Beampattern

Conclusions:

If $\frac{d}{\lambda} \ll \frac{1}{2} \Rightarrow$

- No exact cancelling at $\theta = \pm 90^o$ 两个sensor和一个差不多
- Little difference with single sensor case!

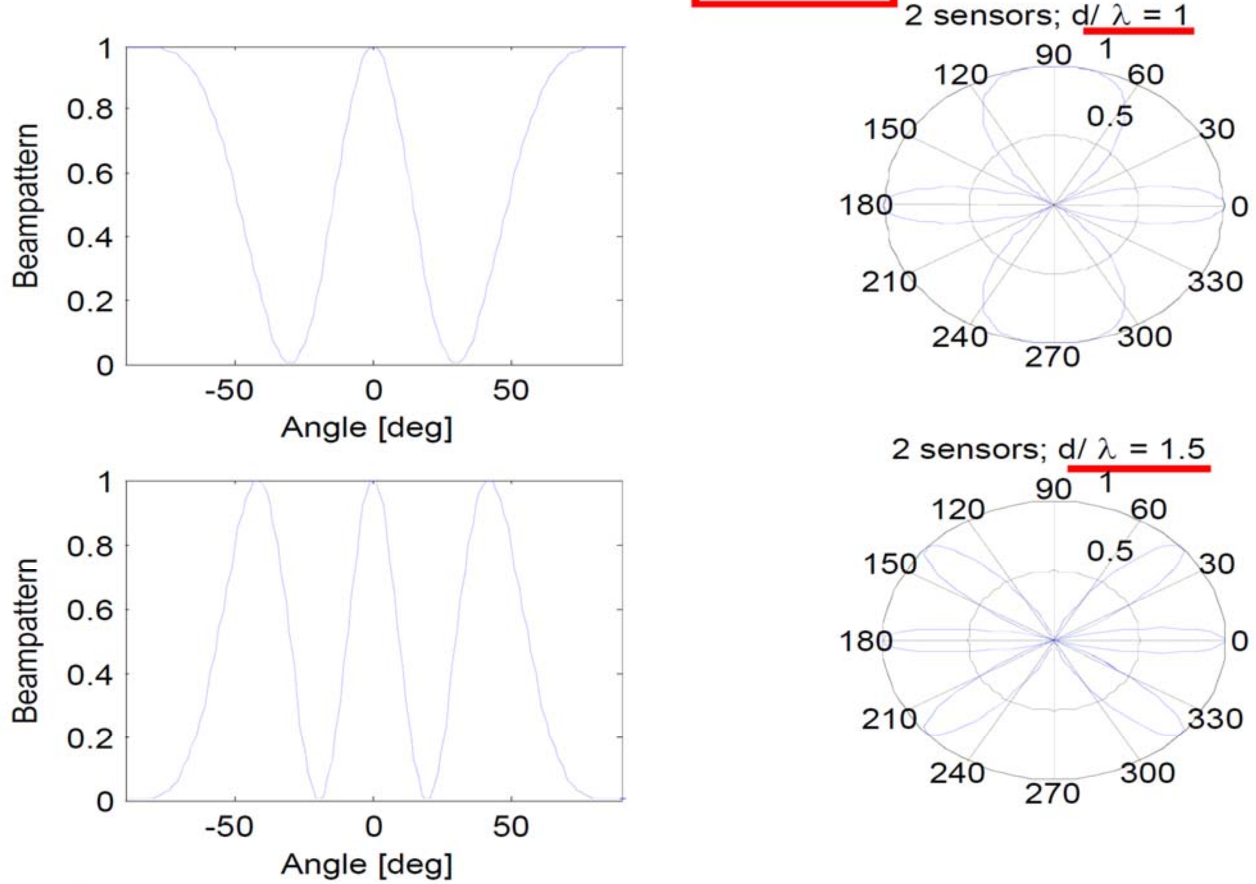
If $\frac{d}{\lambda} = \frac{1}{2} \Rightarrow$

增益大于0.5

- Main lobe beamwidth (DOA 0^o): 60^o
- Nulls at: $\theta = \pm 90^o$

ULA Beampattern

$$J = 2 \text{ and } \boxed{\frac{d}{\lambda} > \frac{1}{2}}$$



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Conclusions: 零点飘移

If $\frac{d}{\lambda} = 1 \;\Rightarrow$
- Nulls migrate to: $\theta = \pm 30^o$

  另外增益为1的地方
- Another two sidelobes at: $\theta = \pm 90^o$

If $\frac{d}{\lambda} > \frac{1}{2} \;\Rightarrow$
- Main lobe beamwidth decreases
- More nulls $\Rightarrow$ Spatial aliasing

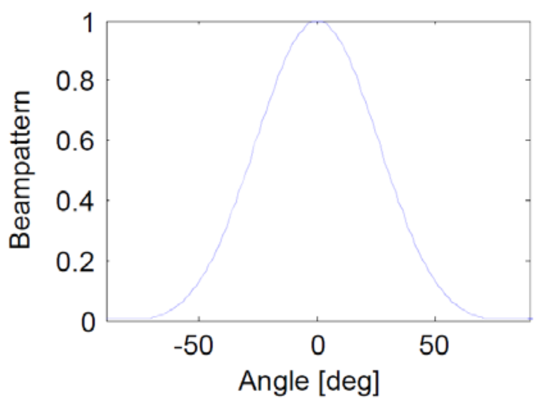
d过大
导致了空间混叠
(类似时间上二采样定理)

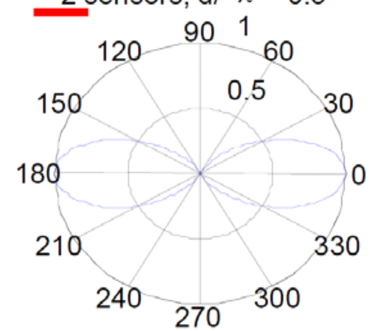
Spatial aliasing:

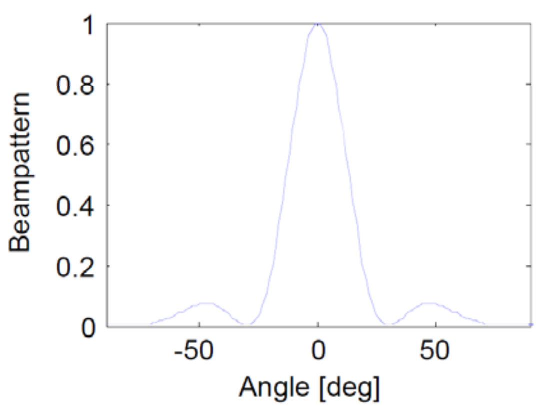
- Ambiguity in source locations
- Same response for sources at different positions
- Occurs if sensors are too far away (relative to λ)

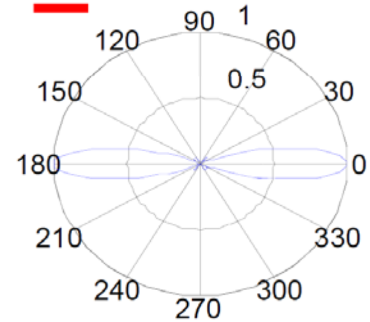
ULA Beampattern

Increase aperture $L = J \cdot d$ by $\boxed{J \uparrow}$, fixed $\frac{d}{\lambda} = \frac{1}{2}$









分辨率增加

ULA Beampattern

Conclusions Beampattern:

- For $J \uparrow \;\Rightarrow\;$ Mainlobe smaller $\;\Rightarrow\;$ 角度分辨率变好 more sensitive
- For $J \uparrow \;\Rightarrow\;$ array aperture $\uparrow$
- For $d/\lambda < 1/2 \;\Rightarrow\;$ No spatial aliasing
- For $d/\lambda \geq 1 \;\Rightarrow\;$ Pattern repeats at $\theta =$ 扫描 $\arcsin(\frac{\lambda}{d})$

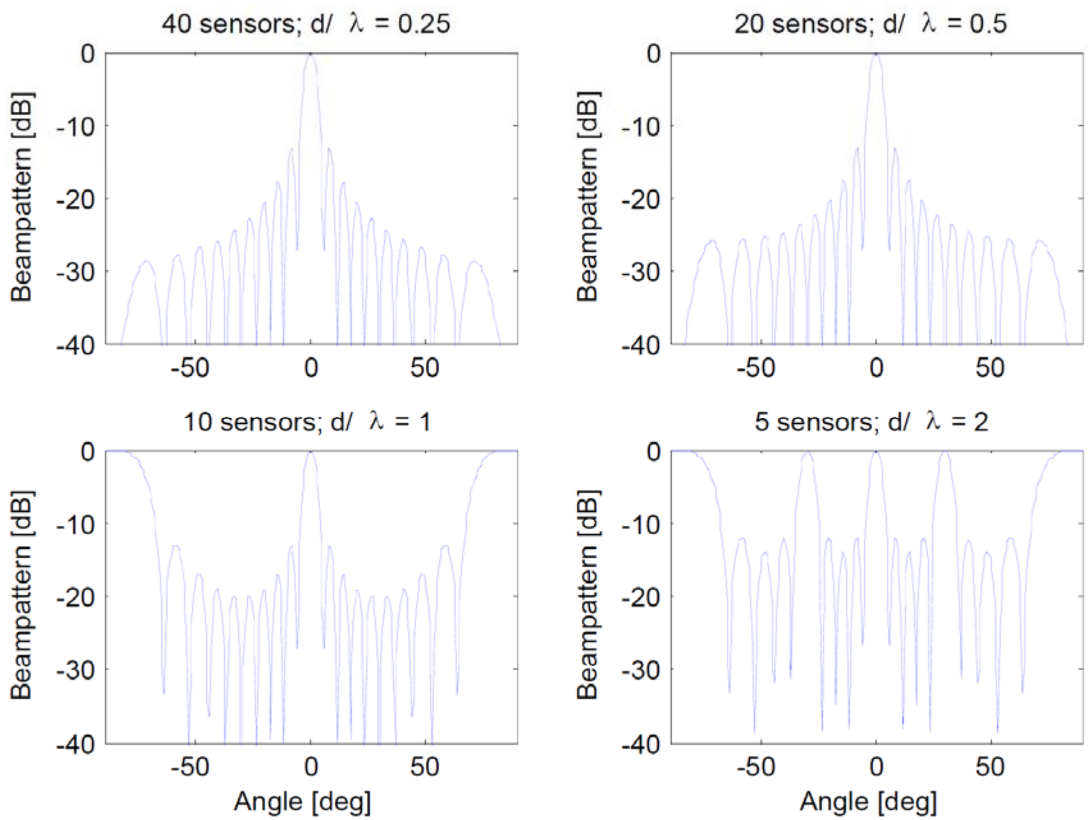
Pattern 对θ求导

- Zeros occur at $\theta = \arcsin\left(i \cdot \frac{1}{J} \cdot \frac{\lambda}{d}\right)$ with $i \in \mathcal{Z}$
- Main lobe at $\theta = 2\arcsin\left(\frac{1}{J} \cdot \frac{\lambda}{d}\right)$

ULA Beampattern

Variable element spacing d, fixed $L = J \cdot d = 10\lambda$

[dB] scale!



增多传感器无意义

$\frac{d}{\lambda} > \frac{1}{2}$

信号混叠，信号分不开

```
plot(theta*180/pi,10*log10(B))
```

ULA Beampattern

Conclusion fixed aperture:

- $d < \frac{\lambda}{2} \Leftrightarrow$ Oversampling:

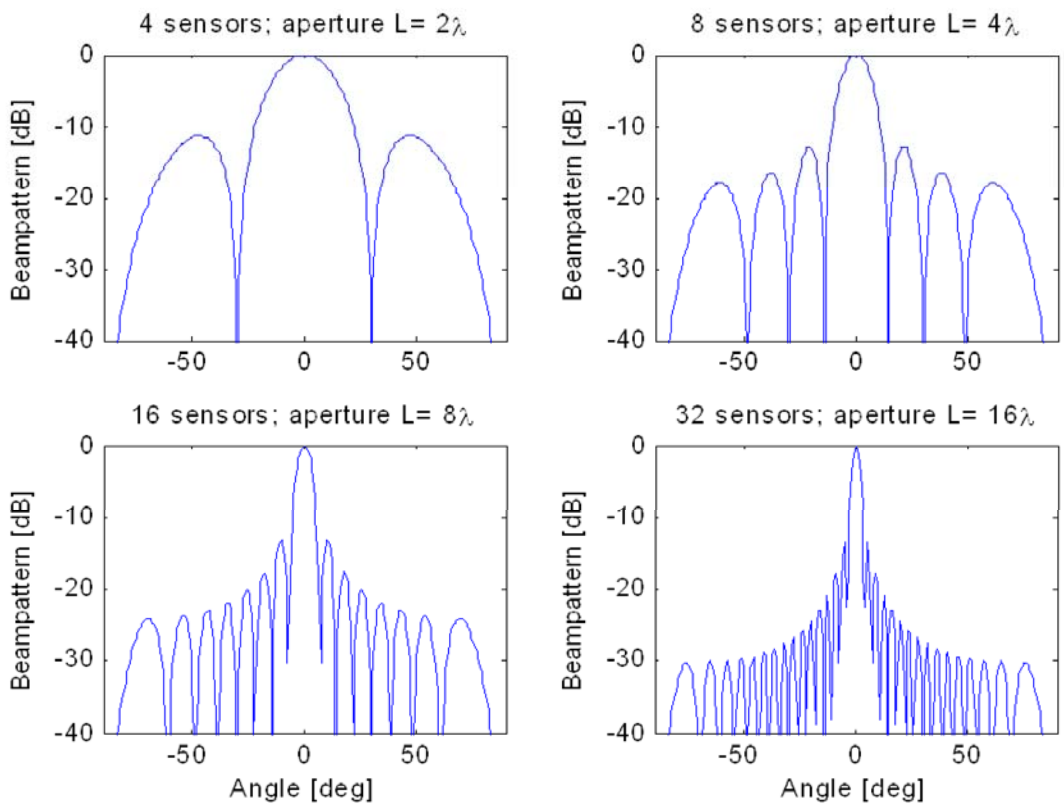
  No additional info 且 sensor 数目更多

- $d > \frac{\lambda}{2} \Leftrightarrow$ Undersampling:

  Grating lobes $=$ spatial ambiguities

ULA Beampattern

Variable aperture size $L = J \cdot d$ **and fixed** $d = \frac{\lambda}{2}$



增加孔径

增加分辨率

ADSP

ULA Beampattern

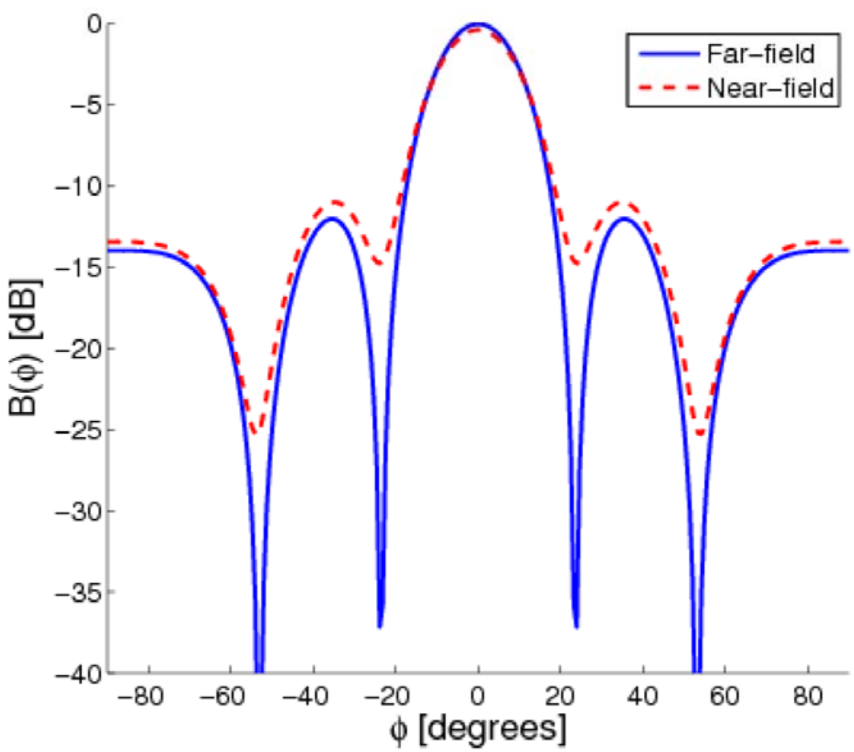
Conclusion fixed element spacing: $\frac{d}{\lambda} = \frac{1}{2}$

- Variable aperture $\Leftrightarrow$ variable resolution
- $L \uparrow$ $\Leftrightarrow$ Improved resolution $\Leftrightarrow$

  Better angle estimation capabilities

ULA Beampattern

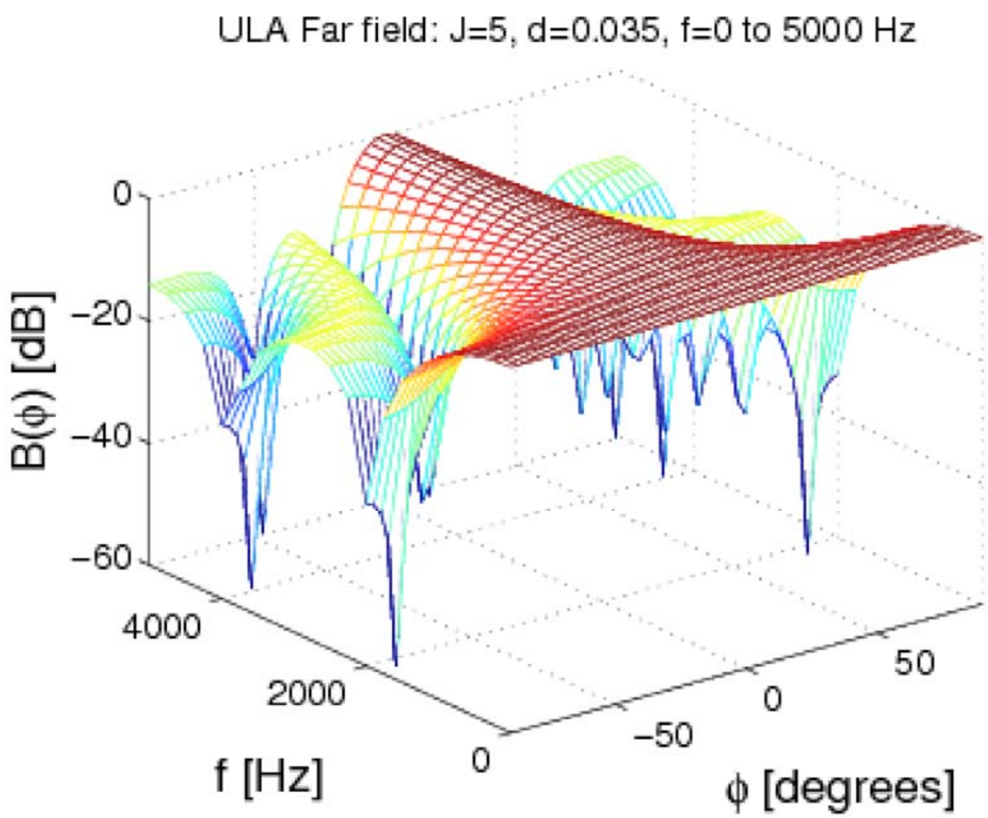
$$\text{Near field: } |\underline{\mathrm{p}}| < \frac{2L^2}{\lambda}$$



近场衰减不够

ULA Beampattern: frequency dependence

Variable frequency, fixed J and d

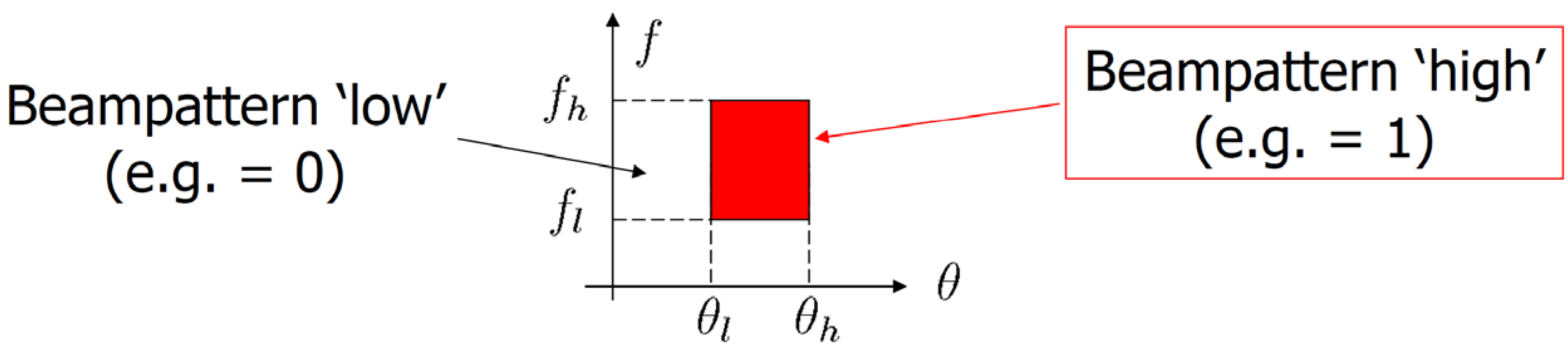


$f = \frac{c}{\lambda}$

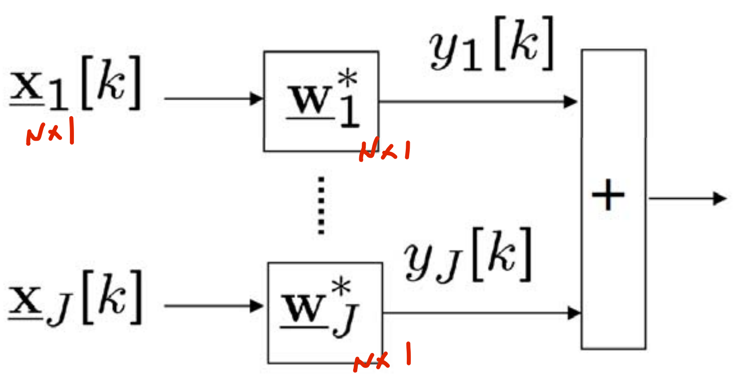
$f \to 0 \sim 5000, \quad \frac{d}{\lambda} = \frac{c}{\lambda} \cdot \frac{d}{c} = f \cdot \frac{0.035}{350} = f \times 10^{-4} \to 0 \sim \frac{1}{2}$ 过采样

ULA Beampattern: frequency dependence

Wish: Frequency "independence" over angular range



Use J different FIR filters (each of length N) for each sensor:



$$y[k] = \sum_{i=1}^{J} \underline{\mathbf{w}}_i^h \underline{\mathbf{x}}_i[k]$$

1×N　N×1

时：N×1

空：J×1

ADSP

ULA Beampattern: frequency dependence

$\underline{a}(f,\theta) = [1, e^{-j2\pi f \cdot \frac{d\sin\theta}{c}}, \cdots e^{-j2\pi f(J-1)\frac{d\sin\theta}{c}}]^t$

Array response: $\underset{N\times 1}{\underline{r}(f,\theta)} = \underset{N\times J}{\underline{\mathbf{w}}^h} \cdot \underset{J\times 1}{\mathbf{a}(f,\theta)} \Rightarrow \sum_{i=0}^{J-1}\sum_{l=0}^{N-1} w_{i,l}^{*[l]} a_i(f,\theta)$

$\Rightarrow$ **Frequence response for ULA:** (T_s is sampling frequency)

$$\underset{1\times 1\,?}{W(f,\theta)} = \sum_{i=0}^{J-1}\sum_{l=0}^{N-1} w_{i,l}^{*} e^{-j2\pi\cdot f\cdot l\cdot T_s} e^{-j2\pi f\frac{d\sin(\theta)}{c}i}$$

Normalized temporal frequency: $f_1 = f\cdot T_s = \frac{f}{f_s}$ $\Rightarrow$ 时间采样频率 f_s

Normalized spatial frequency: $f_2 = \frac{fd\sin(\theta)}{c\,?} = \frac{f}{\frac{c}{d\sin\theta}}$ 空间采样频率 $\frac{c}{d\sin\theta}$

ADSP

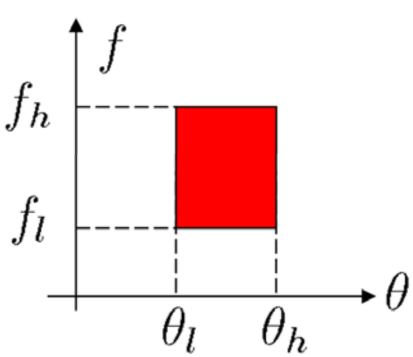
ULA Beampattern: frequency dependence

当f_1与f_2满足时不会混叠

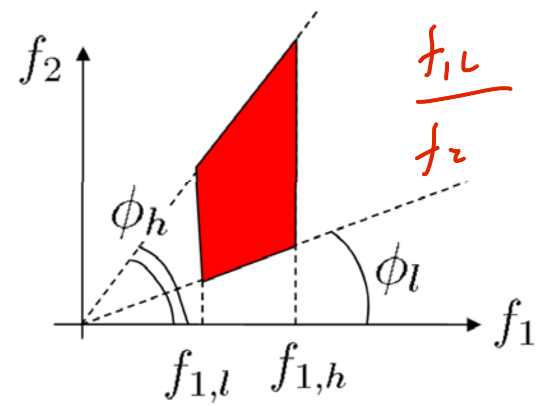
$$\Rightarrow \; W(f_1, f_2) = \sum_{i=0}^{J-1} \sum_{l=0}^{N-1} w_{i,l}^* e^{-j2\pi l f_1} e^{-j2\pi i f_2} \quad : \text{2D-DFT of } w_{i,l}^*$$

$$Note: \; f_2 = \left(\frac{d \sin(\theta)}{cT_s}\right) \cdot f_1 \;\Rightarrow\; \text{Slope through origin of } f_1, f_2 \text{ plane}$$

$$\text{Choose: } T_s = \frac{d}{c} \left(= \frac{d}{\lambda_{min}} \cdot \frac{1}{f_{max}} = \frac{1}{2} \cdot \frac{1}{f_{max}}\right) \qquad \boxed{\Rightarrow \; f_2 = \sin(\theta) \cdot f_1}$$



$$\tan(\phi) = \sin(\theta) = \frac{f_2}{f_1}$$



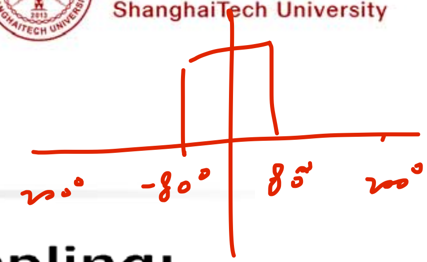
$$\frac{f_{1L}}{f_2} = \frac{f_{2h}}{f_2}$$

$x_i \to \boxed{w_i^*} \to$
$x_j \to \boxed{w_j^*} \to$
$\boxed{+} \to y[k]$

$r(f,\theta) \xrightarrow{F} \Rightarrow w \xrightarrow{F^2}$

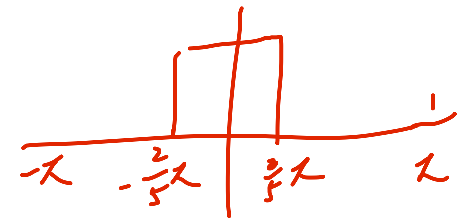
ADSP

Array response with DFT



ULA is related to regular temporal sampling:

传感器采样频率 Spatial sampling frequency : $U_s = \dfrac{1}{d}$



信号本身频率 Spatial frequency : $U = \dfrac{\sin(\theta)}{\lambda}$

归一化频率 Normalized spatial frequency : $u = \dfrac{U}{U_s} = \dfrac{d\sin(\theta)}{\lambda}$, [-1, 1]

$\Rightarrow$ **Steering vector:**

$$\underline{\mathbf{a}}(u) = \left(1, \mathrm{e}^{-\mathrm{j}2\pi u}, \cdots, \mathrm{e}^{-\mathrm{j}2\pi(J-1)u}\right)^t$$

Note: Avoid aliasing $\Rightarrow \quad -\dfrac{1}{2} \le u \le \dfrac{1}{2} \quad \Leftrightarrow \boxed{d \le \dfrac{\lambda}{2}}$

since range of unambiguous angles: $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

ADSP

Array response with DFT

$X[k] = \sum_{n=0}^{N-1} x_p[n]\, e^{-j\frac{2\pi}{N} n}$

Notes:

- Definition of length J DFT:

分辨率

$$F_l = \sum_{i=0}^{J-1} f_i e^{-j\frac{2\pi}{J}il} \text{ for } l = 0, \cdots, J-1 \text{ (resolution } \tfrac{2\pi}{J})$$

- Zero padding for improved resolution: (不改变频域成分)

$$F_l = \sum_{i=0}^{J-1} f_i e^{-j\frac{2\pi}{N}il} \text{ for } l = 0, \cdots, N-1$$

with $N \geq J$ and $f_i \equiv 0$ for $i \geq J \Rightarrow$ resolution $\frac{2\pi}{N}$

Array response with DFT

Array response with $u = \dfrac{d \sin(\theta)}{\lambda}$:

$$r(u) = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(u) = \sum_{i=1}^{J} w_i^* \mathrm{e}^{-\mathrm{j}2\pi(i-1)u} = \sum_{i=0}^{J-1} w_{i+1}^* e^{-j\frac{2\pi}{N} iNu}$$

Zero padded DFT: （注意看做 zero padding 来增加分辨率）

With $N \geq J$ and $w_i \equiv 0$ for $i \geq J \;\; \Rightarrow$

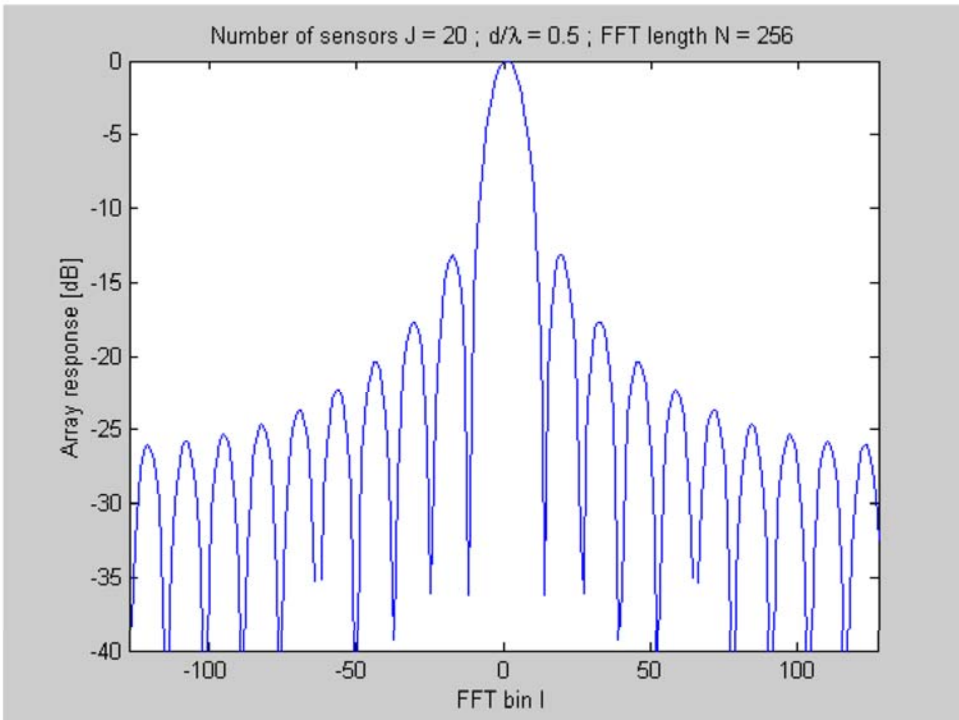
$$r_l = \sum_{i=0}^{J-1} w_{i+1}^* \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}il} \text{ for } l = 0, \cdots, N-1$$

with $l = N \cdot u = N \cdot \dfrac{d \sin(\theta)}{\lambda}$

求 array response 也就 直接 求 w 的 离散 F

Array response with DFT

```
w=(1/J)*ones(J); abs_r=fftshift(abs(fft(w,N)))
```



Compute corresponding angle via: $\theta = \arcsin(\frac{l \cdot \lambda}{N \cdot d})$

将横坐标、l 变成 θ

Data independent beamforming

Conventional approach:

- Beamsteering
- Tapering

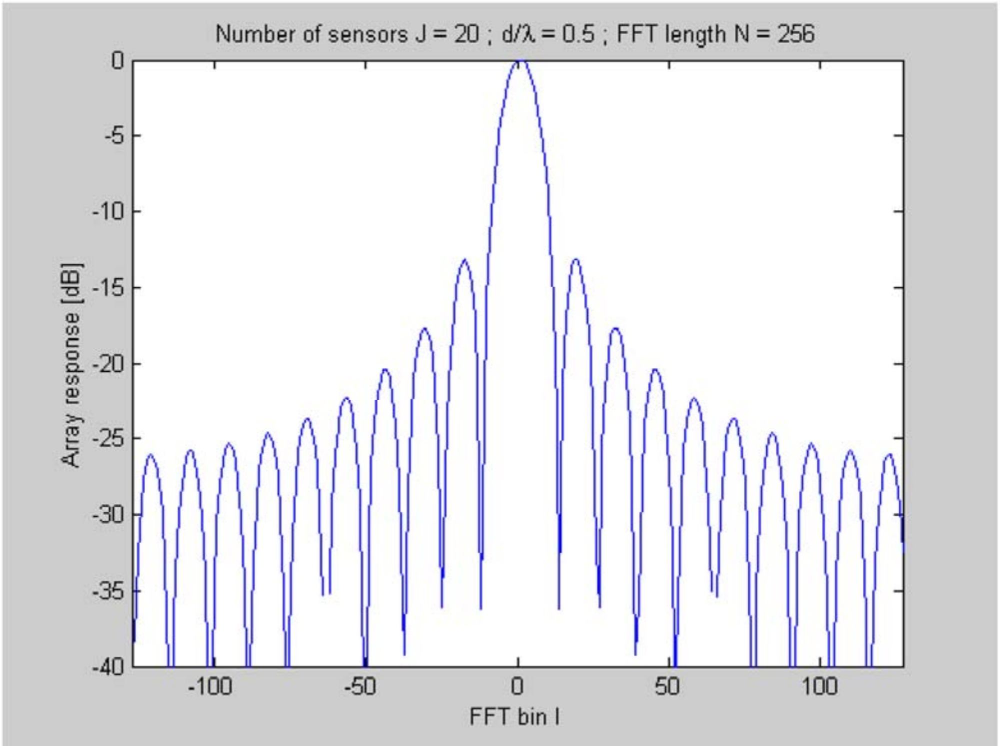
Other data independent approaches:

- Null-steering
- Array response design

ADSP

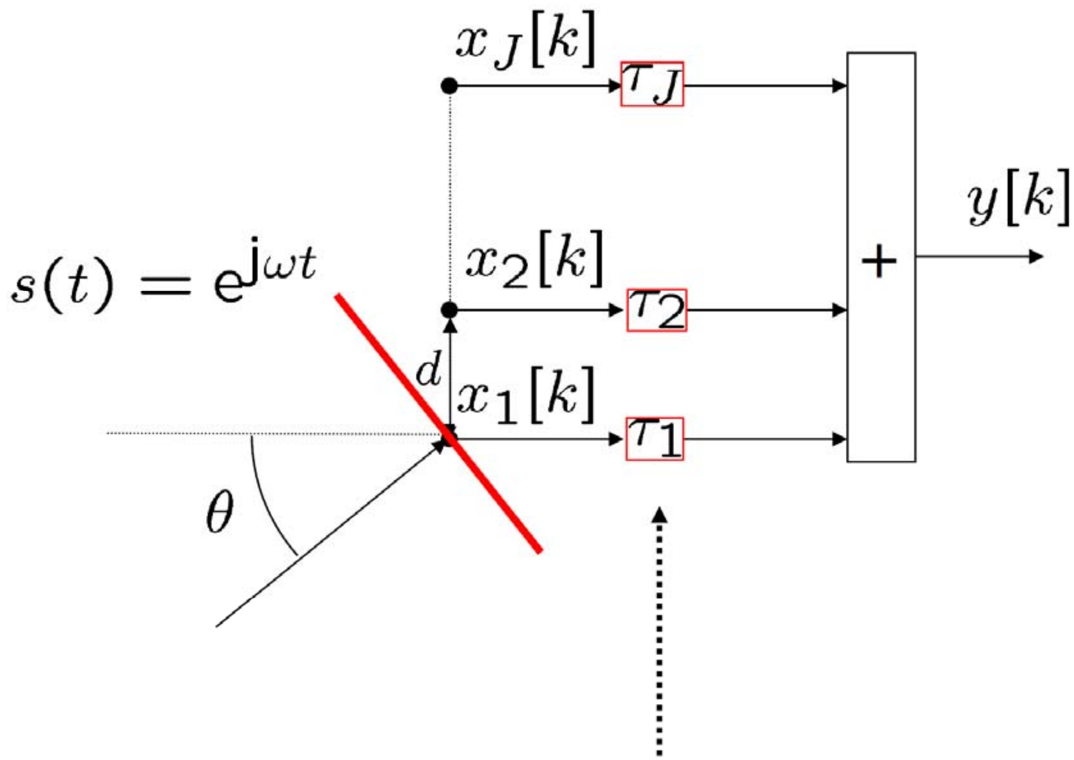
Data independent beamforming

```
w=(1/J)*ones(J); abs_r=fftshift(abs(fft(w,N)))
```



Compute corresponding angle via: $\theta = \arcsin(\frac{l \cdot \lambda}{N \cdot d})$

Data independent beamforming



For ULA choose: $\tau_i = (i-1)\tau \leftrightarrow w_i^* = e^{j(i-1)\omega\tau}$

Data independent beamforming

$$\left.\begin{aligned} y[k] &= \underline{\mathbf{w}}^h \cdot \underline{\mathbf{x}}[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta) \cdot s[k] \\ w_i^* &= \mathrm{e}^{\mathrm{j}(i-1)\omega\tau} \end{aligned}\right\} \Longrightarrow$$

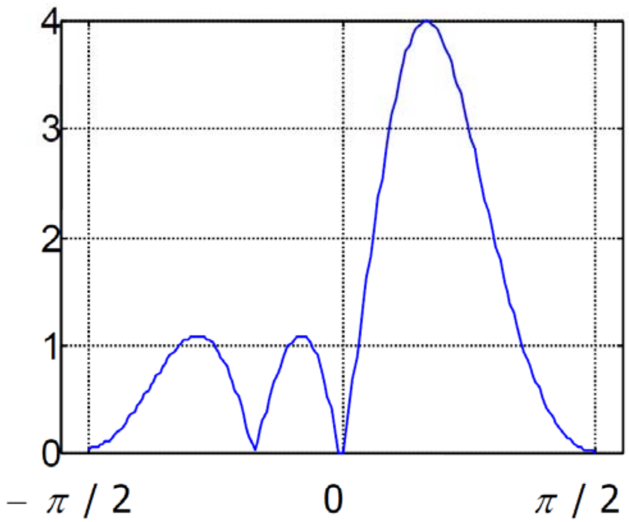
$$B(\theta) = \frac{1}{J^2}\left|\underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta)\right|^2 = \frac{1}{J^2}\left|\sum_{i=1}^{J} \mathrm{e}^{-\mathrm{j}(i-1)\omega\left(\frac{d\sin(\theta)}{c}-\tau\right)}\right|^2$$

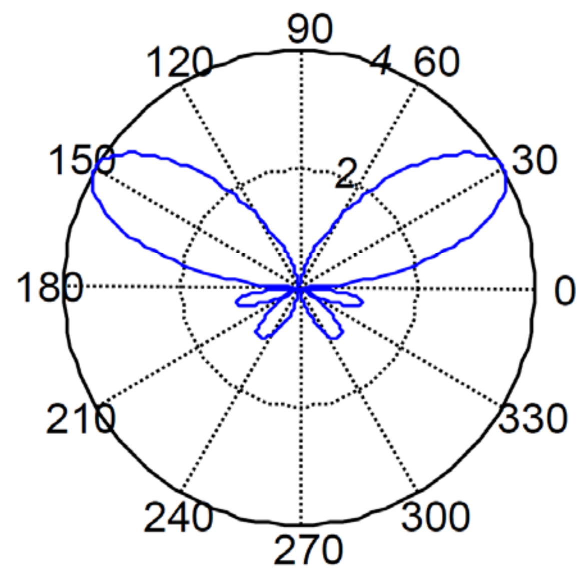
Thus mainlobe beampattern shifted over:

$$\boxed{\theta_0 = \arcsin\left(\frac{c \cdot \tau}{d}\right)}$$

Data independent beamforming

<u>Example:</u> $J = 4; d/\lambda = 1/2; \tau \leftrightarrow 30^o$



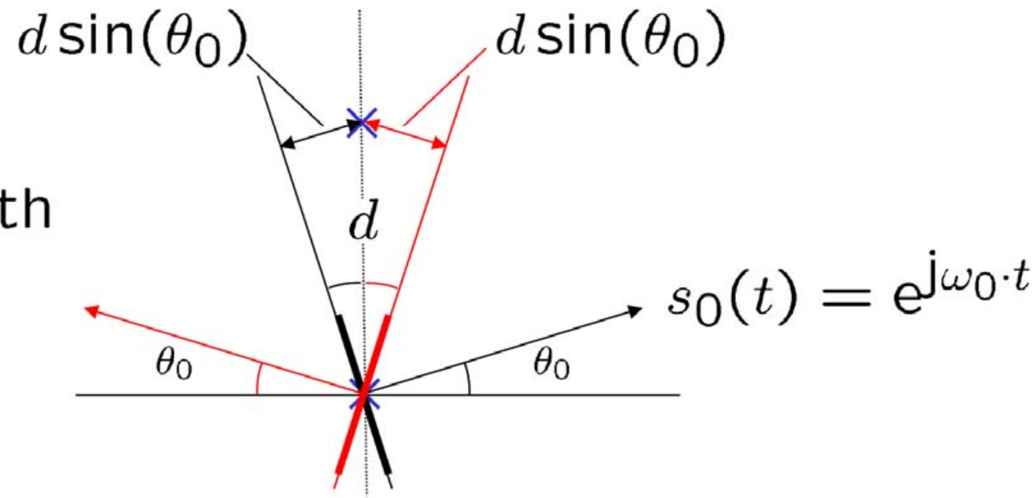


Data independent beamforming

- **Electronic vs Mechanical beamsteering:**

(with omnidirectional sensors)

Ambiguity with s_1 at $\pi - \theta_0$



Delay: $e^{-j\omega_0 \frac{d\sin(\theta_0)}{c}} = e^{-j2\pi \frac{d}{\lambda_0}\sin(\theta_0)}$

$\boxed{\text{Delay is the same for } s_0 \text{ and } s_1}$

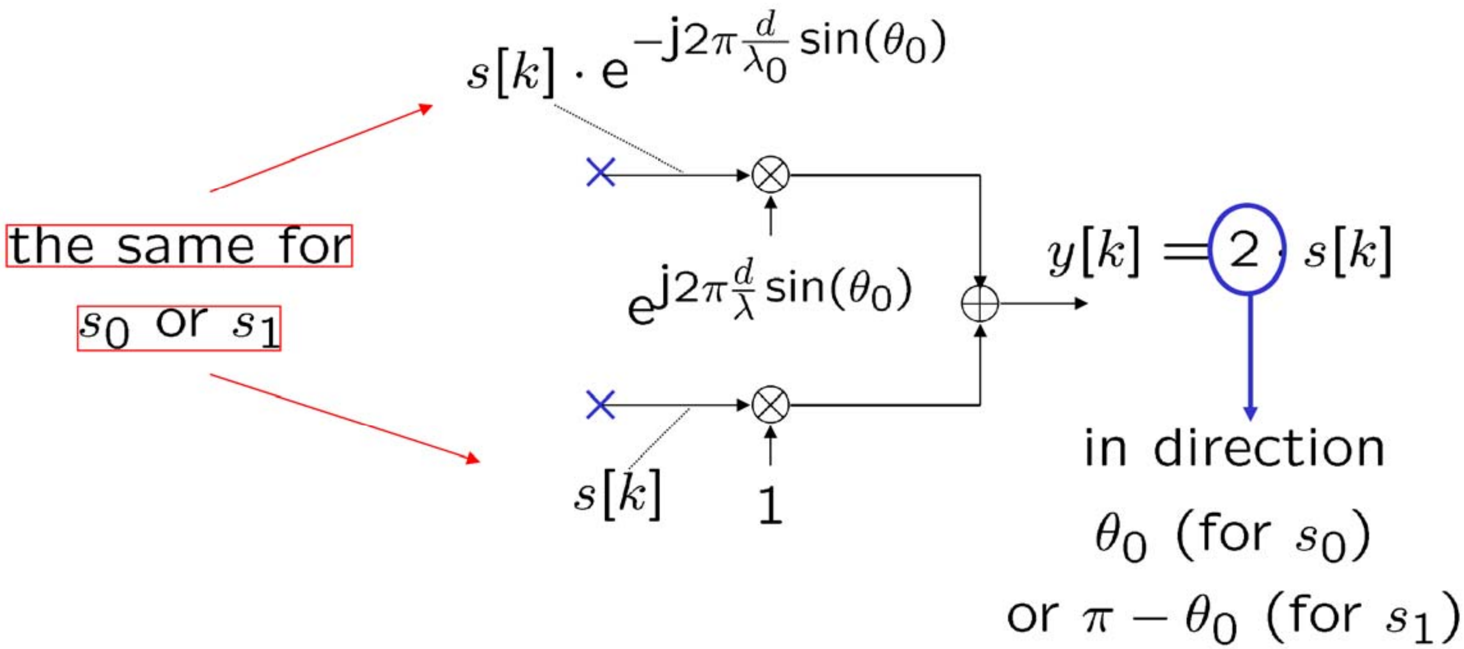
上海科技大学 ShanghaiTech University

ADSP

Data independent beamforming

Electronic beamsteering for ULA:

$$\Rightarrow \quad w_i^* = e^{j2\pi(i-1)\frac{d}{\lambda}\sin(\theta_0)}$$



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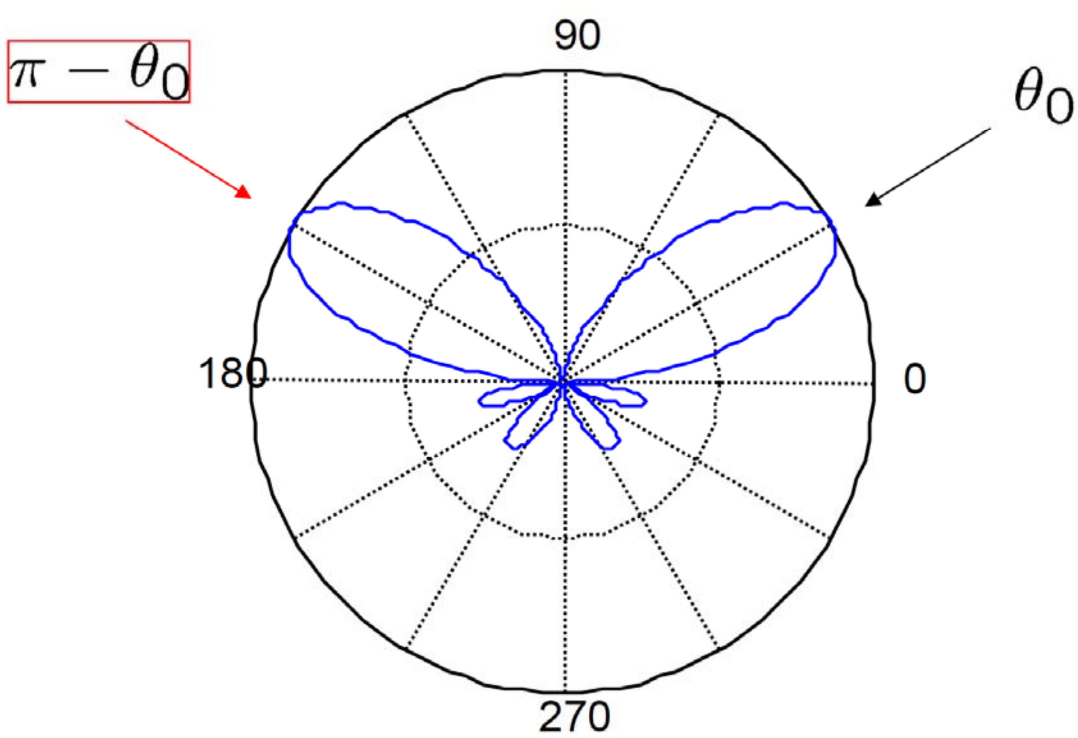
Data independent beamforming

Result electronic beamsteering:



Data independent beamforming

Result mechanical beamsteering: