Advanced Digital Signal Processing (ADSP)

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ADSP

Stochastic process: stationary

Stationary of order N

$$f_x[x_1, \dots, x_N; n_1, \dots, n_N] = f_x[x_1, \dots, x_N; n_1 + k, \dots, n_N + k] \quad \forall k$$

Strict-sense stationary (SSS)

$x[n]$ is stationary for all orders $N=1, 2, \dots$.

An IID sequence is SSS.

Wide-sense stationary (WSS): stationary up to order 2

- Mean is a constant independent of n: $\mathrm{E}\{x[n]\} = \mu_x$
- Variance is a constant independent of n: $\mathrm{var}\{x[n]\} = \sigma_x^2$
- Autocorrelation depends only on l ($l = n_1 - n_2$)

$$r_x[n_1, n_2] = r_x[n_1 - n_2] = r_x\,[l]$$

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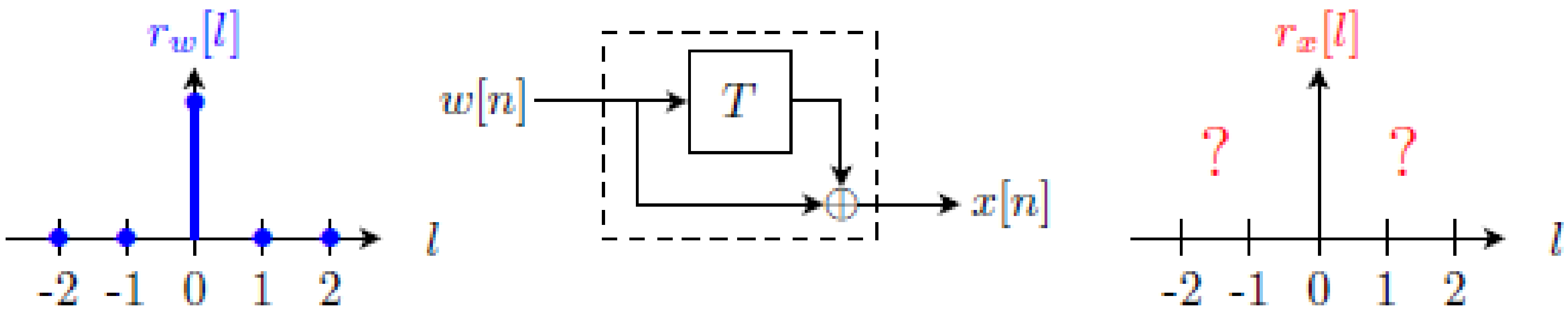
Stochastic process: stationary

Example:

Let $w[n]$ be a zero-mean, uncorrelated Gaussian random sequence with variance $\sigma^2[n] = 1$.

a. Characterize the random sequence $w[n]$.

b. Define $x[\mathrm{n}] = w[n] + w[n-1]$. Determine the mean and autocorrelation of $x[n]$. Also characterize $x[n]$.



ADSP Stochastic process: stationary

Wide-sense stationary (WSS)

Mean : $\mu_x = E\{x[n]\}$

Variance : $\sigma_x^2 = E\{(x[n]-\mu_x)^2\} = E\{x^2[n]\} - \mu_x^2$

Autocorrelation : $r_x[l] = E\{x[n]\cdot x[n-l]\} = E\{x[n+l]x[n]\}$

Autocovariance :
$$\gamma_x[l] = E\{(x[n]-\mu_x)\cdot(x[n-l]-\mu_x)\}$$
$$= E\{(x[n+l]-\mu_x)\cdot(x[n]-\mu_x)\} = r_x[l]-\mu_x^2$$

Properties $r_x[l]$

交流分量　　能量中的直流分量

$$r_x[0] = \mathrm{E}\{x^2[n]\} = \sigma_x^2 + \mu_x^2 \geq 0$$

$r_x[0] \geq r_x[l]$ 　　 $\mathrm{E}\{x^2[n]\}$: Power of $x[n]$

$$r_x[l] = r_x[-l]$$

$E\{(x[n]-x[n-l])^2\} \geq 0 \Rightarrow 2r_x[0] \geq 2r_x[l]$

$E\{x^2[n]+x^2[n-l]-2x[n]x[n-l]\} \geq 0 \Rightarrow 2r_x[0] \geq 2r_x[l]$

ADSP Stochastic process: stationary

Joint wide-sense stationary (WSS) 联合宽平稳

$x[n]$ is WSS, $y[n]$ is WSS, and

Cross-correlation : $r_{xy}[l] = E\{x[n] \cdot y[n-l]\}$

只和时间间隔有关

$$r_{xy}[l] = E\{x[n] \cdot y[n-l]\} \neq E\{x[n+l] \cdot y[n]\}$$

Cross-covariance : $\gamma_{xy}[l] = E\{(x[n] - \mu_x) \cdot (y[n-l] - \mu_y)\}$

$$= r_{xy}[l] - \mu_x \cdot \mu_y$$

Normalized γ_{xy} : $\rho_{xy}[l] = \dfrac{\gamma_{xy}[l]}{\sigma_x \cdot \sigma_y}$

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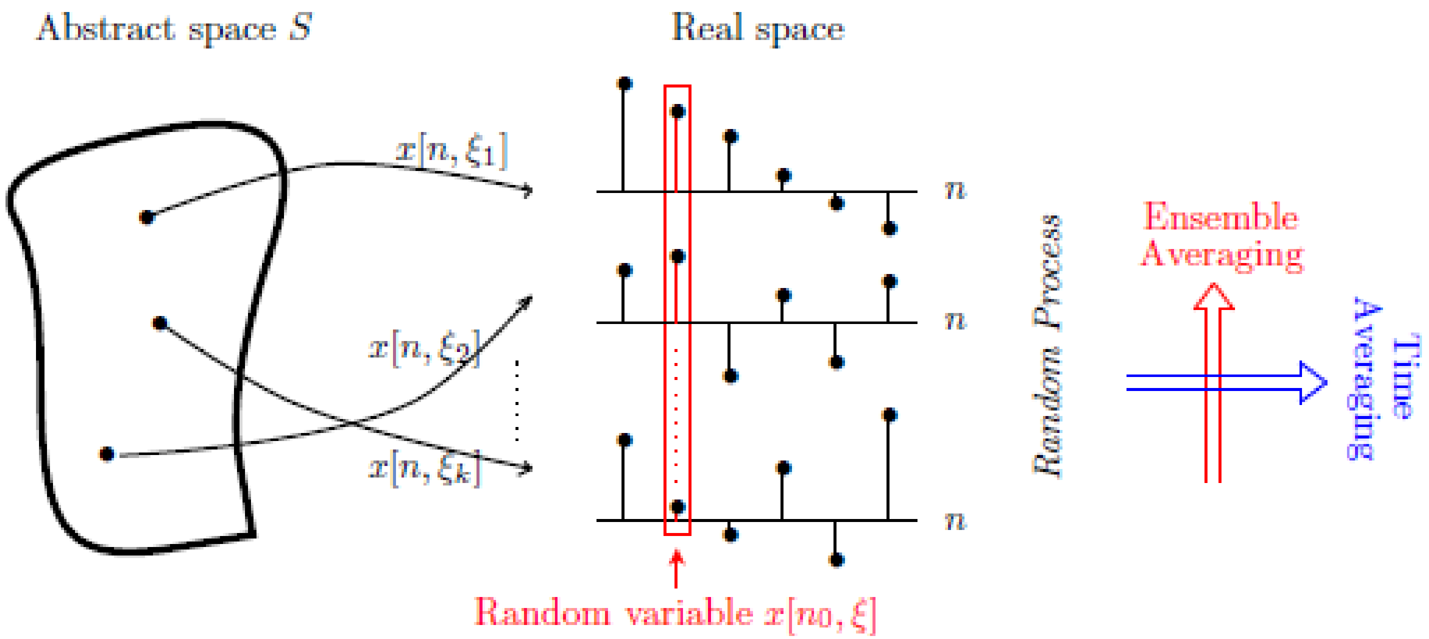
Stochastic process: ergodicity 各态历经



- Ensemble averages: $\mathrm{E}\{\cdot\}$
- Time averages: $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (\cdot)$
- Ergodic: $\mathrm{E}\{\cdot\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (\cdot)$ Ensemble averages = Time averages
- In practice: $\mathrm{E}\{\cdot\} = \frac{1}{2N+1} \sum_{n=-N}^{N} (\cdot)$

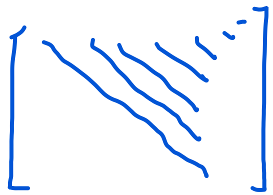
Stochastic process: ergodicity

估计值成立条件是各态历经

Mean : $\hat{\mu}_x = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$

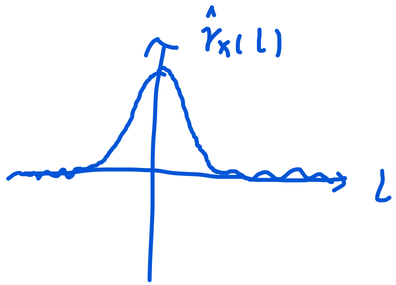
Variance : $\hat{\sigma}_x^2 = \frac{1}{N}\sum_{n=0}^{N-1} (x[n]-\hat{\mu}_x)^2 = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n] - \hat{\mu}_x^2$

Autocorrelation : $\hat{r}_x[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|} x[n]x[n+l]$ for $|l| \le L-1$



Autocovariance : $\hat{\gamma}_x[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|} (x[n]-\hat{\mu}_x)(x[n+l]-\hat{\mu}_x)$

$$= \hat{r}_x[l] - \left(\frac{N-|l|}{N}\right)\hat{\mu}_x^2$$



- Ergodic is also WSS
- Only WSS can be ergodic 二阶各态历经
- WSS does not imply ergodicity of any kind

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Stochastic process: ergodicity

Joint ergodicity

Cross-correlation : $\hat{r}_{xy}[l] = \frac{1}{N} \sum_{n=0}^{N-1-|l|} x[n]y[n+l]$

Cross-covariance : $\hat{\gamma}_{xy}[l] = \frac{1}{N} \sum_{n=0}^{N-1-|l|} (x[n] - \hat{\mu}_x)(y[n+l] - \hat{\mu}_y)$

$$= \hat{r}_{xy}[l] - \hat{\mu}_x \cdot \hat{\mu}_y$$

Normalized $\hat{\gamma}_{xy}$: $\hat{\rho}_{xy}[l] = \frac{\hat{\gamma}_{xy}[l]}{\hat{\sigma}_x \cdot \hat{\sigma}_y}$

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Power spectral density (PSD)

均值为0，保证收敛

PSD and $r_x(l)$ DTFT pair

$x[n]$ stationary, $\mu_x = 0$

$$P_x(e^{j\omega}) = \sum_{l=-\infty}^{\infty} r_x[l] e^{-j\omega l} \quad \circ\!\!-\!\!\circ \quad r_x[l] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_x(e^{j\omega})\, e^{j\omega l} \mathrm{d}\omega$$

no matter $r_x[l]$ is real or complex

Properties PSD :

- $P_x(e^{j\omega})$ real-valued periodic function of frequency (period 2π)
- $x[n]$ real, PSD even: $P_x(e^{j\omega}) = P_x(e^{-j\omega})$ $\quad r_x(l) = r_x(-l)^*$
- PSD nonnegative $P_x(e^{j\omega}) \geq 0$
- Average power: $\frac{1}{2\pi} \int_{-\pi}^{\pi} P_x(e^{j\omega}) \mathrm{d}\omega = r_x(0) = \mathrm{E}\{|x[n]|^2\} \geq 0$ $\quad e^{j\omega l} = 1$

Example: $r_x[l] = a^{|l|}$ with $|a| < 1$. Calculate $P_x(e^{j\omega})$

Estimate PSD:

$$\{\hat{r}_x[l]\}|_{l=-(L-1)}^{L-1} \qquad \hat{P}_x(e^{j\omega}) = \mathrm{fft}(\hat{r}_x[l], M) \text{ with } M \geq 2L-1$$

M若小于(2L-1)会丢失信息

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Power spectral density (PSD)

$r_w = E\{(w[n]-u_1)(w[n-l]-u_2)\}$
$= r_w[l] - u_1u_2$

白噪声: 零均值, 宽平稳, 不相关

宽平稳

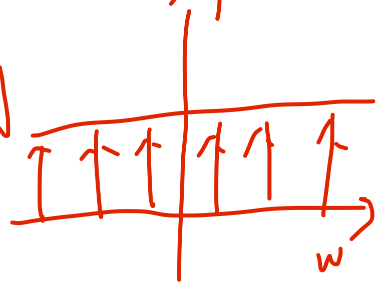
$l \neq 0$, 不相关 $r_w[l] = u_w^2 = 0$

White noise: $w[n] \sim \mathrm{WN}(\mu_w, \sigma_w^2)$

$$\mathrm{E}\{w[n]\} = \mu_w = 0 \quad \text{and} \quad r_w[l] = \mathrm{E}(w[n]w[n-l]) = \sigma_w^2\,\delta(l)$$

$$P_w(e^{j\omega}) = \sigma_w^2 \qquad \forall\omega$$

白噪声功率谱为常数
和每个频率对功率的贡献相同



White Gaussian noise:

- $w[n]$ is white and Gaussian, $w[n] \sim \mathrm{WGN}(\mu_w, \sigma_w^2)$

符合高斯分布的白噪声

IID:

- $w[n]$ are independently and identically distributed with mean μ_w and variance σ_w^2, $w[n] \sim \mathrm{IID}(\mu_w, \sigma_w^2)$ 不同于高斯

ADSP

Power spectral density (PSD)

Cross-Power spectral density

Zero-mean, joint stationary

$$P_{xy}(e^{j\omega}) = \sum_{l=-\infty}^{\infty} r_{xy}[l] e^{-j\omega l} \quad \circ\!-\!\circ \quad r_{xy}[l] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xy}(e^{j\omega})\, e^{j\omega l} \mathrm{d}\omega$$

$$r_{xy}[l] = r_{yx}^{*}[-l] \qquad P_{xy}(e^{j\omega}) = P_{yx}^{*}(e^{j\omega})$$

Coherence function (normalized cross-PSD):

$$C_{xy}(e^{j\omega}) \triangleq \frac{P_{xy}(e^{j\omega})}{\sqrt{P_x(e^{j\omega})}\sqrt{P_y(e^{j\omega})}}$$

Properties coherence function:

- $0 \leq |C_{xy}(e^{j\omega})| \leq 1$
- $C_{xy}(e^{j\omega}) = 1$: $x[n] = y[n]$
- $C_{xy}(e^{j\omega}) = 0$: $x[n]$ and $y[n]$ not correlated

THANKS!

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