Advanced Digital Signal Processing (ADSP)

徐林

上海科技大学
ShanghaiTech University

Beampattern of ULA

main properties

ULA Beampattern

Assumptions:

- Single source $s(t) = \mathrm{e}^{\mathrm{j}\omega t}$
- Frequency relations: $\omega = 2\pi \cdot f = 2\pi \cdot c/\lambda$
- Wavenumber λ
- Speed of propagation: c ($\approx$ 343 [m/sec])
- Directional Of Arrival (DOA): θ
- Far field $\Rightarrow$ Plane wave
- ULA with distance d between sensors
- J omnidirectional sensors
- Array aperture size: $L = J \cdot d$
- No noise, no interferences: $\underline{\mathbf{x}}[k] = \underline{\mathbf{a}}(\theta) \cdot s[k]$
- ASP unit: Single complex weight for each sensor

[No Title]

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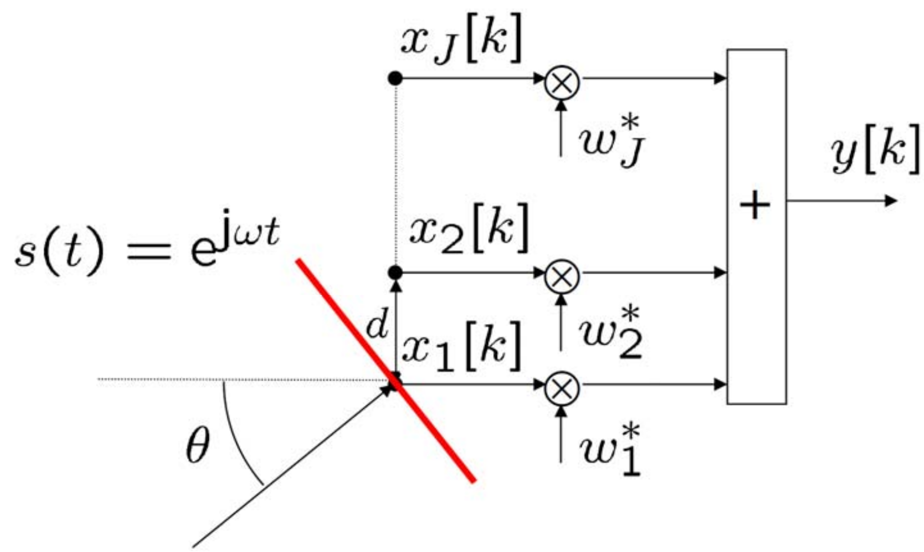

$$\Rightarrow \quad y[k] = \sum_{i=1}^{J} w_i^* x_i[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{x}}[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta) \cdot s[k]$$

$$\text{with: } \left(\underline{\mathbf{a}}(\theta)\right)_i = a_i(\theta) = \mathrm{e}^{-\mathrm{j}2\pi(i-1)\frac{d\sin(\theta)}{\lambda}}$$

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Thus: $y[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{x}}[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta) \cdot s[k] = r(\theta) \cdot s[k]$

Array response: $r(\theta) = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta)$

<u>Other names:</u> *angular repsonse* or *directivity pattern*

Array beam pattern: $B(\theta) = \frac{1}{J^2} \cdot |r(\theta)|^2$

<u>Comparison with FIR:</u>

Frequency response:

$$W(\omega) = \sum_{i=1}^{J} w_i \mathrm{e}^{-\mathrm{j}\omega(i-1)T} = \underline{\mathbf{w}}^t \cdot \underline{\mathbf{a}}(\omega)$$

with $a_i(\omega) = (\underline{\mathbf{a}}(\omega))_i = \mathrm{e}^{-\mathrm{j}\omega \cdot (i-1) \cdot T}$

$\Rightarrow$ Depending on ω, **not** on θ !

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Notes:

- Array response vector: (noise free) response to unit-amplitude plane wave from direction θ
- Nonideal sensor characteristics can be incorporated
- Weights effect both temporal and spatial response
- Vector space interpretation:
  Angle between $\underline{\mathbf{w}}$ and $\underline{\mathbf{a}}$ determine response
- To evaluate beampattern: choose all weights equal

$$\underline{\mathbf{w}} = (1, \cdots, 1)^t$$

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$$B(\theta) \;=\; \frac{1}{J^2}|\underline{\mathbf{1}}^t \cdot \underline{\mathbf{a}}(\theta)|^2 = \frac{1}{J^2}\left|\sum_{i=1}^{J} \mathrm{e}^{-\mathrm{j}2\pi(i-1)\frac{d}{\lambda}\sin(\theta)}\right|^2$$

$$= \frac{1}{J^2}\left|\frac{1-\mathrm{e}^{-\mathrm{j}J2\pi\frac{d}{\lambda}\sin(\theta)}}{1-\mathrm{e}^{-\mathrm{j}2\pi\frac{d}{\lambda}\sin(\theta)}}\right|^2 = \frac{1}{J^2}\left|\frac{\sin(J\pi\frac{d}{\lambda}\sin(\theta))}{\sin(\pi\frac{d}{\lambda}\sin(\theta))}\right|^2$$

Important parameters:

- DOA θ
- Ratio $\frac{d}{\lambda}$ (everything scales with wavelength)
- Number of sensors J
- Element spacing d
- Array aperture $L = J \cdot d$

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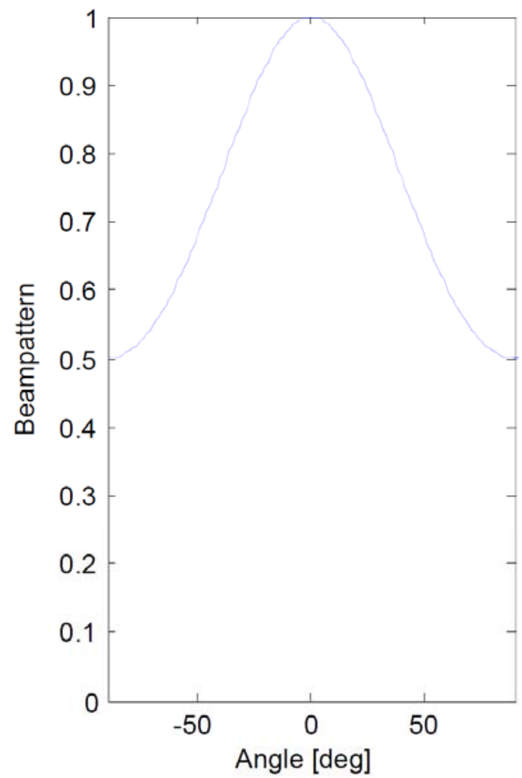
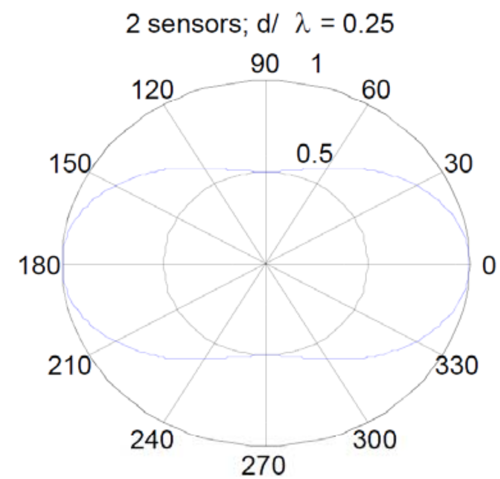
Example: $J = 2$ and $\frac{d}{\lambda} = \frac{1}{4}$ `B = abs(a'*w*w'*a)`

Linear scale!



`plot(theta*180/pi,B)`



`polar(theta,B)`

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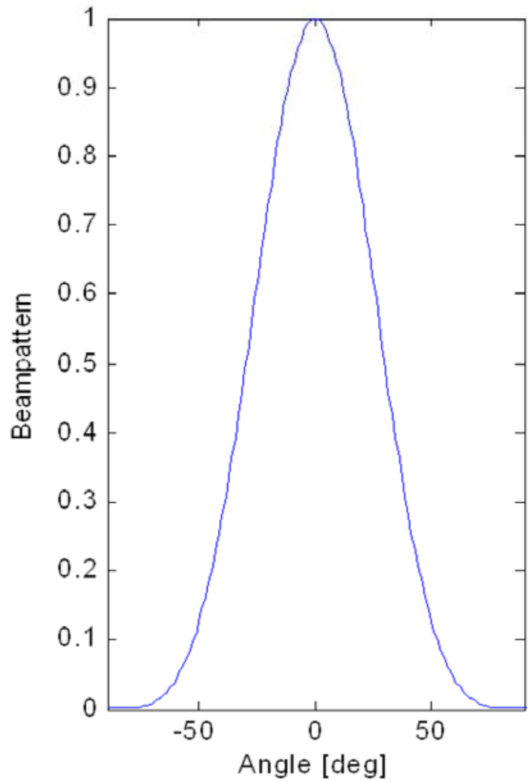
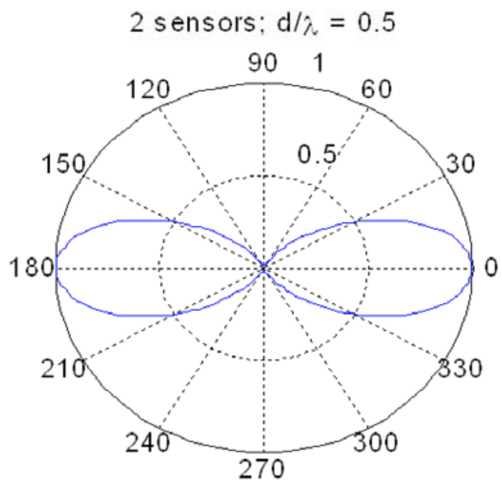
$$J = 2 \text{ and } \boxed{\frac{d}{\lambda} = \frac{1}{2}}$$





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Conclusions:

If $\frac{d}{\lambda} \ll \frac{1}{2} \Rightarrow$

- No exact cancelling at $\theta = \pm 90^o$
- Little difference with single sensor case!

If $\frac{d}{\lambda} = \frac{1}{2} \Rightarrow$

- Main lobe beamwidth (DOA 0^o): 60^o
- Nulls at: $\theta = \pm 90^o$

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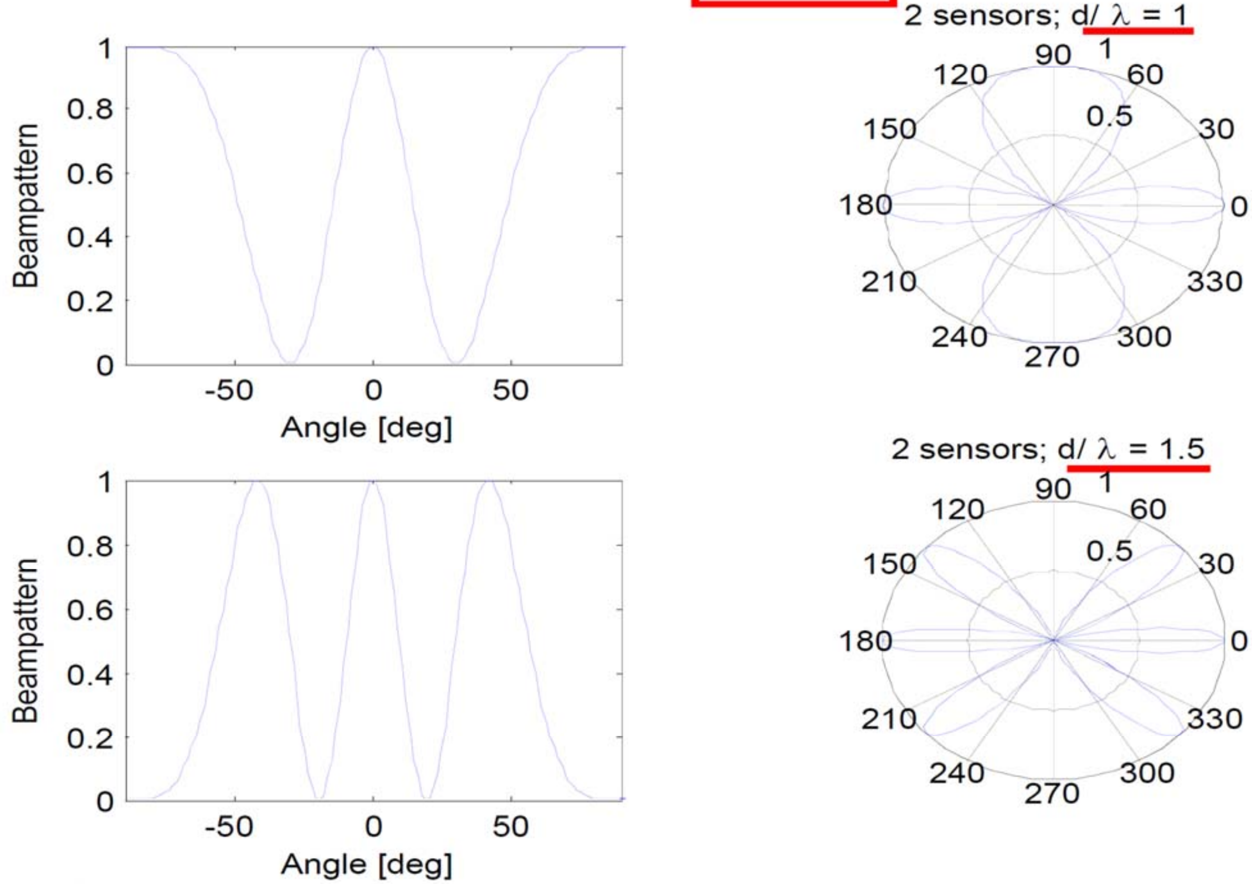
$$J = 2 \text{ and } \boxed{\frac{d}{\lambda} > \frac{1}{2}}$$



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Conclusions:

If $\frac{d}{\lambda} = 1 \;\; \Rightarrow$
- Nulls migrate to: $\theta = \pm 30^o$
- Another two sidelobes at: $\theta = \pm 90^o$

If $\frac{d}{\lambda} > \frac{1}{2} \;\; \Rightarrow$
- Main lobe beamwidth decreases
- More nulls $\Rightarrow$ Spatial aliasing

Spatial aliasing:

- Ambiguity in source locations
- Same response for sources at different positions
- Occurs if sensors are too far away (relative to λ)

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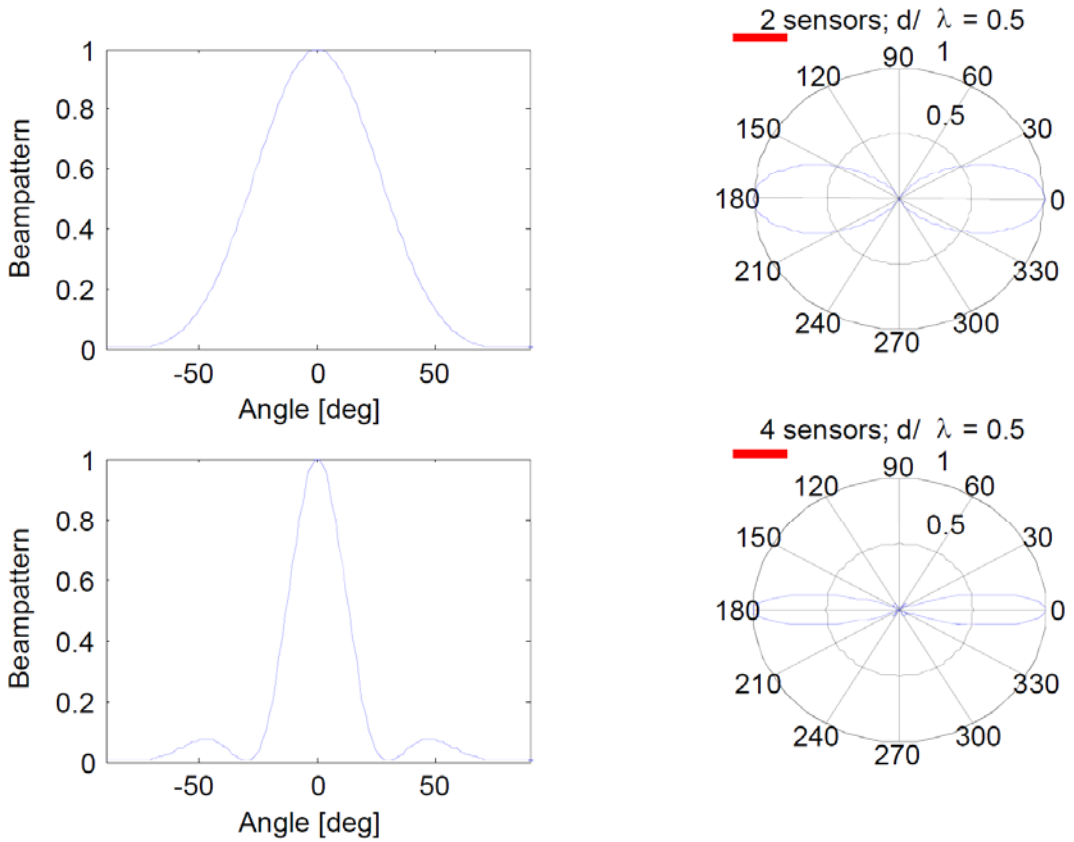
Increase aperture $L = J \cdot d$ by $\boxed{J \uparrow}$, fixed $\dfrac{d}{\lambda} = \dfrac{1}{2}$



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Conclusions Beampattern:

- For $J \uparrow \;\Rightarrow\;$ Mainlobe smaller $\;\Rightarrow\;$ more sensitive
- For $J \uparrow \;\Rightarrow\;$ array aperture $\uparrow$
- For $d/\lambda < 1/2 \;\Rightarrow\;$ No spatial aliasing
- For $d/\lambda \geq 1 \;\Rightarrow\;$ Pattern repeats at $\theta = \arcsin(\frac{\lambda}{d})$
- Zeros occur at $\theta = \arcsin\left(i \cdot \frac{1}{J} \cdot \frac{\lambda}{d}\right)$ with $i \in \mathcal{Z}$
- Main lobe at $\theta = 2\arcsin\left(\frac{1}{J} \cdot \frac{\lambda}{d}\right)$

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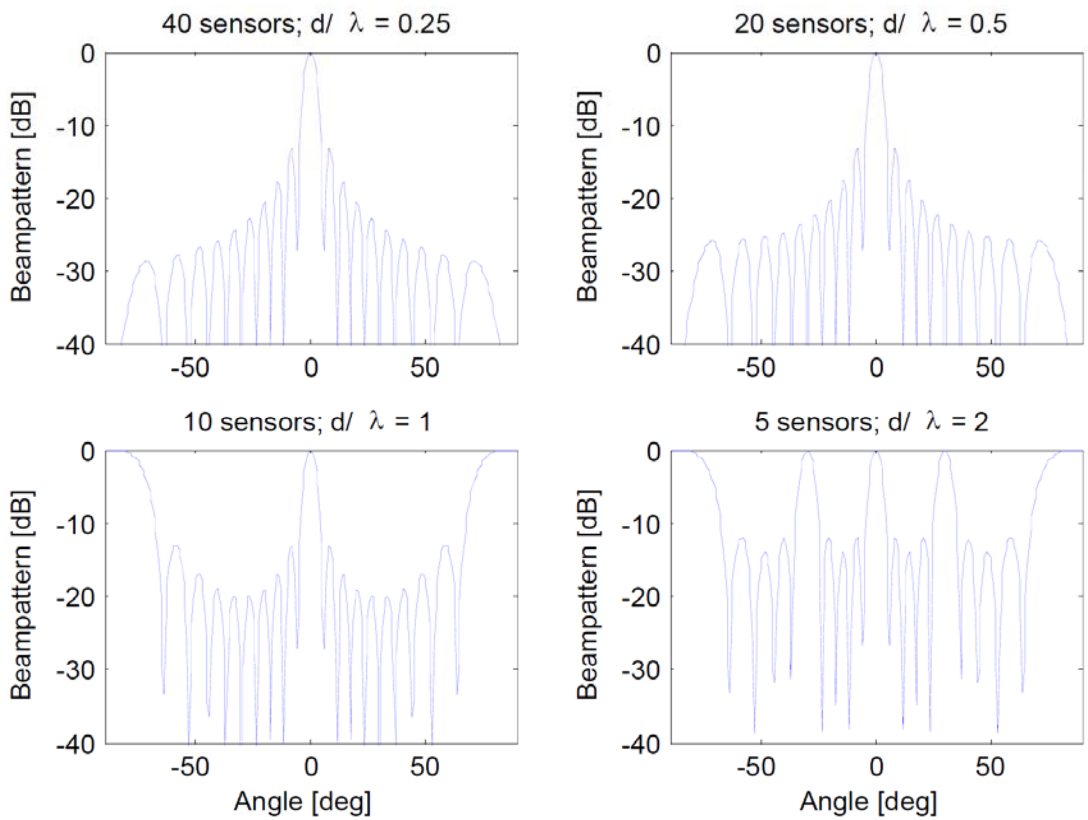
Variable element spacing d, fixed $L = J \cdot d = 10\lambda$

[dB]

scale!




```
plot(theta*180/pi,10*log10(B))
```

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Conclusion fixed aperture:

- $d < \frac{\lambda}{2} \Leftrightarrow$ Oversampling:

  No additional info

- $d > \frac{\lambda}{2} \Leftrightarrow$ Undersampling:

  Grating lobes $=$ spatial ambiguities

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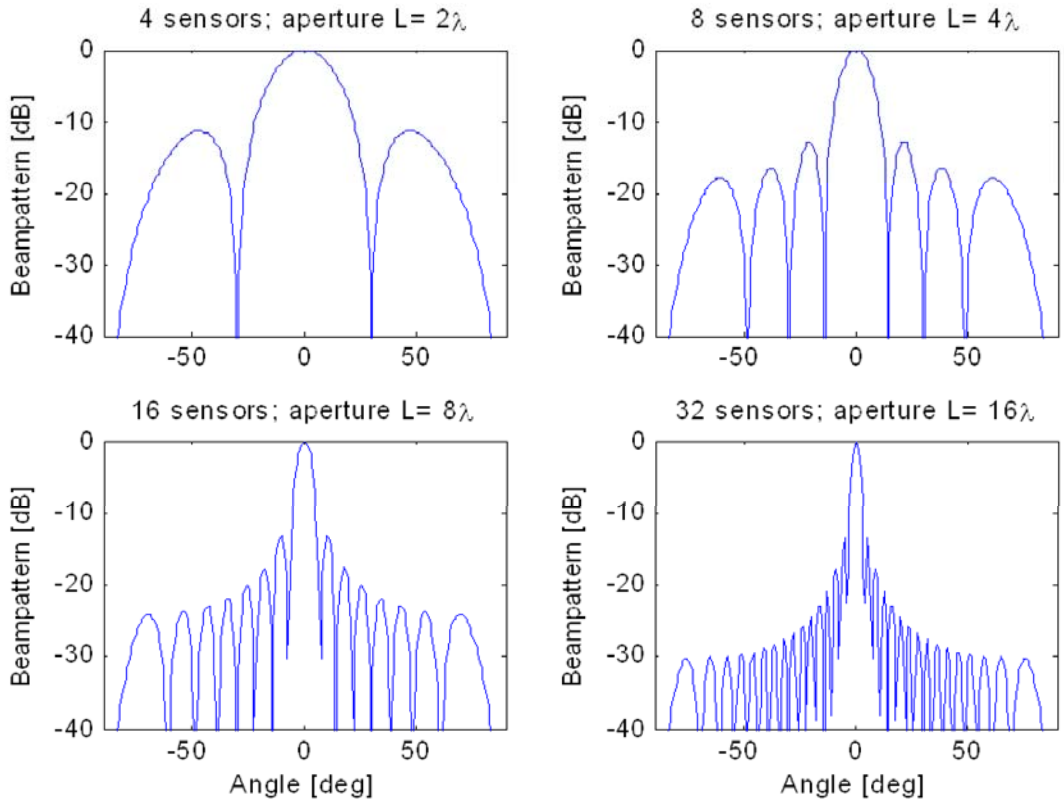
Variable aperture size $L = J \cdot d$ and fixed $d = \frac{\lambda}{2}$



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Conclusion fixed element spacing:

- Variable aperture $\Leftrightarrow$ variable resolution
- $L \uparrow$ $\Leftrightarrow$ Improved resolution $\Leftrightarrow$

  Better angle estimation capabilities

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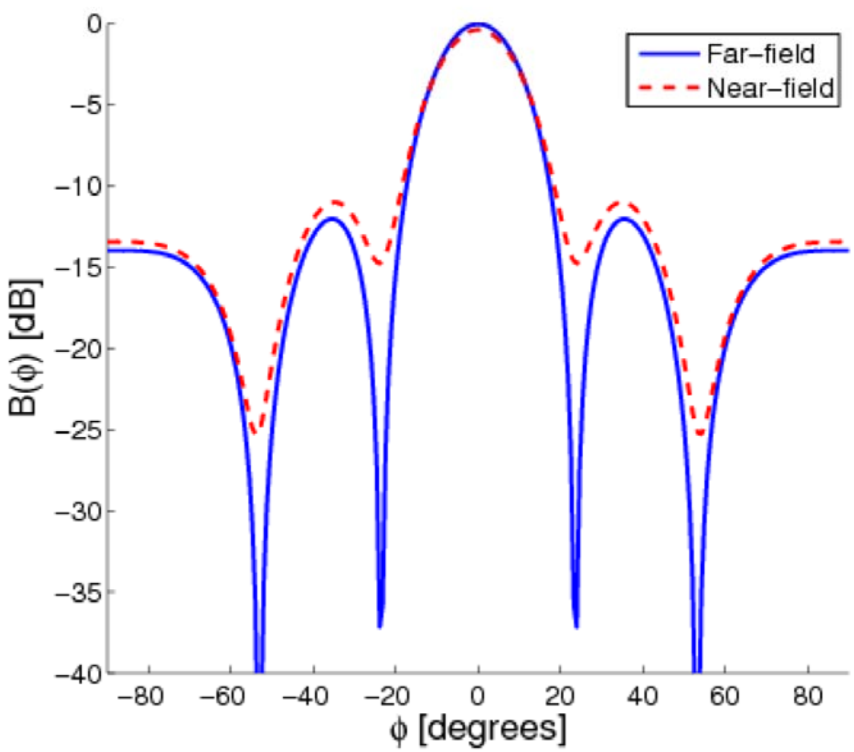
$$\text{Near field: } |\underline{\mathrm{p}}| < \frac{2L^2}{\lambda}$$



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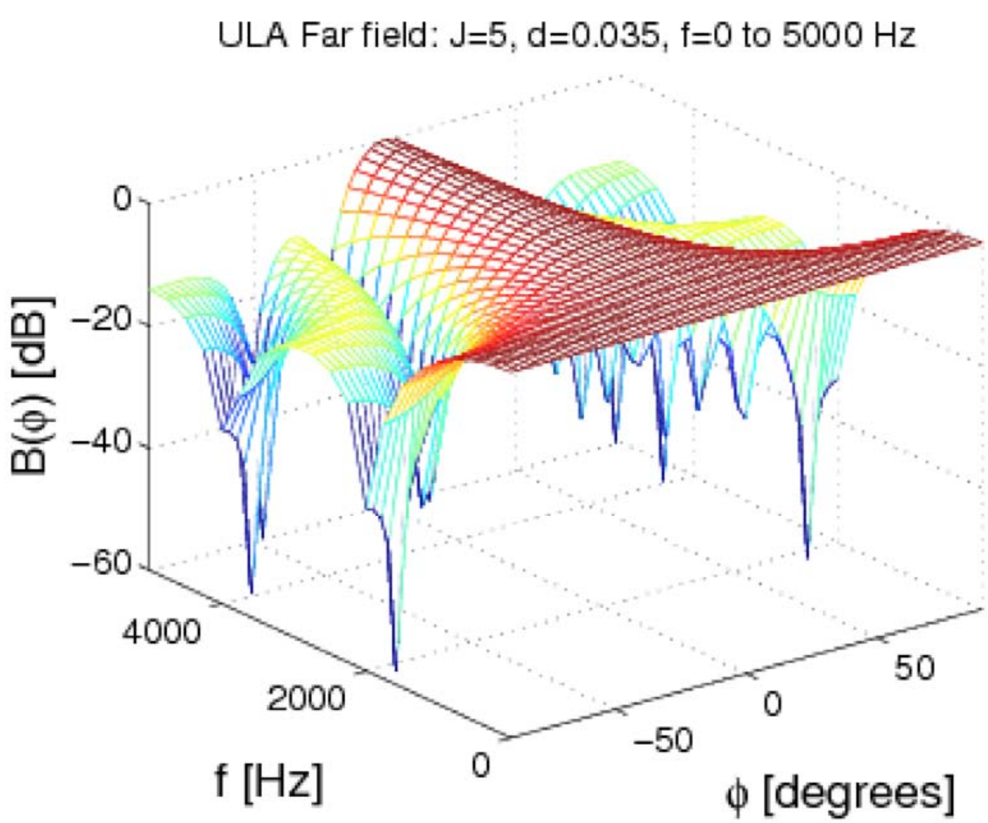
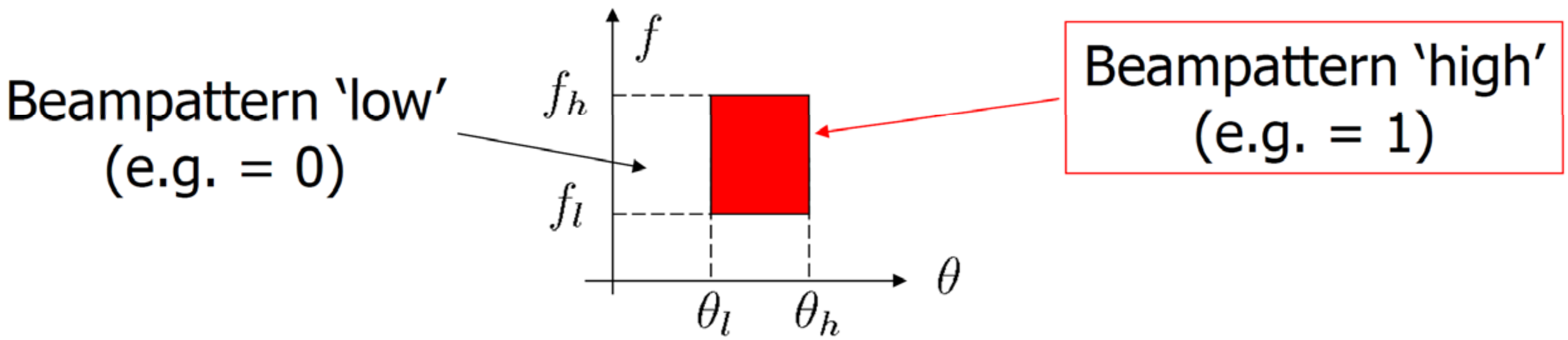
ULA Beampattern: frequency dependence

Variable frequency, fixed J and d

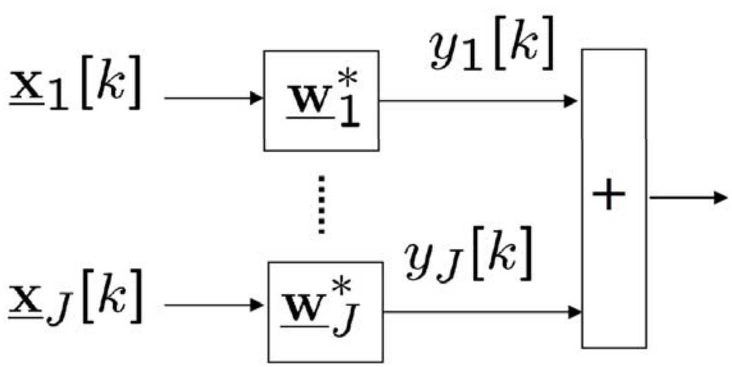


ULA Beampattern: frequency dependence

<u>Wish</u>: Frequency "independence" over angular range



<u>Use J different FIR filters (each of length N) for each sensor:</u>



$$y[k] = \sum_{i=1}^{J} \underline{\mathbf{w}}_i^h \underline{\mathbf{x}}_i[k]$$

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ULA Beampattern: frequency dependence

Array response: $r(f,\theta) = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(f,\theta) \;=\; \sum_{i=0}^{J-1} \underline{\mathrm{w}}_i^*[l] a_i(f,\theta)$

$\Rightarrow$ **Frequence response for ULA:** (T_s is sampling frequency)

$$W(f,\theta) = \sum_{i=0}^{J-1} \sum_{l=0}^{N-1} w_{i,l}^* \mathrm{e}^{-\mathrm{j}2\pi\cdot f\cdot l\cdot T_s} \mathrm{e}^{-\mathrm{j}2\pi f \frac{d\sin(\theta)}{c} i}$$

Normalized temporal frequency: $f_1 = f \cdot T_s$

Normalized spatial frequency: $f_2 = \dfrac{f d \sin(\theta)}{c}$

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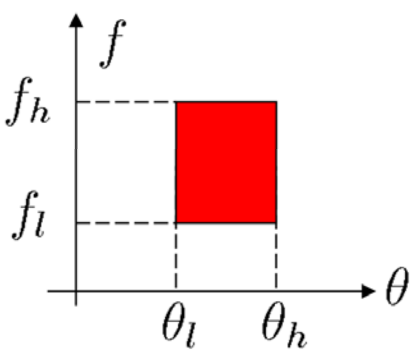
ULA Beampattern: frequency dependence

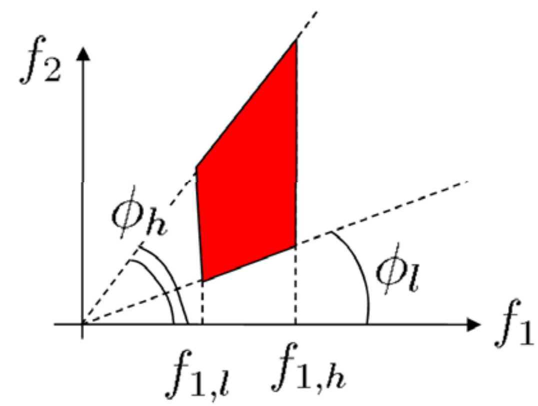
$$\Rightarrow \; W(f_1, f_2) = \sum_{i=0}^{J-1} \sum_{l=0}^{N-1} w_{i,l}^* e^{-j2\pi l f_1} e^{-j2\pi i f_2} \quad : \text{ 2D-DFT of } w_{i,l}^*$$

$$\textit{Note:} \; f_2 = \left(\frac{d \sin(\theta)}{cT_s}\right) \cdot f_1 \;\Rightarrow\; \text{Slope through origin of } f_1, f_2 \text{ plane}$$

$$\text{Choose: } T_s = \frac{d}{c} \left(= \frac{d}{\lambda_{min}} \cdot \frac{1}{f_{max}} = \frac{1}{2} \cdot \frac{1}{f_{max}} \right) \quad \boxed{\Rightarrow \; f_2 = \sin(\theta) \cdot f_1}$$



$$\tan(\phi) = \sin(\theta) = \frac{f_2}{f_1}$$



ULA Beampattern: DFT view

ULA is related to regular temporal sampling:

$$\text{Spatial sampling frequency} \;:\; U_s = \frac{1}{d}$$

$$\text{Spatial frequency} \;:\; U = \frac{\sin(\theta)}{\lambda}$$

$$\text{Normalized spatial frequency} \;:\; u = \frac{U}{U_s} = \frac{d\sin(\theta)}{\lambda}$$

$\Rightarrow$ **Steering vector:**

$$\underline{\mathbf{a}}(u) = \left(1, \mathrm{e}^{-\mathrm{j}2\pi u}, \cdots, \mathrm{e}^{-\mathrm{j}2\pi(J-1)u}\right)^t$$

Note: Avoid aliasing $\Rightarrow \quad -\frac{1}{2} \leq u \leq \frac{1}{2} \quad \Leftrightarrow \boxed{d \leq \frac{\lambda}{2}}$

since range of unambiguous angles: $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

ULA Beampattern: DFT view

Notes:

$$X[k] = \sum_{n=0}^{N-1} x_p[n] \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}kn}$$

- Definition of length J DFT:

$$F_l = \sum_{i=0}^{J-1} f_i \mathrm{e}^{-\mathrm{j}\frac{2\pi}{J}il} \text{ for } l = 0, \cdots, J-1 \text{ (resolution } \tfrac{2\pi}{J}\text{)}$$

- Zero padding for improved resolution:

$$F_l = \sum_{i=0}^{J-1} f_i \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}il} \text{ for } l = 0, \cdots, N-1$$

with $N \geq J$ and $f_i \equiv 0$ for $i \geq J \;\Rightarrow\;$ resolution $\frac{2\pi}{N}$

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ULA Beampattern: DFT view

Array response with $u = \dfrac{d\sin(\theta)}{\lambda}$:

$$r(u) = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(u) = \sum_{i=1}^{J} w_i^* \mathrm{e}^{-\mathrm{j}2\pi(i-1)u}$$

With $N \geq J$ and $w_i \equiv 0$ for $i \geq J \;\; \Rightarrow$

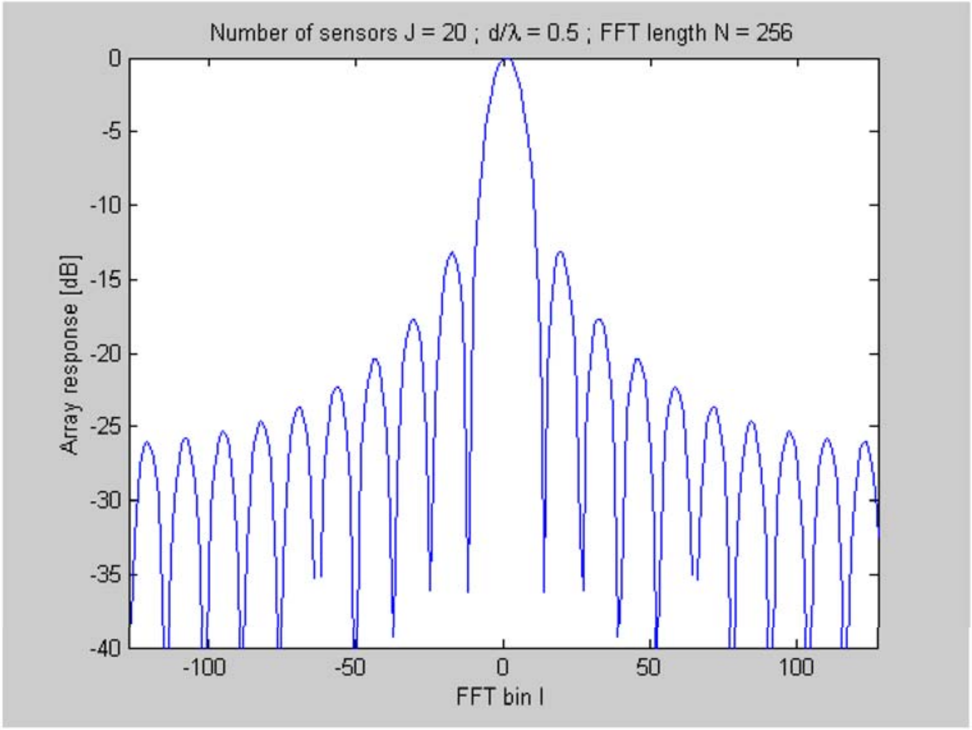
$$r_l = \sum_{i=0}^{J-1} w_{i+1}^* \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}il} \text{ for } l = 0, \cdots, N-1$$

with $l = N \cdot u = N \cdot \dfrac{d\sin(\theta)}{\lambda}$

Zero padded DFT of $\underline{\mathrm{w}}$

ULA Beampattern: DFT view

```
w=(1/J)*ones(J); abs_r=fftshift(abs(fft(w,N)))
```



Compute corresponding angle via: $\theta = \arcsin(\frac{l \cdot \lambda}{N \cdot d})$