Advanced Digital Signal Processing (ADSP)

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ADSP

Part A: Stochastic signal processing

Contents

- Random variable
- Random vector
- Stochastic process
- Second order statistics
- Power spectrum estimation

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Random Vectors

Definition: $\mathrm{x}(\xi) = [x_1(\xi), x_2(\xi), \ldots, x_M(\xi)]^T$, *T denotes the transpose.*

A mapping from an abstract probability space to a vector-valued real space $\mathbb{R}^M$

Distribution function:

$$F_{\mathrm{x}}(x_1, x_2, \ldots, x_M) \triangleq \Pr\{x_1(\xi) \le x_1, x_2(\xi) \le x_2, \ldots, x_M(\xi) \le x_M\}.$$

$$F_{\mathrm{x}}(\mathrm{x}) \triangleq \Pr\{\mathrm{x}(\xi) \le \mathrm{x}\}.$$

Joint probability density function:

$$f_{\mathrm{x}}(\mathrm{x}) = \lim_{\substack{\Delta x_1 \to 0 \\ \vdots \\ \Delta x_M \to 0}} \frac{\Pr\{x_1 < x_1(\xi) \le x_1 + \Delta x_1, x_M < x_M(\xi) \le x_M + \Delta x_M\}}{\Delta x_1 \ldots \Delta x_M}$$

$$\triangleq \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_M} F_{\mathrm{x}}(\mathrm{x}).$$

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Random Vectors

Marginal density function

$$f_{x_j}(x_j) = \underbrace{\int \dots \int}_{M-1} f_{\mathrm{x}}(\mathrm{x}) \mathrm{d}x_1 \dots \mathrm{d}x_{j-1} \mathrm{d}x_{j+1} \dots \mathrm{d}x_M$$

$$F_{\mathrm{x}}(\mathrm{x}) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_M} f_{\mathrm{x}}(\mathbf{v})\, \mathrm{d}v_1 \dots \mathrm{d}v_M = \int_{-\infty}^{\mathrm{x}} f_{\mathrm{x}}(\mathbf{v}) \mathrm{d}\mathbf{v}$$

Independent

$x_1(\xi)$ and $x_2(\xi)$ independent: $\{x_1(\xi) \leq x_1\}$ and $\{x_2(\xi) \leq x_2\}$ jointly independent.

$$\Pr\{x_1(\xi) \leq x_1, x_2(\xi) \leq x_2\} = \Pr\{x_1(\xi) \leq x_1\} \Pr\{x_2(\xi) \leq x_2\}$$

$$F_{x_1,x_2}(x_1, x_2) = F_{x_1}(x_1)\, F_{x_2}(x_2)$$

$$f_{x_1,x_2}(x_1, x_2) = f_{x_1}(x_1)\, f_{x_2}(x_2)$$

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Random Vectors: statistics

Mean Vector

$$\mu_{\mathrm{x}} = \mathrm{E}\{\mathrm{x}(\xi)\} = \begin{bmatrix} \mathrm{E}\{x_1(\xi)\} \\ \vdots \\ \mathrm{E}\{x_M(\xi)\} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_M \end{bmatrix}$$

Correlation matrix

$$\mathrm{R}_{\mathrm{x}} \triangleq \mathrm{E}\{\mathrm{x}(\xi)\mathrm{x}^H(\xi)\} = \begin{bmatrix} r_{11} & \cdots & r_{1M} \\ \vdots & \ddots & \vdots \\ r_{M1} & \cdots & r_{MM} \end{bmatrix}$$

H denotes the conjugate transpose

Diagonal terms: $r_{ii} \triangleq \mathrm{E}\{|x_i(\xi)|^2\}, \quad i = 1, \dots, M$ $\quad r_{x_i}^{(2)}$

Off-diagonal terms: $r_{ij} \triangleq \mathrm{E}\{x_i(\xi)x_j^*(\xi)\} = r_{ji}^*, \quad i \neq j$

r_{ij}: the statistical similarity between $x_i(\xi)$ and $x_j(\xi)$

R_{x} is conjugate symmetric $\quad \mathrm{R}_{\mathrm{x}} = \mathrm{R}_{\mathrm{x}}^H \quad$ Hermitian Matrix

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Random Vectors: statistics

Auto-covariance

$$\Gamma_{\mathrm{x}} \triangleq \mathrm{E}\{[\mathrm{x}(\xi) - \mu_{\mathrm{x}}][\mathrm{x}(\xi) - \mu_{\mathrm{x}}]^H\} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1M} \\ \vdots & \ddots & \vdots \\ \gamma_{M1} & \cdots & \gamma_{MM} \end{bmatrix}$$

Diagonal terms:

$$\gamma_{ii} \triangleq \mathrm{E}\{|x_i(\xi) - \mu_i|^2\} = \mathrm{E}\{|x_i(\xi)|^2\} - \mu_i^2 \quad i = 1, \dots, M \qquad \sigma_{x_i}^{(2)}$$

Off-diagonal terms:

$$\gamma_{ij} \triangleq \mathrm{E}\{[x_i(\xi) - \mu_i][x_j(\xi) - \mu_j]^*\} = \mathrm{E}\{x_i(\xi)x_j^*(\xi)\} - \mu_i\mu_j^* = \gamma_{ji}^*, \quad i \neq j$$

γ_{ij}: the covariance between $x_i(\xi)$ and $x_j(\xi)$

Γ_{x} is also a Hermitian matrix

$$\gamma_{ij} = \mathrm{E}\{x_i(\xi)x_j^*(\xi)\} - \mu_i\mu_j^* = r_{ij} - \mu_i\mu_j^*$$

$$\Gamma_{\mathrm{x}} \triangleq \mathrm{E}\{[\mathrm{x}(\xi) - \mu_{\mathrm{x}}][\mathrm{x}(\xi) - \mu_{\mathrm{x}}]^H\} = \mathrm{R}_{\mathrm{x}} - \mu_{\mathrm{x}}\mu_{\mathrm{x}}^H$$

Random Vectors: statistics

Correlation coefficient between $x_i(\xi)$ and $x_j(\xi)$

$$\rho_{ij} \triangleq \frac{\gamma_{ij}}{\sigma_{x_i}\sigma_{x_j}}$$

ρ_{ij} measures the statistical similarity between $x_i(\xi)$ and $x_j(\xi)$

$$|\rho_{ij}| \leq 1$$

- $\rho_{ij} = 1, x_i(\xi)$ and $x_j(\xi)$ are perfectly correlated;
- $\rho_{ij} = 0$, uncorrelated;
- $\rho_{ij} = -1$, negatively correlated.

Linear correlation

Random Vectors: statistics

Uncorrelated vs. independent

Uncorrelated $$\rho_{ij} \triangleq \frac{\gamma_{ij}}{\sigma_{x_i}\sigma_{x_j}} = \frac{\mathrm{E}\{x_i(\xi)x_j^*(\xi)\} - \mu_i\mu_j^*}{\sigma_{x_i}\sigma_{x_j}} = 0$$

Independent $$f_{x_1,x_2}(x_1,x_2) = f_{x_1}(x_1)\, f_{x_2}(x_2)$$

$$\Downarrow$$

$$\mathrm{E}\{x_i(\xi)x_j^*(\xi)\} = \mathrm{E}\{x_i(\xi)\}\,\mathrm{E}\{x_j^*(\xi)\} = \mu_i\mu_j^*$$

Independent $\Rightarrow$ Uncorrelated

Uncorrelated $\not\Rightarrow$ Independent

Uncorrelated + Gaussian $\Rightarrow$ Independent

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Random Vectors: statistics

Uncorrelated vs. orthogonal

Uncorrelated $\rho_{ij} \triangleq \frac{\gamma_{ij}}{\sigma_{x_i}\sigma_{x_j}} = \frac{\mathrm{E}\{[x_i(\xi) - \mu_i][x_j(\xi) - \mu_j]^*}{\sigma_{x_i}\sigma_{x_j}} = 0$

Orthogonal $\mathrm{E}\{x_i(\xi)x_j^*(\xi)\} = 0$

{Uncorrelated} + $\{\mu_i = 0$ or $\mu_j = 0\}$ $\Rightarrow$ Orthogonal

{Orthogonal} + $\{\mu_i = 0$ or $\mu_j = 0\}$ $\Rightarrow$ Uncorrelated

Part A: Stochastic signal processing

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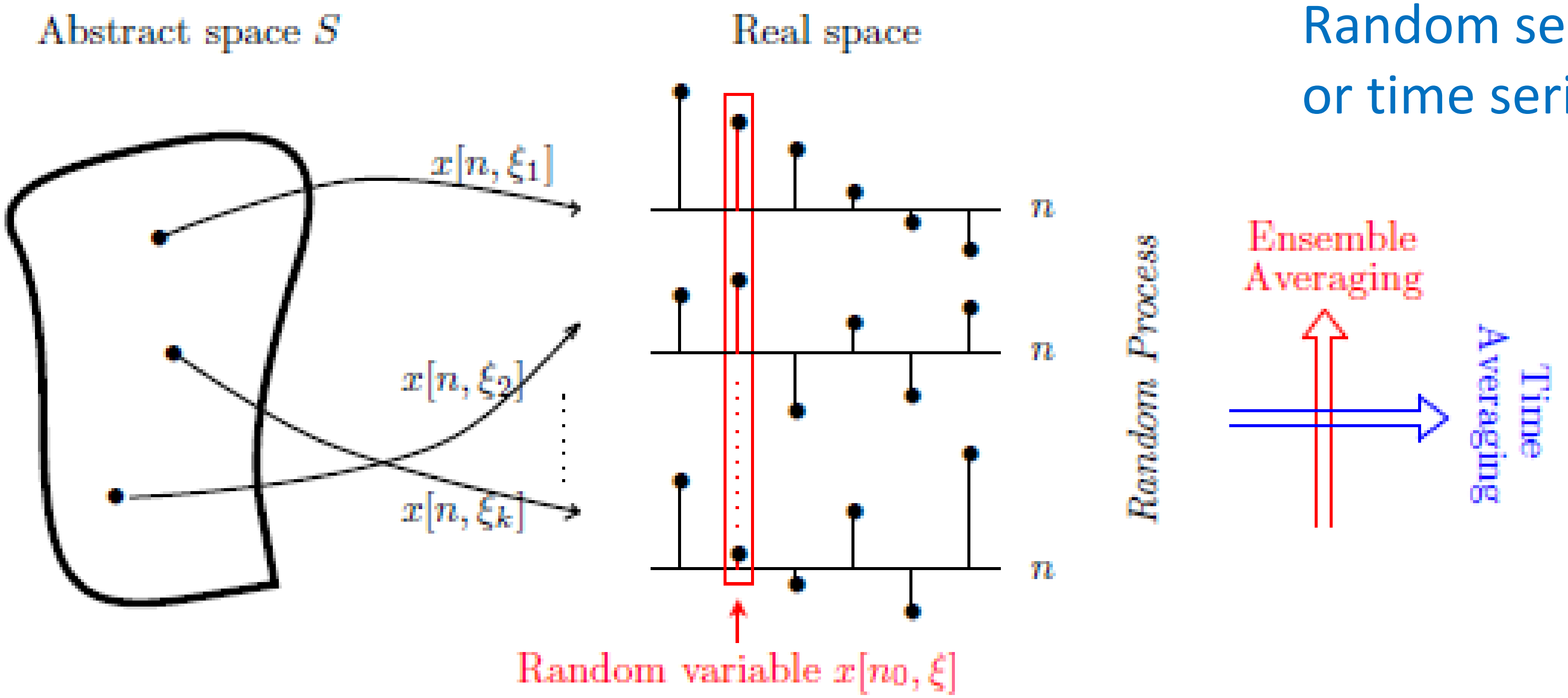
Discrete-time stochastic process



Random sequence or time series

Ensemble: set of all possible sequences $\{x(n,\xi)\}$

- $x(n,\xi)$: a random variable if n is $fixed$ and ξ is a variable.
- $x(n,\xi)$: a sample sequence if ξ is $fixed$ and n is a variable.
- $x(n,\xi)$: a number if both n and ξ are $fixed$.
- $x(n,\xi)$: a stochastic process if both n and ξ are variables.

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Discrete-time stochastic process

k^{th}-order distribution function

$$F_x(x_1, \ldots, x_k; n_1, \ldots, n_k) = \Pr\{x(n_1) \leq x_1, \ldots, x(n_k) \leq x_k\}$$

k^{th}-order pdf

$x\ (n)$ is assumed to be real-valued

$$f_x(x_1, \ldots, x_k; n_1, \ldots, n_k) \triangleq \frac{\partial F_x(x_1, \ldots, x_k; n_1, \ldots, n_k)}{\partial x_1 \ldots \partial x_M} \qquad k \geq 1$$

Notes:

- Probabilistic description requires a lot of information
- Use statistical description in practice
- Skip ξ, thus x[n] for random process or single realization

Part A: Stochastic signal processing

Contents

Stochastic process: 2nd order statistics

Statistical properties of stochastic process $x[n]$ **at time n**

Mean : $\mu_x[n] = E\{x[n]\}$

Variance : $\sigma_x^2[n] = E\{|x[n] - \mu_x[n]|^2\} = E\{|x[n]|^2\} - E\{|\mu_x[n]|^2\}$

Autocorrelation : $r_x[n_1, n_2] = E\{x[n_1] \cdot x^*[n_2]\}$

Autocovariance : $\gamma_x[n_1, n_2] = E\{(x[n_1] - \mu_x[n_1]) \cdot (x[n_2] - \mu_x[n_2])^*\}$
$= r_x[n_1, n_2] - \mu_x[n_1] \cdot \mu_x^*[n_2]$

Random variable $x[\xi]$

Mean : $\mu_x \triangleq E\{x(\xi)\} = \int_{-\infty}^{\infty} x\, f_x(x)\mathrm{d}x$

Variance : ${\sigma_x}^2 \triangleq {\gamma_x}^{(2)} = E\{[x(\xi) - \mu_x]^2\}$

Correlation : $r_{ij} \triangleq \mathrm{E}\{x_i(\xi)x_j^*(\xi)\} = r_{ji}^*, \quad i \neq j$

Covariance : $\gamma_{ij} \triangleq \mathrm{E}\{[x_i(\xi) - \mu_i][x_j(\xi) - \mu_j]^*$
$= \mathrm{E}\{x_i(\xi)x_j^*(\xi)\} - \mu_i\mu_j^*, \quad i \neq j$

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Stochastic process: 2nd order statistics

Statistical relation between two of stochastic process $x[n]$ and $y[n]$

Cross-correlation : $r_{xy}[n_1, n_2] = E\{x[n_1] \cdot y^*[n_2]\}$

Cross-covariance : $\gamma_{xy}[n_1, n_2] = E\{(x[n_1] - \mu_x[n_1]) \cdot (y[n_2] - \mu_y[n_2])^*\}$

$$= r_{xy}[n_1, n_2] - \mu_x[n_1] \cdot \mu_y^*[n_2]$$

Normalized γ_{xy} : $\rho_{xy}[n_1, n_2] = \dfrac{\gamma_{xy}[n_1, n_2]}{\sigma_x[n_1] \cdot \sigma_y[n_2]}$

Stochastic process: some definitions

Independent

If $f_x[x_1, \dots, x_k; n_1, \dots, n_k] = f_1[x_1; n_1] \dots f_k[x_k; n_k] \quad \forall k, n_i, i = 1, \dots, k$

$x[n]$ is a sequence of independent random variables.

IID (Independent and Identically Distributed)

If $f_1[x_1; n_1] = f_2\ [x_2; n_2] = \cdots = f_k[x_k; n_k] \quad \forall k, n_i, i = 1, \dots, k$

$x[n]$ is a IID random sequence.

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Stochastic process: some definitions

Uncorrelated

If $\gamma_x[n_1,n_2] = \begin{cases} \sigma_x^2[n_1] & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases} = \sigma_x^2[n_1]\delta(n_1 - n_2)$

$x[n]$ is a sequence of uncorrelated random variables.

$$\gamma_x[n_1,n_2] = r_x[n_1,n_2] - \mu_x[n_1] \cdot \mu_x^*[n_2]$$

$$r_x[n_1,n_2] = \begin{cases} \sigma_x^2[n_1] + |\mu_x[n_1]|^2 & n_1 = n_2 \\ \mu_x[n_1] \cdot \mu_x^*[n_2] & n_1 \neq n_2 \end{cases}$$

Orthogonal

$$r_x[n_1,n_2] = \begin{cases} \sigma_x^2[n_1] + |\mu_x[n_1]|^2 & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases} = \mathrm{E}\{|x[n_1]|^2\}\,\delta(n_1 - n_2)$$

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Stochastic process: stationary

Stationary of order N

$$f_x[x_1, \dots, x_N; n_1, \dots, n_N] = f_x[x_1, \dots, x_N; n_1 + k, \dots, n_N + k] \quad \forall k$$

Strict-sense stationary (SSS)

$x[n]$ is stationary for all orders *N=1, 2, ….*

An IID sequence is SSS.

Wide-sense stationary (WSS): stationary up to order 2

- Mean is a constant independent of n: $\mathrm{E}\{x(n)\} = \mu_x$
- Variance is a constant independent of n: $\mathrm{var}\{x(n)\} = \sigma_x^2$
- Autocorrelation depends only on l ($l = n_1 - n_2$)

$$r_x(n_1, n_2) = r_x(n_1 - n_2) = r_x\ (l)$$

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Stochastic process: stationary

Wide-sense stationary (WSS):

Example:

Let $w(n)$ be a zero-mean, uncorrelated Gaussian random sequence with variance $\sigma^2(n) = 1$.

a. Characterize the random sequence $w(n)$.

b. Define $x(\mathrm{n}) = w(n) + w(n-1)$. Determine the mean and autocorrelation of $x(\mathrm{n})$. Also characterize $x(\mathrm{n})$.

x[n] 非独立