31.01.2022   = $\underline{\text{EXAMEN}}$ Prob. și Stat. =

$$\boxed{I = \underline{\underline{13}}}$$

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gr: 231

① $(0, 13] \cap \mathbb{N} = \{1, 2, \ldots, 13\} \stackrel{\text{not.}}{=} M$

a) fie $x \in M$ aî $x \,\vdots\, 3$ ⎫⟹ $p = \frac{4}{13} \approx 0,307$
$\quad p = \frac{cf}{cp}$
$\quad cf = 13 \text{ div } 3 = 4$
$\quad cp = 13 - 1 + 1 = 13$ ⎭

b) fie $y \in M$ aî $y = p.p.$ ⎫⟹ $p = \frac{3}{13} \approx 0,23$
$\quad p = \frac{cf}{cp}$
$\quad cf = [\sqrt{13}] = 3$
$\quad cp = 13 - 1 + 1 = 13$ ⎭

c) fie $z \in M$ aî z prim ⎫⟹ $p = \frac{6}{13} \approx 0,461$
$\quad p = \frac{cf}{cp}$
$\quad cf = |\{2, 3, 5, 7, 11, 13\}| = 6$
$\quad cp = 13 - 1 + 1 = 13$ ⎭

② $X \sim \begin{pmatrix} -1 & 0 & 1 \\ \frac{13}{1000} & \frac{1}{100} & c \end{pmatrix}$

a) X - var. aleatoare discretă $\Rightarrow c = 1 - \frac{13}{1000} - \frac{1}{100} \Rightarrow$

$$c = \frac{977}{1000} = 0,977$$

b) $\text{Var}(X) = E(X^2) - (E(X))^2$

$E(X) = -1 \cdot \frac{13}{1000} + 0 \cdot \frac{1}{100} + 1 \cdot \frac{977}{1000} = \frac{964}{1000} = 0,964$

$E(X^2) = (-1)^2 \cdot \frac{13}{1000} + 0^2 \cdot \frac{1}{100} + 1^2 \cdot \frac{977}{1000} = \frac{990}{1000} = 0,99$

$\Rightarrow \text{Var}(X) = \cancel{0,96 +} 0,99 - (0,964)^2 = \frac{990\,000}{1000000} - \frac{929296}{1\,000\,000} =$

$$= \frac{60\,704}{1\,000\,000} \approx 0,06$$

c) fie $y = 2x \Rightarrow V(y) = V(2x) = 2^2 V(x)$

$$= 4 \cdot 0,06 = 0,24$$

③ $X, Y \in [0, 13]$

$f(x) = f(y) = cxy$

a) $\int\limits_{0}^{13} \int\limits_{0}^{13} f(x)\, dx\, dy = 1$ $\qquad$ $\frac{13^2}{2} c \cdot \frac{y^2}{2}\Big|_{0}^{13} = 1 \Rightarrow$

$\int\limits_{0}^{13} \left(\int\limits_{0}^{13} cxy\, dx \right) dy = 1$ $\qquad \Rightarrow \frac{13^2}{2} c \cdot \frac{13^2}{2} = 1 \Rightarrow$

$\int\limits_{0}^{13} \left(cy \cdot \frac{x^2}{2}\Big|_{0}^{13} \right) dy = 1$ $\qquad \frac{13^4}{4} c = 1 \Rightarrow$

$\int\limits_{0}^{13} cy \cdot \frac{13^2}{2}\, dy = 1$ $\qquad c = \frac{4}{13^4}$

b) $P(x \leq 3) = P(0 \leq x \leq 3) = \int_0^3 f_X(x)\,dx = \int_0^3 cx \cdot \frac{13^2}{2}\,dx =$

$$= c \cdot \frac{13^2}{2} \cdot \frac{x^2}{2}\Big|_0^3 = c \cdot \frac{13^2}{2} \cdot \frac{9}{2} = \frac{4}{13^4} \cdot \frac{13^2}{2} \cdot \frac{9}{2}$$

$$= \frac{9}{13^2} = 0{,}053$$

c) X şi Y independente $\Rightarrow$ $f(x) = f_X(x) \cdot f_Y(y)$

$$f_X(x) = \int_0^{13} f(x)\,dy = \int_0^{13} cxy\,dy = cx \frac{y^2}{2}\Big|_0^{13} = cx \cdot \frac{13^2}{2}$$

$$f_Y(y) = \int_0^{13} f(x)\,dx = \int_0^{13} cxy\,dx = cy \frac{x^2}{2}\Big|_0^{13} = cy \cdot \frac{13^2}{2}$$

$$f_X(x) \cdot f_Y(y) = \left(cx \cdot \frac{13^2}{2}\right)\left(cy \cdot \frac{13^2}{2}\right) = c^2 xy \cdot \frac{13^4}{4} =$$

$$= \left(\frac{4}{13^4}\right)^2 \cdot \frac{13^4}{4} \cdot xy = \frac{4}{13^4} \cdot xy = cxy$$

$\Rightarrow$ $\underline{f_X(x) \cdot f_Y(y)} = f(x) = \underline{cxy}$ $\Rightarrow$

$\Rightarrow$ X, Y independente

④ $P(A) = \dfrac{13}{200} = 0,065$; $P(D_H|A) = 0,5$: prob. avers "A"

$P(B) = \dfrac{187}{200} = 0,935$; $P(D_H|B) = 0,8$: prob. avers "B"

→ prob. pred. a priori :

$P(D_H) = P(D_H|A) \cdot P(A) + P(D_H|B) \cdot P(B) = 0,5 \cdot 0,065 +$

$+ 0,8 \cdot 0,935 = 0,0325 + 0,748 \cong \underline{\underline{0,7805}}$

→ prob. pred. a posteriori

$P(H|D_H) = \dfrac{P(D_H|H) \cdot P(H)}{P(D_H)}$ ⟹ $\begin{cases} P(A|D_A) = \dfrac{0,0325}{0,7805} \approx 0,0416 \\[3mm] P(B|D_A) = \dfrac{0,748}{0,7805} \approx 0,9584 \end{cases}$ ⟹

⟹ $P(D_H|b) = P(D_H|A) \cdot P(A|b) + P(D_H|B) \cdot P(B|b) =$

$= 0,5 \cdot 0,0416 + 0,8 \cdot 0,9584 =$

$= 0,0208 + 0,7667 \cong 0,7875$

⑤ 110 aruncări : $\begin{cases} 13 \text{ aversuri} \\ 97 \text{ reversuri} \end{cases}$; total : 110

Datele provin din rep. binomială $(110, \theta)$ ⟹

Verosimilitatea : $P(X_1, \theta) = C_{110}^{13} \cdot \theta^{13} \cdot (1-\theta)^{97}$

a = aversuri (13)

b = reversuri (97)

Tabel de actualizare Bayesiană:

H	a priori	probabilitate	numărător	a posteriori
θ	$d\theta$	$C_{110}^{13}\,\theta^{13}(1-\theta)^{97}$	$C_{110}^{13}\,\theta^{13}(1-\theta)^{97}\,d\theta$	$K\cdot\theta^{13}(1-\theta)^{97}$
total	1		$T=C_{110}^{13}\int_0^1 \theta^{13}(1-\theta)^{97}\,d\theta$	1

Observația are loc cu $a=14$ și $b=98$;
pdf a posteriori: $f(\theta, x_1) = K \cdot \theta^{13}(1-\theta)^{97}$

⑥ $pr = \dfrac{13}{200} = 0,065$

$P(a \le X \le b) = pr = 0,065$

$P(0 \le X \le a) = P(b \le X \le 1) = \dfrac{1-pr}{2} = 0,4675 \Rightarrow$

$\Rightarrow$ $I = [q0,4675 , q_{0,5325}]$ simetric de prob. "pr" pentru normala standard