Advanced Digital Signal Processing (ADSP)

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Summary of last lecture

优化滤波.（由统计量求优化）

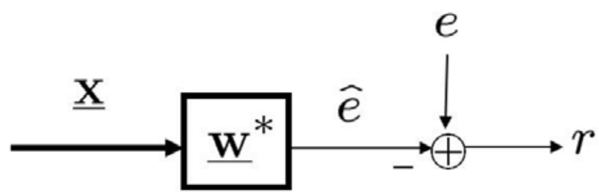
ADSP

Optimum (data-dependent) beamforming

不同的
优化标准

- Minimum mean squared error (MMSE)
- Multiple sidelobe canceller (MSC)
- Max. Signal-to-interference-plus-noise ratio (Max. SINR)
- Minimum variance distortionless response (MVDR)
- Linearly constrained minimum variance (LCMV)
- Generalized sidelobe canceller (GSC)

MMSE: Minimum mean squared error



- Cost function: $\xi = E\{|r|^2\}$
- Solution: $\underline{\mathbf{w}}_{\mathrm{mse}} = \arg\min_{\underline{\mathbf{w}}} \xi = \mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{xe^*}$

  $$\mathbf{R}_x = E\{\underline{\mathbf{x}} \cdot \underline{\mathbf{x}}^h\} \qquad \underline{\mathbf{r}}_{xe^*} = E\{\underline{\mathbf{x}} \cdot e^*\}$$

- Need to know $\mathbf{R}_x$ and $\underline{\mathbf{r}}_{xe^*}$ (from measurements)

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MMSE

Example: ULA, one source, narrowband, farfield

With $\underline{\mathbf{x}} = \underline{\mathbf{a}} \cdot s + \underline{\mathbf{n}}$ and $e = s$ 实际上很难得到

$$\underline{\mathbf{w}}_{mse} = \mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{xe^*} = \left(\underline{\mathbf{a}} \cdot \underline{\mathbf{a}}^h \cdot \sigma_s^2 + \sigma_n^2 \cdot \mathbf{I}\right)^{-1} \cdot \underline{\mathbf{a}} \cdot \sigma_s^2$$

$$\boxed{= \beta \cdot \underline{\mathbf{a}}}$$

类似于匹配滤波

$$\beta = \left(\frac{(\sigma_s^2/\sigma_n^2)}{1 + J \cdot (\sigma_s^2/\sigma_n^2)}\right)$$

$$J_{min} = E\{|e|^2\} - \underline{\mathbf{r}}_{xe^*}^h \mathbf{R}_x^{-1} \underline{\mathbf{r}}_{xe^*} = \boxed{\beta \cdot \sigma_n^2}$$

, $\beta \approx \frac{1}{J}$

Optimum (data-dependent) beamforming

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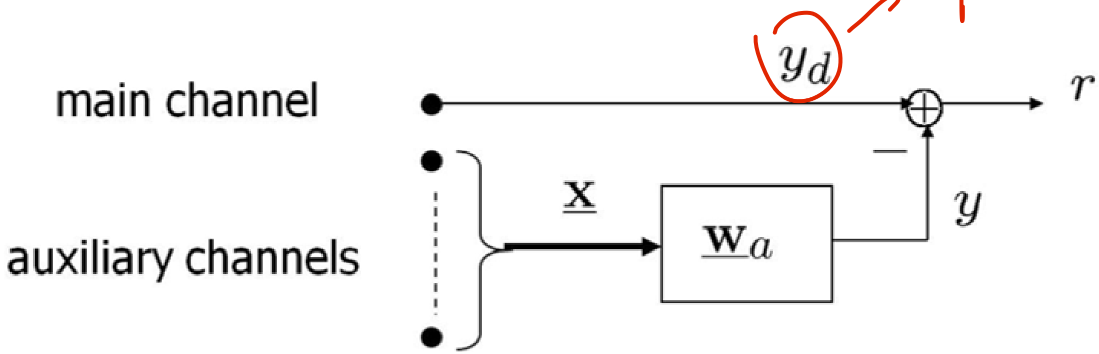
MSC: Multiple sidelobe canceller

作用类似于参考信号 "e"



$$\underline{\mathbf{w}}_o = arg\min_{\underline{\mathbf{w}}_a}\{|y_d - \underline{\mathbf{w}}_a^h \cdot \underline{\mathbf{x}}|^2\} \Rightarrow \boxed{\underline{\mathbf{w}}_o = \mathbf{R}_x^{-1}\cdot \underline{\mathbf{r}}_{xy_d^*}}$$

$$\boxed{P_{out} = \sigma_{y_d}^2 - \underline{\mathbf{r}}_{xy_d}\mathbf{R}_x^{-1}\underline{\mathbf{r}}_{xy_d}}$$

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Max. SINR

$$\underline{\mathbf{x}} = \underline{\mathbf{a}}s + (\underline{\mathbf{i}} + \underline{\mathbf{n}}) = \underline{\mathbf{x}}_d + \underline{\mathbf{x}}_u$$

$$\mathbf{R}_x = \mathbf{R}_{x_d} + \mathbf{R}_{x_u} = \sigma_s^2 \underline{\mathbf{aa}}^h + (\mathbf{R}_i + \sigma_n^2 \mathbf{I})$$

BF output: $y = \underline{\mathbf{w}}^h \underline{\mathbf{x}} = \underbrace{\underline{\mathbf{w}}^h \underline{\mathbf{x}}_d}_{y_d} + \underbrace{\underline{\mathbf{w}}^h \underline{\mathbf{x}}_u}_{y_u}$

$$\text{SINR} = \eta = \frac{|y_d|^2}{|y_u|^2} = \frac{\underline{\mathbf{w}}^h \mathbf{R}_{x_d} \underline{\mathbf{w}}}{\underline{\mathbf{w}}^h \mathbf{R}_{x_u} \underline{\mathbf{w}}} \quad \Rightarrow \quad \underline{\mathbf{w}}_{opt} = arg \max_{\underline{\mathbf{w}}} \{\eta\}$$

$$\boxed{\underline{\mathbf{w}}_{opt} = c\mathbf{R}_{x_u}^{-1} \underline{\mathbf{a}}}$$

Unit gain in look direction: $\underline{\mathbf{w}}_{opt}^h \underline{\mathbf{a}} = 1$, $\boxed{c = \frac{1}{\underline{\mathbf{a}}^h \mathbf{R}_{x_u}^{-1} \underline{\mathbf{a}}}}$

Optimum (data-dependent) beamforming

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MVDR

- Signal model: $\underline{\mathbf{x}} = \underline{\mathbf{a}}s + \underline{\mathbf{n}}$
- Beamformer output: $y = \underline{\mathbf{w}}^h \underline{\mathbf{x}} = y_s + y_n$
- MVDR problem statement: Minimize $E|y_n|^2$ subject to $\underline{\mathbf{w}}^h \underline{\mathbf{a}} = 1$
- Cost function: $\xi = \underline{\mathbf{w}}^h \mathbf{R}_n \underline{\mathbf{w}} + \lambda(\underline{\mathbf{w}}^h \underline{\mathbf{a}} - 1)$

$$\boxed{\underline{\mathbf{w}}_{MVDR} = \frac{\mathbf{R}_n^{-1}\underline{\mathbf{a}}}{\underline{\mathbf{a}}^h \mathbf{R}_n^{-1}\underline{\mathbf{a}}}} = \frac{\mathbf{R}_x^{-1}\underline{\mathbf{a}}}{\underline{\mathbf{a}}^h \mathbf{R}_x^{-1}\underline{\mathbf{a}}}$$

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Optimum (data-dependent) beamforming

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LCMV: Linearly constrained minimum variance

- Directional constraints for robustness against steering errors
  $\underline{\mathbf{w}}^h \underline{\mathbf{a}}(\theta_s + \Delta\theta) = 1$
  $\underline{\mathbf{w}}^h \underline{\mathbf{a}}(\theta_s - \Delta\theta) = 1$

- For robustness against errors in interferer DoA estimation
  $\underline{\mathbf{w}}^h \underline{\mathbf{a}}(\theta_i + \Delta\theta) = 0$
  $\underline{\mathbf{w}}^h \underline{\mathbf{a}}(\theta_i - \Delta\theta) = 0$

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LCMV

- Same signal model as MVDR, $E|y|^2 = \underline{\mathbf{w}}^h \mathbf{R}_x \underline{\mathbf{w}}$
- Constrained opt. problem: $\min_{\underline{\mathbf{w}}} \underline{\mathbf{w}}^h \mathbf{R}_x \underline{\mathbf{w}}$ subject to $\mathbf{C}^h \underline{\mathbf{w}} = \underline{\mathbf{r}}_d$
- Cost function: $\xi = \underline{\mathbf{w}}^h \mathbf{R}_x \underline{\mathbf{w}} + \underline{\lambda}(\mathbf{C}^h \underline{\mathbf{w}} - \underline{\mathbf{r}}_d)$.

$$\boxed{\underline{\mathbf{w}}_{LCMV} = \mathbf{R}_x^{-1}\mathbf{C}(\mathbf{C}^h\mathbf{R}_x^{-1}\mathbf{C})^{-1}\underline{\mathbf{r}}_d}$$

Optimum (data-dependent) beamforming

- Minimum mean squared error (MMSE)
- Multiple sidelobe canceller (MSC)
- Max. Signal-to-interference-plus-noise ratio (Max. SINR) 不能自适应
- Minimum variance distortionless response (MVDR)
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- Generalized sidelobe canceller (GSC)

GSC: Generalized sidelobe canceller

- Alternative (efficient) form of the LCMV beamformer

- Converts a constrained minimization problem into an unconstrained one

GSC

We have J weights to determine and M constraints available. Divide the J dimensional weight space into two subspaces:

- Constraint subspace defined by the columns of $\mathbf{C}$, a $J \times M$ matrix
- A subspace orthogonal to $\mathbf{C}$ defined by the columns of a $J \times (J - M)$ matrix B

We have $\mathbf{C}^h\mathbf{B} = \mathbf{0}_{M \times J-M}$

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GSC

We make use of a known result: if $\mathbf{C}^h\mathbf{B} = \mathbf{0}$ and the matrix $[\mathbf{C} \quad \mathbf{B}]$ is nonsingular, then

$$\mathbf{I} = \underbrace{\mathbf{C}[\mathbf{C}^h\mathbf{C}]^{-1}\mathbf{C}^h}_{\mathbf{P}_c} + \underbrace{\mathbf{B}[\mathbf{B}^h\mathbf{B}]^{-1}\mathbf{B}^h}_{\mathbf{P}_b}$$

$\mathbf{P}_c$ (resp. $\mathbf{P}_b$) is the projection matrix on to the subspace spanned by the columns of $\mathbf{C}$ (resp. $\mathbf{B}$)

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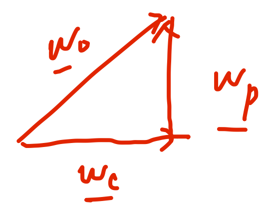
GSC

Let $\underline{\mathbf{w}}_o^h$ be the optimal LCMV solution:

$$\underline{\mathbf{w}}_o^h = \underline{\mathbf{r}}_d^h(\mathbf{C}^h\mathbf{R}_x^{-1}\mathbf{C})^{-1}\mathbf{C}^h\mathbf{R}_x^{-1}$$

$$\begin{aligned}\underline{\mathbf{w}}_o^h &= \underline{\mathbf{w}}_o^h\mathbf{I} \\ &= \underline{\mathbf{w}}_o^h\left(\mathbf{C}[\mathbf{C}^h\mathbf{C}]^{-1}\mathbf{C}^h + \mathbf{B}[\mathbf{B}^h\mathbf{B}]^{-1}\mathbf{B}^h\right) \\ &= \underline{\mathbf{w}}_c^h - \underline{\mathbf{w}}_p^h\end{aligned}$$



$\underline{w}_c = C[C^h C]^{-1} C^h \underline{w}_o$

$\underline{w}_p = (I - C[C^h C]^{-1} C^h)\underline{w}_o$

$$\begin{aligned}\underline{\mathbf{w}}_c^h &= \underline{\mathbf{w}}_o^h\mathbf{C}[\mathbf{C}^h\mathbf{C}]^{-1}\mathbf{C}^h \\ &= \underline{\mathbf{r}}_d^h(\mathbf{C}^h\mathbf{R}_x^{-1}\mathbf{C})^{-1}\mathbf{C}^h\mathbf{R}_x^{-1}\mathbf{C}[\mathbf{C}^h\mathbf{C}]^{-1}\mathbf{C}^h \\ &= \underline{\mathbf{r}}_d^h[\mathbf{C}^h\mathbf{C}]^{-1}\mathbf{C}^h\end{aligned}$$

does not depend on data! R_x 消掉了

GSC

Next, look at the second component, $\underline{\mathbf{w}}_p^h$:

$$\begin{aligned}\underline{\mathbf{w}}_p^h &= -\underline{\mathbf{w}}_o^h\mathbf{B}[\mathbf{B}^h\mathbf{B}]^{-1}\mathbf{B}^h \\ &= -\underline{\mathbf{r}}_d^h(\mathbf{C}^h\mathbf{R}_x^{-1}\mathbf{C})^{-1}\mathbf{C}^h\mathbf{R}_x^{-1}\mathbf{B}[\mathbf{B}^h\mathbf{B}]^{-1}\mathbf{B}^h \\ &\cong -\underline{\mathbf{w}}_a^h\mathbf{B}^h\end{aligned}$$

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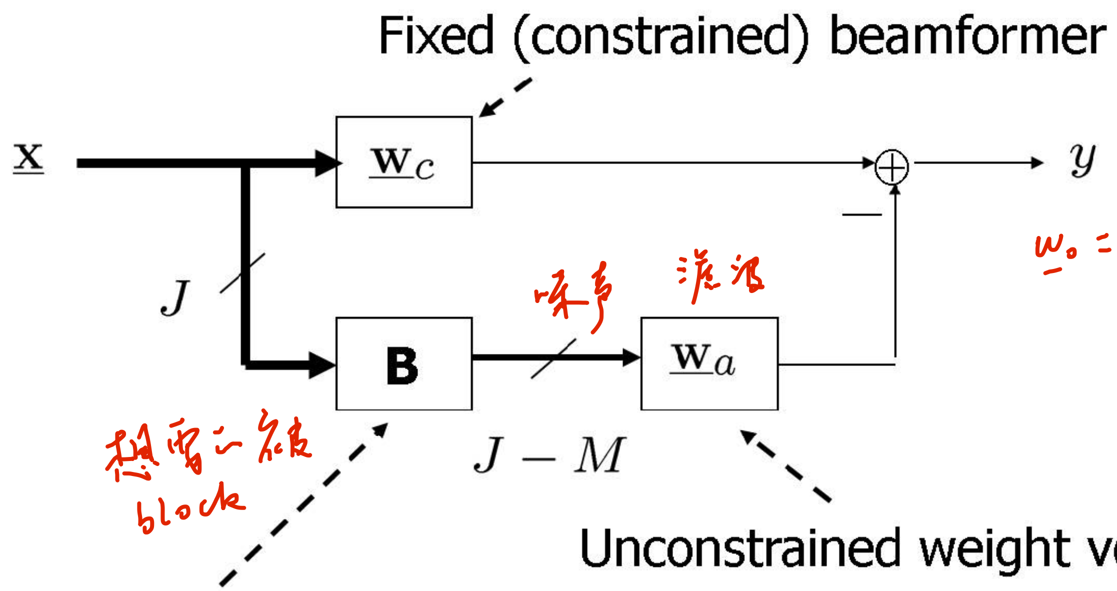
ADSP GSC



$\underline{w}_o = \underline{w}_c - \underline{w}_p$

$= \underline{w}_c - B\underline{w}_a$

Signal blocking matrix B: $C^hB = 0$

Can be constructed by any orthogonalization procedure
(e.g. Gram-Schmidt/ QR decomposition)

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GSC: an example

- Constraint: distortionless response in look direction

- $\mathbf{C} = \underline{\mathbf{a}}(\omega_0, \theta_0) = \left[1, e^{-j\phi_0}, e^{-j2\phi_0}, \cdots, e^{-j(J-1)\phi_0}\right]^t$ $J\times 1$
  with $\phi_0 = \omega_0 \frac{d\sin(\theta_0)}{c}$ and $\underline{\mathbf{r}}_d = 1$

$$
\begin{aligned}
\underline{\mathbf{w}}_c &= \underset{J\times 1}{\mathbf{C}} \left(\overset{J}{\underbrace{\mathbf{C}^h\mathbf{C}}}\right)^{-1} \underline{\mathbf{r}}_d \\
&= \underline{\mathbf{a}} \left(\underline{\mathbf{a}}^h\underline{\mathbf{a}}\right)^{-1} (1) \\
&= \frac{1}{J} \cdot \underline{\mathbf{a}}(\omega_0, \theta_0)
\end{aligned}
$$

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GSC: an example

- Beampattern corresp. to $\underline{\mathbf{w}}_c$

$$
\begin{aligned}
|r(\omega,\theta)| &= |\underline{\mathbf{w}}_c^h \cdot \underline{\mathbf{a}}(\omega,\theta)| \\
&= \left| \frac{1}{J} \sum_{i=1}^{J} e^{j(i-1)\frac{d}{c}(\omega_0 \sin(\theta_0) - \omega \sin(\theta))} \right|
\end{aligned}
$$

- For $\omega = \omega_0 = 2\pi f_0 = 2\pi \frac{c}{\lambda_0}$, beampattern becomes

$$
|r(\omega,\theta)| = \frac{1}{J} \left| \frac{\sin\left(J\pi \frac{d}{\lambda_0}\left[\sin(\theta_0) - \sin(\theta)\right]\right)}{\sin\left(\pi \frac{d}{\lambda_0}\left[\sin(\theta_0) - \sin(\theta)\right]\right)} \right|
$$

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GSC: an example

$M \times J \quad J \times (J-M)$

- Construction of blocking matrix: $\mathbf{C}^h\mathbf{B} = \mathbf{0} \;\Rightarrow$ $\underline{\mathbf{a}}^h\mathbf{B} = \mathbf{0}$ with $J \times (J-1)$ matrix $\mathbf{B}$, $M=1$

- E.g.: $\mathbf{B} = \begin{pmatrix} -1 & -1 & \cdots & -1 \\ e^{-j\phi_0} & 0 & \cdots & 0 \\ 0 & e^{-j2\phi_0} & \cdots & 0 \\ 0 & 0 & \cdots & e^{-j(J-1)\phi_0} \end{pmatrix}$

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GSC: an example

- Thus for m^{th} column $(m = 1, 2, \cdots, J-1)$:
  $\underline{\mathbf{b}}_m = \left(-1, 0, \cdots, 0, e^{-jm\phi_0}, 0, \cdots, 0\right)^t$ 第(m+1)行，第m列
  with beampattern: $|r_m| = |\underline{\mathbf{b}}_m^h \cdot \underline{\mathbf{a}}(\omega, \theta)|$ 共 J-1 个 beampattern
  $\left| -1 + e^{jm\omega\frac{d\sin\theta_0}{c}} e^{-jm\omega\frac{d\sin\theta}{c}} \right|$
- $\Rightarrow$ Beampattern for half wavelength spacing:
  $|r_m(\omega, \theta)| = 2\left|\sin\left(\frac{1}{2}m\pi\left[\sin(\theta_0) - \sin(\theta)\right]\right)\right|$
  (amplitude response of m^{th} column of blocking matrix)

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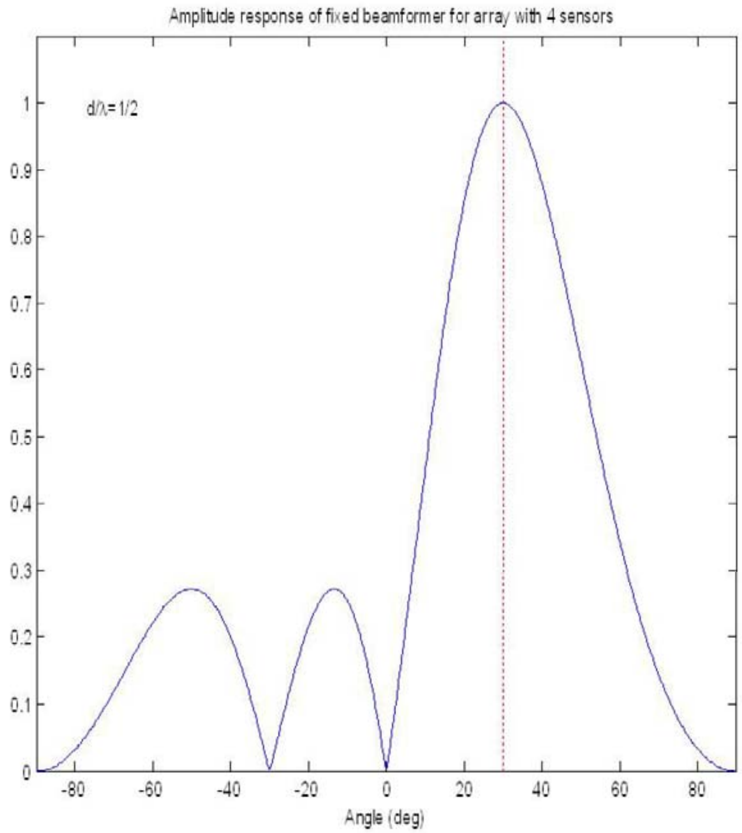
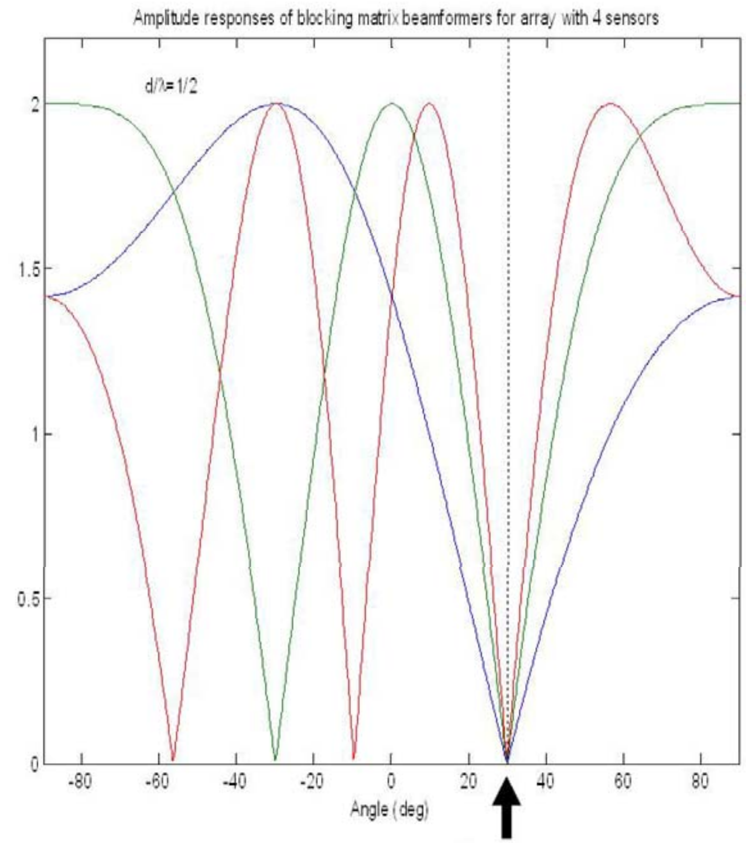
GSC: an example

$|r(\theta)|$



$\theta_{desired}$

$|r_m(\theta)|$



$\theta_{desired}$

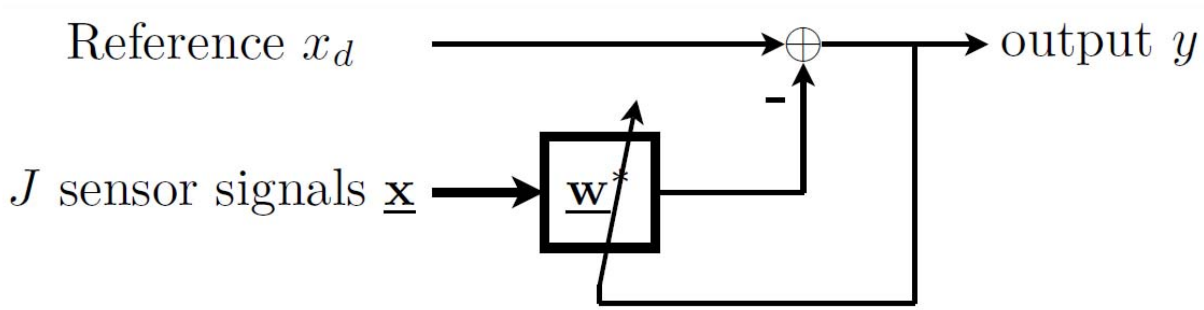
Block

Data dependent beamforming

- Optimum: Assume knowledge of array data statistics. Optimal with respect to a certain optimization criterion (e.g., min MSE, max SINR)

- Adaptive: Estimate the required statistics as data becomes available (when statistics are unknown or time-varying). The derivation is done assuming known statistics. These statistics are estimated on-line.

General adaptive array structure



Wiener: $\underline{\mathbf{w}}_{opt} = arg \min_{\underline{\mathbf{w}}} \{E\{|y|^2\}\}$

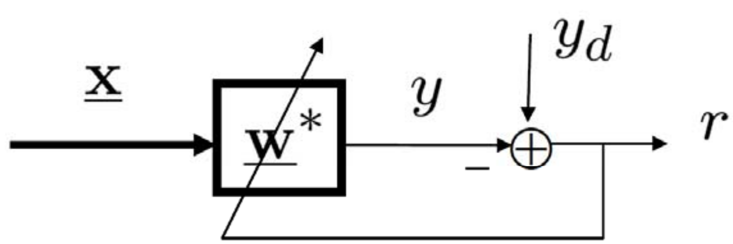
$$\Rightarrow \quad \underline{\mathbf{w}}_{opt} = \mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{xx_d^*}$$

LMS update rule : $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \underline{\mathbf{x}}[k] y^*[k]$

Final value : $\lim_{k \to \infty} E\{\underline{\mathbf{w}}[k]\} = \underline{\mathbf{w}}_{opt}$

Adaptive MMSE

Adaptive MMSE (Wiener):



Optimal (Wiener) solution:

$$\underline{\mathbf{w}}_o = arg \min_{\underline{\mathbf{w}}}\{E\{|r|^2\}\} \;\Rightarrow\; \underline{\mathbf{w}}_o = \mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{xy_d^*}$$

LMS update: $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \underline{\mathbf{x}}[k] r^*[k]$

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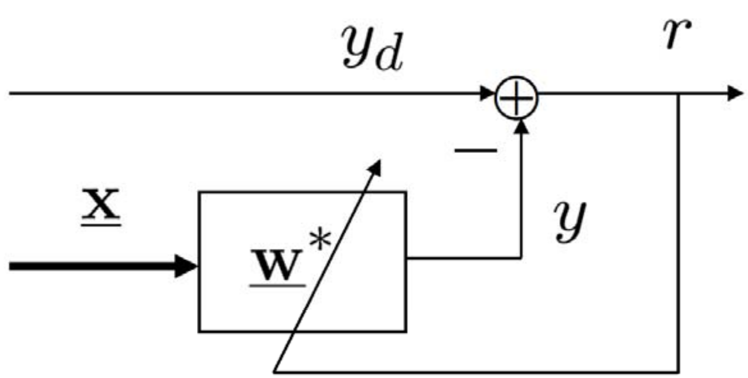
Adaptive MSC



Optimal (Wiener) solution:

(J-1)(J-1)

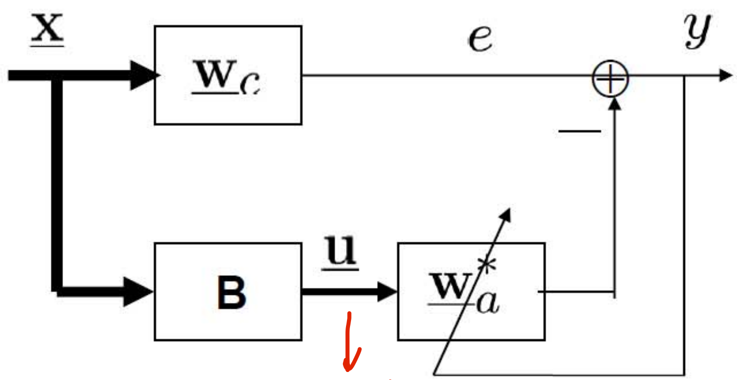
$$\underline{\mathbf{w}}_O = arg\min_{\underline{\mathbf{w}}}\{E\{|r|^2\}\} \Rightarrow \underline{\mathbf{w}}_O = \mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{xy_d^*}$$

LMS update: $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha\underline{\mathbf{x}}[k]r^*[k]$

(J-1)×1

ADSP

Adaptive GSC



仅含噪声

$$\underline{\mathbf{w}}_c = \mathbf{C}\cdot\left(\mathbf{C}^h\mathbf{C}\right)^{-1}\underline{\mathbf{r}}_d$$

$$\mathbf{C}^h\,\mathsf{B} = 0$$

$$\underline{\mathbf{u}} = \mathsf{B}\cdot\underline{\mathbf{x}}$$

Optimal (Wiener) solution:

$$\underline{\mathbf{w}}_o = arg\min_{\underline{\mathbf{w}}_a}\{E\{|y|^2\}\} \;\Rightarrow\; \underline{\mathbf{w}}_o = \mathbf{R}_u^{-1}\cdot\underline{\mathbf{r}}_{ue^*}$$

LMS update: $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha\underline{\mathbf{u}}[k]y^*[k]$

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Adaptive LCMV

Performance index: $J = \underline{\mathbf{w}}^h \cdot \mathbf{R}_x \cdot \underline{\mathbf{w}} + \underline{\lambda}\left(\mathbf{C}^h \underline{\mathbf{w}} - r_d\right)$

Gradient: $\underline{\nabla} = 2\mathbf{R}_x \underline{\mathbf{w}} + \mathbf{C}\underline{\lambda} = 2E\{\underline{\mathbf{x}}\underline{\mathbf{x}}^h\}\underline{\mathbf{w}} + \mathbf{C}\underline{\lambda}$

LMS estimate of gradient (no E { }):

$$\hat{\underline{\nabla}} \approx 2\underline{\mathbf{x}}[k]\underline{\mathbf{x}}^h[k]\underline{\mathbf{w}}[k] + \mathbf{C}\underline{\lambda} = 2\underline{\mathbf{x}}[k]y^*[k] + \mathbf{C}\underline{\lambda}$$

$$\left.\begin{array}{l} \Rightarrow \underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] - 2\alpha\underline{\mathbf{x}}[k]y^*[k] - \alpha\mathbf{C}\underline{\lambda} \\ \text{Furthermore: } \mathbf{C}^h\underline{\mathbf{w}}[k+1] = \underline{\mathbf{r}}_d \end{array}\right\} \Longrightarrow$$

$$\Rightarrow -\alpha\underline{\lambda} = \left(\mathbf{C}^h\mathbf{C}\right)^{-1}\left(\underline{\mathbf{r}}_d - \mathbf{C}^h\underline{\mathbf{w}}[k] + 2\alpha\mathbf{C}^h\underline{\mathbf{x}}[k]y^*[k]\right)$$

$C^h(\underline{w} - 2\alpha\underline{x}y^* - \alpha C\underline{\lambda}) = \underline{r}_d \Rightarrow -\alpha\underline{\lambda} = (C^hC)^{-1}(\underline{r}_d - C^h\underline{w} + 2\alpha C^h\underline{x}y^*)$

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Adaptive GSC

$\underline{w}[k+1] = \underline{w}[k] - 2\alpha \underline{x}[k] y^*[k] + C(C^hC)^{-1}(\underline{r}_d - C^h \underline{w}[k] + 2\alpha C^h \underline{x}[k] y^*[k])$

Final result:

$$\boxed{\underline{\mathbf{w}}[k+1] = \mathbf{P}^{\perp}\cdot\big(\underbrace{\underline{\mathbf{w}}[k] - 2\alpha\underline{\mathbf{x}}[k]y^*[k]}_{\text{无约束时 SGD 的解}}\big) + \underbrace{\mathbf{C}\left(\mathbf{C}^h\mathbf{C}\right)^{-1}\underline{\mathbf{r}}_d}_{\text{初始值}}}$$

→ 初始值

→ 无约束时 SGD 的解

整体和 constrained LMS 相似

with $\mathbf{P}^{\perp} = \mathbf{I} - \mathbf{C}\left(\mathbf{C}^h\mathbf{C}\right)^{-1}\mathbf{C}^h = \mathbf{I} - \mathbf{P}$

and projection matrix $\mathbf{P}$ $\quad \underline{w}[0] = C(C^hC)^{-1}\underline{r}_d$