AASP Appendix

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Eigenvalue problem

Eigenvalue problem

Eigenvalues/vectors of $N \times N$ matrix $\mathbf{R}$:

λ: eigenvalue $\underline{\mathbf{q}}$: eigenvector:

$$\mathbf{R} \cdot \underline{\mathbf{q}} = \lambda \cdot \underline{\mathbf{q}} \quad \Rightarrow \quad (\mathbf{R} - \lambda \mathbf{I}) \cdot \underline{\mathbf{q}} = \underline{\mathbf{0}}$$

Write this down for $i = 0, 1, \cdots, N-1 \quad \Rightarrow$

$$\mathbf{R} \cdot \left(\underline{\mathbf{q}}_0, \underline{\mathbf{q}}_1, \cdots \underline{\mathbf{q}}_{N-1}\right) =$$

$$= \left(\underline{\mathbf{q}}_0, \underline{\mathbf{q}}_1, \cdots \underline{\mathbf{q}}_{N-1}\right) \cdot \begin{pmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_{N-1} \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{R} \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{\Lambda} \quad \Leftrightarrow \mathbf{R} = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^{-1}$$

Eigenvalue problem

TU/e

With $\mathbf{Q} = \left(\underline{\mathbf{q}}_0, \underline{\mathbf{q}}_1, \cdots, \underline{\mathbf{q}}_{N-1}\right)$

and diagonal matrix: $\mathbf{\Lambda} = \begin{pmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_{N-1} \end{pmatrix}$

Property: *Eigenvectors orthogonal*

$\Rightarrow \quad \underline{\mathbf{q}}_i^h \cdot \underline{\mathbf{q}}_j = c \cdot \delta(i-j)$ with c some constant

$\Leftrightarrow \quad \mathbf{Q}^h \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{Q}^h = c \cdot \mathbf{I}$

Diagonalization: $\boxed{\mathbf{Q}^h \mathbf{R} \mathbf{Q} = \mathbf{\Lambda} \quad \Leftrightarrow \quad \mathbf{R} = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^h}$

Eigenvalue problem

TU/e

Example: Input signal is white noise (innovation)

$x[k] = i[k]$ with $E\{i[k]\} = 0$; $E\{i^2[k]\} = \sigma^2$

$\Rightarrow \ \mathbf{R} = \sigma^2 \cdot \mathbf{I}$

Eigenvalues of $N \times N$ matrix:

$\det(\mathbf{R} - \lambda\mathbf{I}) = 0 \ \Rightarrow \ (\sigma^2 - \lambda)^N = 0$

$\Rightarrow \ \lambda_i = \sigma^2$ for $i = 0, 1, \cdots, N-1$

Eigenvectors of $N \times N$ matrix:

$\mathbf{R} \cdot \underline{\mathbf{q}}_i = \lambda_i \cdot \underline{\mathbf{q}}_i \ \Rightarrow \ (\sigma^2 \cdot \mathbf{I}) \cdot \underline{\mathbf{q}}_i = \sigma^2 \cdot \underline{\mathbf{q}}_i \ \Rightarrow$ any $\underline{\mathbf{q}}_i$

Eigenvalue problem

TU/e

Example: Input signal is MA(1)

$$x[k] = i[k] - i[k-1] \text{ with } E\{i[k]\} = 0 \; ; \; E\{i^2[k]\} = \frac{1}{2}$$

$$\Rightarrow \quad \rho[\tau] = E\{x[k]x[k-\tau]\} = \begin{cases} 1 & \text{for } \tau = 0 \\ -\frac{1}{2} & \text{for } |\tau| = 1 \\ 0 & \text{elsewhere} \end{cases}$$

Eigenvalues of 2×2 matrix: $\mathbf{R} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$

$$\det \begin{pmatrix} 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda \end{pmatrix} = 0 \;\Rightarrow\; (\lambda - \frac{1}{2}) \cdot (\lambda - \frac{3}{2}) = 0$$

Eigenvalue problem

TU/e

Eigenvalues $\Rightarrow \lambda_1 = \frac{1}{2}\ ;\ \lambda_2 = \frac{3}{2}$

Eigenvectors of this 2×2 matrix: (c_i some constant)

$$\mathbf{R}\cdot\underline{\mathbf{q}}_i = \lambda_i\cdot\underline{\mathbf{q}}_i \Rightarrow \underline{\mathbf{q}}_0 = c_0\begin{pmatrix}1\\1\end{pmatrix}\ ;\ \underline{\mathbf{q}}_0 = c_1\begin{pmatrix}1\\-1\end{pmatrix}$$

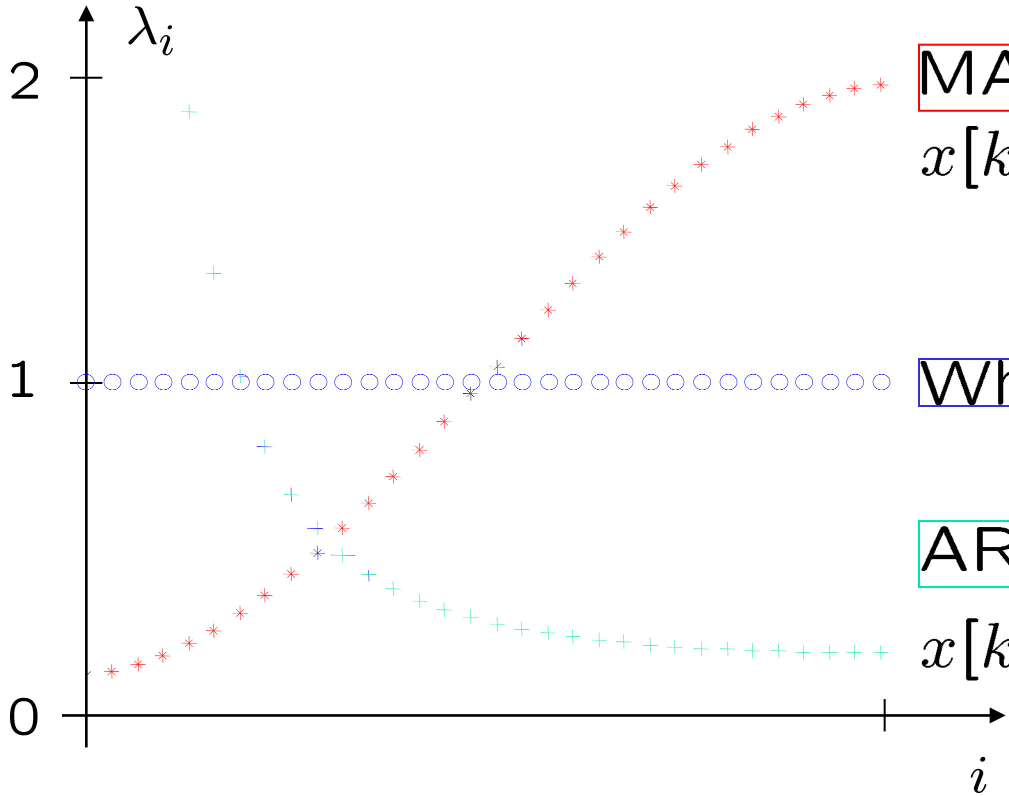
Choose $||\underline{\mathbf{q}}_i|| = 1 \Rightarrow c_0 = c_1 = \pm\frac{1}{2}\sqrt{2}$

Check: Indeed $\underline{\mathbf{q}}_i^h \cdot \underline{\mathbf{q}}_j = \delta(i-j)$

$\underline{\mathbf{q}}_0$ and $\underline{\mathbf{q}}_1$ indeed two ortho**normal** vectors

Eigenvalue problem

Example: N eigenvalues with Matlab (eig.m)



MA(1):
$x[k] = i[k] + a \cdot i[k-1]$

White noise: $x[k] = i[k]$

AR(1):
$x[k] = i[k] + a \cdot x[k-1]$

Generalized inverse

Generalized inverse

Assume $\mathbf{A}$ is $M \times N$

Properties of (unique) $N \times M$ matrix $\mathbf{B}$:

$$\mathbf{ABA} = \mathbf{A} \qquad (\mathbf{AB})^h = \mathbf{AB}$$

$$\mathbf{BAB} = \mathbf{B} \qquad (\mathbf{BA})^h = \mathbf{BA}$$

Generalized (or Moore-Penroose pseudo)

inverse: $\mathbf{B} = \mathbf{A}^\dagger$:

$$\mathbf{A}^\dagger = \mathbf{A}^h \left(\mathbf{AA}^h\right)^{-1} \quad \text{for } M \leq N$$

$$\mathbf{A}^\dagger = \mathbf{A}^{-1} \quad \text{for } M = N$$

$$\mathbf{A}^\dagger = \left(\mathbf{A}^h\mathbf{A}\right)^{-1} \mathbf{A}^h \quad \text{for } M \geq N$$

Generalized inverse

Examples:

Given $M \times N$ matrix $\mathbf{A}$ and $M \times 1$ vector $\underline{\mathbf{b}}$

Find $N \times 1$ vector $\underline{\mathbf{w}}$ from: $\mathbf{A} \cdot \underline{\mathbf{w}} = \underline{\mathbf{b}}$

Case $M < N$: Multiple solutions

Solution with smallest Euclidian norm:

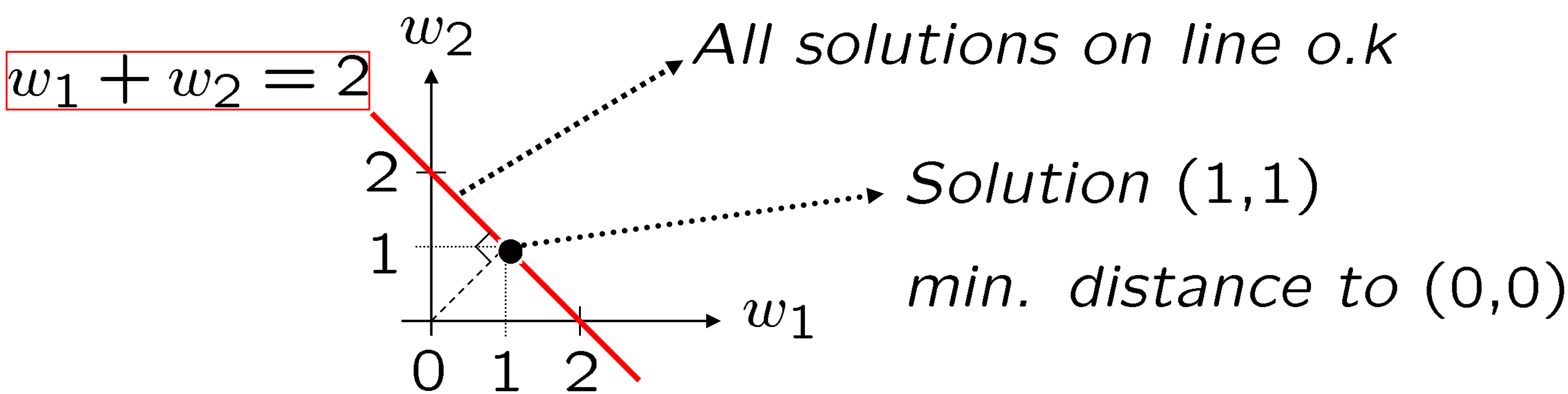
$$\underline{\mathbf{w}} = \mathbf{A}^h \left(\mathbf{A}\mathbf{A}^h\right)^{-1} \cdot \underline{\mathbf{b}} = \mathbf{A}^\dagger \cdot \underline{\mathbf{b}}$$

Note: $\mathbf{A}\mathbf{A}^h$ must be invertable

Generalized inverse

Example: $M = 1, N = 2 \;\rightarrow\;$ Infinite solutions

$$w_1 + w_2 = 2 \;\Rightarrow\; (1,1)\cdot\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = (2)$$



Solution with smallest Euclidian norm: :

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\cdot\left((1,1)\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)^{-1}\cdot(2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Generalized inverse

Case $M = N$: One solution $\underline{\mathbf{w}} = \mathbf{A}^{-1} \cdot \underline{\mathbf{b}}$

Note: $\mathbf{A}$ must be invertable

Example: $M = 2, N = 2 \;\rightarrow\;$ One solution

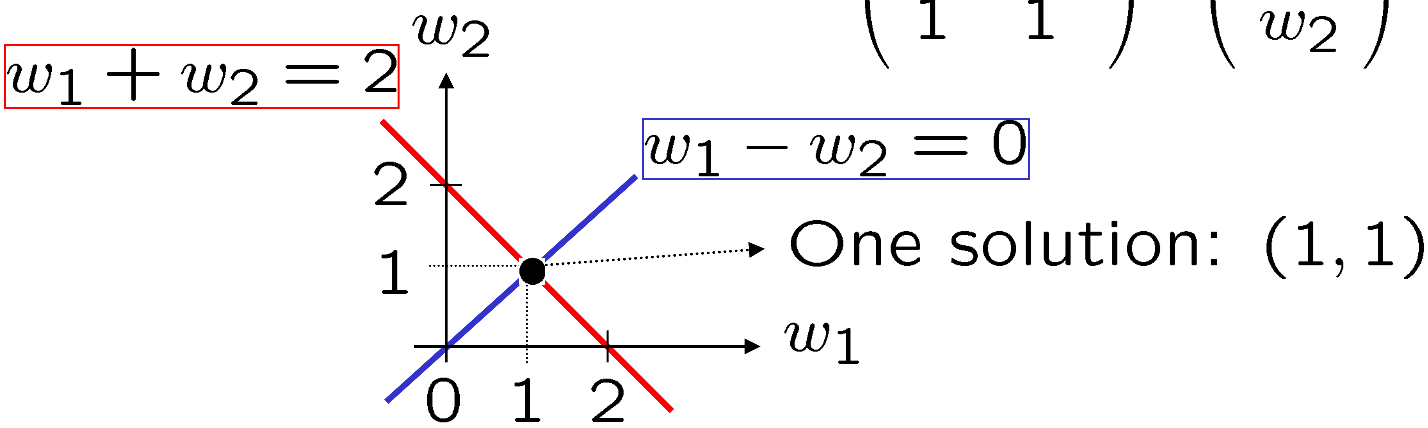
$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Generalized inverse

TU/e

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$



Case $M > N$: Overdetermined set of equations

minimum norm solution: $\underline{\mathbf{w}} = \arg\min_{\underline{\mathbf{w}}} ||\mathbf{A}\underline{\mathbf{w}} - \underline{\mathbf{b}}||^2$

$$\begin{aligned} J &= ||\mathbf{A}\underline{\mathbf{w}} - \underline{\mathbf{b}}||^2 = \left(\underline{\mathbf{w}}^h \mathbf{A}^h - \underline{\mathbf{b}}^h\right) \cdot (\mathbf{A}\underline{\mathbf{w}} - \underline{\mathbf{b}}) \\ &= \underline{\mathbf{w}}^h \mathbf{A}^h \mathbf{A}\underline{\mathbf{w}} - \underline{\mathbf{w}}^h \mathbf{A}^h \underline{\mathbf{b}} - \underline{\mathbf{b}}^h \mathbf{A}\underline{\mathbf{w}} + \underline{\mathbf{b}}^h \underline{\mathbf{b}} \end{aligned}$$

Generalized inverse

TU/e

$$\frac{\mathrm{d}J}{\mathrm{d}\underline{\mathbf{w}}} = 0 \quad \Rightarrow \quad \underline{\mathbf{w}} = \left(\mathbf{A}^h\mathbf{A}\right)^{-1}\mathbf{A}^h\cdot\underline{\mathbf{b}} = \mathbf{A}^\dagger\cdot\underline{\mathbf{b}}$$

Note: $\mathbf{A}^h\mathbf{A}$ must be invertable

Example: $M = 3, N = 2$

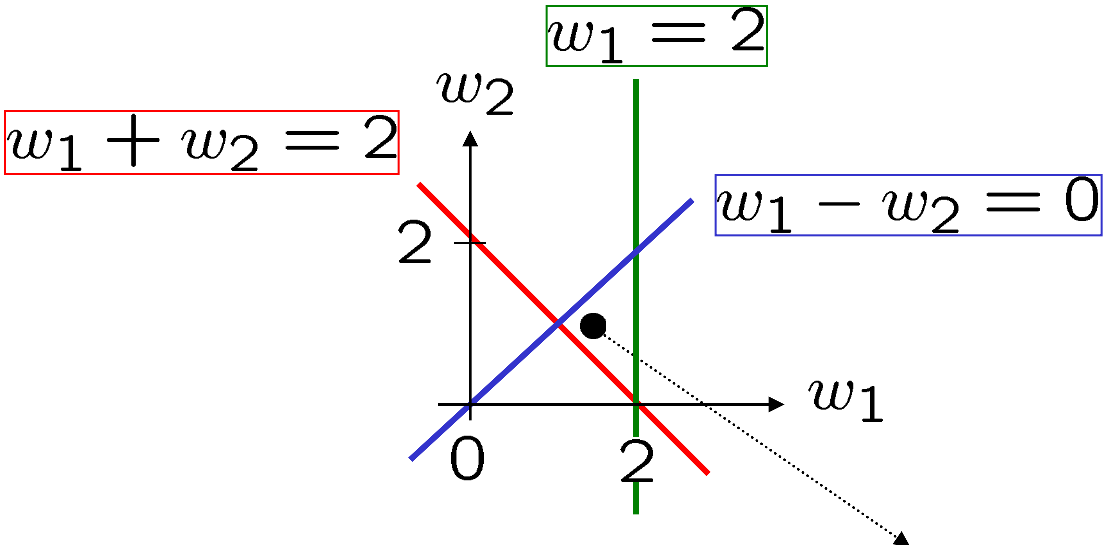
$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \quad \Rightarrow$$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \left(\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\right)^{-1}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix}$$

Generalized inverse

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$



Minimum norm solution: $(\frac{4}{3}, 1)$

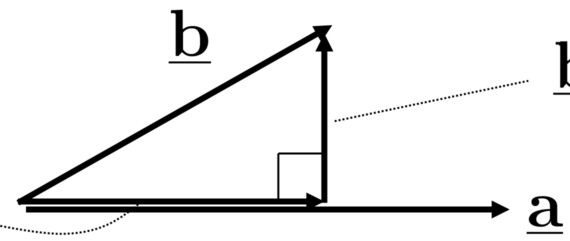
<u>**Prove:**</u> Minimum norm solution has *minimum norm*

Projection matrix

Projection matrix

Example:

$$\hat{\underline{\mathbf{b}}} = \alpha \cdot \underline{\mathbf{a}}$$



$$\underline{\mathbf{b}}^{\perp} = \underline{\mathbf{b}} - \hat{\underline{\mathbf{b}}}$$

$$\hat{\underline{\mathbf{b}}}^{h} \cdot \underline{\mathbf{b}}^{\perp} = 0 \;\Rightarrow\; \alpha = \left(\underline{\mathbf{a}}^{h}\underline{\mathbf{a}}\right)^{-1} \underline{\mathbf{a}}^{h} \cdot \underline{\mathbf{b}}$$

$$\Rightarrow\; \hat{\underline{\mathbf{b}}} = \left\{\underline{\mathbf{a}}\left(\underline{\mathbf{a}}^{h}\underline{\mathbf{a}}\right)^{-1} \underline{\mathbf{a}}^{h}\right\} \cdot \underline{\mathbf{b}}$$

with projection matrix $\mathbf{P}_a = \underline{\mathbf{a}}\left(\underline{\mathbf{a}}^{h}\underline{\mathbf{a}}\right)^{-1} \underline{\mathbf{a}}^{h}$

$\Rightarrow\; \hat{\underline{\mathbf{b}}} = \mathbf{P}_a \cdot \underline{\mathbf{b}}$ and $\underline{\mathbf{b}}^{\perp} = (\mathbf{I} - \mathbf{P}_a)\,\underline{\mathbf{b}}$

Projection matrix

Square matrix $\mathbf{P}$ is **projection** matrix if: $\mathbf{P}^2 = \mathbf{P}$

Orthogonal projection matrix: $\mathbf{P}^h = \mathbf{P}$ and $\mathbf{P}^2 = \mathbf{P}$

General: $N \times M$ matrix $\mathbf{V}$

$\mathbf{V}$ has linearly independent columns

Projection $\underline{\mathbf{b}}$ $(N \times 1)$ onto M-dimensional subspace spanned by columns of $\mathbf{V}$:

$\hat{\underline{\mathbf{b}}} = \mathbf{P}_V \cdot \underline{\mathbf{b}}$ and $\underline{\mathbf{b}}^\perp = (\mathbf{I} - \mathbf{P}_V) \cdot \underline{\mathbf{b}}$

with projection matrix: $\boxed{\mathbf{P}_V = \mathbf{V}\left(\mathbf{V}^h\mathbf{V}\right)^{-1}\mathbf{V}^h}$

Matrix inversion lemma

Matrix inversion lemma

TU/e

Example: Scalar case

$$(a+x\cdot y)^{-1} = \frac{1}{a+x\cdot y} = \cdots = a^{-1} - \frac{a^{-1}xya^{-1}}{1+ya^{-1}x}$$

$$\boxed{(\mathbf{A}+\mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{DA}^{-1}\mathbf{B}+\mathbf{C}^{-1}\right)^{-1}\mathbf{DA}^{-1}}$$

Matrix dimensions:

$$\mathbf{A} : N\times N \ ; \ \mathbf{B} : N\times M \ ; \ \mathbf{C} : M\times M \ ; \ \mathbf{D} : M\times N$$

Special case:

$\mathbf{B} = \underline{\mathbf{x}} : N\times 1$; $\mathbf{C} = 1$: scalar and $\mathbf{D} = \underline{\mathbf{x}}^h$

$$\boxed{\Rightarrow \quad \left(\mathbf{A}+\underline{\mathbf{x}}\underline{\mathbf{x}}^h\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\underline{\mathbf{x}}\underline{\mathbf{x}}^h\mathbf{A}^{-1}}{1+\underline{\mathbf{x}}^h\mathbf{A}^{-1}\underline{\mathbf{x}}}}$$

Derivation RLS update

Derivation RLS update

TU/e

$$\overline{\mathbf{R}}^{-1}[k+1] = \lambda^{-2}\left(\overline{\mathbf{R}}^{-1}[k] - \frac{\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]\underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]}{\lambda^2 + \underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]}\right)$$

$$\Rightarrow$$

$$\overline{\mathbf{R}}^{-1}[k+1]\cdot\underline{\mathbf{x}}[k+1] = \frac{\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]}{\lambda^2 + \underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]} = \underline{\mathbf{g}}[k+1]$$

Furthermore:

$$\underline{\overline{\mathbf{r}}}_{ex}[k+1] = \lambda^2\underline{\overline{\mathbf{r}}}_{ex}[k] + \underline{\mathbf{x}}[k+1]e[k+1]$$

$$\underline{\mathbf{w}}[k+1] = \overline{\mathbf{R}}_x^{-1}[k+1]\cdot\underline{\overline{\mathbf{r}}}_{ex}[k+1]$$

$$\Rightarrow$$

$$\begin{aligned}\underline{\mathbf{w}}[k+1] &= \overline{\mathbf{R}}_x^{-1}[k+1]\cdot\left(\lambda^2\underline{\overline{\mathbf{r}}}_{ex}[k] + \underline{\mathbf{x}}[k+1]e[k+1]\right)\\ &= \lambda^2\overline{\mathbf{R}}_x^{-1}[k+1]\underline{\overline{\mathbf{r}}}_{ex}[k] + \overline{\mathbf{R}}_x^{-1}[k+1]\underline{\mathbf{x}}[k+1]e[k+1]\end{aligned}$$

Derivation RLS update

with

$$\lambda^2 \overline{\mathbf{R}}^{-1}[k+1] = \overline{\mathbf{R}}^{-1}[k] - \frac{\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]\underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]}{\lambda^2 + \underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]\underline{\mathbf{x}}[k+1]}$$

and $\overline{\mathbf{R}}^{-1}[k+1] \cdot \underline{\mathbf{x}}[k+1] = \underline{\mathbf{g}}[k+1]$

$\Rightarrow$

$$\begin{aligned}
\underline{\mathbf{w}}[k+1] &= \left(\overline{\mathbf{R}}^{-1}[k] - \underline{\mathbf{g}}[k+1]\underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]\right) \cdot \underline{\overline{\mathbf{r}}}_{ex}[k] + \\
&\quad + \underline{\mathbf{g}}[k+1]e[k+1]
\end{aligned}$$

$$\begin{aligned}
\underline{\mathbf{w}}[k+1] &= \overline{\mathbf{R}}^{-1}[k]\underline{\overline{\mathbf{r}}}_{ex}[k] - \underline{\mathbf{g}}[k+1]\underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}^{-1}[k]\underline{\overline{\mathbf{r}}}_{ex}[k] + \\
&\quad + \underline{\mathbf{g}}[k+1]e[k+1] \\
&= \underline{\mathbf{w}}[k] + \left\{\underline{\mathbf{g}}[k+1]\right\} \cdot \left[e[k+1] - \underline{\mathbf{x}}^t[k+1]\underline{\mathbf{w}}[k]\right] \\
&= \underline{\mathbf{w}}[k] + \{\text{Kalman gain}\} \cdot [\text{a priori residual}]
\end{aligned}$$

Circulant Toeplitz matrix approximation

Circulant - Toeplitz

TU/e

Psd and autocorrelation Fourier pair

Alternative decorrelation in Frequency domain

DFT: For $l = 0, 1, \cdots, N-1 \;\; \rightarrow \;\; P_l = \sum_{i=0}^{N-1} c_i \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}il}$

In vector-matrix notation: $\underline{\mathbf{P}} = \mathbf{F}\cdot\underline{\mathbf{c}}$ with:

$$\underline{\mathbf{c}} = (c_0, \cdots, c_{N-1})^t \; ; \; \underline{\mathbf{P}} = (P_0, \cdots, P_{N-1})^t$$

Properties $N \times N$ Fourier (DFT) matrix:

$$(\mathbf{F})_{k,l} = \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}kl} \text{ for } k, l = 0, 1, \cdots, N-1$$

$$\mathbf{F} = \mathbf{F}^t \;\; \Rightarrow \;\; \mathbf{F}^h = \mathbf{F}^*$$

$$\mathbf{F}\mathbf{F}^h = N\mathbf{I} \;\; \Rightarrow \;\; \mathbf{F}^{-1} = \frac{1}{N}\mathbf{F}^h$$

Circulant - Toeplitz

TU/e

DFT has **circular** property

$\Rightarrow$ *circulant matrix diagonalized by* $\mathbf{F}$

Circulant matrix: $\mathbf{C} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & \cdots & c_{N-2} \\ \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{pmatrix}$

$$\mathbf{C} = \mathbf{F}\mathbf{P}\mathbf{F}^{-1} = \frac{1}{N}\mathbf{F}\mathbf{P}\mathbf{F}^h \; ; \; \mathbf{P} = diag\{\underline{\mathbf{P}}\} = diag\{\mathbf{F}\cdot\underline{\mathbf{c}}\}$$

$\Rightarrow$ Eigenvalues circulant matrix obtained by DFT

Circulant - Toeplitz

Asymptotic equivalence eigenvalues:

Assume autocorrelation: $\rho[\tau] = 0$ for $|\tau| > \tau_{max}$

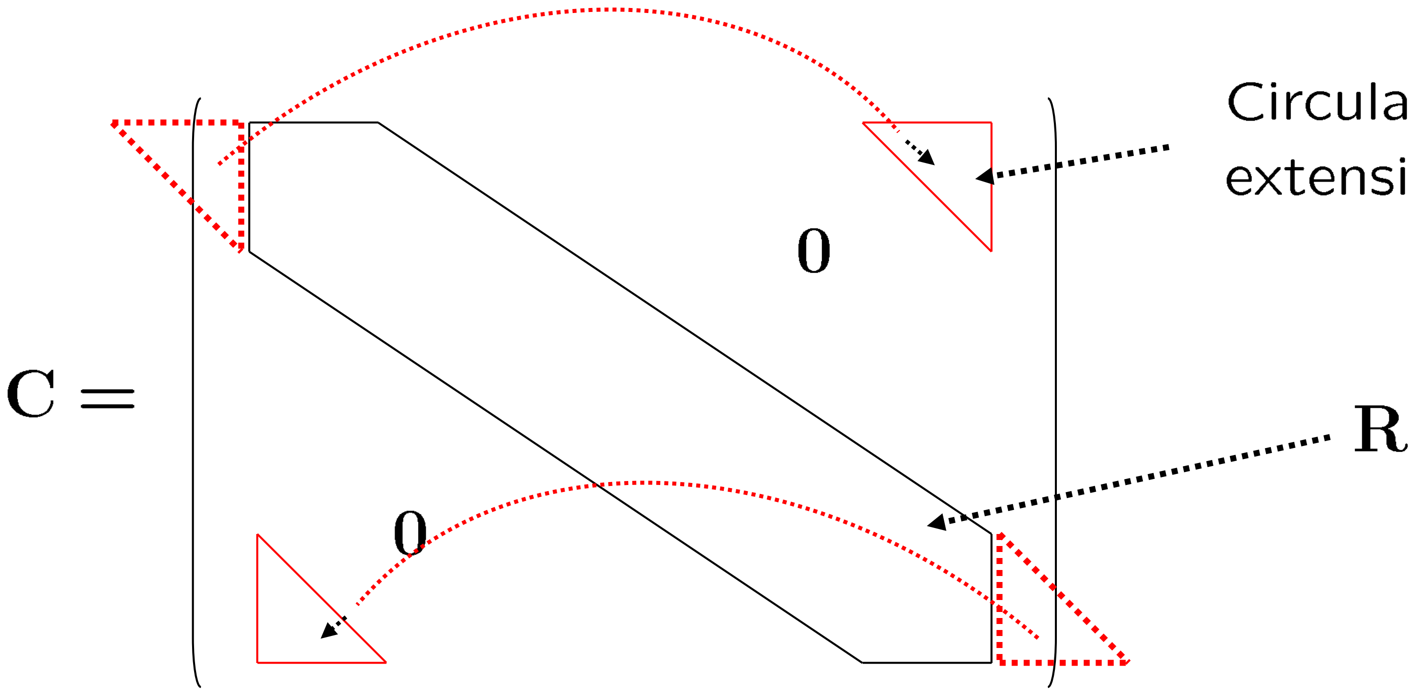
$N \times N$ autocorrelation matrix $\mathbf{R}$ Toeplitz structure:

$$\mathbf{R} = \begin{pmatrix} \rho[0] & \cdots & \rho[\tau_{max}] & & \mathbf{0} \\ \vdots & \ddots & & \ddots & \\ \rho[\tau_{max}] & & \ddots & & \rho[\tau_{max}] \\ & \ddots & & \ddots & \vdots \\ \mathbf{0} & & \rho[\tau_{max}] & \cdots & \rho[0] \end{pmatrix}$$

Circulant - Toeplitz

TU/e

For $\tau_{max} \ll N$ approximate $\mathbf{R}$ by circulant matrix:



$$c[\tau] = \begin{cases} \rho[\tau] & \text{for } \tau = 0, \cdots, \tau_{max} \\ \rho[N-\tau] & \text{for } \tau = N - \tau_{max}, \cdots, N-1 \\ 0 & \text{elsewhere} \end{cases}$$

Circulant - Toeplitz

TU/e

Definition psd: $P(e^{j\theta}) = \sum_{\tau=-\tau_{max}}^{\tau_{max}} \rho[\tau] e^{j\tau\theta}$

Circular extended autocorrelation matrix $\mathbf{C}$:

Eigenvalues: $\Rightarrow \quad P_l \approx P(e^{j\theta})|_{\theta = l \cdot \frac{2\pi}{N}}$

Saying it another way:

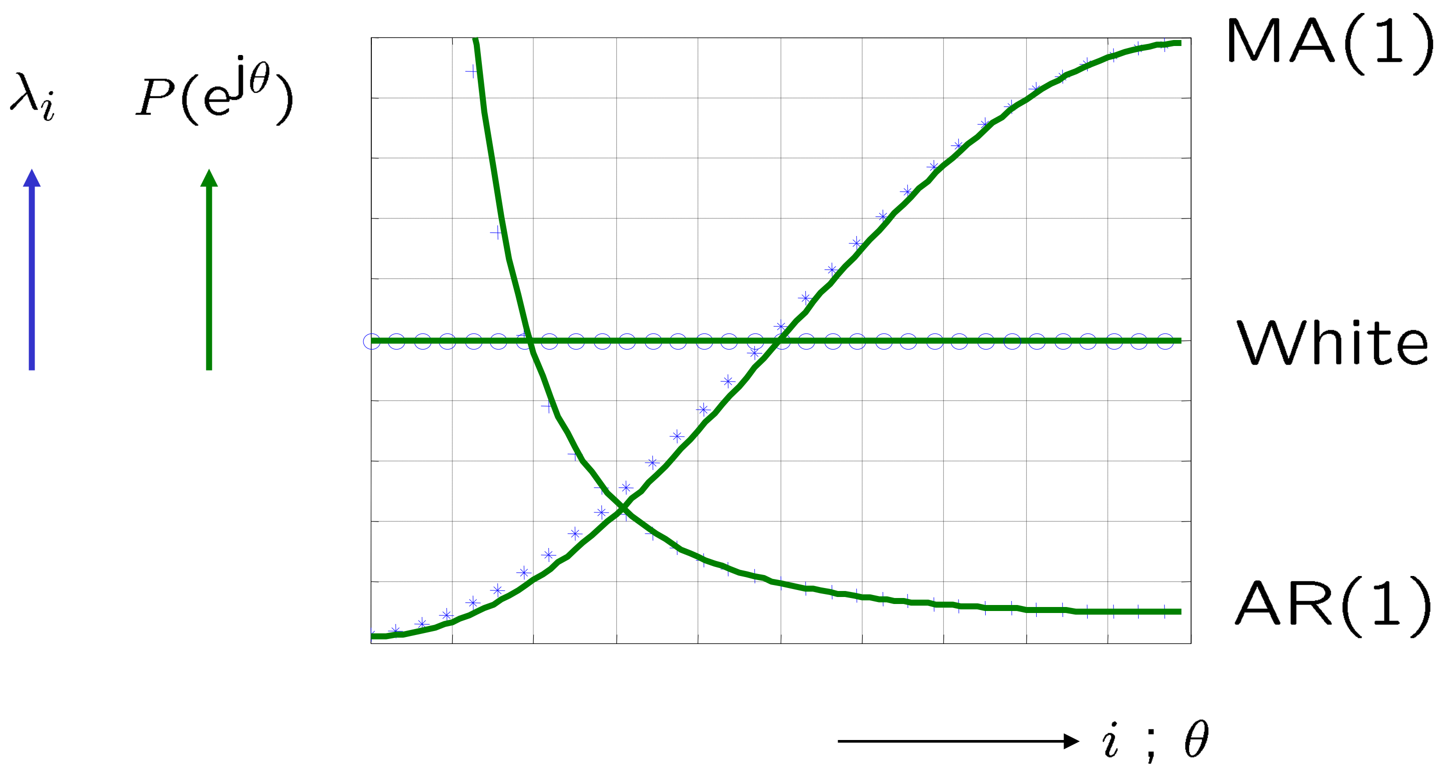
$$
\begin{aligned}
\frac{1}{N} E\{\underline{\mathbf{X}}[k]\underline{\mathbf{X}}^h[k]\} &= \frac{1}{N} \mathbf{F} E\{\underline{\mathbf{x}}[k]\underline{\mathbf{x}}^t[k]\}\mathbf{F}^h = \frac{1}{N}\mathbf{F}\mathbf{R}\mathbf{F}^h \\
&\approx \frac{1}{N}\mathbf{F}\mathbf{C}\mathbf{F}^h = \mathbf{P} \\
&= diag\{P_0, \cdots \; P_{N-1}\}
\end{aligned}
$$

Circulant - Toeplitz

TU/e

Example:

Psd's and (ordered) eigenvalues in one plot



Signal subspace techniques

Signal subspace techniques

TU/e

Goal:

Determine which sinusoidals present in noise

Signal model: J sensors, P sources $(P < J)$

$$\text{for } i = 1, 2, \cdots, J \; : \; x_i[k] = \sum_{p=1}^{P} a_i(\theta_p) \cdot s_p[k] + n_i[k] \;\; \Rightarrow$$

$$\boxed{\underline{\mathbf{x}}[k] = \mathbf{A} \cdot \underline{\mathbf{s}}[k] + \underline{\mathbf{n}}[k]}$$

$$\begin{aligned} \underline{\mathbf{x}}[k] &= (x_1[k], \cdots, x_J[k])^t \\ \underline{\mathbf{s}}[k] &= (s_1[k], \cdots \; s_P[k])^t \\ \underline{\mathbf{n}}[k] &= (n_1[k], \cdots, n_J[k])^t \end{aligned}$$

Signal subspace techniques

TU/e

Covariance structure:

$$\mathbf{R}_x = E\{\underline{\mathbf{x}} \cdot \underline{\mathbf{x}}^h\} = \mathbf{A}\mathbf{R}_s\mathbf{A}^h + \mathbf{R}_n$$

with $\mathbf{R}_s = E\{\underline{\mathbf{s}} \cdot \underline{\mathbf{s}}^h\} = \text{diag}\{\sigma_{s_1}^2, \cdots, \sigma_{s_P}^2\}$; $\mathbf{R}_n = \sigma_n^2 \mathbf{I}$

$J \times P$ steering matrix $\mathbf{A} = (\underline{\mathbf{a}}(\theta_1), \underline{\mathbf{a}}(\theta_2), \cdots, \underline{\mathbf{a}}(\theta_P))$

What about rank of the matrices?:

$$\left\{\begin{array}{l} \text{rank}\{\mathbf{A}\} = P \;;\; \text{rank}\{\mathbf{R}_s\} = P \;\rightarrow\; \boxed{\text{rank}\{\mathbf{A}\mathbf{R}_s\mathbf{A}^h\} = P} \\ \text{rank}\{\mathbf{R}_n\} = J \qquad\qquad J \times J \end{array}\right.$$

$$\boxed{\Rightarrow \;\; \text{rank}\{\mathbf{R}_x\} = J}$$

Signal subspace techniques

Eigenvalue decomposition of **source signal** part:

$$\left(\mathbf{A}\mathbf{R}_s\mathbf{A}^h\right)\cdot\underline{\mathbf{u}}_i = \lambda_{s_i}\cdot\underline{\mathbf{u}}_i \text{ for } i = 1, 2, \cdots, J$$

$$\text{rank}\{\mathbf{A}\mathbf{R}_s\mathbf{A}^h\} = P \;\Rightarrow\; \lambda_{s_{P+1}} = \cdots = \lambda_{s_J} = 0$$

order eigenvalues: $\lambda_{s_1} \geq \lambda_{s_2} \geq \cdots \geq \lambda_{s_P} > 0$

Shorthand notation: $\left(\mathbf{A}\mathbf{R}_s\mathbf{A}^h\right)\cdot\mathbf{U}_s = \mathbf{U}_s\cdot\mathbf{\Lambda}_s$

with $\mathbf{U}_s = (\underline{\mathbf{u}}_1, \cdots, \underline{\mathbf{u}}_P)$ and $\mathbf{\Lambda}_s = \left(\lambda_{s_1}, \cdots, \lambda_{s_P}\right)$

Signal subspace techniques

TU/e

Eigenvalue decomposition of **input signal** x:

Use previous eigenvectors $\underline{\mathbf{u}}_i$ for $i = 1, 2, \cdots, J$

$$\Rightarrow \quad \mathbf{R}_x \cdot \underline{\mathbf{u}}_i = \left(\mathbf{A}\mathbf{R}_s\mathbf{A}^h\right) \cdot \underline{\mathbf{u}}_i + \sigma_n^2 \mathbf{I} \cdot \underline{\mathbf{u}}_i = \left(\lambda_{s_i} + \sigma_n^2\right) \cdot \underline{\mathbf{u}}_i$$

Thus eigenvalues of $\mathbf{R}_x$ divide into two groups:

$$\lambda_{x_i} = \begin{cases} \lambda_{s_i} + \sigma_n^2 & \text{for } i = 1, \cdots, P \\ \sigma_n^2 & \text{for } i = P+1, \cdots, J \end{cases}$$

Signal subspace techniques

$$\mathbf{R}_x = \mathbf{U}_x \mathbf{\Lambda}_x \mathbf{U}_x^h = \sum_{i=1}^{J} \lambda_{x_i} \underline{\mathbf{u}}_i \underline{\mathbf{u}}_i^h$$

$$= \sum_{i=1}^{P} \left(\lambda_{s_i} + \sigma_n^2\right) \underline{\mathbf{u}}_i \underline{\mathbf{u}}_i^h + \sum_{i=P+1}^{J} \sigma_n^2 \underline{\mathbf{u}}_i \underline{\mathbf{u}}_i^h$$

$$= \mathbf{U}_s \mathbf{\Lambda}_{s,n} \mathbf{U}_s^h + \mathbf{U}_n \mathbf{\Lambda}_n \mathbf{U}_n^h$$

$$\mathbf{U}_x = (\underline{\mathbf{u}}_1, \cdots, \underline{\mathbf{u}}_J) \ ; \ \mathbf{\Lambda}_x = \text{diag}\{\lambda_{x_1}, \cdots, \lambda_{x_J}\}$$

Signal subspace:

$$\mathbf{U}_s = (\underline{\mathbf{u}}_1, \cdots, \underline{\mathbf{u}}_P) \ ; \ \mathbf{\Lambda}_{s,n} = \text{diag}\{\lambda_{s_1} + \sigma_n^2, \cdots, \lambda_{s_P} + \sigma_n^2\}$$

Noise subspace:

$$\mathbf{U}_n = \left(\underline{\mathbf{u}}_{P+1}, \cdots, \underline{\mathbf{u}}_J\right) \ ; \ \mathbf{\Lambda}_n = \text{diag}\{\sigma_n^2, \cdots, \sigma_n^2\}$$

Signal subspace techniques

TU/e

$$\left.\begin{array}{l}\text{Since } \mathbf{U}_x = (\underline{\mathbf{u}}_1, \cdots, \underline{\mathbf{u}}_J) = (\mathbf{U}_s, \mathbf{U}_n) \\ \text{and all eigenvectors orthogonal } \underline{\mathbf{u}}_i \perp \underline{\mathbf{u}}_j\end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{\mathbf{U}_s \perp \mathbf{U}_n \;\Leftrightarrow\; \mathbf{U}_s^h \cdot \mathbf{U}_n = \mathbf{0} \;\Leftrightarrow\; \mathbf{U}_n^h \cdot \mathbf{U}_s = \mathbf{0}}$$

Conclusion:

Any vector from signal subspace is orthogonal to noise subspace

Matrix splitting

Matrix splitting

Split full rank $J \times J$ matrix $\mathbf{A} = (\mathbf{B}|\mathbf{C})$

- $\mathbf{A}$ has J independent colums
- Rank $J \times J$ matrix $\mathbf{A}$ is J
- Rank $J \times M$ matrix $\mathbf{B}$ is M $(< J)$
- Rank $J \times (J - M)$ matrix $\mathbf{C}$ is $J - M$

$\Rightarrow$ Any vector $\underline{\mathbf{w}}_a \in$ column space $\mathbf{A}$:

$$\underline{\mathbf{w}}_a = \mathbf{A} \cdot \begin{pmatrix} \underline{\mathbf{w}}_b \\ \underline{\mathbf{w}}_c \end{pmatrix} = (\mathbf{B}|\mathbf{C}) \cdot \begin{pmatrix} \underline{\mathbf{w}}_b \\ \underline{\mathbf{w}}_c \end{pmatrix} = \mathbf{B}\underline{\mathbf{w}}_b + \mathbf{C}\underline{\mathbf{w}}_c$$

Note: $\begin{pmatrix} \underline{\mathbf{w}}_b \\ \underline{\mathbf{w}}_c \end{pmatrix} = \mathbf{A}^{-1} \cdot \underline{\mathbf{w}}_a$

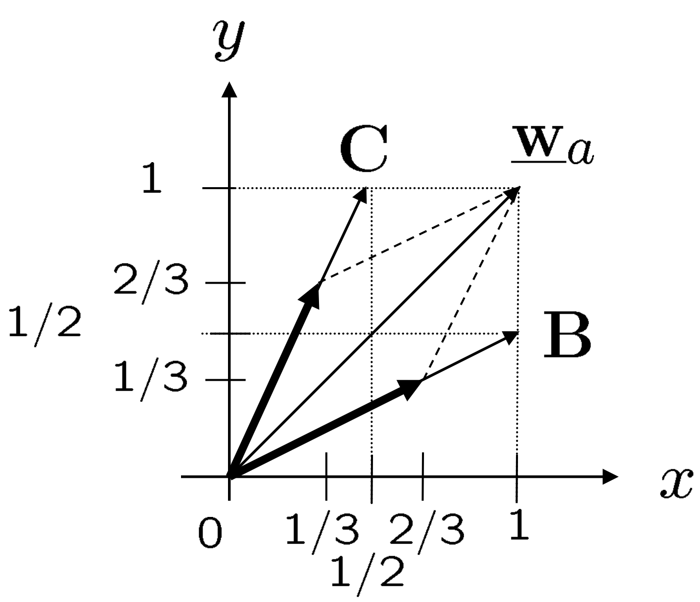
$\underline{\mathbf{w}}_b \in$ column space $\mathbf{B}$ and $\underline{\mathbf{w}}_c \in$ column space $\mathbf{C}$

Matrix splitting

TU/e

Example: $\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = (\mathbf{B}|\mathbf{C})$ and $\underline{\mathbf{w}}_a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\underline{\mathbf{w}}_a = \mathbf{B}\underline{\mathbf{w}}_b + \mathbf{C}\underline{\mathbf{w}}_c \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \cdot \underline{\mathbf{w}}_b + \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \cdot \underline{\mathbf{w}}_c$$



with $\begin{pmatrix} \underline{\mathbf{w}}_b \\ \underline{\mathbf{w}}_c \end{pmatrix} = \mathbf{A}^{-1} \cdot \underline{\mathbf{w}}_a = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \cdot \frac{2}{3} + \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \cdot \frac{2}{3}$$

End appendix