RLS

RLS algorithm

Initialization:

$$\underline{\overline{\mathbf{r}}}_{ex}[0] = \underline{\mathbf{0}} \;;\; \overline{\mathbf{R}}_x^{-1}[0] = \delta^{-1}\mathbf{I} \quad \text{with } \delta \text{ small}$$

For $k \geq 0$:

$$\underline{\mathbf{g}}[k+1] = \frac{\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]}{\lambda^2 + \underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]}$$

$$\overline{\mathbf{R}}^{-1}[k+1] = \lambda^{-2}\left(\overline{\mathbf{R}}^{-1}[k] - \underline{\mathbf{g}}[k+1]\cdot\underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]\right)$$

$$\underline{\overline{\mathbf{r}}}_{ex}[k+1] = \lambda^2\underline{\overline{\mathbf{r}}}_{ex}[k] + \underline{\mathbf{x}}[k+1]e[k+1]$$

$$\underline{\mathbf{w}}[k+1] = \overline{\mathbf{R}}_x^{-1}[k+1]\cdot\underline{\overline{\mathbf{r}}}_{ex}[k+1]$$

RLS

Compare RLS with "LMS/Newton"

"LMS/Newton": $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \mathbf{R}_x^{-1} \underline{\mathbf{x}}[k] r[k]$

Update RLS can be rewritten as (see Appendix):

$$
\begin{aligned}
\underline{\mathbf{w}}[k+1] &= \underline{\mathbf{w}}[k] + \underline{\mathbf{g}}[k+1]\left(e[k+1] - \underline{\mathbf{x}}^t[k+1]\underline{\mathbf{w}}[k]\right) \\
&= \underline{\mathbf{w}}[k] + \underline{\mathbf{g}}[k+1] r[k+1]
\end{aligned}
$$

with gain vector:

$$
\underline{\mathbf{g}}[k+1] = \frac{\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]}{\lambda^2 + \underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]}
$$

RLS

Effective memory: $1/(1-\lambda)$

λ: 0.99~1

$\bar{R}^{-1}[k]$: should be Hermitian, positive definiteness

$$r_{ij}[k] = r_{ji}^{*}[k]$$

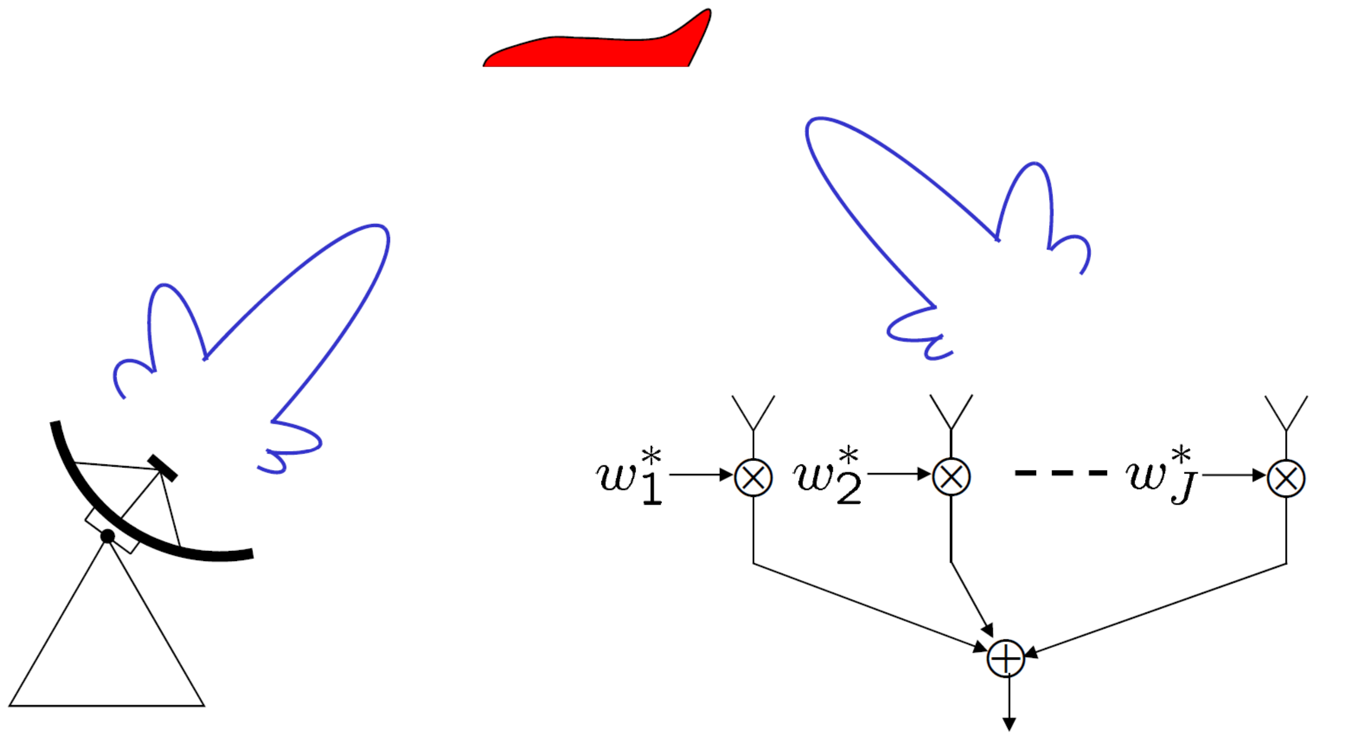
$$\bar{R}^{-1}[k] = [\bar{R}^{-1}[k] + (\bar{R}^{-1}[k])^{H}]/2$$

ADSP Advanced digital signal processing

Main content ADSP course

- Part A: Stochastic Signal Processing
- Part B: Adaptive Signal Processing
- **Part C: Array Signal Processing (ASP) (including DOA)**
- Part D: Adaptive Array Signal Processing (AASP)

Introduction



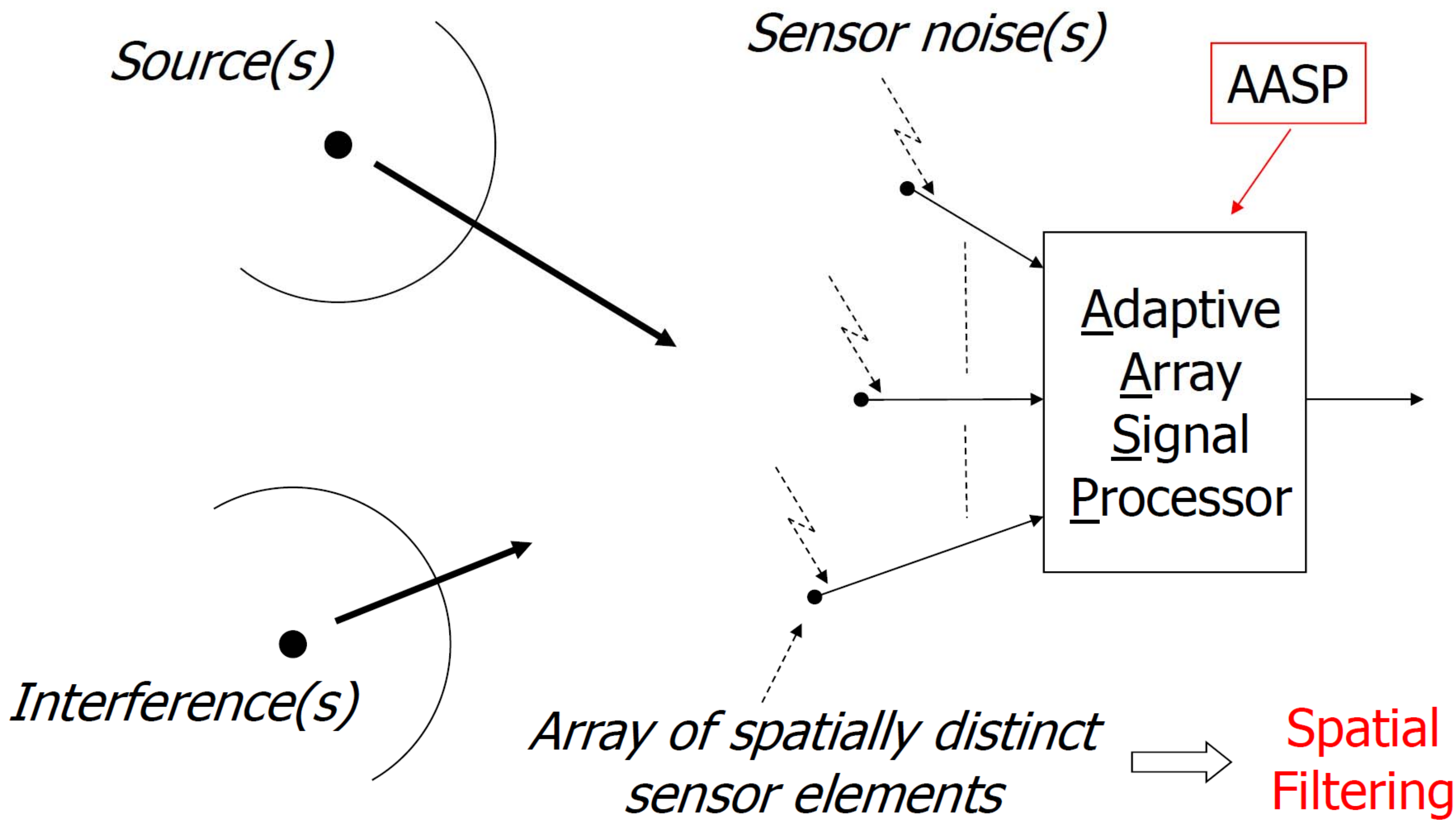
Parabolic dish antenna
(continuous aperture)

Sensor array antenna
(discrete spatial aperture)

上海科技大学 ShanghaiTech University

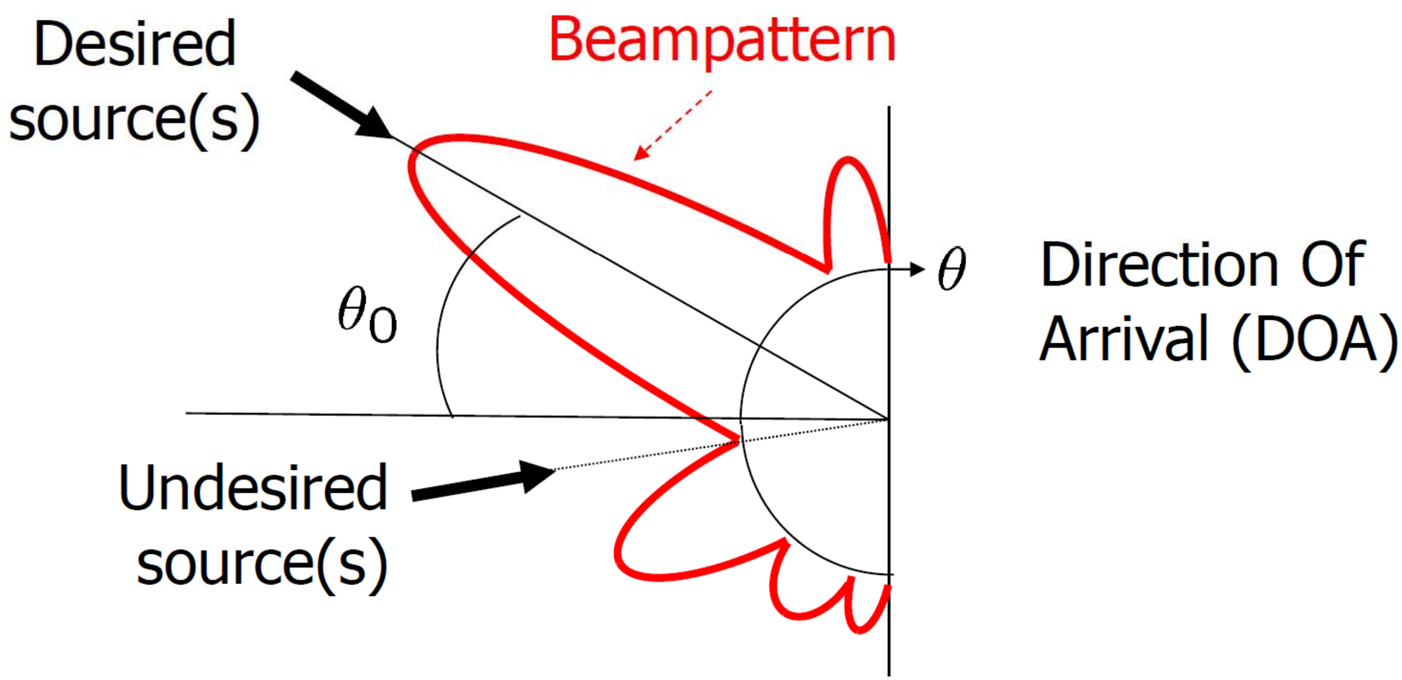
ADSP

Introduction



Introduction

Result of spatial filtering:



Introduction

Beamforming:

Spatio temporal filtering to either direct or block the radiation or reception of signals in specified directions.

Result Array Signal Processing:

Spatial filtering: Separate signals with possible overlapping frequency content but from different spatial locations.

Introduction

Furthermore:

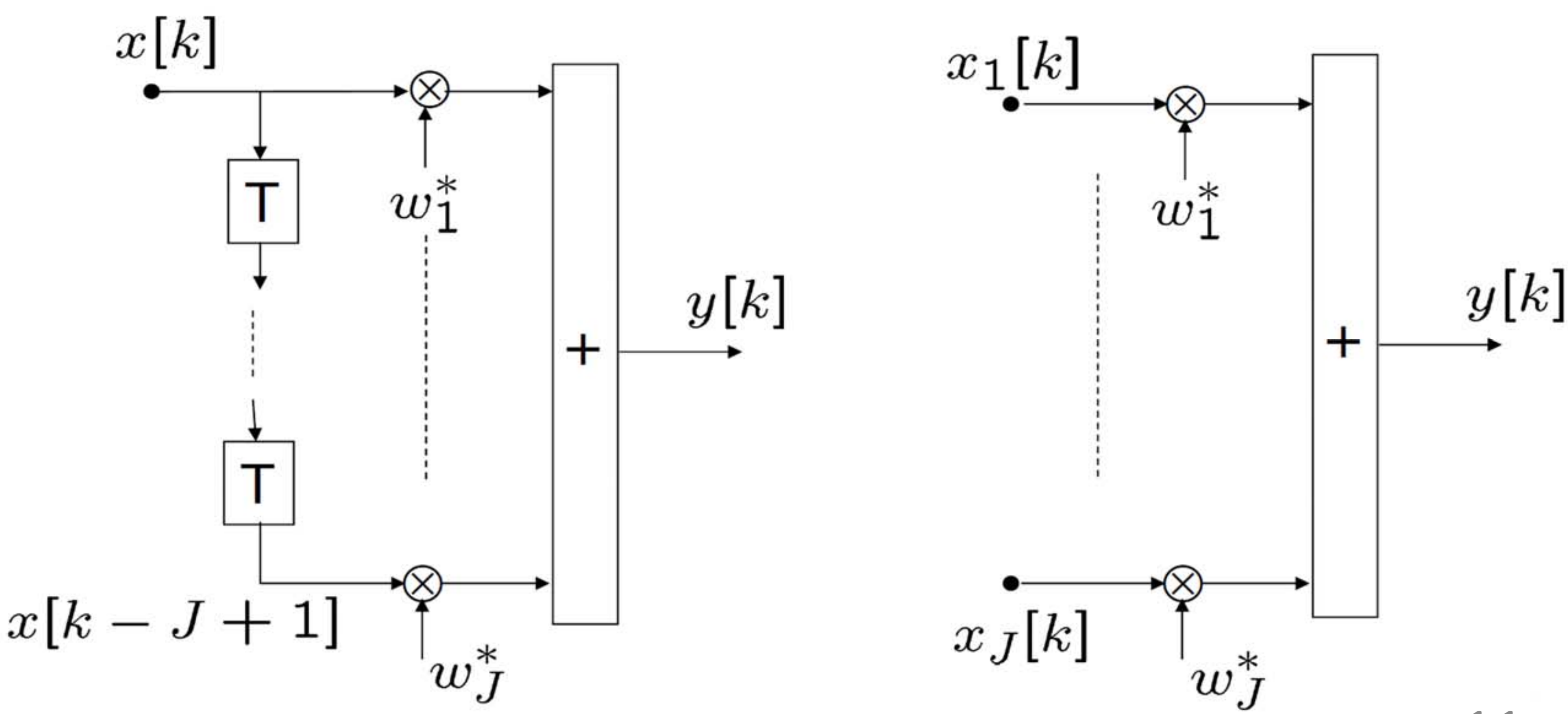
- Able to ‘look’ in several directions simultaneously
- Signal enhancement: averaging different sensor measurements improves SNR
- Flexible spatial discrimination: size of spatial aperture can be adapted.
- Adaptivity --> able to adapt response

AASP is versatile and flexible

Introduction

To provide insight into various aspects of AASP we use familiar methods and techniques from FIR filtering.

FIR: Array:



Introduction

However main differences ASP and FIR filtering:

- Source can have several parameters of interest (e.g. range, azimuth and elevation angle, polarization, temporal frequency content)
- Different signal often mutually correlated (multipath)
- Spatial sampling often nonuniform and multidimensional
- Uncertainty must often be included in characterization of individual sensor response and location (robust ASP techniques required)

Introduction

General objective of AASP:

Detect or enhance desired signal(s) (increase SNR), while simultaneously reducing unwanted interference.

Physical form array varies according to medium:

Microphone for pressure variations in air
Hydrophone for pressure variations in water (sonar)
Geophone for land base seismology
Radar of electromagnetic waves
Etc.

Information of interest:

- Signal itself (teleconferencing, communications)
- Location of source (=DOA) (radar, sonar)
- Number of sources

Introduction

AASP applications:

Radar: Phased-array, air traffic control

Sonar: Source localization and classification

Communication: Directional transmission and reception

Imaging: Ultrasonic, optical, tomographic

Geophysical exploration: Earth crust mapping, oil exploration

Astrophysical exploration: High resolution imaging of universe

Biomedical: Fetal heart monitoring

Acoustic: Hearing aids, transparent communication

Different scenario's

Bandwidth source
Source position
Array geometry
Discrete-time signal representation
Array signal model
ASP unit

Assumptions:
Superposition principle applies to propagating wave signals
Homogeneous, lossless medium, neglect dispersion,

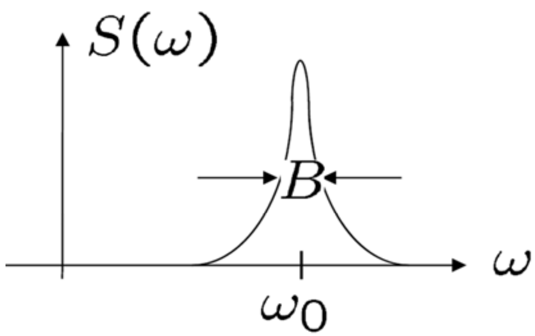
diffraction, changes in propagation speed

Scenarios

Bandwidth source

Analytical representation: $s(t) = A(t)\mathrm{e}^{\mathrm{j}(\omega_0 t + \phi(t))}$

Narrowband: $A(t)$ and $\phi(t)$ vary slower than $\mathrm{e}^{\mathrm{j}\omega_0 t}$



Narrowband: $|\tau| \ll 1/B \quad \Rightarrow$

$$A(t-\tau) \approx A(t) = 1 \text{ (usually)}$$

$$\phi(t-\tau) \approx \phi(t) = 0 \text{ (usually)}$$

$$\Rightarrow \quad s(t-\tau) = A(t-\tau)\mathrm{e}^{\mathrm{j}\phi(t-\tau)}\mathrm{e}^{\mathrm{j}\omega_0(t-\tau)} \approx \mathrm{e}^{-\mathrm{j}\omega_0\tau} \cdot s(t)$$

Thus for narrow band: Time delay $\Rightarrow$ phase shift

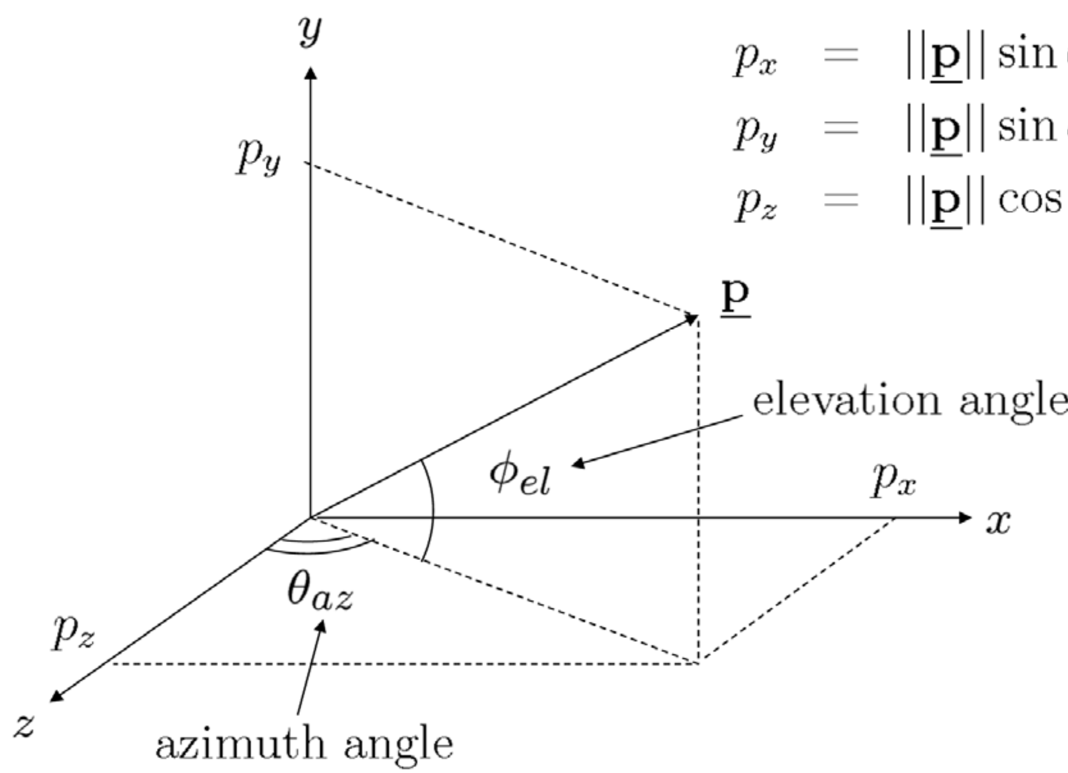
In this course: **mainly narrowband**

Scenarios

Source position

$$\underline{\mathbf{p}} = (p_x, p_y, p_z)^t$$

$$\begin{aligned} p_x &= ||\underline{\mathbf{p}}|| \sin\theta_{az} \cos\phi_{el} \\ p_y &= ||\underline{\mathbf{p}}|| \sin\phi_{el} \\ p_z &= ||\underline{\mathbf{p}}|| \cos\theta_{az} \cos\phi_{el} \end{aligned}$$

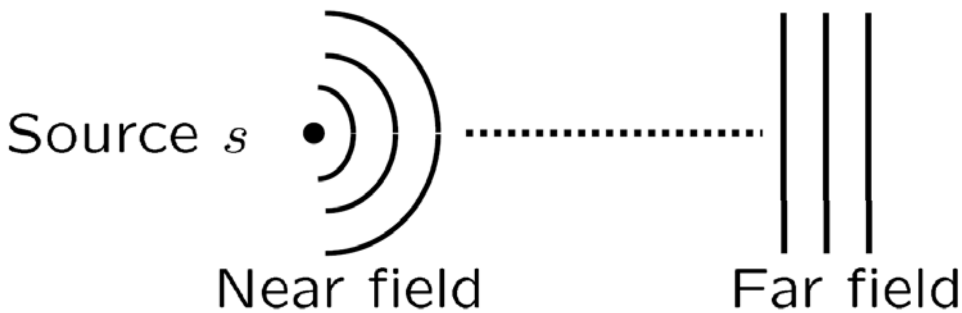


Scenarios

Array aperture: Volume (1D length) that collects incoming energy

Far field:
- Distance source - array $\gg$ array aperture
- Plane wavefront

Near field:
- Distance source - array $\ll$ array aperture
- Spherical wavefront



In this course mainly **Far field**

Scenarios

Propagation for **near field** (single frequency) source

$$s(t,\underline{\mathbf{p}}) = \frac{A}{||\underline{\mathbf{p}}||^2} e^{j\omega(t - \frac{||\underline{\mathbf{p}}||}{c})}$$

with $\omega = 2\pi f$ and $f = \frac{c}{\lambda}$

$\lambda =$ wavelength; $c =$ speed in medium

Note: For acoustic sound in air $c \approx 334$[m/sec]

$\Longrightarrow$ Amplitude decays proportional to distance from source

Scenarios

In **far field** at position $\underline{\mathbf{p}}_i$ monochromic plane wave:

$$s(t, \underline{\mathbf{p}}_i) = Ae^{\mathrm{j}\omega(t-\tau_i)} = Ae^{\mathrm{j}\omega(t-\frac{\underline{\mathbf{v}}^t}{c}\cdot\underline{\mathbf{p}}_i)} = Ae^{\mathrm{j}(\omega t-\underline{\mathbf{k}}^t\cdot\underline{\mathbf{p}}_i)}$$

Direction vector $\underline{\mathbf{v}}$; Wave number vector $\underline{\mathbf{k}} = \frac{\omega}{c}\cdot\underline{\mathbf{v}}$

- Propagation expressed as function of time and space
- Information is preserved while propagating

$\Rightarrow$ Band-limited signal can be reconstructed over all space and time by either:
- *temporally sampling* at given location in space
- *spatially sampling* at given instant of time

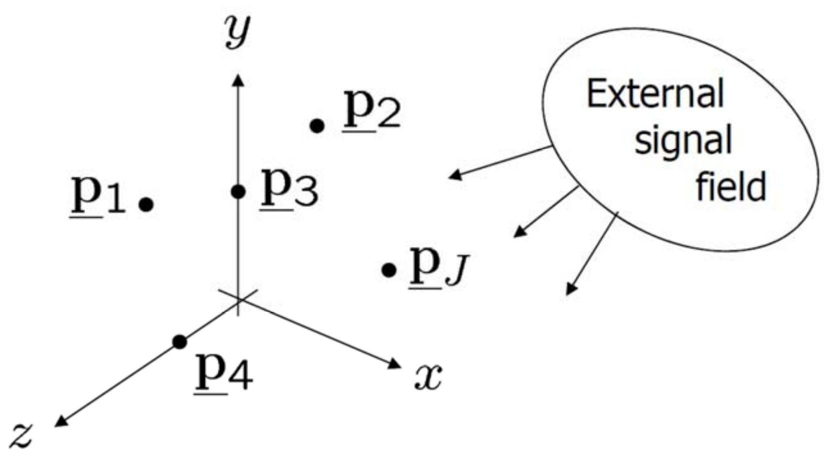
Basis for all aperture and sensor array processing techniques

Scenarios

Array geometry

Array can be uniform, nonuniform, linear, circular, ...

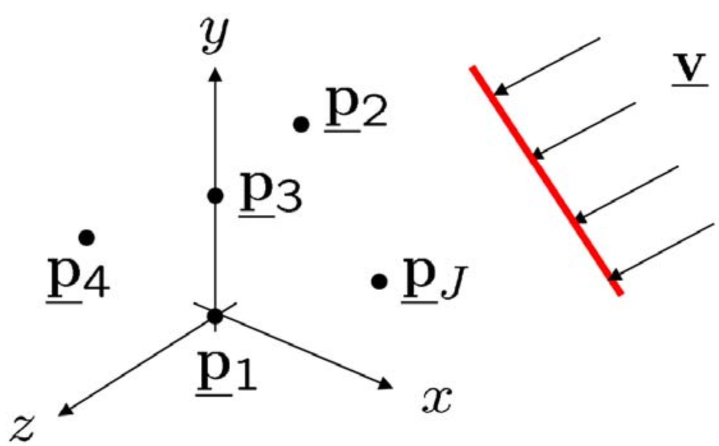
Sensors at J locations $\underline{\mathbf{p}}_i$



In this course mainly: **Uniform Linear Array (ULA)**

Scenarios

Propagation between two points for **plane wave***:*



Assume reference $\underline{\mathbf{p}}_1$:

$s(t)$ arrives at $\underline{\mathbf{p}}_1$

$\underline{\mathbf{p}}_1$ in origin

Analog signal at location $\underline{\mathbf{p}}_i$: $s(t-\tau_i) = s(t)\mathrm{e}^{-\mathrm{j}\omega\tau_i}$

with delay: $\tau_i = \underline{\mathbf{v}}^t \cdot \underline{\mathbf{p}}_i / c$ Direction vector v

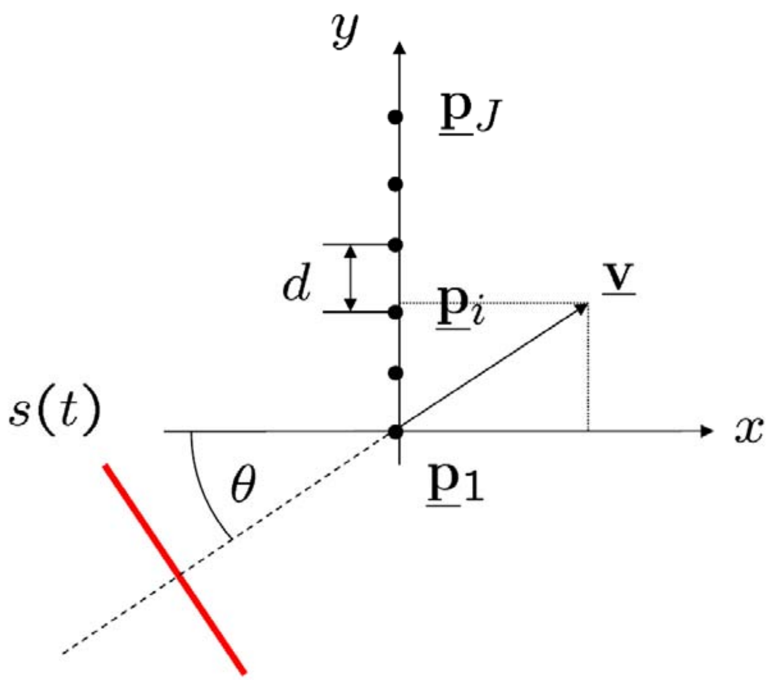
with $\omega = 2\pi f$ and $f = \dfrac{c}{\lambda}$

$\lambda =$ wavelength; $c =$ speed in medium

Scenarios

Note: Location is 3D quantity

In practice: Direction Of Arrival (DOA) 2D



ULA:

If reference $\underline{\mathbf{p}}_1$ at $(0,0)$

$$\Rightarrow \quad \underline{\mathbf{p}}_i = (0, (i-1)\cdot d)^t$$

Directional vector:

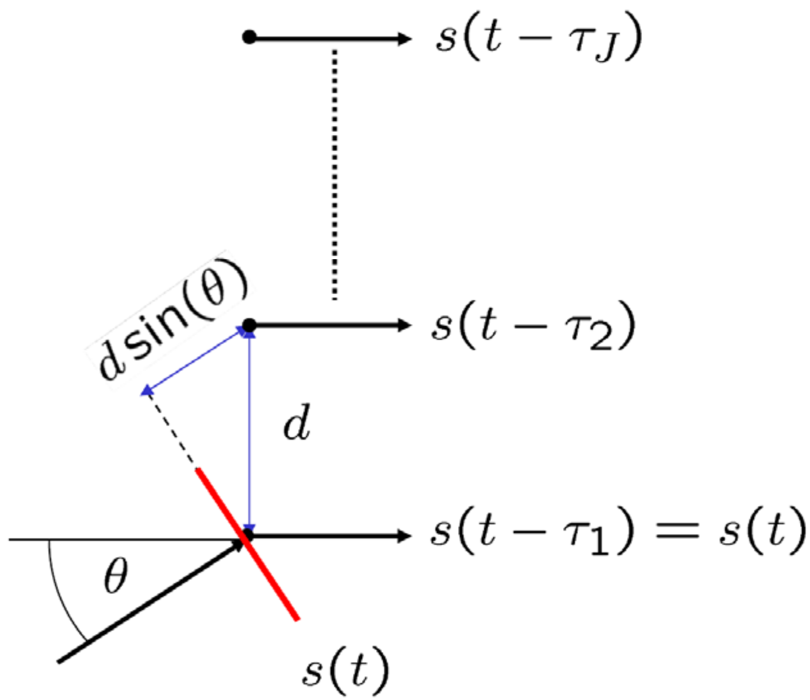
$$\underline{\mathbf{v}} = (\cos(\theta), \sin(\theta))^t$$

$$|\underline{\mathbf{v}}| = 1$$

Scenarios

Example: **Plane wave, ULA**

For far field only one parameter (DOA) characterizing position



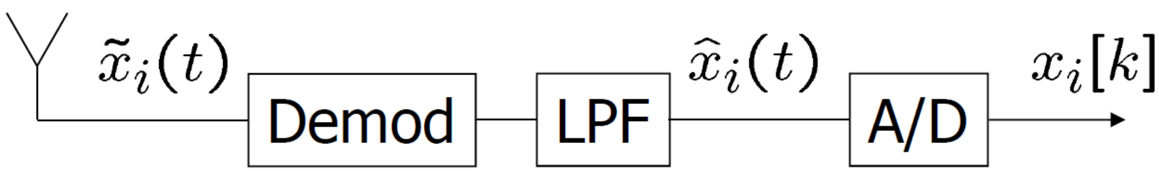
$$\Rightarrow \tau_i = \tau_i(\theta)$$
$$= \underline{\mathbf{v}}^t \underline{\mathbf{p}}_i / c$$
$$= (i-1)\frac{d\sin(\theta)}{c}$$

Scenarios

Discrete-time signal representation

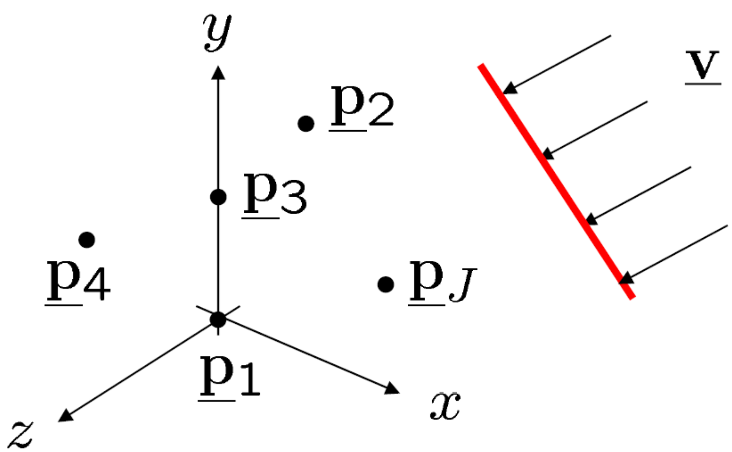
- Analog sensor signal at J locations: $\tilde{x}_i(t)$
- In each sensor $i = 1 \cdots J$ ideal demodulation and LPF takes place to baseband signal $\hat{x}_i(t)$
- After A/D:

  Complex valued discrete-time signal $x_i[k]$



Scenarios

Propagation between two points for **plane wave***:*



Assume reference $\underline{\mathbf{p}}_1$:

$s(t)$ arrives at $\underline{\mathbf{p}}_1$

$\underline{\mathbf{p}}_1$ in origin

Analog signal at location $\underline{\mathbf{p}}_i$: $s(t-\tau_i) = s(t)\mathrm{e}^{-\mathrm{j}\omega\tau_i}$

with delay: $\tau_i = \underline{\mathbf{v}}^t \cdot \underline{\mathbf{p}}_i / c$

$\Rightarrow$ Discrete-time signal at $\underline{\mathbf{p}}_i$: $s[k] \cdot \mathrm{e}^{-\mathrm{j}\omega\tau_i}$

Scenarios

$\Rightarrow$ Discrete-time signal in sensor i (for ULA):

$$
\begin{aligned}
s[k] \cdot \mathrm{e}^{-\mathrm{j}\omega\tau_i} &\equiv s[k] \cdot a_i(\omega, \theta) \\
\text{with } a_i(\omega, \theta) &= \mathrm{e}^{-\mathrm{j}\omega\tau_i} \\
&= \mathrm{e}^{-\mathrm{j}\omega(i-1)\frac{d \sin(\theta)}{c}} \\
&= \mathrm{e}^{-\mathrm{j}2\pi\frac{c}{\lambda}(i-1)\frac{d \sin(\theta)}{c}} \\
&= \mathrm{e}^{-\mathrm{j}2\pi(i-1)\frac{d \sin(\theta)}{\lambda}}
\end{aligned}
$$

Note: Usually simplified notation

$$a_i(\omega, \theta) \rightarrow a_i(\theta)$$

Scenarios

Array signal model

Array sensor vector: $\underline{\mathbf{x}}[k] = (x_1[k], x_2[k], \cdots, x_J[k])^t$

Noise vector: $\underline{\mathbf{n}}[k] = (n_1[k], n_2[k], \cdots, n_J[k])^t$

Steering vector: $\underline{\mathbf{a}}(\theta) = (a_1(\theta), a_2(\theta), \cdots, a_J(\theta))^t$

with $a_i(\theta) = \mathrm{e}^{-\mathrm{j}\omega\tau_i(\theta)}$

Note: for ULA: $a_i(\theta) = \mathrm{e}^{-\mathrm{j}2\pi(i-1)\frac{d\sin(\theta)}{\lambda}}$

Scenarios

Case: Noisy observation, one source, J sensors

for $i = 1, 2, \cdots, J$: $x_i[k] = a_i(\theta) \cdot s[k] + n_i[k]$ $\Rightarrow$

$$\boxed{\underline{\mathbf{x}}[k] = \underline{\mathbf{a}}(\theta) \cdot s[k] + \underline{\mathbf{n}}[k]}$$

Covariance structure:

$$\mathbf{R}_x = E\{\underline{\mathbf{x}} \cdot \underline{\mathbf{x}}^h\} = \sigma_s^2 \cdot (\underline{\mathbf{a}} \cdot \underline{\mathbf{a}}^h) + \mathbf{R}_n$$

with $\sigma_s^2 = E\{|s|^2\}$ and $\mathbf{R}_n = E\{\underline{\mathbf{n}} \cdot \underline{\mathbf{n}}^h\}$

For spatially white noise: $\mathbf{R}_n = \sigma_n^2 \mathbf{I}$

Note: Time indices are skipped for simplicity

Scenarios

Case: Noisy observation, P sources, J sensors

for $i = 1, 2, \cdots, J$: $x_i[k] = \sum_{p=1}^{P} a_i(\theta_p) \cdot s_p[k] + n_i[k] \Rightarrow$

$$\boxed{\underline{\mathbf{x}}[k] = \mathbf{A} \cdot \underline{\mathbf{s}}[k] + \underline{\mathbf{n}}[k]}$$

$J \times P$ steering matrix $\mathbf{A} = (\underline{\mathbf{a}}(\theta_1), \underline{\mathbf{a}}(\theta_2), \cdots, \underline{\mathbf{a}}(\theta_P))$

$P \times 1$ signal vector $\underline{\mathbf{s}}[k] = (s_1[k], s_2[k], \cdots, s_P[k])^t$

Covariance structure:

$$\mathbf{R}_x = E\{\underline{\mathbf{x}} \cdot \underline{\mathbf{x}}^h\} = \mathbf{A}\mathbf{R}_s\mathbf{A}^h + \mathbf{R}_n$$

with $\mathbf{R}_s = E\{\underline{\mathbf{s}} \cdot \underline{\mathbf{s}}^h\}$ and $\mathbf{R}_n = E\{\underline{\mathbf{n}} \cdot \underline{\mathbf{n}}^h\} = \sigma_n^2 \mathbf{I}$

Scenarios

General case:

Noisy observation, desired + undesired signals:

$$\boxed{\underline{\mathbf{x}}[k] = \underline{\mathbf{x}}_d[k] + \underline{\mathbf{x}}_u[k] + \underline{\mathbf{n}}[k]}$$

$\underline{\mathbf{x}}_d[k]$: P independent desired sources

$\underline{\mathbf{x}}_u[k]$: Q independent undesired sources

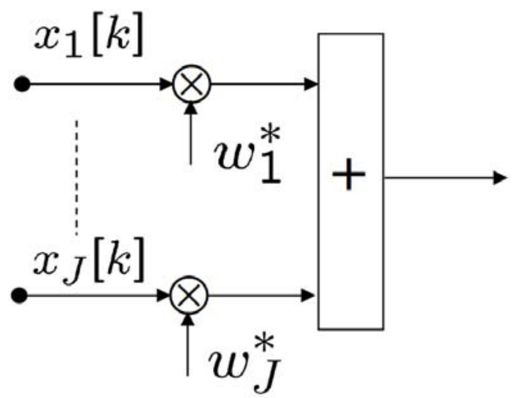
$\underline{\mathbf{n}}[k]$: Spatially white noise

Covariance structure:

$$\mathbf{R}_x = E\{\underline{\mathbf{x}} \cdot \underline{\mathbf{x}}^h\} = \mathbf{R}_{x_d} + \mathbf{R}_{x_u} + \mathbf{R}_n$$

Scenarios

Case: <u>Single complex weight for each sensor</u>



$$y[k] = \sum_{i=1}^{J} w_i^* x_i[k]$$

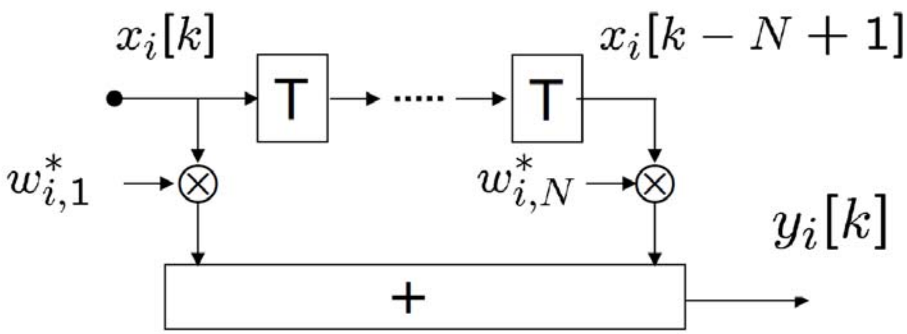
Short notation: $y[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{x}}[k]$

$$\underline{\mathbf{x}}[k] = (x_1[k], \cdots, x_J[k])^t$$

$$\underline{\mathbf{w}} = (w_1, \cdots, w_J)^t$$

Scenarios

Case: <u>FIR filter for each sensor</u>



$$y_i[k] = \sum_{l=1}^{N} w_{i,l}^* x_i[k-l+1]$$

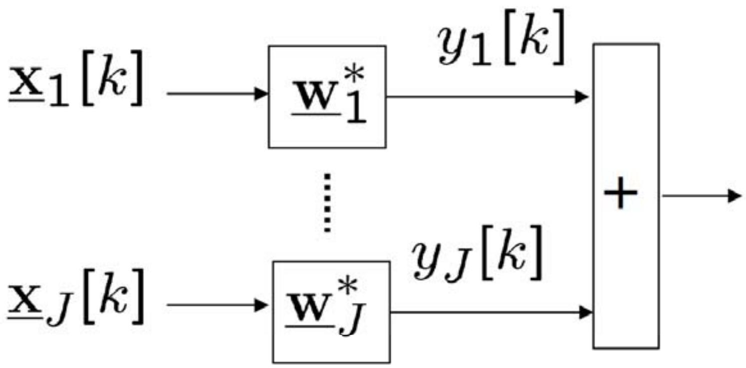
Short notation for $i = 1, 2, \cdots, J$:

$$\underline{\mathbf{x}}_i[k] \longrightarrow \boxed{\underline{\mathbf{w}}_i^*} \longrightarrow \qquad y_i[k] = \underline{\mathbf{w}}_i^h \cdot \underline{\mathbf{x}}_i[k]$$

$$\underline{\mathbf{x}}_i[k] = (x_i[k], \cdots, x_i[k-N+1])^t$$

$$\underline{\mathbf{w}}_i = (w_{i,1}, \cdots, w_{i,N})^t$$

Scenarios

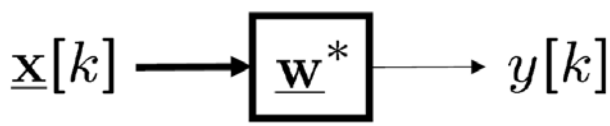


$$y[k] = \sum_{i=1}^{J} \underline{w}_i^h \underline{x}_i[k]$$

Short notation: $y[k] = \underline{w}^h \cdot \underline{x}[k]$

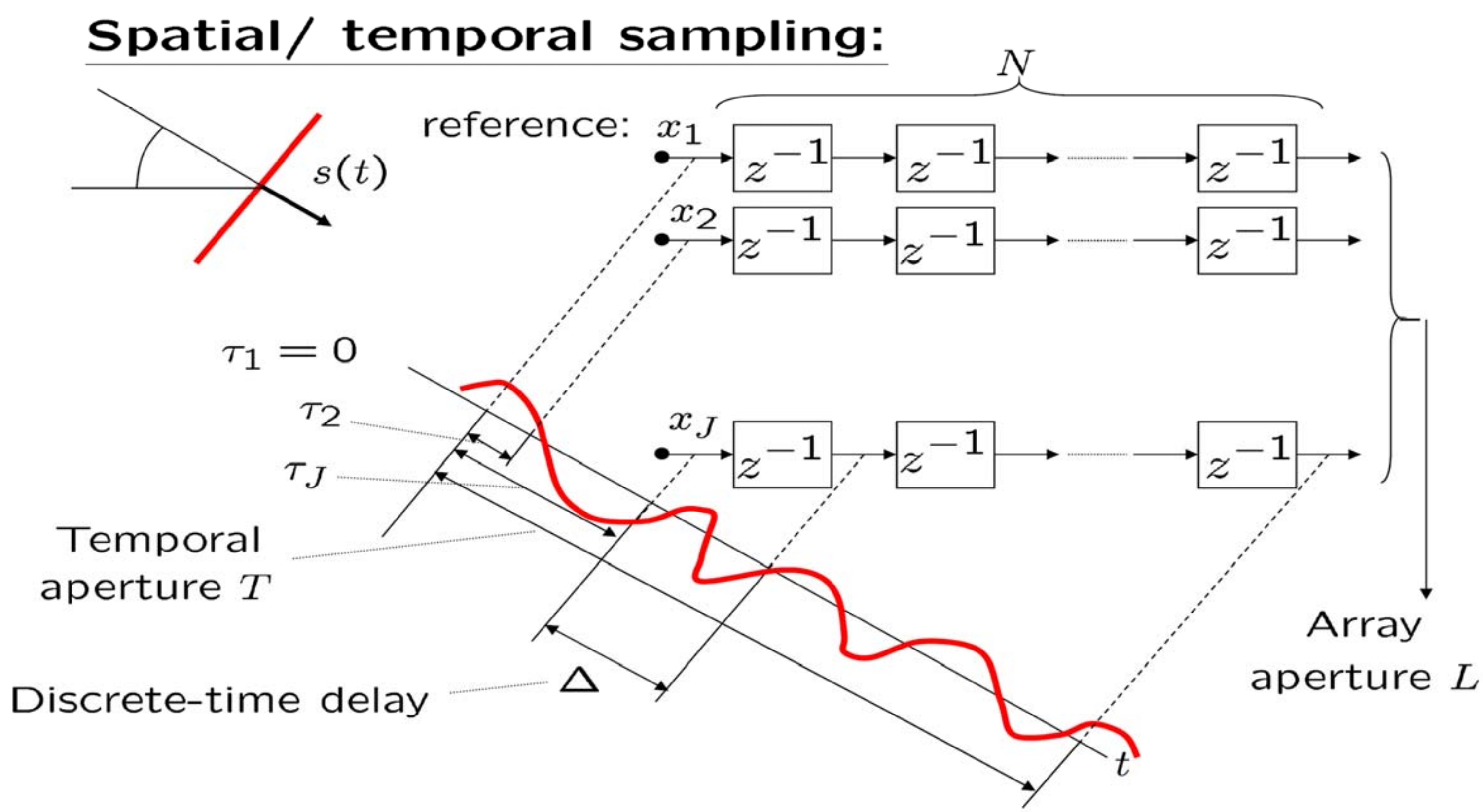
$$\underline{x}[k] = (\underline{x}_1[k], \cdots \underline{x}_J[k])^t$$

$$\underline{w} = (\underline{w}_1, \cdots, \underline{w}_J)^t$$



ADSP

Scenarios



ADSP

Scenarios

Notes:

- Propagating source signal is sampled at $J \cdot N$ nonuniformly spaced points
- Temporal aperture: $T(\theta)$
- Array aperture L: array length in wave length
  For ULA: $L = J \cdot d$
- FIR provide not simple frequency depending weighting of each channel. Weights effect **both** temporal and spatial response

Scenarios

How to cope with broadband signals: