Advanced Digital Signal Processing (ADSP)

徐林

上海科技大学
ShanghaiTech University

ADSP

Power spectral density (PSD)

Harmonic process: $$x[n] = \sum_{k=1}^{M} A_k \cos(\omega_k n + \phi_k)$$

$\{\phi_k\}$ pairwise independent random variables uniform in $[0, 2\pi]$.

$$E\{x[n]\} = 0 \quad \forall n \qquad\qquad r_x[l] = \frac{1}{2}\sum_{k=1}^{M} A_k^2 \cos(\omega_k l) \quad -\infty < l < \infty$$

$$P_x[e^{j\omega}] = \sum_{k=-M}^{M} 2\pi\left(\frac{A_k^2}{4}\right) \delta(\omega - \omega_k) = \sum_{k=-M}^{M} \frac{\pi}{2} A_k^2 \delta(\omega - \omega_k) \quad -\pi < \omega < \pi$$

$\omega_k/(2\pi)$ rational number, spectral lines equidistant (harmonically related)

Example: $x[n] = \cos(0.1\pi n + \phi_1) + 2\sin(1.5n + \phi_2)$
ϕ_1 and ϕ_2 IID, uniform in $[0, 2\pi]$

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Power spectral density (PSD)

Cross-Power spectral density

Zero-mean, joint stationary

$$P_{xy}(e^{j\omega}) = \sum_{l=-\infty}^{\infty} r_{xy}[l] e^{-j\omega l} \quad \circ\!\!-\!\!\circ \quad r_{xy}[l] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xy}(e^{j\omega})\, e^{j\omega l} \mathrm{d}\omega$$

$P_{xy}(e^{j\omega})$ complex value

Periodicity : $X(\mathrm{e}^{\mathrm{j}\theta}) = X(\mathrm{e}^{\mathrm{j}\theta + l\cdot 2\pi}) \quad l \in \mathbb{N}$

Symmetry :

$x[n]$	$X(\mathrm{e}^{\mathrm{j}\theta})$
Real, even	Real, even
Real, odd	Imaginary, odd
Imaginary, even	Imaginary, even
Imaginary, odd	Real, odd

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$R_{xy}(e^{j\omega})$

$$P_{xy}(e^{j\omega}) = \sum_{l=-\infty}^{\infty} r_{xy}[l] e^{-j\omega l} \quad \circ\!\!-\!\!\circ \quad r_{xy}[l] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xy}(e^{j\omega})\, e^{j\omega l} \mathrm{d}\omega$$

$P_{xy}(e^{j\omega})$ complex value

$$r_{xy}[l] = r_{yx}^{*}[-l] \qquad P_{xy}(e^{j\omega}) = P_{yx}^{*}(e^{j\omega})$$

Coherence function (normalized cross-PSD):

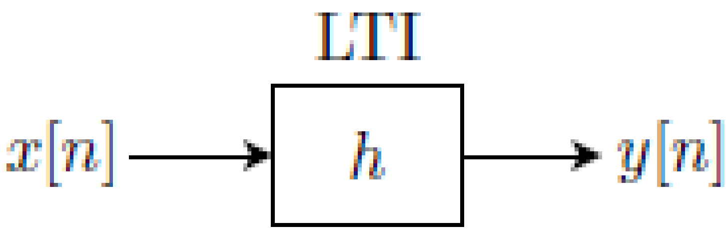
$$C_{xy}(e^{j\omega}) \triangleq \frac{P_{xy}(e^{j\omega})}{\sqrt{P_x(e^{j\omega})}\sqrt{P_y(e^{j\omega})}}$$

Properties coherence function:

- $0 \leq |C_{xy}(e^{j\omega})| \leq 1$
- $C_{xy}(e^{j\omega}) = 1$: $x[n] = y[n]$
- $C_{xy}(e^{j\omega}) = 0$: $x[n]$ and $y[n]$ not correlated

ADSP Part A: Stochastic signal processing

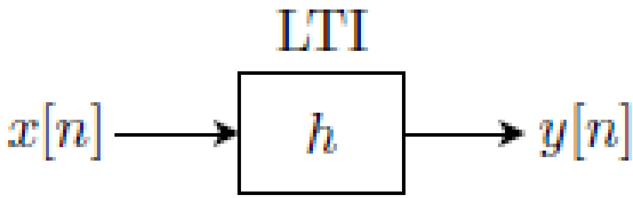
Linear system with stationary random input



- $x[n]$: stationary
- $h[n]$: BIBO stable

$\Rightarrow$ $y[n] = h[n] \star x[n]$: stationary

Linear system with stationary random input



Time domain analysis

Output mean value

$$\mu_y = \sum_{k=-\infty}^{\infty} h[k]E\{x[n-k]\} = \mu_x \sum_{k=-\infty}^{\infty} h[k] = \mu_x \cdot H(e^{j0}) \quad = \text{constant}$$

$H(e^{j0})$ is the DC gain of the system

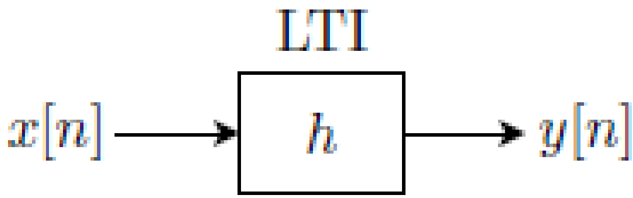
Input output cross-correlation

$$r_{xy}[l] = \mathrm{E}\{x[n+l]y^*[n]\} = \sum_{k=-\infty}^{\infty} h^*[k]\mathrm{E}\{x[n+l]x^*[n-k]\}$$

$$= \sum_{k=-\infty}^{\infty} h^*[k] r_x[l+k] = \sum_{m=-\infty}^{\infty} h^*[-m] r_x[l-m]$$

$$r_{xy}[l] = h^*[-l] \star r_x[l] \qquad r_{yx}[l] = h[l] \star r_x[l]$$

ADSP Linear system with stationary random input



Output auto-correlation

$$r_y[l] = \mathrm{E}\{y[n]y^*[n-l]\} = \sum_{k=-\infty}^{\infty} h[k]\mathrm{E}\{x[n-k]y^*[n-l]\}$$

$$= \sum_{k=-\infty}^{\infty} h[k]r_{xy}[l-k] = h[l] \star r_{xy}[l]$$

$$r_{xy}[l] = h^*[-l] \star r_x[l]$$

$$r_y[l] = h[l] \star h^*[-l] \star r_x[l] = r_h[l] \star r_x[l]$$

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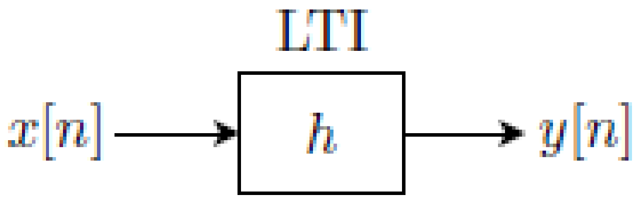
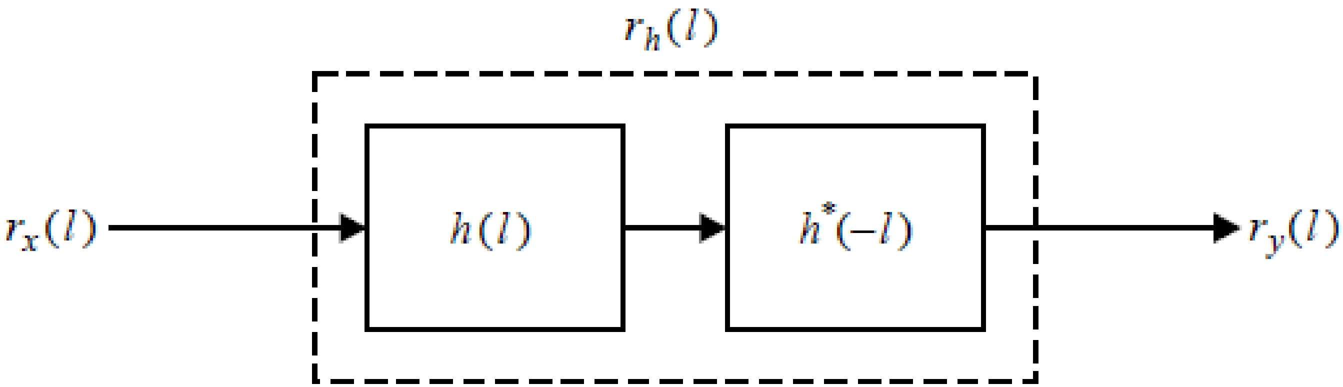
Linear system with stationary random input

Output auto-correlation



$$r_h[l] = h[l] \star h^*[-l]$$



$$r_y[l] = h[l] \star h^*[-l] \star r_x[l] = r_h[l] \star r_x[l]$$

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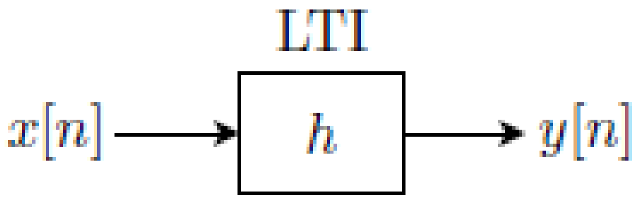
Linear system with stationary random input

Output power



$$P_y = r_y[0] = r_h[l] \star r_x[l]_{l=0} = \sum_{k=-\infty}^{\infty} r_h[k] r_x[-k] = \sum_{k=-\infty}^{\infty} r_h[k] r_x[k]$$

$$r_y[l] = r_h[l] \star r_x[l]$$

If $x[n]$ is real

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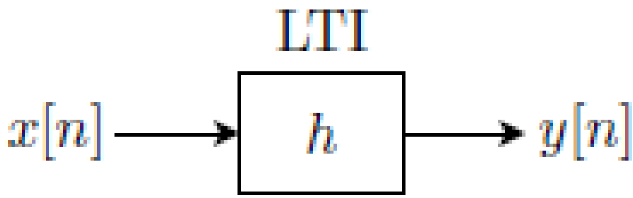
Linear system with stationary random input



Frequency domain

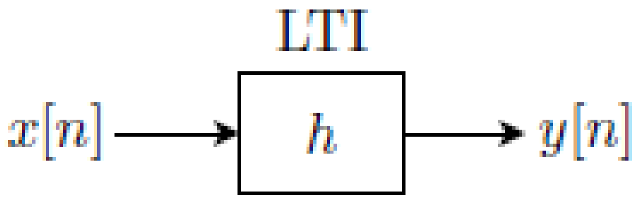
$$Z\{h[n]\} = H(z) \qquad Z\{h^*[-n]\} = H^*(\frac{1}{z^*})$$

$$r_{xy}[l] = h^*[-l] \star r_x[l] \qquad R_{xy}(z) = H^*(\frac{1}{z^*})R_x(z)$$

$$r_{yx}[l] = h[l] \star r_x[l] \qquad R_{yx}(z) = H(z)R_x(z)$$

$$r_y[l] = h[l] \star h^*[-l] \star r_x[l] \qquad R_y(z) = H(z)H^*(\frac{1}{z^*})R_x(z)$$

Linear system with stationary random input



Frequency domain

$h[n]$ stable, $z = e^{j\omega}$ within the ROC of $\mathrm{H}[z]$ and $\mathrm{H}[z^{-1}]$

$$R_{xy}(\mathrm{e}^{j\omega}) = H^*(e^{j\omega})R_x(e^{j\omega})$$

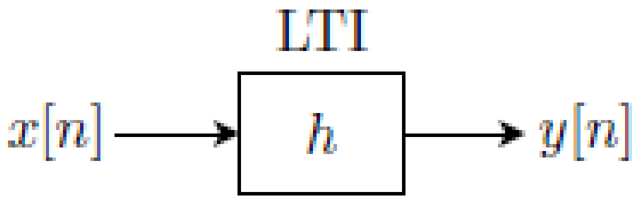
$$R_{yx}(e^{j\omega}) = H(e^{j\omega})R_x(e^{j\omega})$$

$$R_y(e^{j\omega}) = H(e^{j\omega})H^*(e^{j\omega})R_x\left(e^{j\omega}\right) = |H(e^{j\omega})|^2 R_x(e^{j\omega})$$

Power: $P_y = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 P_x\left(e^{j\omega}\right)\mathrm{d}\omega$

$$= r_y(0) = \sum_{l=-\infty}^{\infty} r_h[l]\, r_x\,[l] = E\{|y[n]|^2\}$$

Linear system with stationary random input



Frequency domain

$$H(e^{j\omega}) = \begin{cases} 1 & \omega_c - \dfrac{\Delta\omega}{2} \le \omega \le \omega_c + \dfrac{\Delta\omega}{2} \\ 0 & \text{elsewhere} \end{cases}$$

$$P_y = \mathrm{E}\{|x[n]|^2\} = \frac{1}{2\pi}\int_{\omega_c-\frac{\Delta\omega}{2}}^{\omega_c+\frac{\Delta\omega}{2}} |H(e^{j\omega})|^2 P_x\left(e^{j\omega}\right)\mathrm{d}\omega = \frac{\Delta\omega}{2\pi} P_x\left(e^{j\omega}\right)|_{\omega=\omega_c}$$