Advanced Digital Signal Processing (ADSP)

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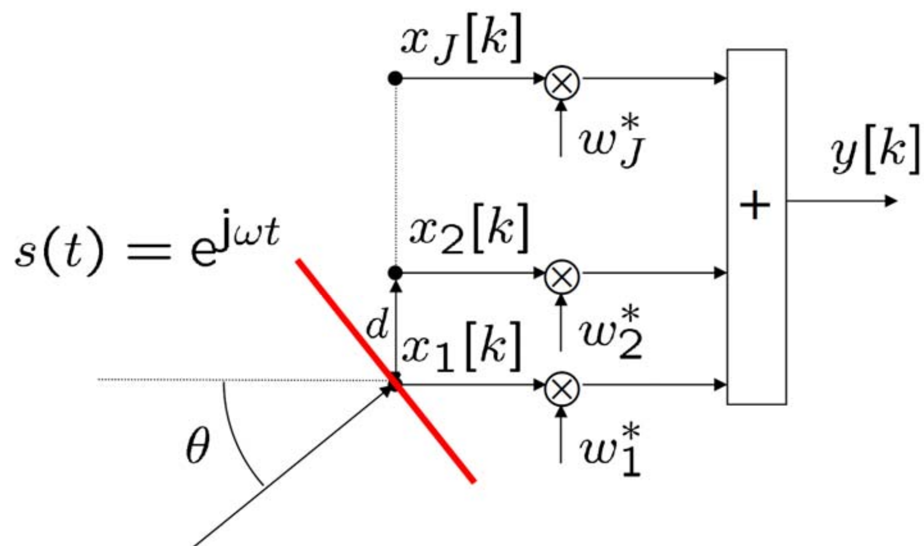
Recall last lecture: ULA Beampattern



共轭转置

$$\Rightarrow \quad y[k] = \sum_{i=1}^{J} w_i^* x_i[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{x}}[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta) \cdot s[k]$$

$$\text{with:} \quad (\underline{\mathbf{a}}(\theta))_i = a_i(\theta) = \mathrm{e}^{-\mathrm{j}2\pi(i-1)\frac{d\sin(\theta)}{\lambda}}$$

Recall last lecture: ULA Beampattern

采样空间频率 $\frac{1}{d}$

信号本身空间频率 $\frac{\sin\theta}{\lambda}$

Array response: $r(\theta) = \mathbf{w}^h \cdot \mathbf{a}(\theta)$

Array beam pattern: $B(\theta) = \frac{1}{J^2} \cdot |r(\theta)|^2$

$$B(\theta) = \frac{1}{J^2}|\underline{1}^t \cdot \underline{\mathrm{a}}(\theta)|^2 = \frac{1}{J^2}\left|\sum_{i=1}^{J} \mathrm{e}^{-\mathrm{j}2\pi(i-1)\frac{d}{\lambda}\sin(\theta)}\right|^2$$

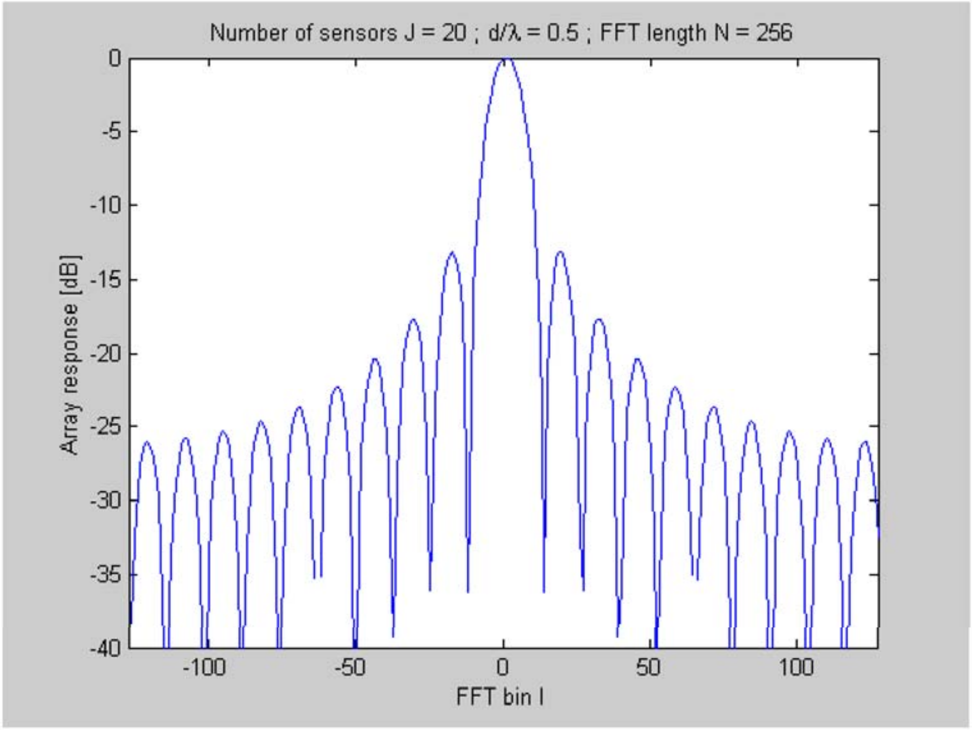
$$= \frac{1}{J^2}\left|\frac{1-\mathrm{e}^{-\mathrm{j}J2\pi\frac{d}{\lambda}\sin(\theta)}}{1-\mathrm{e}^{-\mathrm{j}2\pi\frac{d}{\lambda}\sin(\theta)}}\right|^2 = \frac{1}{J^2}\left|\frac{\sin(J\pi\frac{d}{\lambda}\sin(\theta))}{\sin(\pi\frac{d}{\lambda}\sin(\theta))}\right|^2$$

Important parameters:

- DOA θ
- Ratio $\frac{d}{\lambda}$ (everything scales with wavelength)
- Number of sensors J
- Element spacing d
- Array aperture $L = J \cdot d$

Recall last lecture: ULA Beampattern

```
w=(1/J)*ones(J); abs_r=fftshift(abs(fft(w,N)))
```



Compute corresponding angle via: $\theta = \arcsin(\frac{l \cdot \lambda}{N \cdot d})$

Data independent beamforming

$J: \frac{\sigma_s^2}{\sigma_n^2}$

$\Uparrow$

max. SNR

$\Uparrow$

Conventional approach:

- Beamsteering (Delay - and - sum beamformer) $\Rightarrow$ match filter
- Tapering

Other data independent approaches:

- Null-steering
- Array response design

Data independent beamforming

Beamsteering

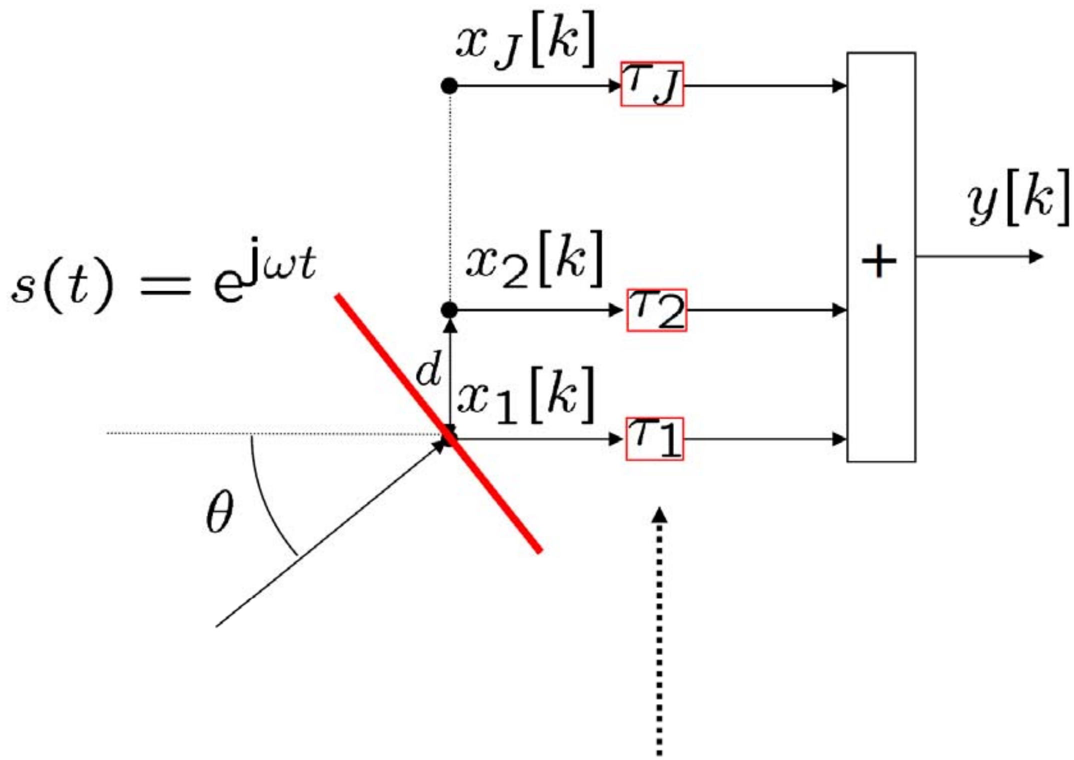
Goal:

Design w_i such that they apply delays

$\Rightarrow$ beampattern is rotated

Other name : **Delay and Sum Beamformer** (DSB)
补偿 Compensate propagation path length differences of direct path from source to each sensor, to obtain properly aligned direct path signal at the output.

Data independent beamforming



For ULA choose: $\tau_i = (i-1)\tau \leftrightarrow w_i^* = e^{j(i-1)\omega\tau}$

Data independent beamforming

$$\left.\begin{array}{l} y[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{x}}[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta) \cdot s[k] \\ w_i^* = \mathrm{e}^{\mathrm{j}(i-1)\omega\tau} \end{array}\right\} \Longrightarrow$$

$$B(\theta) = \frac{1}{J^2}\left|\underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta)\right|^2 = \frac{1}{J^2}\left|\sum_{i=1}^{J} \mathrm{e}^{-\mathrm{j}(i-1)\omega(\frac{d\sin(\theta)}{c}-\tau)}\right|^2 \leq \frac{J^2}{J^2}$$

when $\tau = \frac{d \sin\theta}{c}$, 取等号

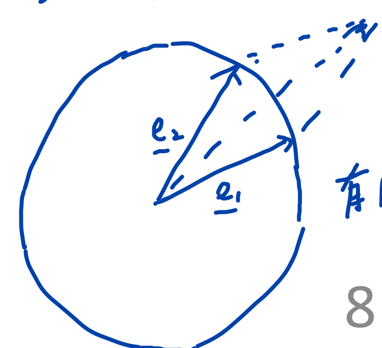
Thus mainlobe beampattern shifted over:

$$\boxed{\theta_0 = \arcsin(\frac{c\cdot\tau}{d})}$$

$\frac{d \sin\theta_0}{c} = \tau$

$\Rightarrow \theta_0 = \arcsin(\frac{c \cdot \tau}{d})$

when $J = 2$, we can see



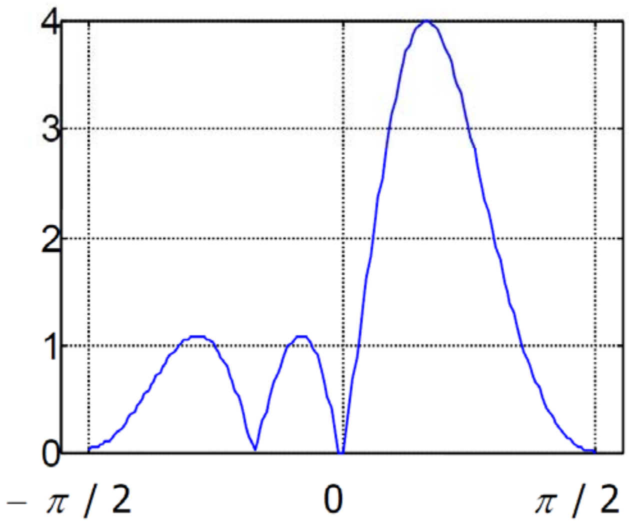
有 $\|e_1 + e_2\| = 2$

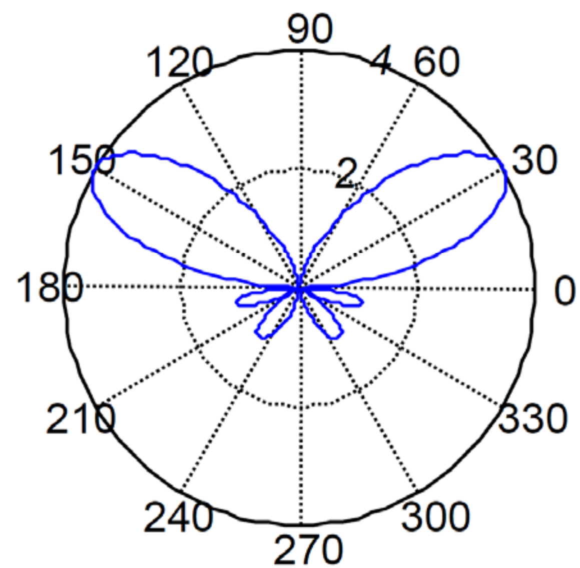
$\|e_1 + e_2\| < 2$, 当 e_1 和 e_2 角度相同

Data independent beamforming

Example: $J = 4; d/\lambda = 1/2; \tau \leftrightarrow 30^o$

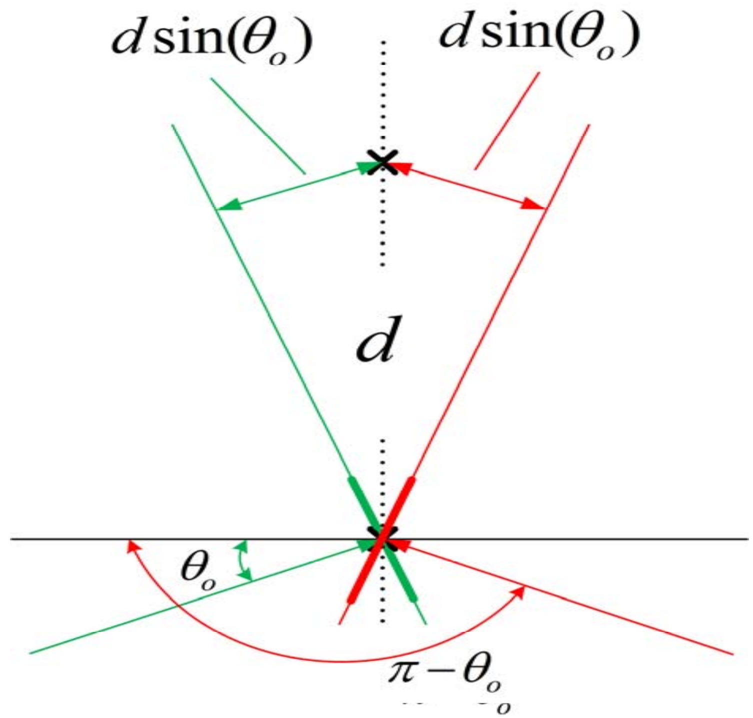
$\frac{\pi}{6}$ 与 $\pi - \frac{\pi}{6}$





Data independent beamforming

缺点：无法区分 θ 和 180°-θ



$s_0(t) = e^{j\omega_0 \cdot t}$

Ambiguity with s_1 at $\pi - \theta_0$

Delay: $e^{-j\omega_0 \frac{d\sin(\theta_0)}{c}} = e^{-j2\pi \frac{d}{\lambda_0}\sin(\theta_0)}$

$\boxed{\text{Delay is the same for } s_0 \text{ and } s_1}$

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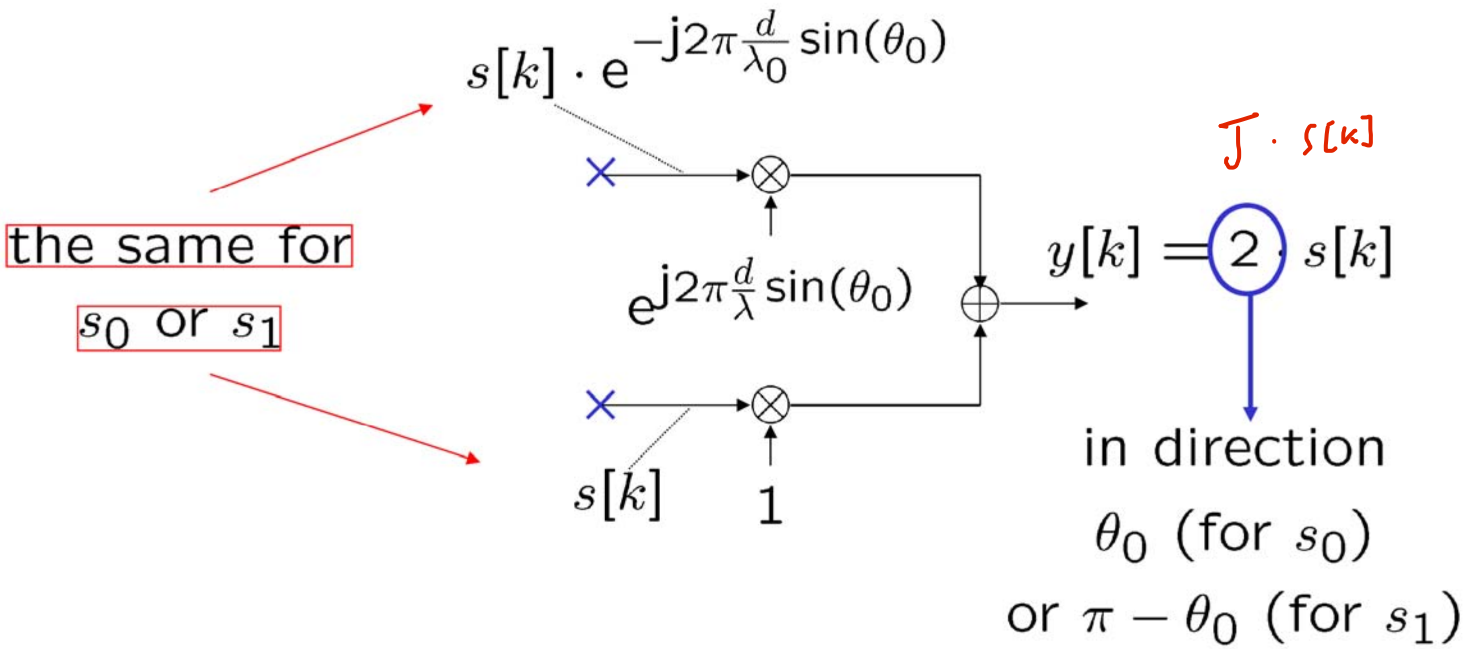
Data independent beamforming

Electronic beamsteering for ULA:

$$\Rightarrow \quad w_i^* = \mathrm{e}^{\mathrm{j}2\pi(i-1)\frac{d}{\lambda}\sin(\theta_0)}$$



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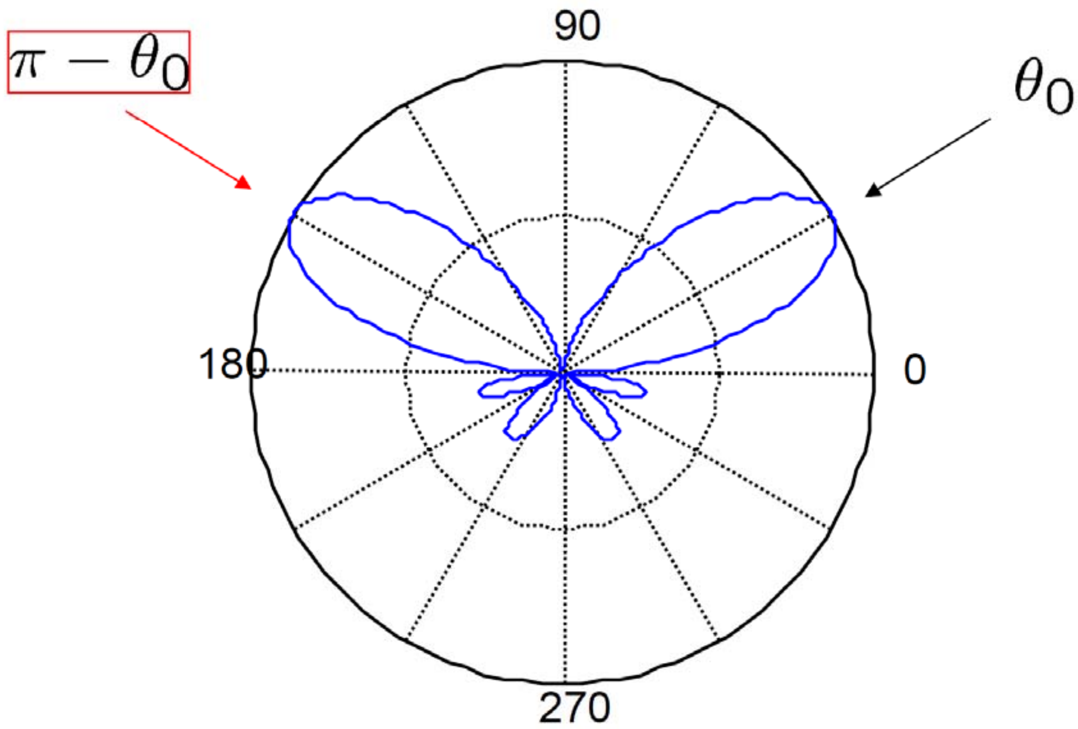
Result electronic beamsteering:



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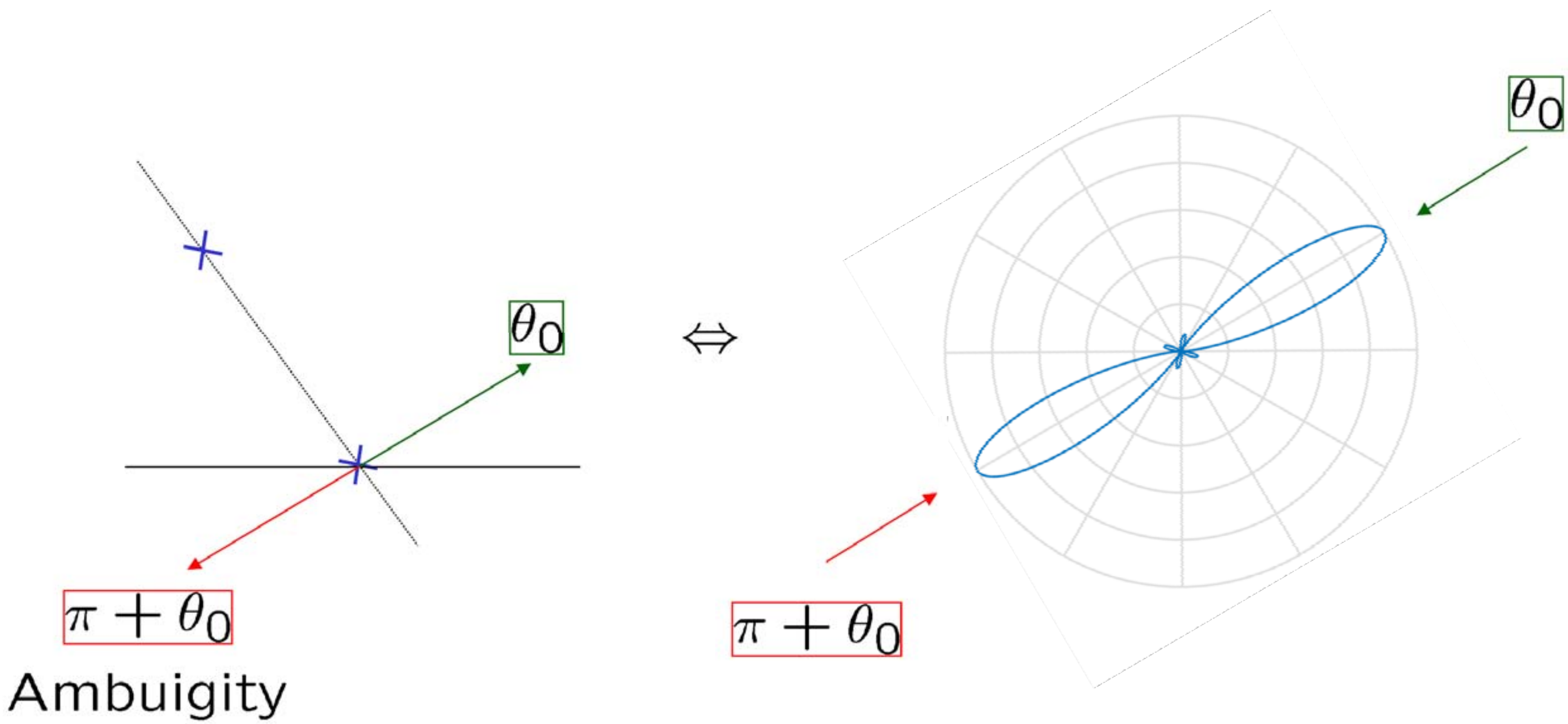
Data independent beamforming

Result mechanical bear



Data independent beamforming

Another view to beamsteering: **Known θ**

传感器噪声

$$y[k] = \underline{\mathbf{w}}^h \cdot \underline{\mathbf{x}}[k] = \underline{\mathbf{w}}^h \cdot (\underline{\mathbf{a}}(\theta)s[k] + \underline{\mathbf{n}}[k])$$

$$= \underline{\mathbf{w}}^h \cdot \underline{\mathbf{a}}(\theta)s[k] + \underline{\mathbf{w}}^h \cdot \underline{\mathbf{n}}[k] = y_s[k] + y_n[k]$$

$$E\{y^2[k]\} = E\{y_s^2[k]\} + E\{y_n^2[k]\} = E\{s^2[k]\}\, \underline{w}^h \cdot \underline{a}\, \underline{a}^h \underline{w} + \underline{w}^h E\{\underline{n}[k]\, \underline{n}^h[k]\} \cdot \underline{w}$$

$$P_y = \sigma_s^2 \cdot \underline{\mathbf{w}}^h(\underline{\mathbf{a}}(\theta) \cdot \underline{\mathbf{a}}^h(\theta))\underline{\mathbf{w}} + \sigma_n^2 \cdot \underline{\mathbf{w}}^h \underline{\mathbf{w}}$$

$$= P_s + P_n$$

Input SNR (for each sensor): $SNR_i = \dfrac{\sigma_s^2}{\sigma_n^2}$

Output SNR: $SNR_o = \dfrac{P_s}{P_n} = \dfrac{\sigma_s^2}{\sigma_n^2} \cdot \dfrac{\underline{w}^h(\underline{a}\, \underline{a}^h)\underline{w}}{\underline{w}^h \underline{w}}$

Data independent beamforming

$w = [1, e^{-j2\pi(i-1)\cdot\frac{d\sin\theta}{\lambda}} \ldots]^t$

$$\Rightarrow SNR_o = \frac{\underline{\mathbf{w}}^h(\underline{\mathbf{a}}(\theta)\cdot\underline{\mathbf{a}}^h(\theta))\underline{\mathbf{w}}}{\underline{\mathbf{w}}^h\underline{\mathbf{w}}}\cdot\frac{\sigma_s^2}{\sigma_n^2} = G(\theta)\cdot SNR_i$$

$\underline{w}^h \underline{w} = J$ is a constant

噪声指传感器自带的噪声

$$\underline{\mathbf{w}} = \beta\cdot\underline{\mathbf{a}}(\theta) \;\Rightarrow\; \text{Clearly } SNR_o \text{ maximized}$$

$$\frac{\underline{w}^h(\underline{a}\,\underline{a}^h)\underline{w}}{\underline{w}^h\underline{w}} = \frac{\|\underline{w}^h\underline{a}\|^2}{J} = \frac{\left(\sum_{i=1}^{J} e^{-j\omega(i-1)(\tau_i-\tau)}\right)^2}{J} \le \frac{J^2}{J} = J$$

Conclusion: Maximizing SNR

当 $\tau_i = \tau$ 时取等号，即 $\underline{w} = \beta\,\underline{a}(\theta)$

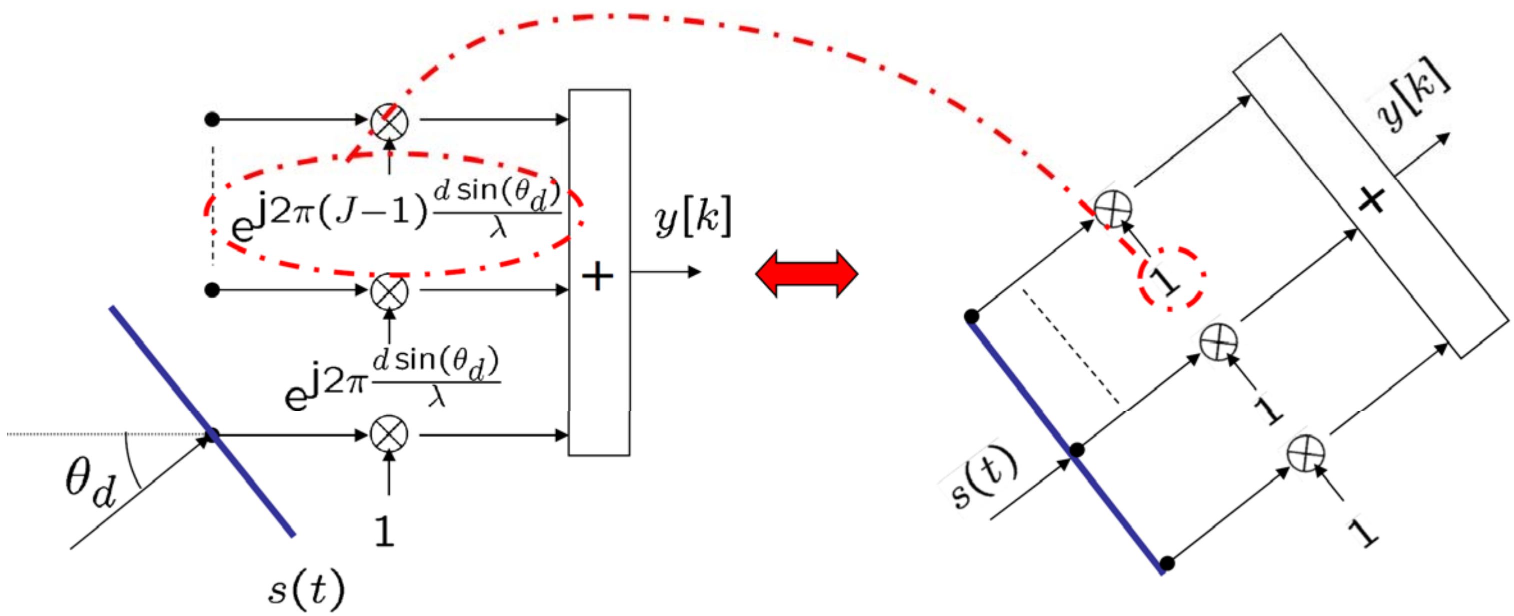
$\Rightarrow$ Spatial filter that matches DOA

$=$ **Matched filter**

Data independent beamforming

Matched filter: choose $\tau = \dfrac{d\sin(\theta_d)}{c} \Rightarrow \underline{\mathbf{w}} = \underline{\mathbf{a}}(\theta_d)$

$\Leftrightarrow$ "Rotate" array to DOA θ_d



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Another view to matched filter:

Match weight vector $\underline{\mathbf{w}}$ to align J sensor signals:

$$\underline{\mathbf{w}}_{mf} = \underline{\mathbf{a}}(\theta_d)$$

$\underline{x}[k] = \underline{a} \cdot s[k]$

matched filter

$\Rightarrow$ Output signal $y[k] = \underline{\mathbf{w}}_{mf}^h \cdot \underline{\mathbf{x}}[k] = \underline{\mathbf{a}}^h \cdot \underline{\mathbf{x}}[k]$ $= \underline{a}^h \underline{a} \cdot s[k] = J \cdot s[k]$

$\Rightarrow$ Array gain $G = \dfrac{\underline{\mathbf{w}}^h(\underline{\mathbf{a}} \cdot \underline{\mathbf{a}}^h)\underline{\mathbf{w}}}{\underline{\mathbf{w}}^h\underline{\mathbf{w}}} = J$ ($= \#$ sensors)

传感器数目

Conclusion:

In sense of max SNR_o, the spatial matched filter is optimum for spatially white noise

Data independent beamforming

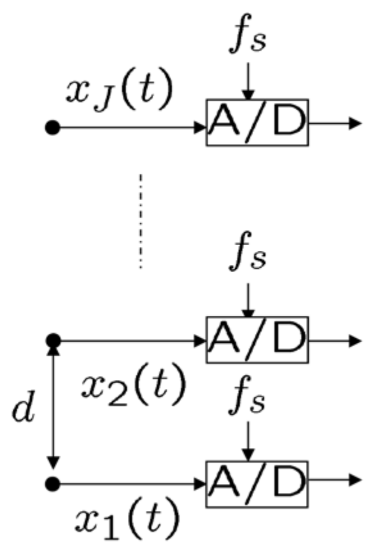
Notes on beamsteering or Delay and Sum Beamformer (DSB):

- Source location (or DOA) required
- Position sensors has to be known exactly (to obtain delays)
- DSB aligns only direct path signal

单源？

Data independent beamforming

Note on discrete-time beamsteering



Steering delay: $\tau = \dfrac{d \sin(\theta)}{c}$

Sample rate: $f_s = \dfrac{1}{T_s}$

$\Rightarrow$ Steering possible for $\tau = \alpha \cdot T_s$

$(\alpha = 0, 1, 2, \cdots)$

Conclusion:

Beam can only be steered to: $\theta_s = \arcsin\left(\dfrac{c \cdot \alpha \cdot T_s}{d}\right)$

with $\alpha = 0, 1, 2, \cdots$

Data independent beamforming

Assume: $d = \frac{\lambda}{2}; \lambda = \frac{c}{f}$ and choose $f_s = 2{\cdot}\gamma{\cdot}f$

$$\frac{c\cdot\alpha\cdot T_s}{d} = \frac{c\cdot\alpha\cdot\frac{1}{2\gamma f}}{\frac{\lambda}{2}} = \frac{c\cdot\alpha}{2\gamma\cdot\frac{c}{\lambda}\cdot\frac{\lambda}{2}} = \frac{\alpha}{\gamma}$$

$$\Rightarrow \quad \theta_s = \arcsin\left(\frac{c\cdot\alpha\cdot T_s}{d}\right) = \arcsin\left(\frac{\alpha}{\gamma}\right) \quad \Leftrightarrow \quad |\alpha| \leq \gamma$$

Conclusion: Beam can only be steered to

$1 + 2\lfloor\gamma\rfloor$ different angles! $\quad \frac{\alpha}{\gamma} = \frac{0}{\gamma}, \pm\frac{1}{\gamma}, \pm\frac{2}{\gamma}, \pm\frac{\gamma}{\gamma}$

<u>*Example:*</u> $f_s = 4 \cdot f$

$\Rightarrow$ beam can be steerded to $0^o, \pm 30^o, \pm 90^o$

可以插值，改变 T_s

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Data independent beamforming

If more directions are needed:

- **Interpolation** by factor K (thus $T_s \rightarrow T_s/K$)

Previous example: Interpolation factor $K = 4$

Results in 17 steering directions: $0^o, \pm 7.2^o, \pm 14.5^o,$

$$, \pm 22^o, \pm 30^o, \pm 38.7^o, \pm 48.6^o, \pm 60^o, \pm 90^o$$

- Use **fractional delays** $z^{-i/K}$

Data independent beamforming

Tapering

Goal: Control shape of response i.e. to form beam

Thus window (weighted) sensor signals to

compromise between resolution (main lobe

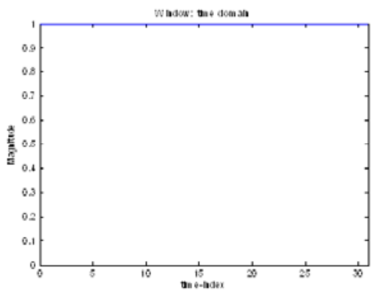
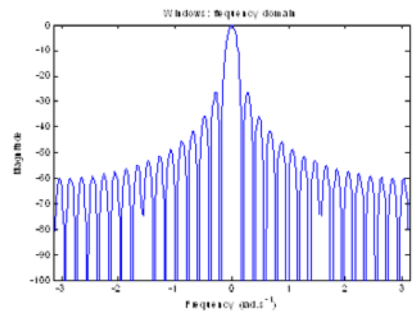
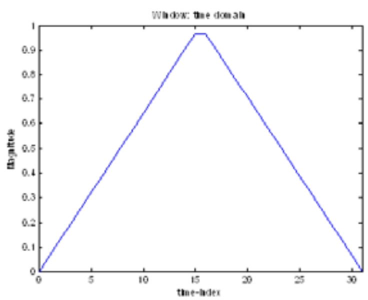
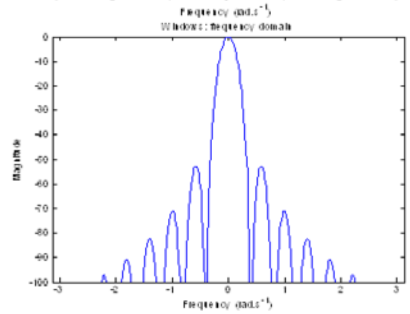
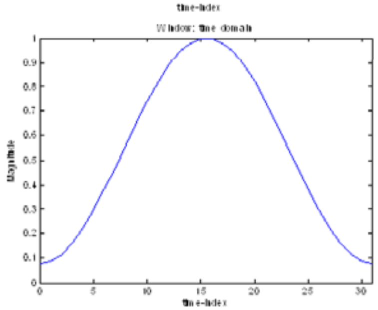
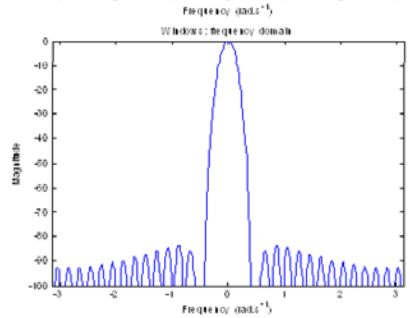
width) and leakage ($=$ side lobe level)

$\Rightarrow \quad \underline{\mathbf{w}}_t = \underline{\mathbf{t}} \odot \underline{\mathbf{w}}$ with $\underline{\mathbf{t}} =$ taper window

and $\odot$ element-by-element multiplication

类似于加窗

Data independent beamforming

Taper window	*Window* N=32	*theta-domain*
	Rectangular	
	Triangular	
	Hanning	

Lower sidelopes: less leakage

Wider main lope: Less resolution

Data independent beamforming

加窗类似于减小孔径，削弱两侧传感器

Note:

Lower sidelobes $\longrightarrow$ wider mainlobe

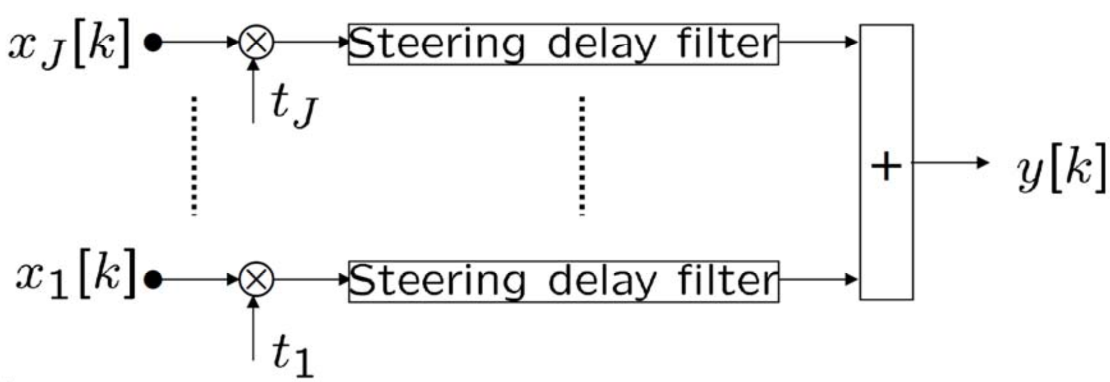
$\Leftrightarrow$ less resolution $\Leftrightarrow$ loss in array aperture

Compare e.g. rectangular with Triangular

sensors at left and right of centre are weighted less

Data independent beamforming

Conventional beamforming (=DSB)



Notes:

- Taper weights $t_1, \cdots t_J$ used to shape beampattern
- FIR filters approximate propagation delays
  (linear phase over frequency band of interest)

Data independent beamforming

Conclusion conventional beamformer:

Simply aligns phases of the desired signal component of the received waveforms

Required knowledge: DOA of desired signal

Performance: In environment consisting of only uncorrelated noise and no directional interferers, delay-and-sum beamformer provides maximum SNR!

match filter

Properties: Similar to periodogram;
see "DOA estimation: spatial spectrum"

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Data independent beamforming

Properties:

+ Robust to model assumptions
+ Simple implementation
+ Maximum likelihood for one desired signal case

- Resolution limited by array aperture (=array length in wavelengths) $\Delta\ \phi = 1/J$ for ULA 空间分辨率
- Resolution improved by increasing J

Data independent beamforming

零点

Null-steering 多源

Another way of data independent weight design

Data independent beamforming

Goal: Calculate J weights to meet M constraints

e.g. Amplify desired and null undesired sources

array response

desire 设 1

undesire 设 0

$$M \begin{cases} \underset{1\times J}{\underline{\mathbf{a}}^h(\omega_1, \theta_1)} \cdot \underset{J\times 1}{\underline{\mathbf{w}}} = r_d(\omega_1, \theta_1) \\ \vdots \\ \underline{\mathbf{a}}^h(\omega_M, \theta_M) \cdot \underline{\mathbf{w}} = r_d(\omega_M, \theta_M) \end{cases}$$

$J \times 1$ weight vector $\underline{\mathbf{w}} = (w_1, \cdots, w_J)^t$

$y = \underline{w}^h \underline{a}_1 s_1 + \underline{w}^h \underline{a}_2 s_2$; $P = E[y^2] = \sigma_{s_1}^2 \underline{w}^h \underline{a}_1 \underline{a}_1^h \underline{w} + \sigma_{s_2}^2 \underline{w}^h \underline{a}_2 \underline{a}_2^h \underline{w}$

Part C: Array signal processing

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$$\frac{\underline{w}^h \underline{a}_1 \underline{a}_1^h \underline{w}}{\underline{w}^h \underline{a}_2 \underline{a}_2^h \underline{w}}$$

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Data independent beamforming

Notation:

$M \times 1$ desired beampattern vector:

$$\underline{\mathbf{r}}_d(\omega,\theta) = (r_d(\omega_1,\theta_1),\cdots,r_d(\omega_M,\theta_M))^h \;\Rightarrow\; \underline{\mathbf{r}}_d$$ 由0,1构成的向量

$J \times M$ beam steering matrix:

$$\mathbf{A}(\omega,\theta) = (\underline{\mathbf{a}}_1(\omega_1,\theta_1),\cdots,\underline{\mathbf{a}}_1(\omega_M,\theta_M)) \;\Rightarrow\; \mathbf{A}$$

$J \times M$

$\Rightarrow$ Compact notation: $\mathbf{A}^h \cdot \underline{\mathbf{w}} = \underline{\mathbf{r}}_d$

$\underline{w}$ 并非是纯虚数的补偿设定，而是任意的

Case: $M < J$ (less constraints then weights)

$\Longrightarrow$ $\boxed{\underline{\mathbf{w}} = \mathbf{A}\cdot\left(\mathbf{A}^h\cdot\mathbf{A}\right)^{-1}\cdot\underline{\mathbf{r}}_d}$ ("Smallest distance to origin")

Data independent beamforming

Example: Null a signal at 90^o with 2 sensor ULA

$$J = 2, M = 1, \theta_u = 90^o = \pi/2, r(\theta_u) = 0, d/\lambda = 1/2$$

$$\Rightarrow \ \mathbf{A} = \begin{pmatrix} 1 & \mathrm{e}^{-\mathrm{j}\pi \sin(\pi/2)} \end{pmatrix}^h = \begin{pmatrix} 1 & -1 \end{pmatrix}^h \text{ and } \underline{\mathbf{r}}_d = (0)$$

$$\Rightarrow \ \underline{\mathbf{w}} = \mathbf{A} \cdot \left(\mathbf{A}^h \cdot \mathbf{A}\right)^{-1} \cdot \underline{\mathbf{r}}_d = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Indeed nulls signal at 90^o ... however also all others

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0 \ \Rightarrow \ w_1 = w_2 \ \textit{(Line through origin)}$$

One degree of freedom left.

Use this to overcome $w_1 = w_2 = 0$

Data independent beamforming

Example: $(d/\lambda = 1/2)$

Two complex plane waves, same temporal frequency (one desired, other undesired), ULA with 3 sensors:

$J = 3, M = 2, \theta_d = 0^o, \theta_u = 30^o, r(\theta_d) = 1, r(\theta_u) = 0$

$$\mathbf{A}^h = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \mathrm{e}^{\mathrm{j}\pi \sin(\pi/6)} & \mathrm{e}^{\mathrm{j}2\pi \sin(\pi/6)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \mathrm{j} & -1 \end{pmatrix}$$

$$\underline{\mathbf{r}}_d = (1, 0)^t \Rightarrow \underline{\mathbf{w}} = \mathbf{A} \cdot \left(\mathbf{A}^h \cdot \mathbf{A}\right)^{-1} \cdot \underline{\mathbf{r}}_d = \frac{1}{8} \begin{pmatrix} 3 - \mathrm{j} \\ 2 \\ 3 + \mathrm{j} \end{pmatrix}$$

此时的 w 不同于补偿滤波中的 w → 纯虚数

Part C: Array signal processing

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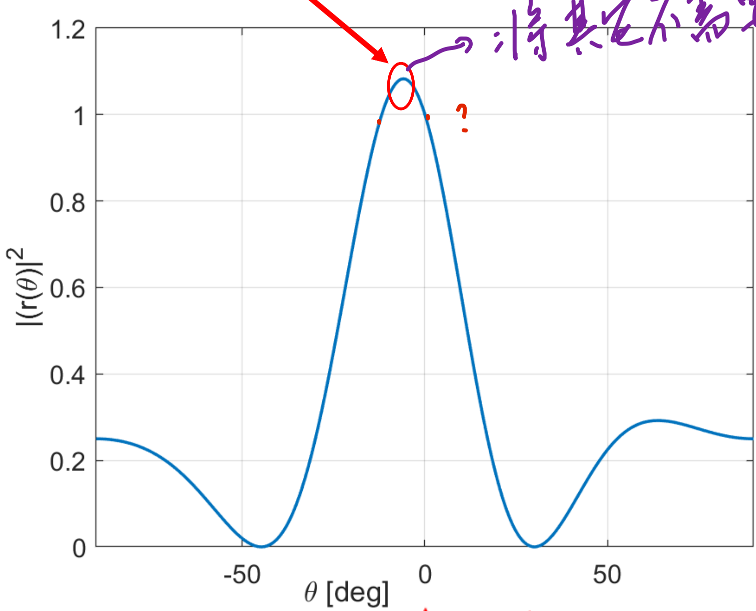
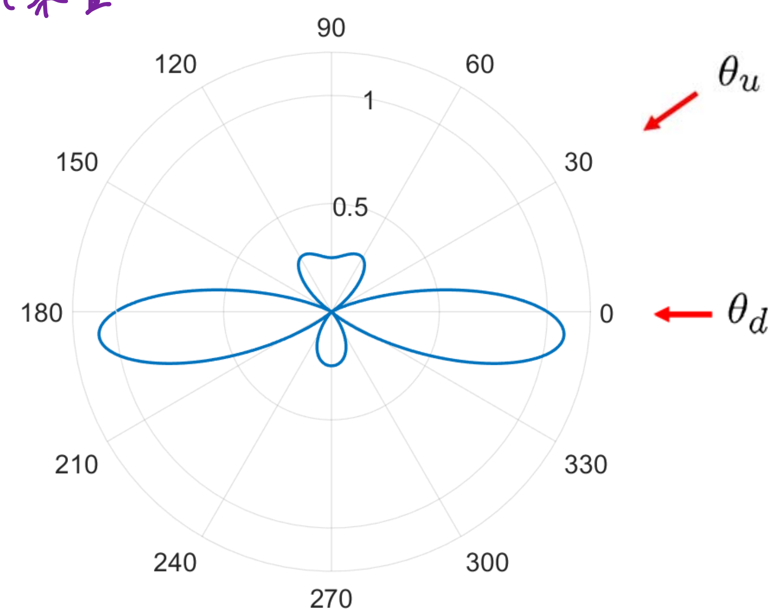
Data independent beamforming

此处的 Noise 指 interference，指 undesired 源

Bad SNR here! 不能取得最优SNR

将其它不需要处的源（求置0）处 放大系数大于1





$y = w^h a_1 s_1 + w^h a_2 s_2$ $\quad \theta_d \quad \theta_u$ $\quad P = E[y^2] = \sigma_{s_1}^2 w^h a_1 a_1^h w + \sigma_{s_2}^2 w^h a_2 a_2^h w$

非匹配的 pattern why mean SNR ?

$$\frac{w^h a_1 a_1^h w}{w^h a_2 a_2^h w}$$

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Data independent beamforming

<u>Conclusion null steering beamformer:</u>

Null steering beamformers are used to cancel plane waves arriving from *known* directions

Required knowledge:
DOA's of desired signal and interferers 多源 DOA 已知

Performance:
SNR is not maximized, but nulls can be put in DOA's of interferers

Properties: (see next slide)

Data independent beamforming

Properties:

- Result not robust to frequency jammer
- J weights can set J predefined conditions

  (e.g. 1 desired, J-1 undesired)
- Method need much a priori information

  (e.g. DOA's, frequency source (d/λ), etc.)
- If $M < J$ one can add extra constraints

  (e.g. minimize output power, etc.)
- Use FIR filters for broadband

Data independent beamforming

Array resonse design

Case: # constraints > # weights $M > J$, J means sensors

$\Rightarrow$ Solution via **overdetermined least squares**:

Procedure: $\underline{\mathbf{w}} = arg \min_{\underline{\mathbf{w}}} \{J\}$ $\boxed{\Leftarrow J \text{ is error not } \# \text{ sensors}}$

$$\text{with: } J = \left| \mathbf{A}^h \cdot \underline{\mathbf{w}} - \underline{\mathbf{r}}_d \right|^2$$

$$\begin{aligned} J &= \left(\underline{\mathbf{w}}^h \cdot \mathbf{A} - \underline{\mathbf{r}}_d^h\right) \cdot \left(\mathbf{A}^h \cdot \underline{\mathbf{w}} - \underline{\mathbf{r}}_d\right) \\ &= \underline{\mathbf{w}}^h \mathbf{A}\mathbf{A}^h \underline{\mathbf{w}} - \underline{\mathbf{w}}^h \mathbf{A} \underline{\mathbf{r}}_d - \underline{\mathbf{r}}_d^h \mathbf{A}^h \underline{\mathbf{w}} + \underline{\mathbf{r}}_d^h \underline{\mathbf{r}}_d \end{aligned}$$

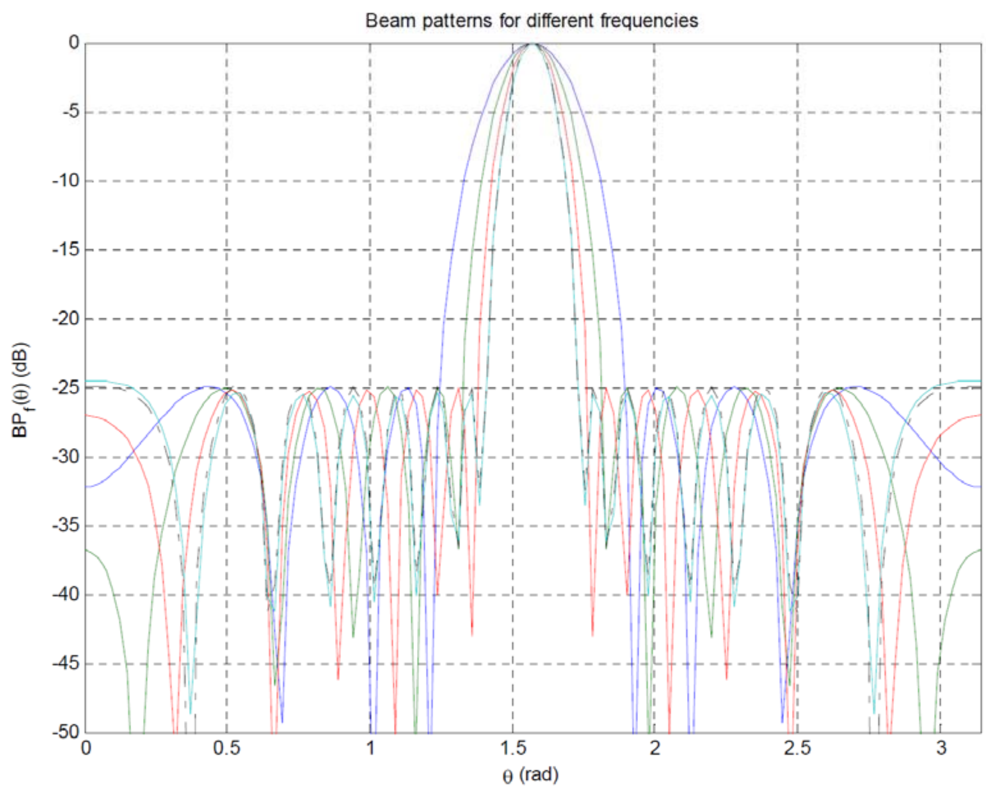
$$\frac{\mathrm{d}J}{\mathrm{d}\underline{\mathbf{w}}} = 0 \quad \Rightarrow \quad \boxed{\underline{\mathbf{w}} = \left(\mathbf{A} \cdot \mathbf{A}^h\right)^{-1} \cdot \mathbf{A} \cdot \underline{\mathbf{r}}_d} \quad \text{(See Appendix)}$$

类似于 wiener , $\underline{w} = R_x^{-1} \underline{r_{ed}}$

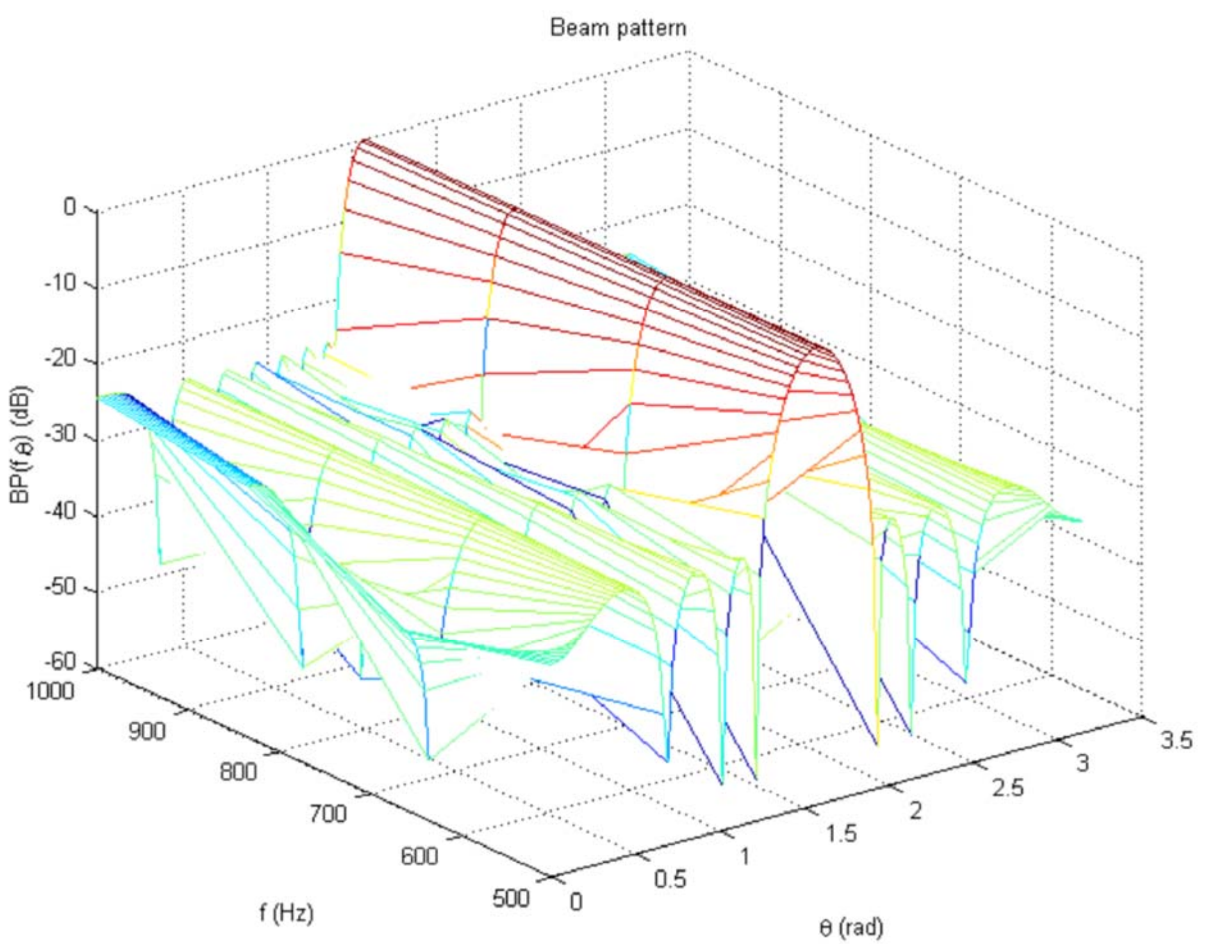
ADSP

Data independent beamforming

Example: Constraint $B(\theta) = 1$ at $\theta = \pi/2$ and < -25dB outside this area



Data independent beamforming



Data independent beamforming

Data independent:

- Beamsteering 单源, matched filter, max SNR
  (delay and sum, phased array)
- Tapering 改变主瓣与旁瓣形状
- Null-steering / Array response design 多源 (interference)

(Beamsteering and Tapering:) "Conventional" approach

Data dependent (statistical optimum):

- Minimum Sidelobe Canceller
- Wiener
- Max SNR
- Linear Constraint Minimum Variance
- Generalized Sidelobe Canceller

(all of the above:) **Part III**

Data independent beamforming

DOA estimation → part III

Goal:

Estimate Direction Of Arrival (DOA's) of sources (and interferences) from noisy observations, in order to locate and track them