Advanced Digital Signal Processing (ADSP)

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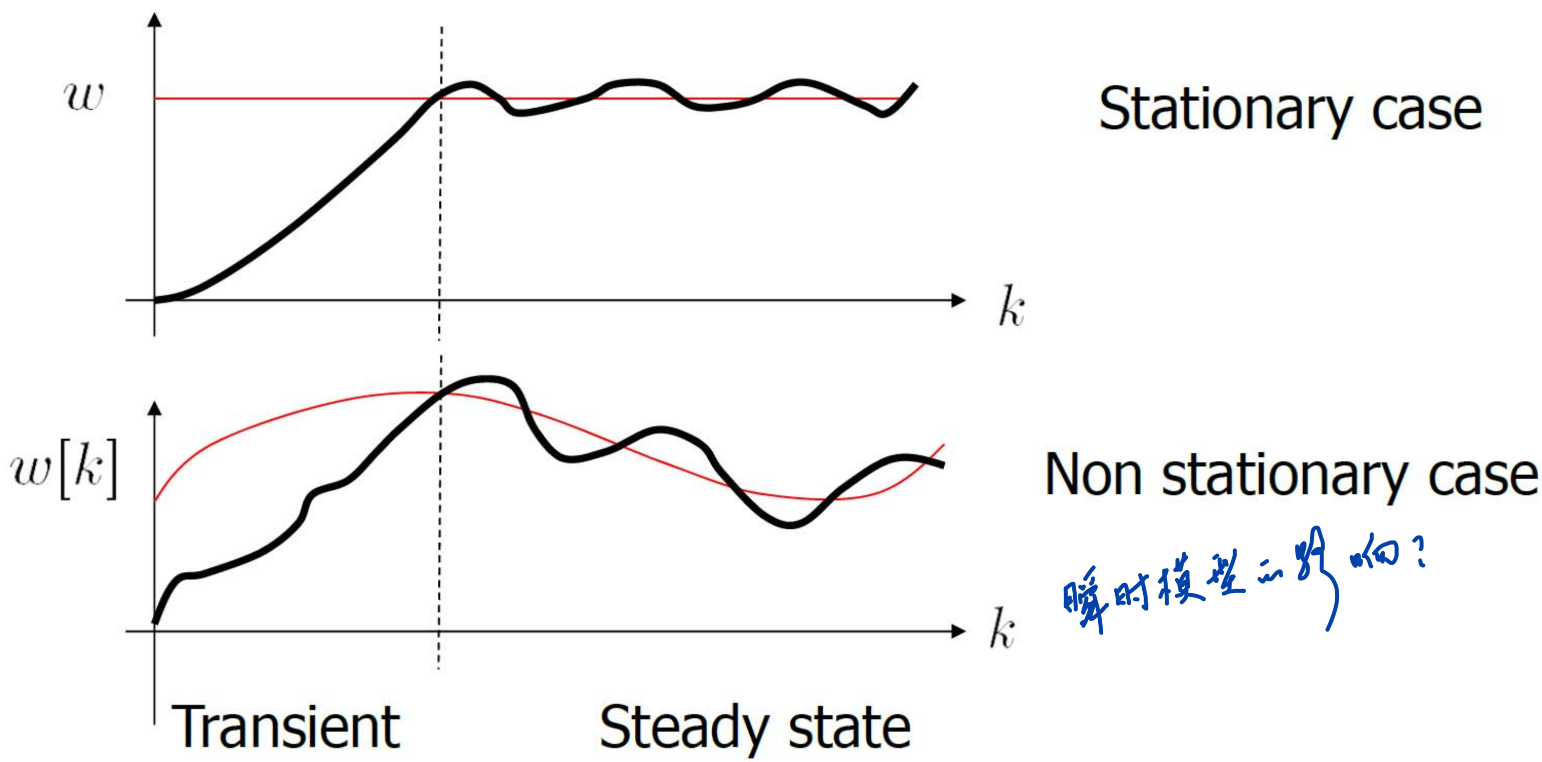
Convergence LMS

Acquisition and tracking:



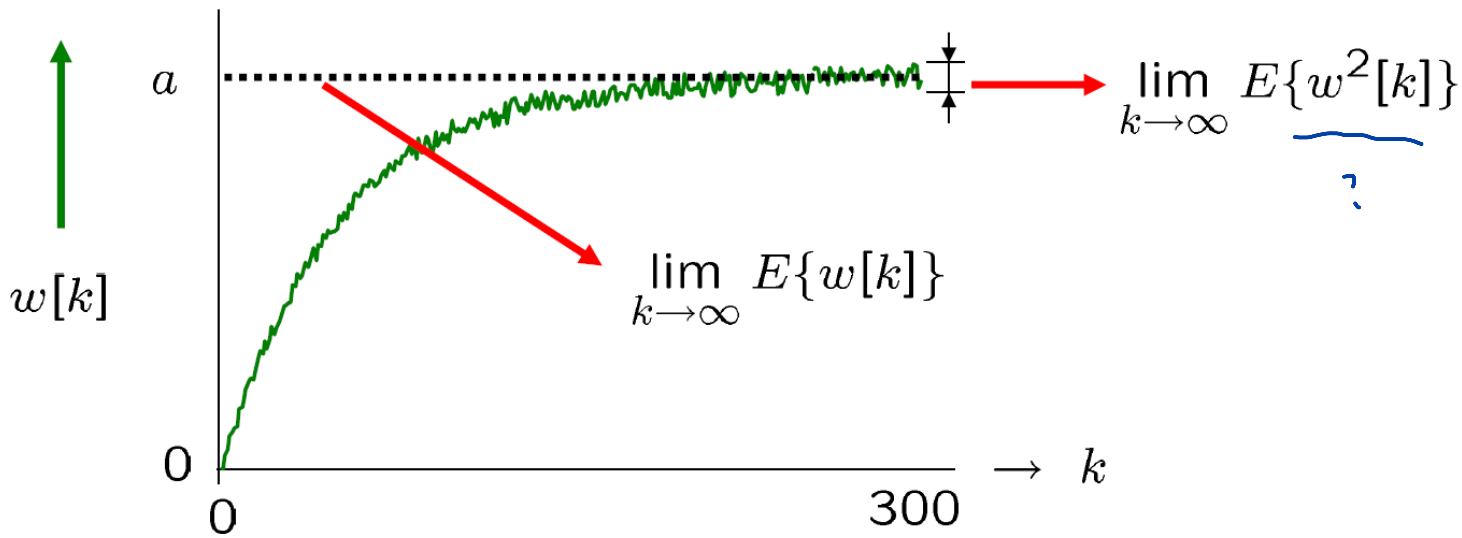
瞬时模型不好啊？

ADSP

Convergence LMS

Consequence of not using $E\{\cdot\}$? 不使用E[·]的推论

Example: LMS, $N = 1$, $w[0] = 0$ $w_o = a$



Questions about convergence:

$$\lim_{k\to\infty} E\{w[k]\} = w_o = a \text{ and } \lim_{k\to\infty} E\{w^2[k]\} < \infty?$$

Convergence LMS

First compare difference with (optimal) Wiener weights:

$$\underline{\mathbf{d}}[k] = \underline{\mathbf{w}}[k] - \underline{\mathbf{w}}_o \quad \text{with} \quad \underline{\mathbf{w}}_o = \mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{ex}$$

$$\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \left(\underline{\mathbf{x}}[k]e[k] - \underline{\mathbf{x}}[k]\underline{\mathbf{x}}^t[k]\underline{\mathbf{w}}[k]\right)$$

$$\underline{\mathbf{w}}[k+1] - \underline{\mathbf{w}}_o = \left(\mathbf{I} - 2\alpha\underline{\mathbf{x}}[k]\underline{\mathbf{x}}^t[k]\right)\underline{\mathbf{w}}[k] - \underline{\mathbf{w}}_o + 2\alpha\underline{\mathbf{x}}[k]e[k]$$

$$\underline{\mathbf{d}}[k+1] = \left(\mathbf{I} - 2\alpha\underline{\mathbf{x}}[k]\underline{\mathbf{x}}^t[k]\right)\underline{\mathbf{d}}[k] + 2\alpha\underline{\mathbf{x}}[k]r_{min}[k]$$

$- 2\alpha \underline{x}[k] \underline{x}^t[k] \underline{w}_o + 2\alpha \underline{x}[k] e[k]$

$2\alpha \underline{x}[k] \left(e[k] - \underline{x}^t[k] \underline{w}_o \right)$

with $r_{min}[k] = e[k] - \underline{\mathbf{x}}^t[k]\underline{\mathbf{w}}_o$

$\hat{e}_{min}$

ADSP

Convergence LMS

迭代：SGD: $\underline{d}[k+1] = (I - 2\alpha R_x)\underline{d}[k]$; $\underline{d}[k] = (I - 2\alpha R_x)^k \underline{d}[0]$

Convergence in the mean: Why use mean $E[\cdot]$?

$$E\{\underline{\mathbf{d}}[k+1]\} = E\left\{\left(\mathbf{I} - 2\alpha\underline{\mathbf{x}}[k]\underline{\mathbf{x}}^t[k]\right)\underline{\mathbf{d}}[k]\right\}$$

0

$r_{min} \perp \underline{x}$

$$+2\alpha\left(E\{\underline{\mathbf{x}}[k]e[k]\} - E\{\underline{\mathbf{x}}[k]\underline{\mathbf{x}}^t[k]\}\underline{\mathbf{w}}_O\right)$$

With independence assumption:

$$E\{\underline{\mathbf{x}}[k]\underline{\mathbf{x}}^t[k]\underline{\mathbf{d}}[k]\} \approx E\{\underline{\mathbf{x}}[k]\underline{\mathbf{x}}^t[k]\}\cdot E\{\underline{\mathbf{d}}[k]\}$$

$k \to \infty$ $(1 - 2\alpha\lambda_i)^k = 0$

$\Rightarrow 0 < |1 - 2\alpha\lambda_i| < 1$

$$\Rightarrow E\{\underline{\mathbf{d}}[k+1]\} = (\mathbf{I} - 2\alpha\mathbf{R}_x)E\{\underline{\mathbf{d}}[k]\}$$

for all i: $\alpha < \frac{1}{\lambda_i} \Rightarrow \alpha < \frac{1}{\lambda_{max}}$

开始迭代：$E\{\underline{d}[k]\} = (I - 2\alpha R_x)^k E\{\underline{d}[0]\}$

Average convergence behaviour LMS same as SGD

$$0 < \alpha < 1/\lambda_{max}: \quad \lim_{k\to\infty} E\{\underline{\mathbf{w}}[k]\} = \underline{\mathbf{w}}_O \;;\; \tau_{av,i} \approx 1/2\alpha\lambda_i$$

$\Rightarrow$ Depends on coloration input process!

$\ln(e^{-\frac{k}{\tau_i}}) = \ln(1 - 2\alpha\lambda_i)^k$

白信号和有色信号不同点在于？

$\Rightarrow \tau_i = -\frac{1}{\ln(1-2\alpha\lambda_i)} \sim \frac{1}{2\alpha\lambda_i}$

Convergence LMS

Mean-square convergence:

$$J_{LMS} \;=\; E\{r^2\} = E\{(e - \underline{\mathbf{w}}^t\underline{\mathbf{x}})(e - \underline{\mathbf{x}}^t\underline{\mathbf{w}})\}$$

With $\underline{\mathbf{d}} = \underline{\mathbf{w}} - \underline{\mathbf{w}}_o$ or $\underline{\mathbf{w}} = w_o + \underline{\mathbf{d}} \;\;\Rightarrow$

$$J_{LMS} = E\{\left((e - \underline{\mathbf{w}}_o^t\underline{\mathbf{x}}) - \underline{\mathbf{d}}^t\underline{\mathbf{x}}\right)\left((e - \underline{\mathbf{x}}^t\underline{\mathbf{w}}_o) - \underline{\mathbf{x}}^t\underline{\mathbf{d}}\right)\}$$

$$J_{LMS} = E\{(e - \underline{\mathbf{w}}_o^t\underline{\mathbf{x}})(e - \underline{\mathbf{x}}^t\underline{\mathbf{w}}_o)\} + E\{\underline{\mathbf{d}}^t\underline{\mathbf{x}}\underline{\mathbf{x}}^t\underline{\mathbf{d}}\}$$
$$-E\{\underline{\mathbf{d}}^t\underline{\mathbf{x}}(e - \underline{\mathbf{x}}^t\underline{\mathbf{w}}_o)\} - E\{(e - \underline{\mathbf{w}}_o^t\underline{\mathbf{x}})\underline{\mathbf{x}}^t\underline{\mathbf{d}}\}$$

Independence assumption $\;\Rightarrow$

$$E\{\underline{\mathbf{d}}^t\underline{\mathbf{x}}(e - \underline{\mathbf{x}}^t\underline{\mathbf{w}}_o)\} \;\approx\; E\{\underline{\mathbf{d}}^t\} \cdot \left(E\{\underline{\mathbf{x}}e\} - E\{\underline{\mathbf{x}}\underline{\mathbf{x}}^t\}\underline{\mathbf{w}}_o\right)$$
$$=\; E\{\underline{\mathbf{d}}^t\} \cdot (\underline{\mathbf{r}}_{ex} - \mathbf{R}_x\underline{\mathbf{w}}_o) = 0$$

Similar $E\{(e - \underline{\mathbf{w}}_o^t\underline{\mathbf{x}})\underline{\mathbf{x}}^t\underline{\mathbf{d}}\} \rightarrow 0$

Convergence LMS

Compare with MMSE expression

$$\Rightarrow \; J_{LMS} \; \approx \; E\{(e - \underline{\mathbf{w}}_o^t \underline{\mathbf{x}})(e - \underline{\mathbf{x}}^t \underline{\mathbf{w}}_o)\} + E\{\underline{\mathbf{d}}^t \underline{\mathbf{x}\mathbf{x}}^t \underline{\mathbf{d}}\}$$

Fixed *Adaptive*

Wiener error: $J_{min} = E\{(e - \underline{\mathbf{w}}_o^t \underline{\mathbf{x}})(e - \underline{\mathbf{x}}^t \underline{\mathbf{w}}_o)\} = E\{r_{min}^2\}$

Excess error: $J_{ex} \; = \; E\{\underline{\mathbf{d}}^t \underline{\mathbf{x}\mathbf{x}}^t \underline{\mathbf{d}}\} \approx E\{\underline{\mathbf{d}}^t E\{\underline{\mathbf{x}\mathbf{x}}^t\} \underline{\mathbf{d}}\}$

$$= \; E\{\underline{\mathbf{d}}^t \mathbf{R}_x \underline{\mathbf{d}}\}$$

Dynamic behaviour adaptive filter: $\tilde{J}[k] = \dfrac{J_{ex}[k]}{J_{min}[k]}$?

Depends on coloration input process!

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Convergence LMS

Conclusion convergence LMS:

Convergence gradient based algorithms, like SGD heavily relates on corolation input process and initiation of adaptive weights!

This also follows from rewriting gradient as:

$\underline{\nabla} = -2 (\underline{r}_{ex} - R_x \underline{w}[k])$

$$\underline{\nabla} = -2\mathbf{R}_x \cdot \left(\mathbf{R}_x^{-1}\underline{\mathbf{r}}_{ex} - \underline{\mathbf{w}}[k]\right)$$

$\Rightarrow$ Gradient (=update) depends on coloration input

Solution:

Alternative that decorrelates input $\rightarrow$ Newton

输入去相关

ADSP Part B: Adaptive signal processing

Focus on single channel adaptive algorithms using FIR structure

- Applications Adaptive Algorithms
- Minimum Mean Square Error (MMSE)
- Constrained MMSE
- Least Square (LS)
- Steepest Gradient Descent (SGD)
- Three LMS variants: NLMS, Complex LMS, Constrained LMS
- Newton
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- Frequency Domain Adaptive Filter (FDAF)
- Summary

Newton

取消彩色之影响

Principle: Undo coloration effect SGD

SGD:

$$\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] - \alpha \underline{\bigtriangledown} \text{ with } \underline{\bigtriangledown} = -2\left(\underline{\mathbf{r}}_{ex} - \mathbf{R}_x \underline{\mathbf{w}}[k]\right)$$

Newton:

$$\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] - \alpha \mathbf{R}_x^{-1} \underline{\bigtriangledown} \Rightarrow$$

白化

$$\boxed{\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \mathbf{R}_x^{-1} \cdot \left(\underline{\mathbf{r}}_{ex} - \mathbf{R}_x \underline{\mathbf{w}}[k]\right)}$$

Note: General update rule $\underline{\mathbf{w}} = \underline{\mathbf{w}} - \alpha \underline{\mathbf{U}}$

$\underline{\mathbf{U}}$ must be such that each iteration J decreases

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Newton

Convergence Newton algorithm: 消除了R_x对收敛的影响

$$\underline{\mathbf{d}}[k+1] = \left(\mathbf{I} - 2\alpha \mathbf{R}_x^{-1}\mathbf{R}_x\right)\underline{\mathbf{d}}[k] = (1-2\alpha)\underline{\mathbf{d}}[k]$$

Conclusion:

$\underline{d}[k] = (1-2\alpha)^k \underline{d}[0]$

For $|1-2\alpha| < 1 \Leftrightarrow 0 < \alpha < 1$

$$\lim_{k\to\infty} E\{\underline{\mathbf{d}}[k]\} = \underline{\mathbf{0}} \Leftrightarrow \lim_{k\to\infty} E\{\underline{\mathbf{w}}[k]\} = \underline{\mathbf{w}}_o = \mathbf{R}_x^{-1}\underline{\mathbf{r}}_{ex}$$

用E{·}是为了和"LMS"保持一致进行对比

$\mathbf{R}_x^{-1}$ causes whitening of input x :

w的每一个元素

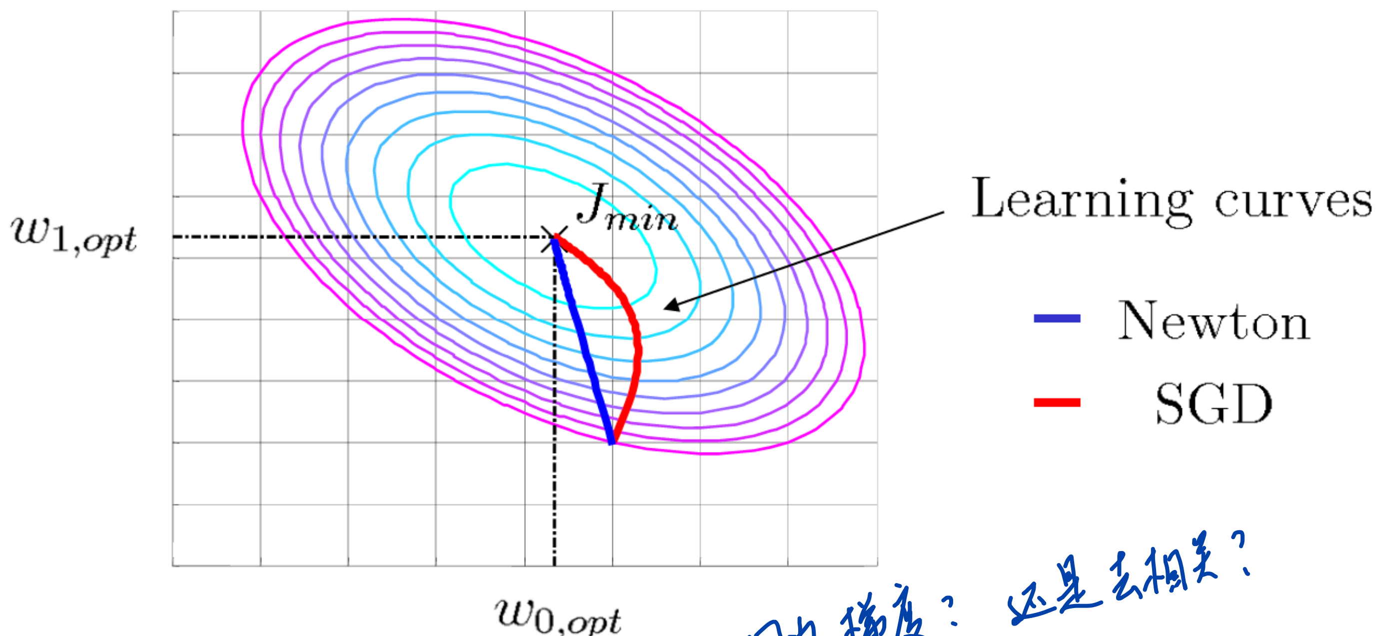
- All weights have same convergence!
- Equivalent to SGD with white noise input!

R_x的特征值相等?

Newton

Learning curves Newton vs. SGD in contour plot

Coloured input process with $\Gamma_x = \lambda_{max}/\lambda_{min} = 3$



因为梯度？还是去相关？

Note: SGD-curve each iteration orthogonal to contourplot J

Newton-curve point each iteration towards J_{min}

白化改变了梯度？

① first view: $\underline{\nabla} = \underline{\nabla}_{SGD} = -2(\underline{r}_{ex} - R_x \underline{w})$

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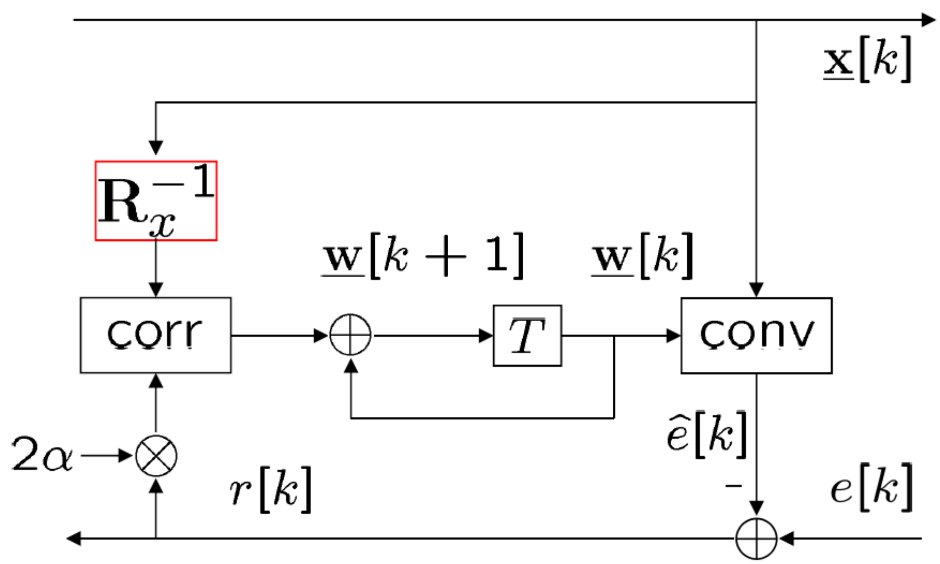
Newton

② *Another view:* $\hat{\underline{\nabla}}_{LMS} = -2\,\underline{x}[k]\,r[k]$

By replacing $\underline{\nabla} \rightarrow \hat{\underline{\nabla}}_{LMS} = \underline{\mathbf{x}}[k]r[k]$

白化

$\Rightarrow$ "LMS/Newton": $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \mathbf{R}_x^{-1}\underline{\mathbf{x}}[k]r[k]$



$\mathbf{R}_x^{-1}$ causes whitening of input x

Newton

收敛快（需要的步数少）
每一步计算复杂

Practical problems Newton:

Autocorrelation matrix $\mathbf{R}_x$:

- Not known in general
- May change in time (non-stationary process)
- Inversion is very expensive (many MIPS)

Complexity Newton: Huge

$\Rightarrow$ Need for efficient solution with estimate of $\mathbf{R}_x$

$\Rightarrow$ RLS; FDAF; etc.

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RLS

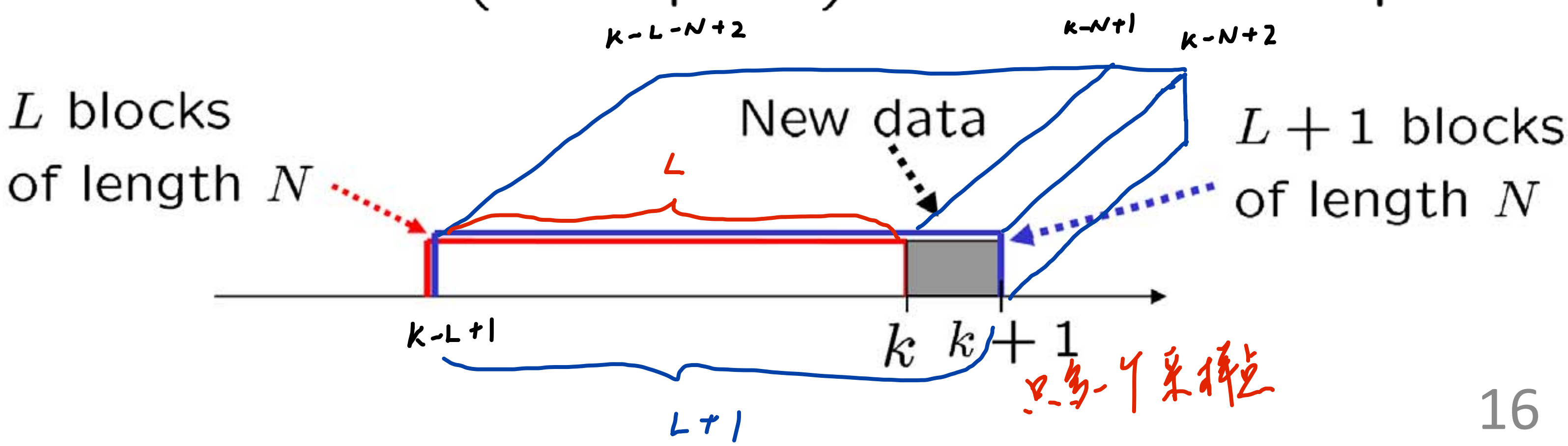
Recursive Least Squares

For L fixed, least squares problem becomes:

$$\min_{\underline{\mathbf{w}}} |\underline{\mathbf{e}} - \mathbf{X}\cdot\underline{\mathbf{w}}|^2 \;\Rightarrow\; \underline{\mathbf{w}}_{LS} = \underbrace{(\mathbf{X}^t\mathbf{X})^{-1}}_{\bar{R}_x^{-1}} \cdot \underbrace{(\mathbf{X}^t\underline{\mathbf{e}})}_{\bar{r}_{ex}}$$

RLS concept for $k \to k+1$: 每次迭代使用"L+1"blocks数据 (L+1)·N
除了 k=0时?

Find recursive (=adaptive) solution for LS problem



RLS

Current solution (time k) :

Based on L data vectors, each of length N

$$
\begin{aligned}
\underline{\mathbf{w}}_{LS}^{L}[k] &= (\overline{\mathbf{R}}_x^L[k])^{-1} \cdot \underline{\overline{\mathbf{r}}}_{ex}^L[k] \\
&= \Big(\underbrace{(\mathbf{X}^L[k])^t}_{N\times L}\underbrace{\mathbf{X}^L[k]}_{L\times N}\Big)^{-1} \cdot \underbrace{(\mathbf{X}^L[k])^t}_{N\times L}\underbrace{\underline{\mathbf{e}}^L[k]}_{L\times 1}
\end{aligned}
$$

$$
\mathbf{X}^L[k] = \begin{pmatrix} \underline{\mathbf{x}}^t[k] \\ \underline{\mathbf{x}}^t[k-1] \\ \vdots \\ \underline{\mathbf{x}}^t[k-L+1] \end{pmatrix} \quad \underline{\mathbf{e}}^L[k] = \begin{pmatrix} e[k] \\ e[k-1] \\ \vdots \\ e[k-L+1] \end{pmatrix}
$$

Similar result for L+1

"RLS" 原"LS"中求迭代

Compute solution at time $k+1$:

$$
\begin{aligned}
\underset{N\cdot 1}{\underline{\mathbf{w}}_{LS}^{L+1}[k+1]} &= (\overline{\mathbf{R}}_x^{L+1}[k+1])^{-1} \cdot \underline{\mathbf{r}}_{ex}^{L+1}[k+1] \\
&= \Big(\underbrace{(\mathbf{X}^{L+1}[k+1])^t}_{N\times(L+1)} \underbrace{\mathbf{X}^{L+1}[k+1]}_{(L+1)\times N} \Big)^{-1}_{N\times N} \cdot \\
&\quad \cdot \underbrace{(\mathbf{X}^{L+1}[k+1])^t}_{N\times(L+1)} \underbrace{\underline{\mathbf{e}}^{L+1}[k+1]}_{(L+1)\cdot 1} \quad N\cdot 1
\end{aligned}
$$

With $\quad \mathbf{X}^{L+1}[k+1] = \begin{pmatrix} \underline{\mathbf{x}}^t[k+1] \\ \underline{\mathbf{x}}^t[k] \\ \vdots \\ \underline{\mathbf{x}}^t[k-L+1] \end{pmatrix}_{(L+1)\cdot N}$

$$
\underline{\mathbf{e}}^{L+1}[k+1] = (e[k+1], e[k], \cdots, e[k-L+1])^t_{(L+1)\times 1}
$$

RLS

$\overline{R}_x^L[k] = X^L[k]^t X[k] = \sum_{i=0}^{L-1} \underline{x}[k-i]\,\underline{x}^t[k-i]$

Observe:

$$\overline{\mathbf{R}}_x^{L+1}[k+1] = \sum_{i=0}^{L} \underline{\mathbf{x}}[k+1-i]\underline{\mathbf{x}}^t[k+1-i] \quad \text{共}(L+1)\text{项}$$

$X^{L+1}[k+1]^t X^{L+1}[k+1]$

$$\overset{\sum_{i=1}^{L}}{} = \overline{\mathbf{R}}_x^{L}[k] + \underbrace{\underline{\mathbf{x}}[k+1]\cdot\underline{\mathbf{x}}^t[k+1]}_{N\times N\ \text{矩阵}} \quad \to i=0$$

$$\underline{\overline{\mathbf{r}}}_{ex}^{L+1}[k+1] = \sum_{i=0}^{L} \underline{\mathbf{x}}[k+1-i]e[k+1-i]$$

$$= \underline{\overline{\mathbf{r}}}_{ex}^{L}[k] + \underline{\mathbf{x}}[k+1]e[k+1]$$

From matrix inversion lemma (see Appendix): $\Rightarrow$

$N\cdot(L+1)\cdot(L+1)\cdot N$?

$$\overline{\mathbf{R}}_x^{-1}[k+1] = \overline{\mathbf{R}}_x^{-1}[k] - \frac{\overline{\mathbf{R}}_x^{-1}[k]\underline{\mathbf{x}}[k+1]\underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}_x^{-1}[k]}{1+\underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}_x^{-1}[k]\underline{\mathbf{x}}[k+1]}$$

$N(L+1)\cdot(L+1)\cdot N$?

Finally: $\underline{\mathbf{w}}[k+1] = \overline{\mathbf{R}}_x^{-1}[k+1]\cdot\underline{\overline{\mathbf{r}}}_{ex}[k+1]$

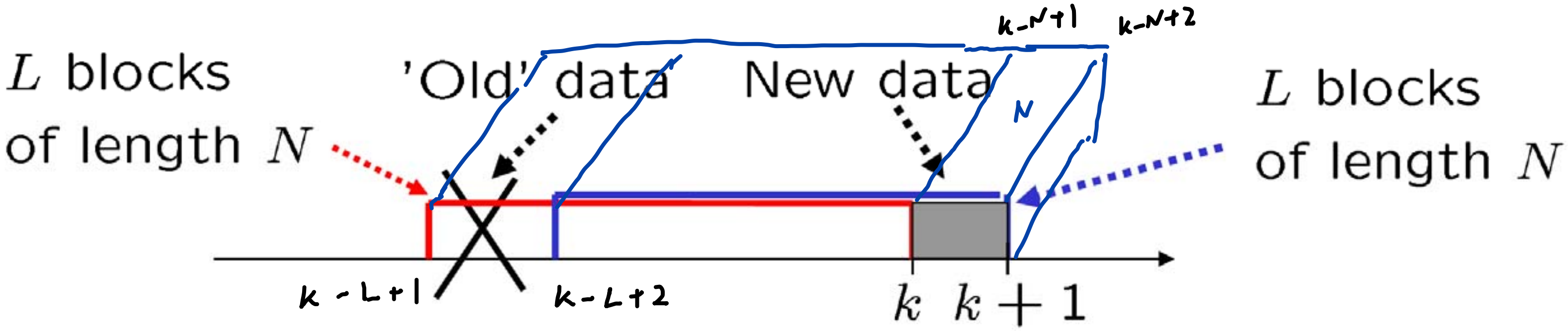
RLS

For adaptivity $\rightarrow$ more effective data window

每次迭代使用"L" blocks 数据

$L \cdot N$

Sliding window: Keep window length L constant



Note: Now we can write for autocorrelation

$N \cdot L \cdot L \cdot N$

$$\overline{\mathbf{R}}_x[k+1] = \overline{\mathbf{R}}_x[k] - \underline{\mathbf{x}}[k-L+1] \cdot \underline{\mathbf{x}}^t[k-L+1] + \underline{\mathbf{x}}[k+1] \cdot \underline{\mathbf{x}}^t[k+1]$$

$$\bar{R}_x^{L+1}[k+1] = \bar{R}_x^{L}[k] + \underline{x}[k+1]\,\underline{x}^t[k+1]$$

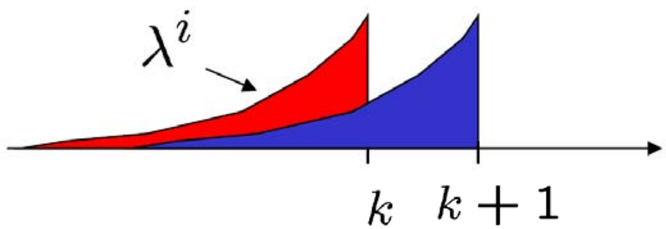
$N \cdot (L+1) \cdot (L+1) \cdot N$

$\Rightarrow$ Still very complex

RLS

Exponential window: Scale down data by factor λ

每次迭代使用数据量不同，第 i 次使用 (i+1) blocks 数据



$0 < \lambda < 1$: forget factor $(i+1)\cdot N$

$\frac{1}{1-\lambda}$: 'memory' of algorithm

$$\mathbf{X}[k] = \begin{pmatrix} \lambda^0 \underline{\mathbf{x}}^t[k] \\ \lambda^1 \underline{\mathbf{x}}^t[k-1] \\ \vdots \\ \underbrace{\lambda^k \underline{\mathbf{x}}^t[0]} \end{pmatrix}_{(k+1)\cdot N\ ?} \quad \underline{\mathbf{e}}[k] = \begin{pmatrix} \lambda^0 e[k] \\ \lambda^1 e[k-1] \\ \vdots \\ \lambda^k e[0] \end{pmatrix}_{(k+1)\cdot 1}$$

所有累积量均含于其中

$\lambda^{k-1} \underline{x}^t[1]$?

$$J[k] = \sum_{i=0}^{k} \lambda^i r^2[k-i] = \left(\underline{\mathbf{e}}^t[k] - \underline{\mathbf{w}}^t[k]\mathbf{X}^t[k]\right)\left(\underline{\mathbf{e}}[k] - \mathbf{X}[k]\underline{\mathbf{w}}[k]\right)$$

$$\bar{R}_x[k] = \sum_{i=0}^{k} \lambda^i \underline{x}[k-i] \cdot \lambda^i \underline{x}^t[k-i] = \sum_{i=0}^{k} \lambda^{2i} \underline{x}[k-i]\, \underline{x}^t[k-i]$$

$\bar{R}_x[k] = \sum_{i=0}^{k} \lambda^{2i} \underline{x}[k-i]\, \underline{x}^t[k-i]$

RLS

$\bar{R}_x[k+1] = \sum_{i=0}^{k+1} \lambda^{2i} \underline{x}[k+1-i]\, \underline{x}^t[k+1-i] = \sum_{i=1}^{k+1} [\cdot] + \sum_{i=0}^{0} \cdots$

$= \lambda^2 \bar{R}_x[k] + \lambda^0 \underline{x}[k+1]\, \underline{x}^t[k+1]$

Observe:

$$\overline{\mathbf{R}}_x[k+1] = \lambda^2 \overline{\mathbf{R}}_x[k] + \underline{\mathbf{x}}[k+1] \cdot \underline{\mathbf{x}}^t[k+1]$$

$$\underline{\overline{\mathbf{r}}}_{ex}[k+1] = \lambda^2 \underline{\overline{\mathbf{r}}}_{ex}[k] + \underline{\mathbf{x}}^t[k+1] e[k+1]$$

From matrix inversion theorem (see Appendix):

$$\overline{\mathbf{R}}^{-1}[k+1] = \lambda^{-2} \left(\overline{\mathbf{R}}^{-1}[k] - \underline{\mathbf{g}}[k+1] \cdot \underline{\mathbf{x}}^t[k+1] \overline{\mathbf{R}}^{-1}[k] \right)$$

with gain vector: $\underline{\mathbf{g}}[k+1] = \dfrac{\overline{\mathbf{R}}^{-1}[k] \underline{\mathbf{x}}[k+1]}{\lambda^2 + \underline{\mathbf{x}}^t[k+1] \overline{\mathbf{R}}^{-1}[k] \underline{\mathbf{x}}[k+1]}$

New weight vector:

$$\underline{\mathbf{w}}[k+1] = \overline{\mathbf{R}}_x^{-1}[k+1] \cdot \underline{\overline{\mathbf{r}}}_{ex}[k+1]$$

RLS

RLS algorithm

Initialization:

$$\underline{\overline{\mathbf{r}}}_{ex}[0] = \underline{\mathbf{0}} \ ; \ \overline{\mathbf{R}}_x^{-1}[0] = \delta \mathbf{I} \ \text{ with } \delta \text{ small}$$

For $k \geq 0$:

$$\underline{\mathbf{g}}[k+1] = \frac{\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]}{\lambda^2 + \underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]}$$

$$\overline{\mathbf{R}}^{-1}[k+1] = \lambda^{-2}\left(\overline{\mathbf{R}}^{-1}[k] - \underline{\mathbf{g}}[k+1]\cdot\underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]\right)$$

$$\underline{\overline{\mathbf{r}}}_{ex}[k+1] = \lambda^2\underline{\overline{\mathbf{r}}}_{ex}[k] + \underline{\mathbf{x}}[k+1]e[k+1]$$

$$\underline{\mathbf{w}}[k+1] = \overline{\mathbf{R}}_x^{-1}[k+1]\cdot\underline{\overline{\mathbf{r}}}_{ex}[k+1]$$

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RLS

Compare RLS with "LMS/Newton"

"LMS/Newton": $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \mathbf{R}_x^{-1} \underline{\mathbf{x}}[k] r[k]$

Update RLS can be rewritten as (see Appendix):

[No Title]

$$\begin{aligned} \underline{\mathbf{w}}[k+1] &= \underline{\mathbf{w}}[k] + \underline{\mathbf{g}}[k+1]\left(e[k+1] - \underline{\mathbf{x}}^t[k+1]\underline{\mathbf{w}}[k]\right) \\ &= \underline{\mathbf{w}}[k] + \underline{\mathbf{g}}[k+1] r[k+1] \end{aligned}$$

with gain vector:

$$\underline{\mathbf{g}}[k+1] = \frac{\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]}{\lambda^2 + \underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]}$$

RLS

Notes on RLS:

$\underline{\mathbf{w}}[\infty] = \underline{\mathbf{w}}_o$

Complexity: $O(N^2)$ per time update

Window length increases when time increases!

Exhibits unstable roundoff error accumulation

误差的累积

通过平均的方法

消除累积误差

RLS basis for many practical algorithms

Decorrelation takes place in algorithm

去相关

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FDAF

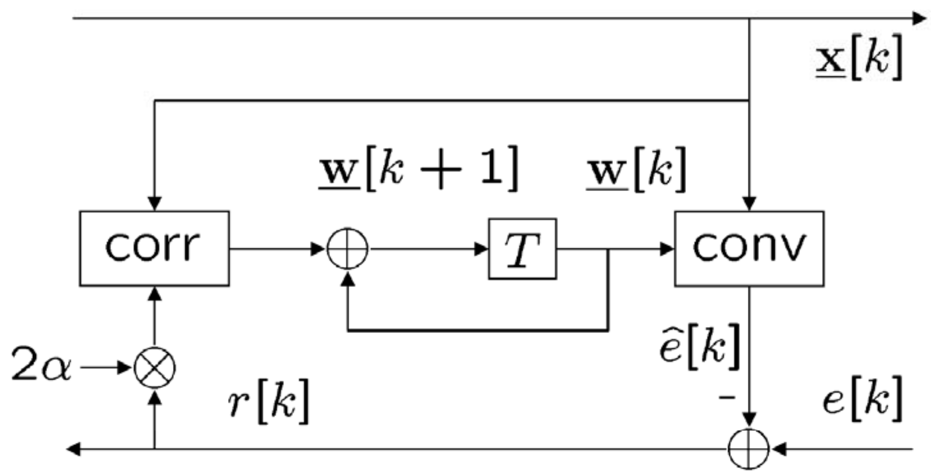
Frequency Domain Adaptive Filter

Alternative for LMS/Newton and RLS

FDAF

Frequency Domain Adaptive Filter

First translate LMS to frequency domain:



LMS weight update:

$$\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \underline{\mathbf{x}}[k] r[k]$$

Filter output:

$$\hat{e}[k] = \underline{\mathbf{x}}^t[k] \cdot \underline{\mathbf{w}}[k]$$

FDAF

Apply filter operation in frequency domain:

$$\mathbf{F}\cdot\underline{\mathbf{x}}[k] = \underline{\mathbf{X}}[k] = \left(X_0[k], X_1[k], \cdots, X_{N-1}[k]\right)^t$$

$$\mathbf{F}^{-1}\cdot\underline{\mathbf{w}}[k] = \underline{\mathbf{W}}[k] = \left(W_0[k], W_1[k], \cdots, W_{N-1}[k]\right)^t$$

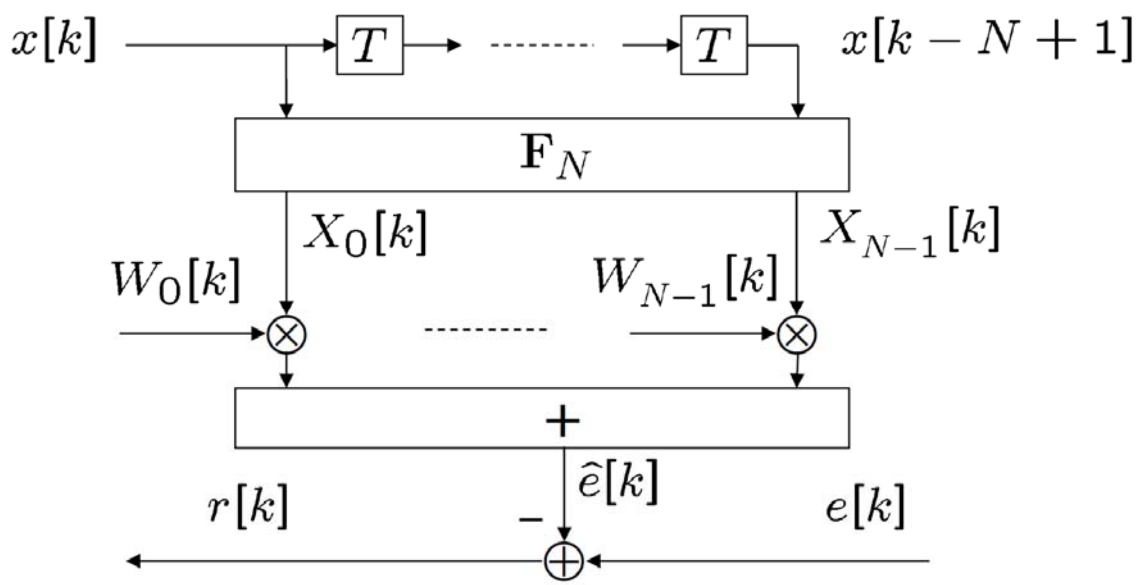
$$Note:\ \mathbf{F}^{-1} = \frac{1}{N}\mathbf{F}^*$$

Filter output:

$$\begin{aligned}\hat{e}[k] &= \sum_{i=0}^{N-1} x[k-i]w_i[k] = \underline{\mathbf{x}}^t[k]\cdot\underline{\mathbf{w}}[k] \\ &= \underline{\mathbf{x}}^t[k]\mathbf{F}\cdot\mathbf{F}^{-1}\underline{\mathbf{w}}[k] = (\mathbf{F}\underline{\mathbf{x}}[k])^t\cdot\left(\mathbf{F}^{-1}\underline{\mathbf{w}}[k]\right) \\ &= \underline{\mathbf{X}}^t[k]\cdot\underline{\mathbf{W}}[k] = \sum_{l=0}^{N-1} X_l[k]W_l[k]\end{aligned}$$

ADSP

FDAF



Notes:

- Weights perfom inverse transform
- Use DFT symmetry to reduce complexity
- Separate frequency bins 'uncorrelated' (large N)

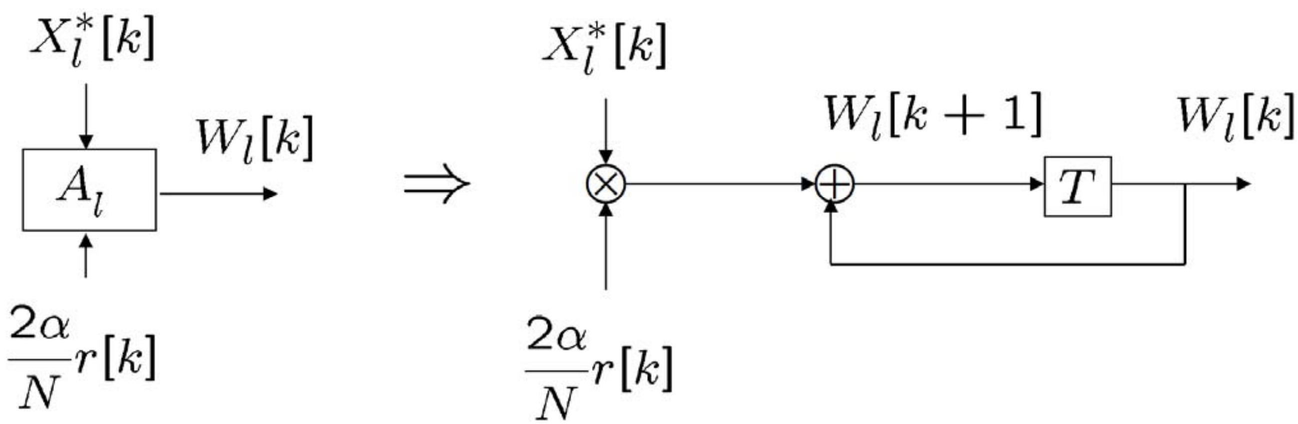
FDAF

Apply LMS update in frequency domain:

Multiply both sides by $\mathbf{F}^{-1}$ $\Rightarrow$

$$\mathbf{F}^{-1}\underline{\mathbf{w}}[k+1] = \mathbf{F}^{-1}\underline{\mathbf{w}}[k] + 2\alpha\mathbf{F}^{-1}\underline{\mathbf{x}}[k]r[k] \quad \Rightarrow$$

$$\underline{\mathbf{W}}[k+1] = \underline{\mathbf{W}}[k] + \frac{2\alpha}{N}\underline{\mathbf{X}}^*[k]r[k]$$



FDAF

Improve convergence properties easily by

Decorrelation by power normalization:

FDAF algorithm:

$$\underline{\mathbf{W}}[k+1] = \underline{\mathbf{W}}[k] + \frac{2\alpha}{N}\mathbf{P}^{-1}\underline{\mathbf{X}}^{*}[k]r[k]$$

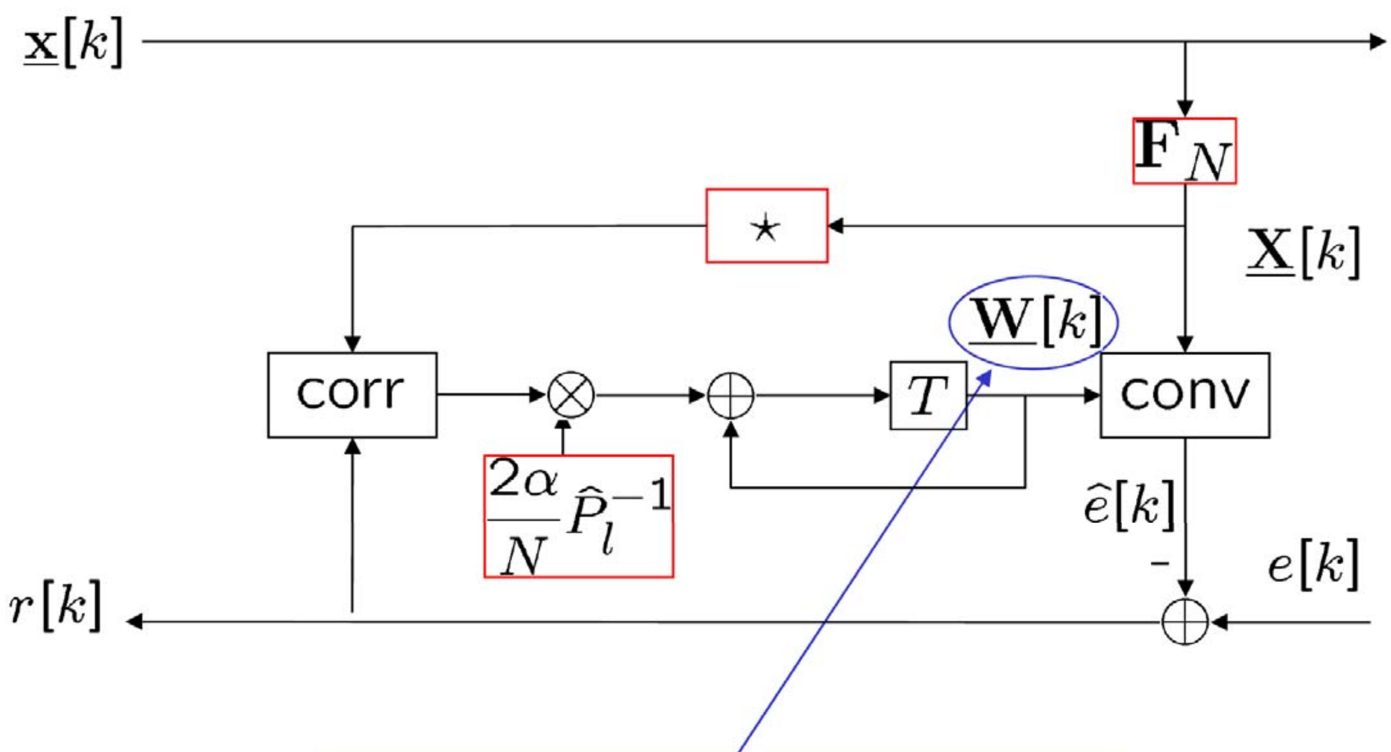
$$\mathbf{P} = diag\{\underline{\mathbf{P}}\} \text{ with } P_l = \frac{1}{N}E\{|X_l[k]|^2\}$$

In practice (e.g.):

$$\widehat{P}_l[k+1] = \beta\widehat{P}_l[k] + (1-\beta)\frac{|X_l[k]|^2}{N} \;\forall l$$

FDAF

Simplified realization scheme:



Note: $\underline{\mathbf{W}}[k] = \mathbf{F}^{-1} \cdot \underline{\mathbf{w}}[k]$

FDAF

Average behaviour FDAF:

With $\underline{\mathbf{D}} = \mathbf{F}^{-1}\underline{\mathbf{d}} = \mathbf{F}^{-1}(\underline{\mathbf{w}} - \underline{\mathbf{w}}_o) = \underline{\mathbf{W}} - \underline{\mathbf{W}}_o$

FDAF can be rewritten as:

$$\underline{\mathbf{D}}[k+1] \;=\; \left(\mathbf{I} - \frac{2\alpha}{N}\mathbf{P}^{-1}\underline{\mathbf{X}}^*[k]\underline{\mathbf{X}}^t[k]\right)\underline{\mathbf{D}}[k] + \frac{2\alpha}{N}\mathbf{P}^{-1}\underline{\mathbf{X}}^*[k]r_{min}[k]$$

Different bins uncorrelated: $\Rightarrow \dfrac{E\{\underline{\mathbf{X}}^*[k]\underline{\mathbf{X}}^t[k]\}}{N} \approx \mathbf{P}$

Thus $E\{\underline{\mathbf{D}}[k+1]\} \;\approx\; (1-2\alpha)\,E\{\underline{\mathbf{D}}[k]\}$

FDAF

Notes FDAF:

- Both update algorithm and filter are transformed
- FFT (or DFT) is fixed transform --> easy but not exact

Conclusions FDAF:

$$\lim_{k\to\infty} E\{\underline{\mathbf{D}}[k]\} = \underline{\mathbf{0}} \quad \Leftrightarrow \quad \lim_{k\to\infty} E\{\underline{\mathbf{W}}[k]\} = \underline{\mathbf{W}}_o = \mathbf{F}^{-1}\underline{\mathbf{w}}_o$$

$\mathbf{P}_x^{-1}$: update of each bin is power normalized

All weights have (in average) similar convergence!

Equivalent to NLMS with white noise input!

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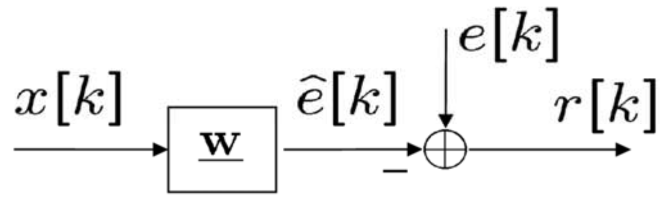
ADSP Part B: Adaptive signal processing

Focus on single channel adaptive algorithms using FIR structure

- Applications Adaptive Algorithms
- Minimum Mean Square Error (MMSE)
- Constrained MMSE
- Least Square (LS)
- Steepest Gradient Descent (SGD)
- Three LMS variants: NLMS, Complex LMS, Constrained LMS
- Newton
- Recursive Least Squares (RLS)
- Frequency Domain Adaptive Filter (FDAF)
- Summary

Summary



	MMSE	LS
Auto correlation	$\mathbf{R}_x = E\{\underline{\mathbf{x}}[k] \cdot \underline{\mathbf{x}}^t[k]\}$	$\overline{\mathbf{R}}_x = \mathbf{X}^t \cdot \mathbf{X}$
Cross correlation	$\underline{\mathbf{r}}_{ex} = E\{e[k] \cdot \underline{\mathbf{x}}[k]\}$	$\underline{\overline{\mathbf{r}}}_{ex} = \mathbf{X}^t \cdot \underline{\mathbf{e}}$
Error J	$E\{r^2[k]\}$	$\sum_{i=0}^{L-1} r^2[k-i]$
Criterion	$\min_{\underline{\mathbf{w}}}\{E\{r^2[k]\}\}$	$\min_{\underline{\mathbf{w}}} \lvert\underline{\mathbf{e}} - \mathbf{X} \cdot \underline{\mathbf{w}}\rvert^2$
Opt. solution $\underline{\mathbf{w}}_o$	$\mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{ex}$	$\overline{\mathbf{R}}_x^{-1} \cdot \underline{\overline{\mathbf{r}}}_{ex}$
Min. error J_{min}	$E\{e^2\} - \underline{\mathbf{r}}_{ex}^t \mathbf{R}_x^{-1} \underline{\mathbf{r}}_{ex}$	$\underline{\mathbf{e}}^t \underline{\mathbf{e}} - \underline{\overline{\mathbf{r}}}_{ex}^t \overline{\mathbf{R}}_x^{-1} \underline{\overline{\mathbf{r}}}_{ex}$

ADSP

Summary

Set of constraints: $\mathbf{C}^t \cdot \underline{\mathbf{w}} = \underline{\mathbf{f}}$

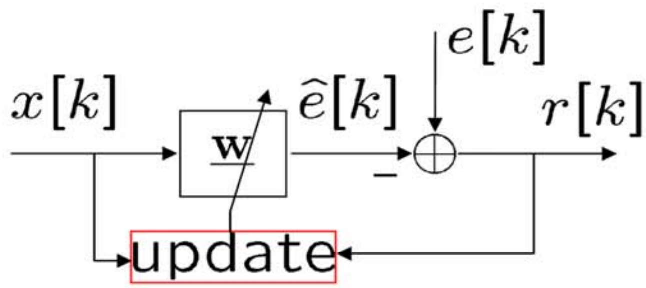
Solution for $N \geq M$: $\underline{\mathbf{w}}^c = \mathbf{C}\left(\mathbf{C}^t\mathbf{C}\right)^{-1}\underline{\mathbf{f}}$

Solution for $N > M$ with MMSE:

$$\underline{\mathbf{w}}_o^c = \underline{\mathbf{w}}_o + \mathbf{R}_x^{-1}\mathbf{C}\left(\mathbf{C}^t\mathbf{R}_x^{-1}\mathbf{C}\right)^{-1}\left(\underline{\mathbf{f}} - \mathbf{C}^t\underline{\mathbf{w}}_o\right)$$

Equivalently: $\underline{\mathbf{w}}_o^c = \mathbf{R}_x^{-1}\mathbf{C}\left(\mathbf{C}^t\mathbf{R}_x^{-1}\mathbf{C}\right)^{-1}\underline{\mathbf{f}}$

Summary



Simple adaptive algorithms (no decorrelation):

$$\text{SGD} \;:\; \underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha\left(\underline{\mathbf{r}}_{ex} - \mathbf{R}_x\underline{\mathbf{w}}[k]\right)$$

$$\text{(Complex)(N)LMS} \;:\; \underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + \frac{2\alpha}{\hat{\sigma}_x^2}\underline{\mathbf{x}}[k]r^*[k]$$

Constrained LMS : $\mathrm{C}^t \cdot \underline{\mathrm{w}} = \underline{\mathrm{f}}$

$$\underline{\mathbf{w}}[k+1] = \tilde{\mathbf{P}} \cdot \left(\underline{\mathbf{w}}[k] + 2\alpha\underline{\mathbf{x}}[k]r[k]\right) + \underline{\mathbf{w}}[0]$$

$$\tilde{\mathbf{P}} = \mathbf{I} - \mathbf{C}\left(\mathbf{C}^t\mathbf{C}\right)^{-1}\mathbf{C}^t \text{ and } \underline{\mathbf{w}}[0] = \mathbf{C}\left(\mathbf{C}^t\mathbf{C}\right)^{-1}\underline{\mathbf{f}}$$

Summary

Algorithms with improved convergence:

"LMS/Newton" : $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \mathbf{R}_x^{-1} \cdot \underline{\mathbf{x}}[k] r[k]$

Newton : $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \mathbf{R}_x^{-1} \cdot (\underline{\mathbf{r}}_{ex} - \mathbf{R}_x \underline{\mathbf{w}}[k])$

RLS : $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + \underline{\mathbf{g}}[k+1] r[k+1]$

with $\underline{\mathbf{g}}[k+1] = \dfrac{\overline{\mathbf{R}}^{-1}[k] \underline{\mathbf{x}}[k+1]}{\lambda^2 + \underline{\mathbf{x}}^t[k+1] \overline{\mathbf{R}}^{-1}[k] \underline{\mathbf{x}}[k+1]}$

FDAF : $\underline{\mathbf{W}}[k+1] = \underline{\mathbf{W}}[k] + \dfrac{2\alpha}{N} \mathbf{P}^{-1} \underline{\mathbf{X}}^*[k] r[k]$

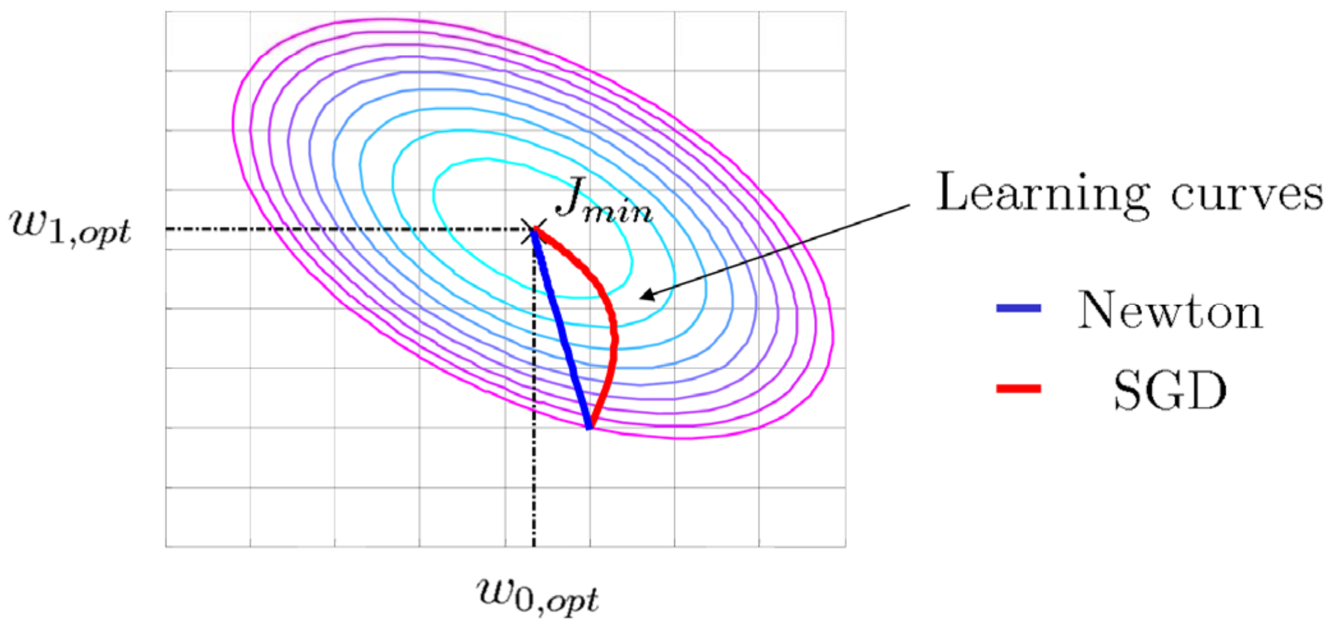
etc.

ADSP

Summary

Learning curves Newton vs. SGD in contour plot

Coloured input process with $\Gamma_x = \lambda_{max}/\lambda_{min} = 3$



Note: SGD-curve each iteration orthogonal to contourplot J

Newton-curve point each iteration towards J_{min}