Estimation and detection using the matched filter

Practicals 4 and 5 of Fundamental of Estimation and Detection Course

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"I certify that this work is original, that we have cited all the sources used, and that it contains no plagiarism."

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Table des matières

1   Introduction

We consider an image sequence $f_k(x, y), k \in [1, K], x \in [1, M], y \in [1, N]$. In our case, in regard to a reference image, each following image in the sequence can be registered on the reference by a rotation of angle θ_k and a translation by a vector $\tau_k = (x - x_k, y - y_k)$. In our situation, we know there is no rotation in the sequence so we want to estimate the parameters τ_k for each image.

$$\forall \, k > 1, \quad f_k(x, y) = f_1(x - x_k, y - y_k) + b(x, y)$$
$$\forall \, k > 1, \quad f_k(x_i, y_j) = f_1(x_i - x_k, y_j - y_k) + b_k(x_i, y_j)$$

Here, f is probabilist due to the nature of b, while f_1 is determinist. We sample the signal so it is now a serie of discret value, taken at pixel position (i, j).

In the case of a normalized signal, because the log-likelyhood shows a mean-square difference, one can show that its derivative is proportional to the scalar product of the observed signal f_k and the model signal f_1. If the noise is gaussian, then MLE is optimal, and thus the matched filter is optimal. If the noise is correlated (covariance matrix Γ), one can substitute the signal to whitened the noise : consider $\Gamma^{-1}.f$ and $\Gamma^{-1}.r(x, y)$.

The estimator is then :

$$\hat{\tau}_k = arg\ max\left[f_k^T \Gamma^{-1} f_1\right] \tag{1}$$

And if the noise is uncorrelated, then $\Gamma^{-1} = Id$.

2   Registration algorithm

We consider a white noise, and we want to express the scalar product :

$$f_k^T \cdot f_1 = \sum_{x,y} f_k . f_1 = a.A(\tau_k, \tau_1) + b'(\tau_k), \quad b'(\tau_k) = b^T f_1(\tau_k) \tag{2}$$

We suppose that the ambiguity function A is symetrical with respect to θ_0 and thus is unbiased.

The ambiguity function is equal to the autocorrelation function of the reference signal f_1 with τ the translation and τ_k the true translation for image k : $A(\tau, \tau_k) = C(\tau, \tau_k)$.

By computing the autocorrelation in the Fourier domain, we can estimate the translation of the centers of the two images. To test the obtain translation vector, one could simply roll the unregistered image according to that vector and then substract the two images and observe a uniform noise. If some shapes are still visible then the shift was not right.

2.1 Results

Here the image is of shape (300 x 500) thus the center is located at (150,250). We select the second image of the sequence and computes the correlation between the reference image and the selected image to find a translation vector (1,4).
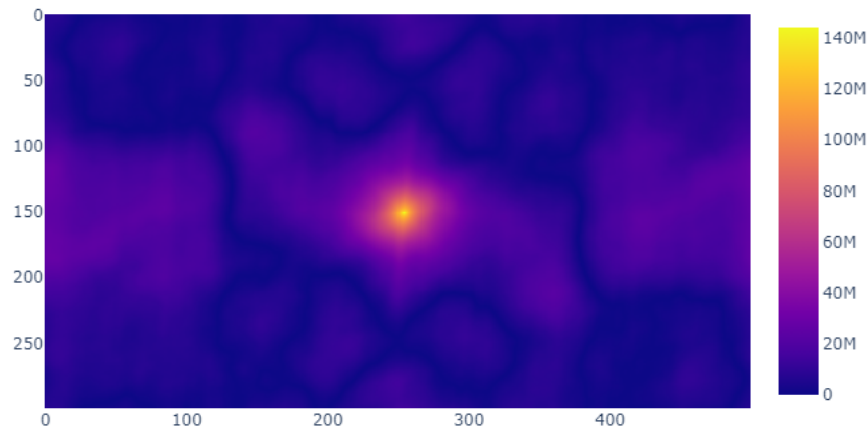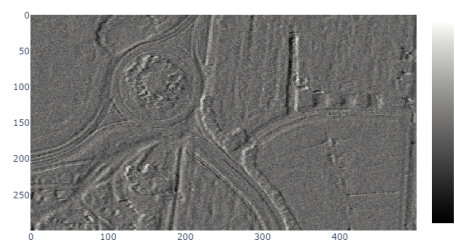

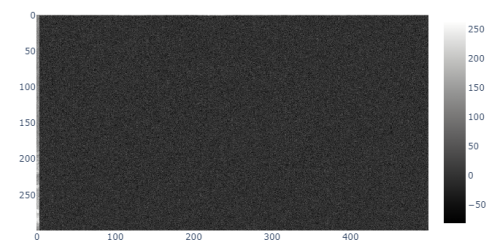FIGURE 1 – Correlation between reference image and second unregistered image

We then roll the selected image according to that translation vector and substract the two : we observe a white noise over the screen so we conclude that the estimated translation was correct.

(a) Substraction before rolling

(b) Substraction after rolling

FIGURE 2 – Registration check

Because the defined algorithm has been verified, we present here the translation vector (in red) onto the reference image.
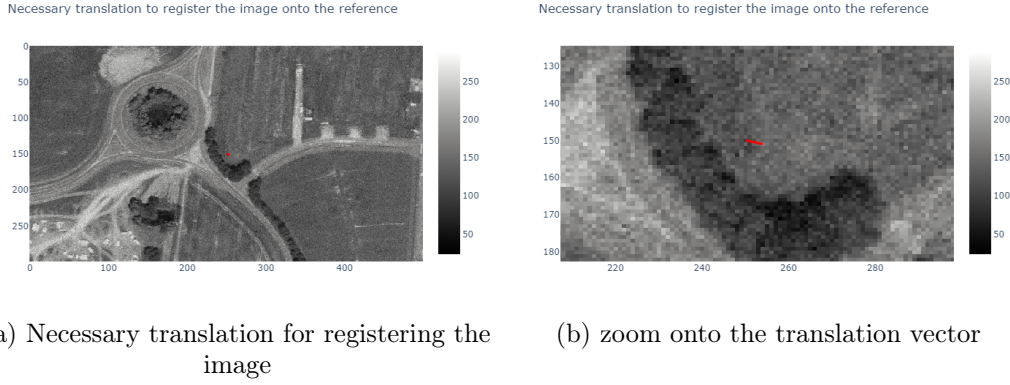
(a) Necessary translation for registering the image

(b) zoom onto the translation vector

FIGURE 3 – Translation vector

2.2 Blocks

For sake of memory resources or because of time calculation, it is not always possible do compute the Fourier transform of the whole image. In this case, one has to compute the shift from a small region of the reference image. We will take a 30×30 pixels wide region.

The goal is then to choose the region of the image that leads to the highest precision for shift estimation.

Here we present 5 blocks that we chose according to be different parts of the image. We also plot the Harris corner detection at 1% of the maximum of the Harris score.

FIGURE 4 – Blocks on the reference image

We then need to find the different position of the block onto the reference and the unregistered image. For that, we compute the correlation between the block and

the reference image, which leads to the position of the block on the reference, and we compute the correlation of the block and the unregistered image, which gives the corresponding position on the unregistered image. By computing the difference one should find the neccessary shift to register the image.
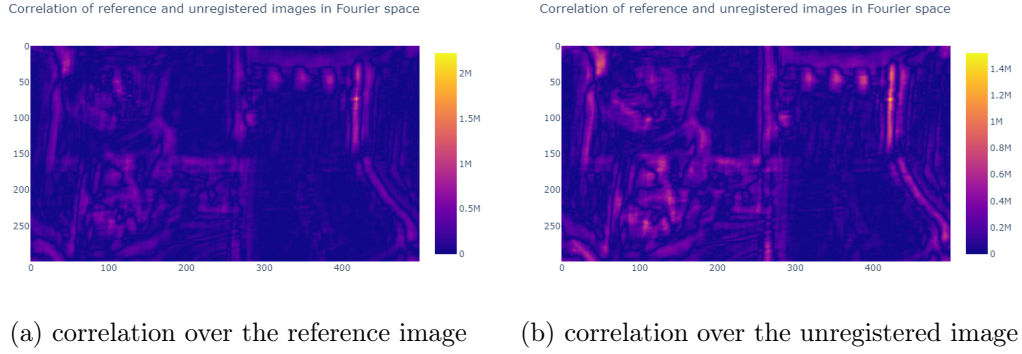

(a) correlation over the reference image

(b) correlation over the unregistered image

FIGURE 5 – Correlation over 30 x 30 pixels blocks

We then compare the translation vectors with the true one, checked with the substraction test.

	Unregistered image 2		Unregistered image 3		Unregistered image 4		Unregistered image 5		Unregistered image 6	
	t_y	t_x	t_y	t_x	t_y	t_x	t_y	t_x	t_y	t_x
true translation	5	10	1	4	5	3	9	2	11	1
translation block 1	5	10	1	4	5	3	9	2	11	1
translation block 2	5	10	1	4	5	3	9	2	11	1
translation block 3	165	12	161	7	165	6	169	5	171	3
translation block 4	5	10	1	4	5	3	9	2	11	1
translation block 5	155	74	151	68	155	66	247	277	250	276

FIGURE 6 – Computed translation vector

We observe that the blocks 1, 2 and 4 always give the correct translation vector. This is also the blocks the most filled with Harris corners, which are points with a maximum gradient in intensity. We thus suggest that to find the optimal block, one should focus on a block with high intensity variations.

2.3 Optimality

To find the optimal block for shift estimation, one can compute the Fisher information matrix (see the course p.66) and finally get the CRLB for x_k, y_k :

$$VAR(\hat{x}_k) = \frac{SNR^{-1}}{4\pi^2} \cdot \frac{\overline{\Delta\nu\nu^2}}{\overline{\Delta\mu\mu^2} \times \overline{\Delta\nu\nu^2} - \left(\overline{\Delta\mu\nu^2}\right)^2}$$

$$VAR(\hat{y}_k) = \frac{SNR^{-1}}{4\pi^2} \cdot \frac{\overline{\Delta\mu\mu^2}}{\overline{\Delta\mu\mu^2} \times \overline{\Delta\nu\nu^2} - \left(\overline{\Delta\mu\nu^2}\right)^2}$$

Where μ is the fourier variable for x and ν is the fourier variable for y, and $\sqrt{\overline{\Delta\mu^2}}$ represents the spectral "width" of the signal along the x-axis. We have :

$$\sqrt{\overline{\Delta\mu^2}} = \iint \nu^2 |\hat{f}(\mu,\nu)|^2 d\nu d\mu$$

Here, because the noisy signal is separable, we have $\overline{\Delta\mu\nu^2} = 0$. The we get :

$$VAR(\hat{x}_k) = \frac{SNR^{-1}}{4\pi^2 \overline{\Delta\mu\mu^2}}$$
$$VAR(\hat{y}_k) = \frac{SNR^{-1}}{4\pi^2 \overline{\Delta\nu\nu^2}}$$

3 Detection using the matched filter

In radar detection, one must test two hypotheses :
— Hypothesis H0 $x(t) = b(t)$
— Hypothesis H1 $x(t) = as(t) + b(t)$
where :
— $x(t)$ is the measured signal,
— $s(t)$ is the emitted pulse, known by the user,
— a is the attenuation factor,
— $b(t)$ is the noise on the measure.

We assume that we know where the target may be. We only want to know if the target is or not at this position. We first only consider the detection problem. In reality, we don't know if there is a target, neither its possible position. We must detect and estimate at the same time. That is what we will study in the image processing application.

If the noise $b(t)$ is white, the optimal detector (matched filter) is a correlation between the measured signal and the emitted pulse :

$$F(\theta) = \int x(t)s(t-\theta)dt$$

We only need to calculate this expression at one point (chosen here as 0) if we assume that the position is known. Thus, we have to test :

$$F(0) = \int x(t)s(t)dt$$

This value is then compared to a threshold in order to take a decision.

3.1 Noisy gaussian pulse

The SNR (Signal to Noise Ratio) of the detection is defined by :

$$\text{SNR} = \frac{a^2 E_s}{\sigma^2} \quad \text{SNR}_{\text{dB}} = 10 \log\left(\frac{a^2 E_s}{\sigma^2}\right)$$

where E_s is the energy of the emitted pulse : $E_s = \int |s(t)|^2 dt$ and σ^2 the variance of the noise $b(t)$. We assume that the temporal shape of the pulse is gaussian and the attenuation factor $a = 1$. We suppose also that $b(t)$ is a gaussian white noise.
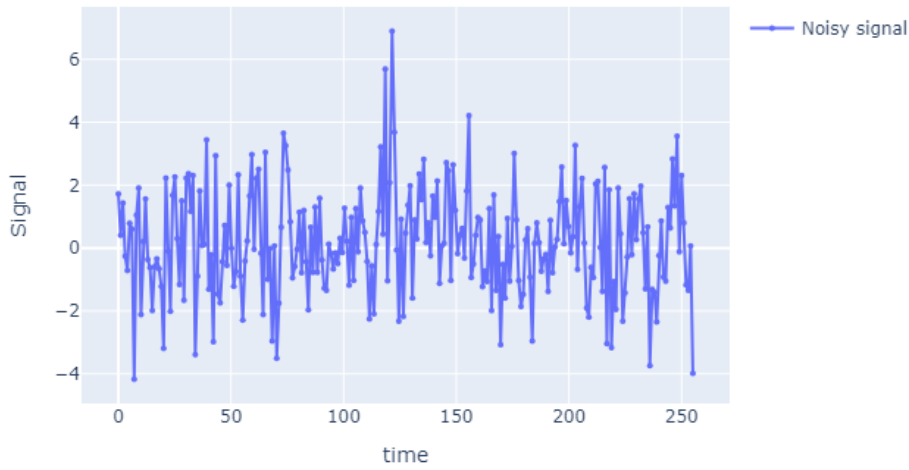
FIGURE 7 – Noisy signal
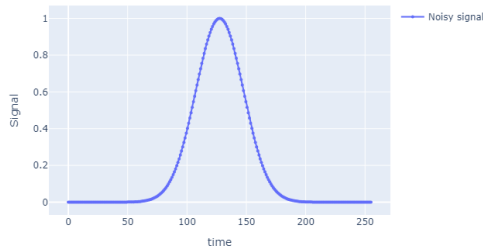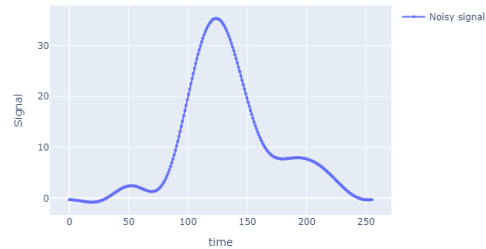
(a) Clean signal

(b) Filtered signal

FIGURE 8 – Clean and filtered signal

We run the script several times, and we observe that the mean of the Fake Alarm signal seem always inferior to the difference between the means of the FA signal and
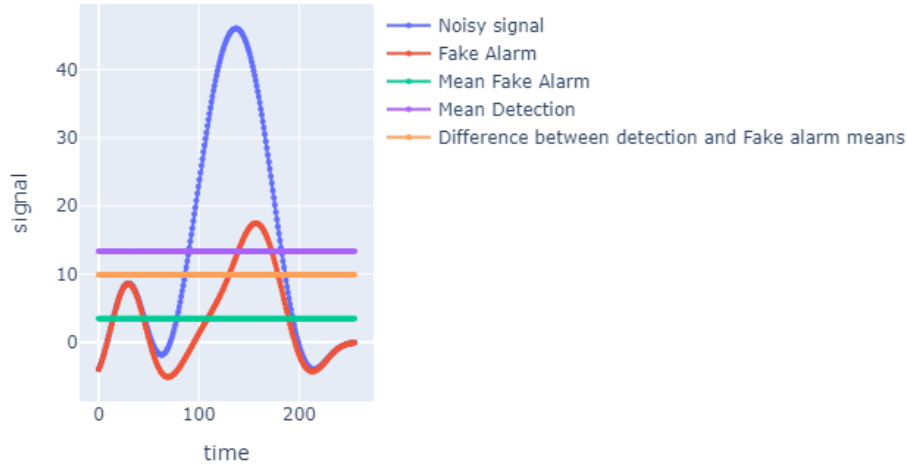
FIGURE 9 – Fake alarm and detection

the Detection signal (which remain constant over iterations, approximately equals to 10). On the contrary, the mean of the Detection signal is sometimes superior, sometimes inferior to this threshold.

As a result, one can define the difference between the means as a threshold to compare the Fake Alarm mean to.

Over the iteration, one can also observe that the mean of the Detection signal seems always superior to the mean of the Fake Alarm signal. Thus, one could try to see if those means converge toward a certain value over the iterations in order to see if another threshold could be defined.

However, one could notice that here, the noisy signal is made of the clean signal without any translation over time. Thus, it is not needed to compute the whole correlation of the signal : one could simply look at the value of the correlation function at $\theta = 0$ (i.e. no translation at all). In this case, it seems that 25 is a suitable threshold.

Notice that in the case of a translated clean signal, one could look at the maximum of the correlation function to estimate the value of the translation in the case of a detection.

3.2 ROC curves

The Receiver Operating Characteristic is a curve of the detection probability versus false alarm probability for all the threshold possible values.

3.3   OCR in a text

Here we wish to detect a character in a text. In this case we don't know its position. We must address the "detection-estimation" problem. The matched filter can be used for that purpose :
— a peak in the matched filter output indicate if the character is detected,
— the peak position gives an estimate of the character position.

First, we need to extract a perfect caracter from the perfect image in order to perform a correlation between the image and the selected caracter. Then, if the process is a success, we will perform the correlation between the same caracter and a noisy image to test the robustness of the matched filter.

In order to test both the detection and estimation performed by the match filter, we decide to draw a circle centered on each estimated position of detected correlation on the image.
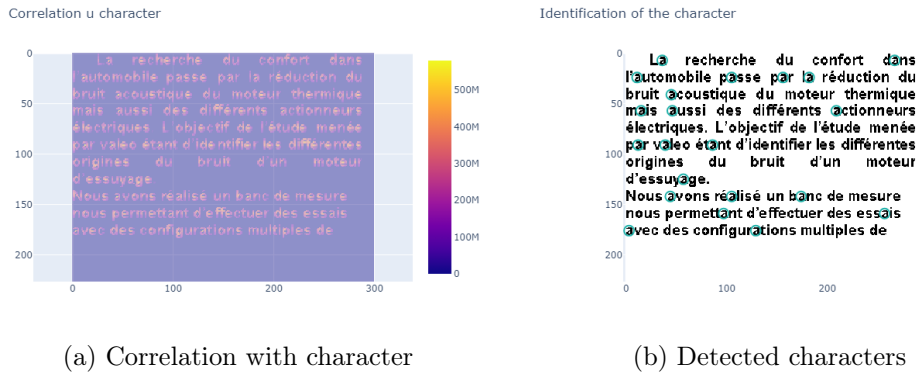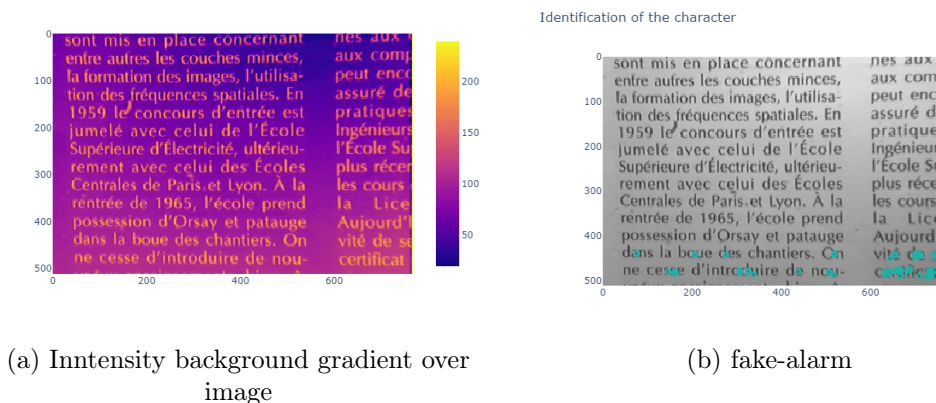

(a) Correlation with character

(b) Detected characters

FIGURE 10 – Detection in perfect text

(a) Inntensity background gradient over image

(b) fake-alarm

FIGURE 11 – Fake alarm on noisy text

Table des figures