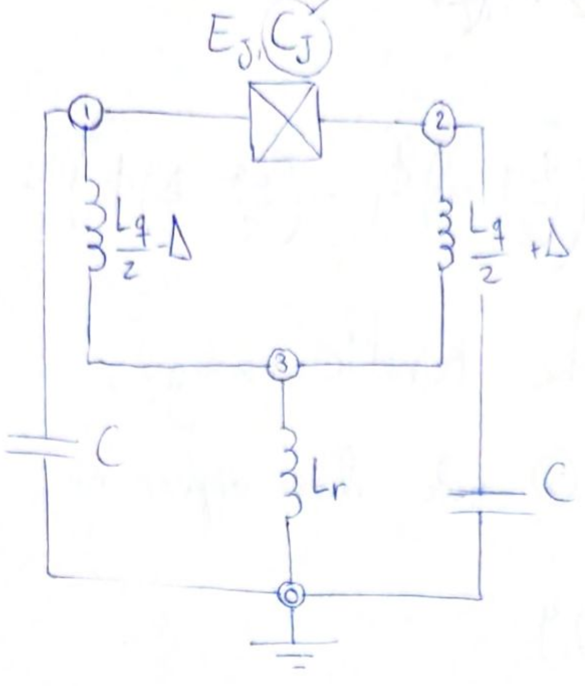
$\to C_J + C_{sh}$ de los parámetros experimentales.



La energía cinética del circuito en la base de flujos de nodos es: $\Phi_n = \{\phi_0, \phi_1, \phi_2, \phi_3\}$

$$T = \frac{1}{2} \dot{\Phi}_n^T \, \mathcal{C}_n \, \dot{\Phi}, \qquad \mathcal{C}_n = \begin{bmatrix} C+C_J & -C_J & 0 \\ -C_J & C+C_J & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where:

In this simplified model of the experimental circuit, there are no capacitances connected to the ③ node, and hence the circuit has one degree of freedom less.

Applying Kirchoff's Law of intensity to the third node results in:

$$\frac{\phi_3 - \phi_0}{L_r} = \frac{\phi_1 - \phi_3}{\frac{L_q}{2} - \Delta} + \frac{\phi_2 - \phi_3}{\frac{L_q}{2} + \Delta} \longrightarrow$$

$$\left(\cancel{\to \phi_3 = \frac{4 L_r}{\ell^2} \left[\frac{L_q}{2} (\phi_1 + \phi_2) + \Delta (\phi_1 - \phi_2) \right]} \right)$$

$$\ell^2 = L_q (L_q + 4 L_r) - 4\Delta^2 = \cancel{\ldots}$$

~~We can eliminate~~ $\phi_3 = \frac{4 L_r}{\ell^2}\left(\left(\frac{L_q}{2}+\Delta\right)\phi_1 + \left(\frac{L_q}{2}-\Delta\right)\phi_2\right)$.

This result does not affect the kinetic energy, indeed, but the node in the ③ node does appear in the inductive part of the energy:

flux

$$U_L = \frac{1}{2}\bar{\Phi}_n^T L_n \bar{\Phi}_n , \qquad L = \begin{bmatrix} \left(\frac{L_q}{2}-\Delta\right)^{-1} & 0 & -\left(\frac{L_q}{2}-\Delta\right)^{-1} \\ 0 & \left(\frac{L_q}{2}+\Delta\right)^{-1} & -\left(\frac{L_q}{2}+\Delta\right)^{-1} \\ -\left(\frac{L_q}{2}-\Delta\right)^{-1} & -\left(\frac{L_q}{2}+\Delta\right)^{-1} & \end{bmatrix}$$

$$L_r^{-1} + \left(\frac{L_q}{2}+\Delta\right)^{-1} + \left(\frac{L_q}{2}-\Delta\right)^{-1}$$

To simplify notation

$L_+ = \frac{L_q}{2} + \Delta$, $L_- = \frac{L_q}{2} - \Delta$, $\gamma = \frac{4 L_r}{\ell^2}$ $\Rightarrow$ $\phi_3 = \gamma L_+ \phi_1 + \gamma L_- \phi_2$

$\rightarrow$ $\ell^2 = 4 L_r/\gamma = 4\cdot(L_+ \cdot L_- + L_+ \cdot L_r + L_- \cdot L_r)$

To eliminate de dependence on ϕ_3 in the inductive energy we perform the transformation to ~~$\bar{\Phi}_{n'} = \{\phi_1, \phi_2\}$~~

$$\bar{\Phi}_{n^*} = P_1 \cdot \bar{\Phi}_{n'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma L_+ & \gamma L_- & 0 \end{bmatrix} \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \\ 0 \end{bmatrix} \qquad \bar{\Phi}_{n'} = \{\phi_1, \phi_2, 0\}$$

¿I am not shure if this wold de the correct notation?

Some usefull relationships between the inductive coeficients:

$$\frac{1}{\gamma} = \frac{(L_r^{-1} + L_+^{-1} + L_-^{-1})}{L_+^{-1} \cdot L_-^{-1}} \rightarrow \frac{L_+^{-1} L_-^{-1}}{\gamma} = L_r^{-1} + L_+^{-1} + L_-^{-1}$$

Now, the inductive energy of the circuit can be written in the new basis $\overline{\Phi}_{n'}$ as:

$$U_L = \frac{1}{2} \overline{\Phi}_n^T L_n \overline{\Phi}_n = \frac{1}{2} \overline{\Phi}_{n'}^T \cdot \underbrace{P_i^T L_n P_i}_{L_{n'}} \overline{\Phi}_{n'}$$

Recalling that $L_n = \left[\begin{array}{c|c|c} L_-^{-1} & 0 & -L_-^{-1} \\ 0 & L_+^{-1} & -L_+^{-1} \\ \hline -L_-^{-1} & -L_+^{-1} & L_r^{-1} + L_-^{-1} + L_+^{-1} \end{array}\right]$ $= \frac{L_+^{-1} \cdot L_-^{-1}}{\gamma}$

thus,

$$L_{n'} = P_i^T \cdot L_n \cdot P_i = \begin{bmatrix} 1 & 0 & \gamma L_+ \\ 0 & 1 & \gamma L_- \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_-^{-1} & 0 & -L_-^{-1} \\ 0 & L_+^{-1} & -L_+^{-1} \\ -L_-^{-1} & -L_+^{-1} & \frac{L_+^{-1} \cdot L_-^{-1}}{\gamma} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma L_+ & \gamma L_- & \end{bmatrix}$$

$$P_i^T \cdot L_n = \left[\begin{array}{c|c|c} L_-^{-1} - \gamma L_+ \cdot L_-^{-1} & -\gamma & \cancel{L_-^{-1}}\ 0 \\ -\gamma & L_+^{-1} - \gamma L_- L_+ & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$$

Finally,

$$L_{n'} = P_1^T \cdot L_n \cdot P_1 = \begin{bmatrix} L_-^{-1} - \gamma L + L_-^{-1} & -\gamma & 0 \\ -\gamma & L_+^{-1} - \gamma L - L_+^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma L_+ & \gamma L_- & 0 \end{bmatrix} = \begin{bmatrix} L_-^{-1} - \gamma L + L_-^{-1} & -\gamma & 0 \\ -\gamma & L_+^{-1} - \gamma L - L_+^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

To proceed with the simplifications, now that we have eliminated the flux at the ③ node, we can now make another change of variables to diagonalize the kinetic energy.

$$T = \frac{1}{2} \dot{\Phi}_n^T \cdot C_n \cdot \dot{\Phi}_n = \frac{1}{2} \dot{\Phi}_{n'}^T \cdot C_{n'} \cdot \dot{\Phi}_{n'} \; ; \; \Phi_{n'} = \{\phi_1, \phi_2\}$$

$$C_{n'} = \begin{bmatrix} C + C_J & -C_J \\ -C_J & C + C_J \end{bmatrix}$$

Looking at the structure of this matrix, it is clear that ~~the~~ its eigenstates are either a symmetric or an antisymmetric combination of the node variables. The simplest option is to switch to the new modes:

$$\phi_+ = \phi_1 + \phi_2 \quad , \quad \phi_- = \phi_1 - \phi_2 \quad ,$$

arising from the transformation : $\overline{\Phi}_\pm = \{\phi_+ , \phi_-\}$

$$\overline{\Phi}_{n'} = P_2 \cdot \overline{\Phi}_\pm \quad , \quad = \underbrace{\boxed{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}}_{P_2} \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix}$$

[As a check, indeed $\phi_1 = \frac{1}{2}[\phi_+ + \phi_-] = \frac{1}{2}[2\phi_1]$ ✓]

Hence, the kinetic energy can be written in terms of a new capacitance matrix :

$$T = \frac{1}{2} \dot{\overline{\Phi}}_{n'}^T C_{n'} \cdot \dot{\overline{\Phi}}_{n'} = \frac{1}{2} \dot{\overline{\Phi}}_\pm^T \underbrace{P_2^T \cdot C_{n'} \cdot P_2}_{C_\pm} \dot{\overline{\Phi}}_\pm$$

$$C_\pm = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} C+C_J & -C_J \\ -C_J & C+C_J \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} =$$

$$= \frac{1}{4}\begin{bmatrix} C & C \\ C+2C_J & -C-2C_J \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2C & 0 \\ 0 & 2C+4C_J \end{bmatrix} \cdot \frac{1}{4}$$

$$C_{\pm} = \begin{bmatrix} C/2 & 0 \\ 0 & C/2 + C_J \end{bmatrix}$$

Now, the inductive energy in this basis:

$$U_L = \frac{1}{2} \bar{\Phi}_{n'}^{T} L_{n'} \bar{\Phi}_{n'} = \frac{1}{2} \bar{\Phi}_{\pm}^{T} \underbrace{P_2^{T} L_{n'} P_2}_{L_{\pm}} \bar{\Phi}_{\pm}$$

$$L_{\pm} = P_2^{T} L_{n'} P_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} L_-^{-1} - \gamma L_+ L_-^{-1} & -\gamma \\ -\gamma & L_+^{-1} - \gamma L_- L_+^{-1} \end{bmatrix} \cdot P_2$$

To proceed it is usefull to write the inductance matrix with a comon denominator ℓ^2.

The off diagonal terms are easy, since

$$-\gamma = \frac{-4L_r}{\ell^2}.$$

The diagonal terms can be written as:

$$L_-^{-1} - \gamma L_+ L_-^{-1} = \frac{1}{\ell^2}\left(\ell^2 L_-^{-1} - 4 L_r L_+ L_-^{-1}\right)$$

noting that $\ell^2 = 4(L_r L_+ + L_r L_- + L_+ L_-) \Rightarrow \ell^2 L_-^{-1} = 4\left(L_r L_+ L_-^{-1} + L_r + L_+\right)$

thus,

$$\frac{1}{\ell^2}\left(\ell^2 L_-^{-1} - 4L_r L_+ L_-^{-1}\right) = \frac{1}{\ell^2}\left(4\cancel{L_r} L_+ L_-^{-1} - 4\cancel{L_r L_+} L_-^{-1} + 4L_r + 4L_+\right)$$

that is,

$$L_-^{-1} - 8L_+L_-^{-1} = \frac{4\,(L_r+L_+)}{\ell^2}$$

and

$$L_+^{-1} - 8L_-L_+^{-1} = \frac{4\,(L_r+L_-)}{\ell^2}.$$

Following the previous transformation

$$L_\pm = P_2^T L_n' P_2 = \cancel{\frac{1}{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\cdot\underbrace{\frac{\cancel{4}}{\ell^2}\begin{bmatrix}L_r+L_+ & -L_r\\ -L_r & L_r+L_-\end{bmatrix}}_{L_n'}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\cancel{\frac{1}{2}} =$$

$$= \frac{1\cancel{\times 4}}{\ell^2}\begin{bmatrix}L_+ & L_-\\ 2L_r+L_+ & -2L_r-L_-\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{\ell^2}\begin{bmatrix}L_++L_- & L_+-L_-\\ L_+-L_- & 4L_r+L_++L_-\end{bmatrix}$$

in the original variables $L_+ = \frac{L_q}{2}+\Delta$, $L_- = \frac{L_q}{2}-\Delta$:

$$L_\pm = \frac{1}{\ell^2}\begin{bmatrix}L_q & 2\Delta\\ 2\Delta & 4L_r+L_q\end{bmatrix}.$$

The last term, the Josephson energy, can be written directly, as there is only one junction; $\mathcal{U}_J = -E_J \cdot \cos\left[\frac{2\pi}{\Phi_0}(\phi_2 - \phi_1) + \varphi_0\right]$

and since $\phi_2 - \phi_1 = \phi_-$ it is already in our set of variables.

Finally, the Lagrangian of the system can be written as:

$$\mathcal{L} = T - \mathcal{U} = \frac{1}{2}\dot{\vec{\phi}}_\pm^T \cdot \mathcal{C}_\pm \dot{\vec{\phi}}_\pm - \frac{1}{2}\vec{\phi}_\pm^T \mathcal{L}_\pm \vec{\phi}_\pm - \mathcal{U}_J$$

where, again:

$$\mathcal{C}_\pm = \begin{bmatrix} C/2 & 0 \\ 0 & C/2 + C_J \end{bmatrix} \quad \mathcal{L}_\pm = \frac{1}{\rho^2}\begin{bmatrix} L_q & 2\Delta \\ 2\Delta & L_q + 4L_r \end{bmatrix}$$

We can now define the node charges as the canonically conjugate ~~of the~~ momenta of the fluxes:

$$q_+ = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_+} \quad , \quad q_- = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_-}$$

Performing a Legendre transformation and replacing the canonical variables with quantum operators, finally allows us to obtain the quantum Hamiltonian:

$$H(\bar{Q}_{\pm}, \bar{\phi}_{\pm}) = \frac{1}{2} \bar{Q}_{\pm}^T \mathcal{C}_{\pm}^{-1} \bar{Q}_{\pm} + \frac{1}{2} \bar{\phi}_{\pm}^T L_{\pm} \bar{\phi}_{\pm} - U_J$$

$$H = \underbrace{\frac{1}{2} \frac{Q_+^2}{C/2} + \frac{1}{2} \frac{L_q}{\ell^2} \phi_+^2}_{\text{resonator}} + \underbrace{\frac{1}{2} \frac{Q_-^2}{C/2 + C_J} + \frac{1}{2} \frac{L_q + 4L_r}{\ell^2} \phi_-^2 - E_J \cdot \cos\left[\frac{2\pi}{\Phi_0}(\phi_- - \phi_{ext})\right]}_{\text{fluxonium}}$$

$$\left. + \frac{2\Delta}{\ell^2} \phi_+ \phi_- \right\} \text{Coupling.}$$

That is, our system can be decomposed in a resonator with effective parameters $C_+ = C/2$, $L_+ = \ell^2/L_q$; a fluxonium with effective parameters $C_- = C/2 + C_J$, $L_- = \ell^2/(L_q + 4L_r)$, E_J, and ~~an~~ inductive coupling $g\phi_+\phi_-$, $g = \frac{2\Delta}{\ell^2}$.