Introdução à Teoria dos Grafos

Prof. Alexandre Noma

Aula passada...

```
Dijkstra(G, w, s):

para cada vértice v_i em  G.V  faça
     v_i.d = INFINITO
     v_i.p = NIL
s.d = 0
Q = G.V
------------------------------------------------------------
enquanto Q != VAZIO faça
     u_i = ExtraiMinimo(Q)
     para cada v_i em G.Adj[u_i] faça
          se u_i.d + w(u,v) < v_i.d
             entao v_i.d = u_i.d + w(u,v)
                   v_i.p = u
```

Aula passada...

Dijkstra(**G**, **w**, **s**):	**Consumo de tempo:**
para cada vértice v_i em G.V faça	**O(n)**
v_i.**d** = INFINITO	**O(n)**
v_i.**p** = NIL	**O(n)**
s.d = 0	**O(1)**
Q = G.V	**???**
enquanto **Q** != VAZIO faça	**O(n)**
u_i = **ExtraiMinimo**(**Q**)	**O(n) * ???**
para cada v_i em G.**Adj**[u_i] faça	**O(m)**
se u_i.**d** + w(u,v) < v_i.**d**	**O(m)**
entao v_i.**d** = u_i.**d** + w(u,v)	**O(m)**
v_i.**p** = u	**O(m)**

Total: **T(n,m) = ???**

Exercício Programa

- 09-filaDePrioridade.py
  (implementação simples e ineficiente com **vetor de índices**)

- Ex. Implementação eficiente com HEAP

Fila de prioridade (Heap)

- ConstroiHeap(Q)
  - Constrói um Heap em um vetor
- ExtraiMínimo(Q)
  - remove e devolve o elemento de Q com a menor chave
- Vazio(Q)
  - devolve verdadeiro se fila vazia, falso caso contrário
- DiminuiChave(Q, x, k)
  - diminui o valor da chave de x para o novo valor k.

Fila de prioridade (Heap)

- ConstroiHeap(Q) consome O(n) unidades de tempo.
  - Constrói um Heap em um vetor

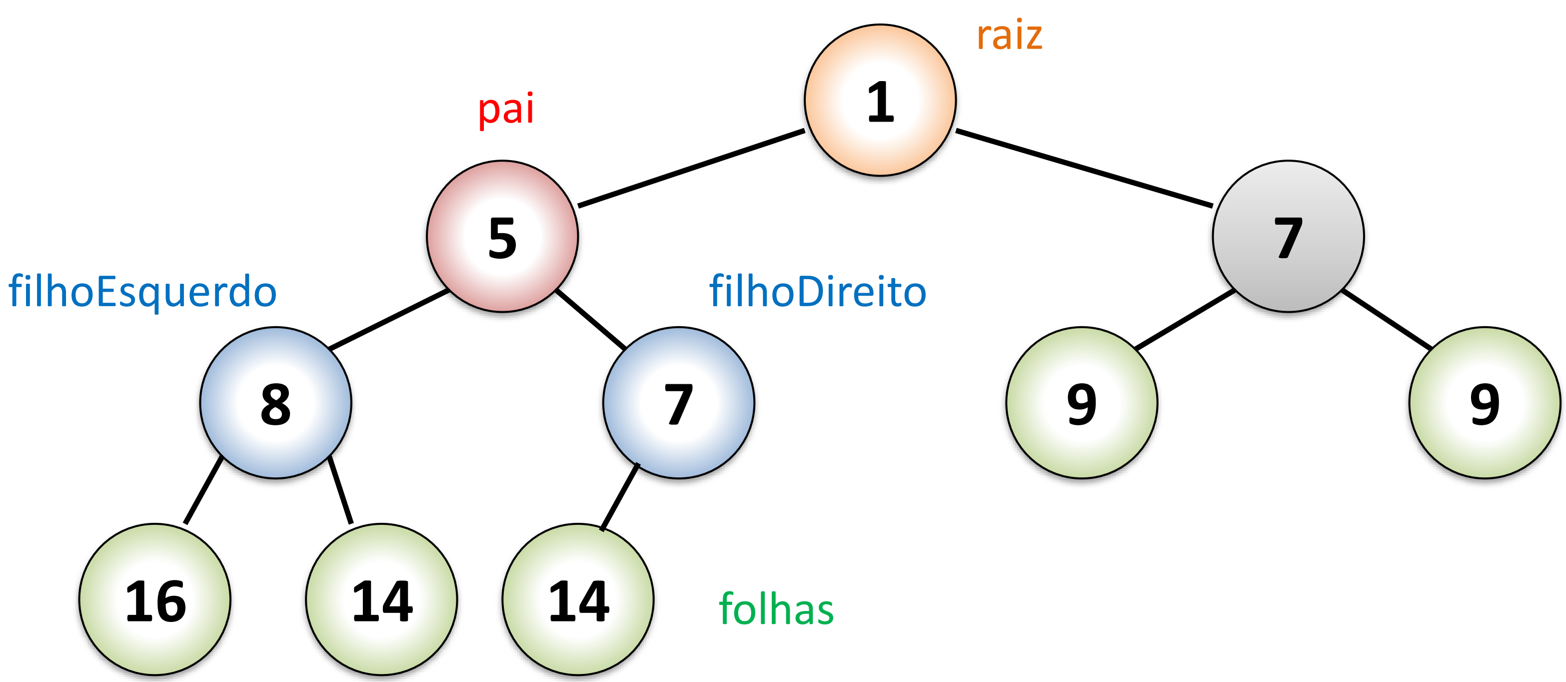
- ExtraiMínimo(Q) consome O(log n) unidades de tempo.
  - remove e devolve o elemento de Q com a menor chave

- Vazio(Q) consome O(1) unidades de tempo.
  - devolve verdadeiro se fila vazia, falso caso contrário

- DiminuiChave(Q, x, k) consome O(log n) unidades de tempo.
  - diminui o valor da chave de x para o novo valor k.

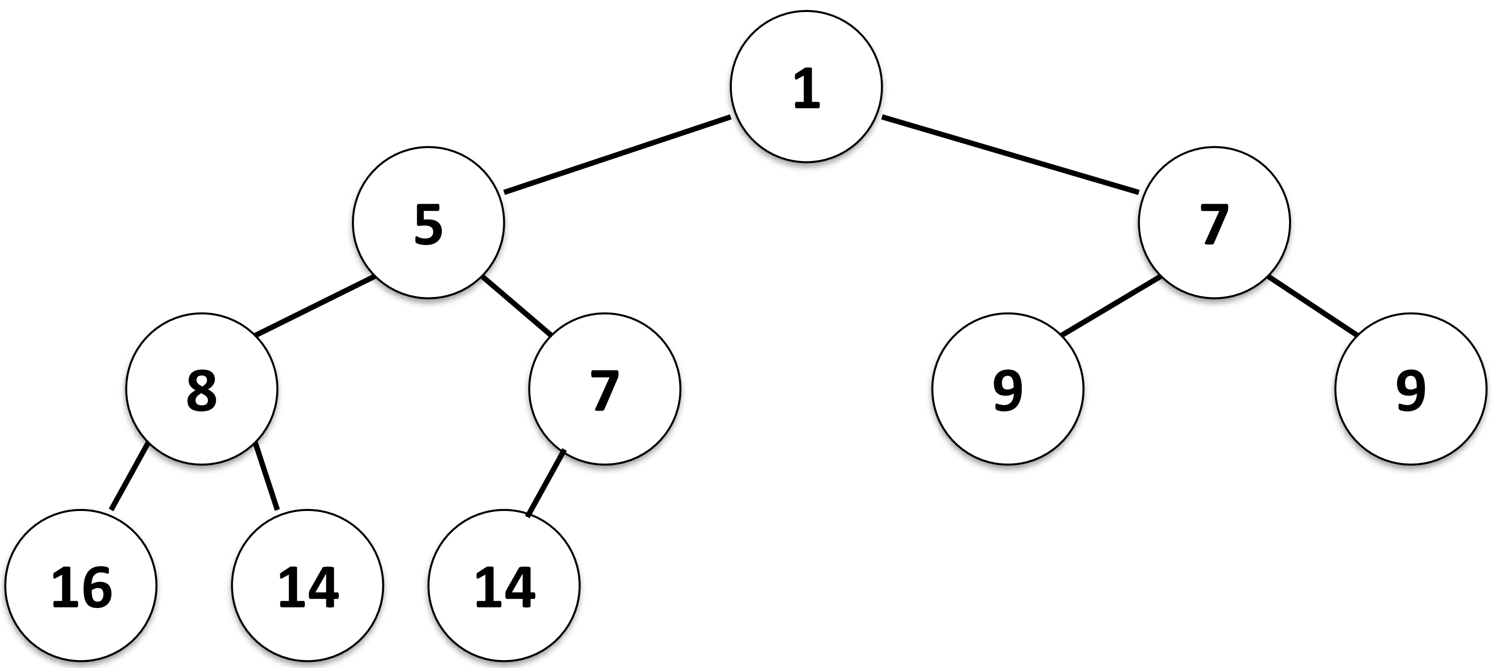
(min) Heap

- Propriedade
  - Chave do **pai** é menor ou igual a dos seus **filhos**



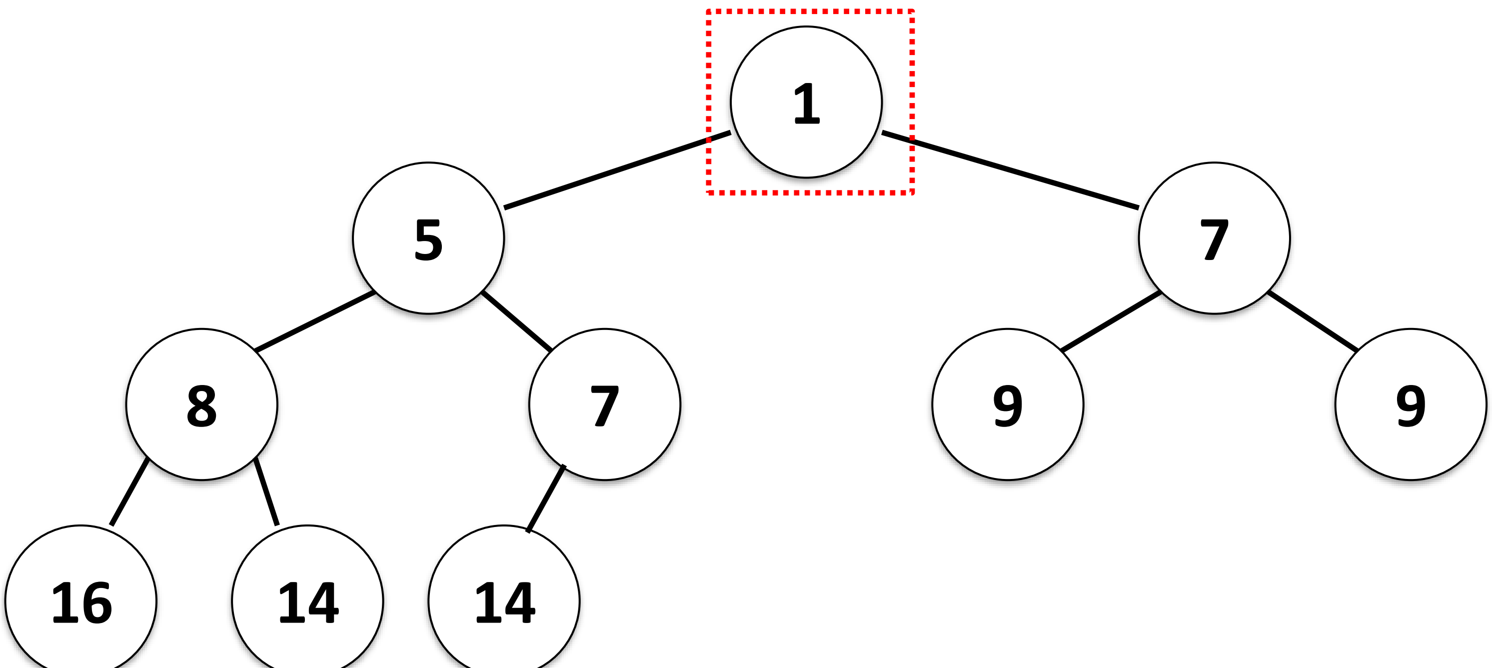
(min) Heap

	0	1	2	3	4	5	6	7	8	9
vetor	**1**	**5**	**7**	**8**	**7**	**9**	**9**	**16**	**14**	**14**



(min) Heap

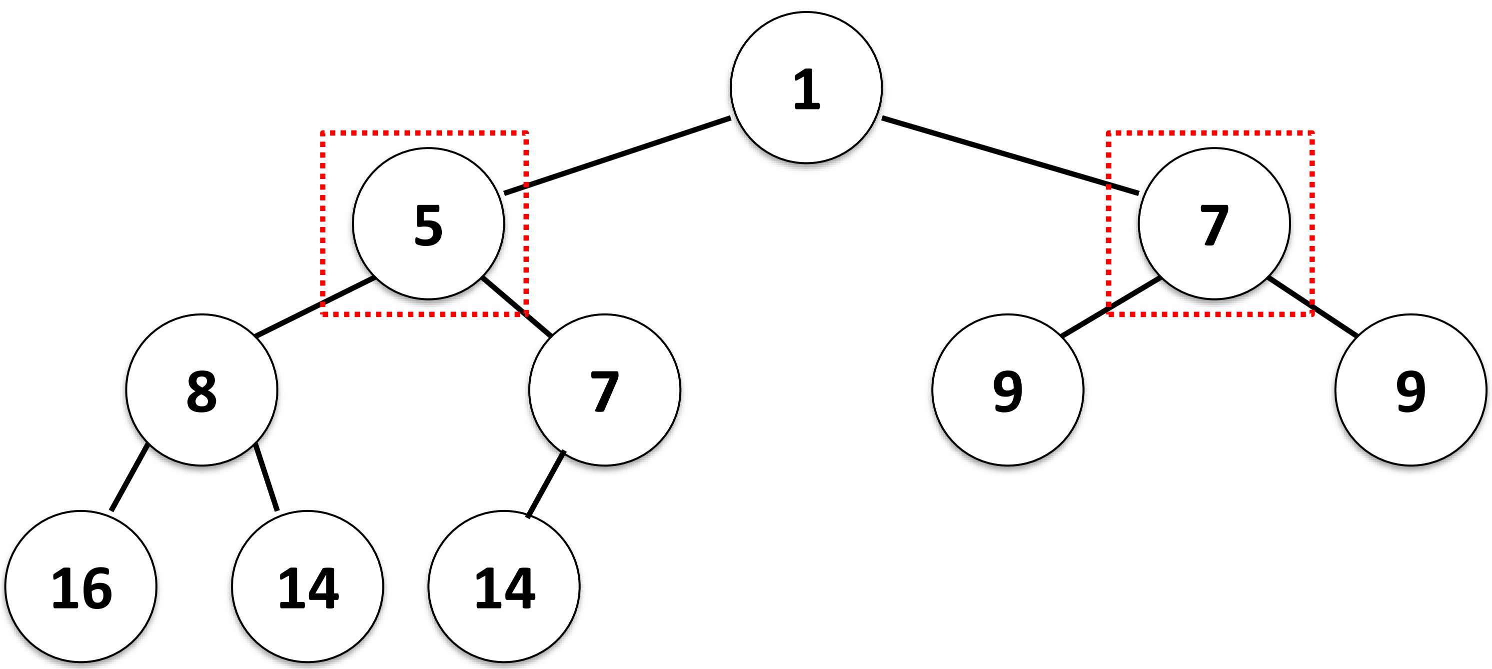
	0	1	2	3	4	5	6	7	8	9
vetor	**1**	**5**	**7**	**8**	**7**	**9**	**9**	**16**	**14**	**14**



(min) Heap

vetor

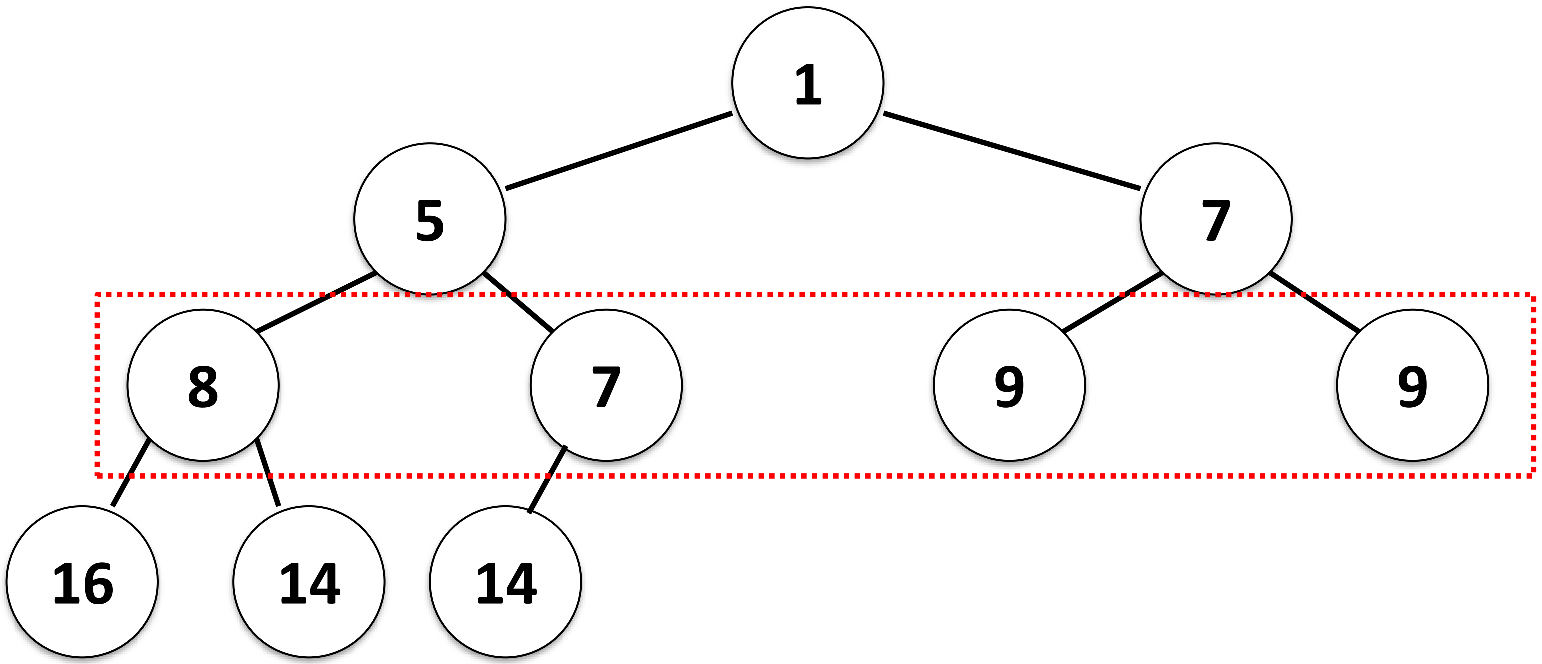
0	1	2	3	4	5	6	7	8	9
1	**5**	**7**	**8**	**7**	**9**	**9**	**16**	**14**	**14**



(min) Heap

vetor

0	1	2	3	4	5	6	7	8	9
1	**5**	**7**	**8**	**7**	**9**	**9**	**16**	**14**	**14**

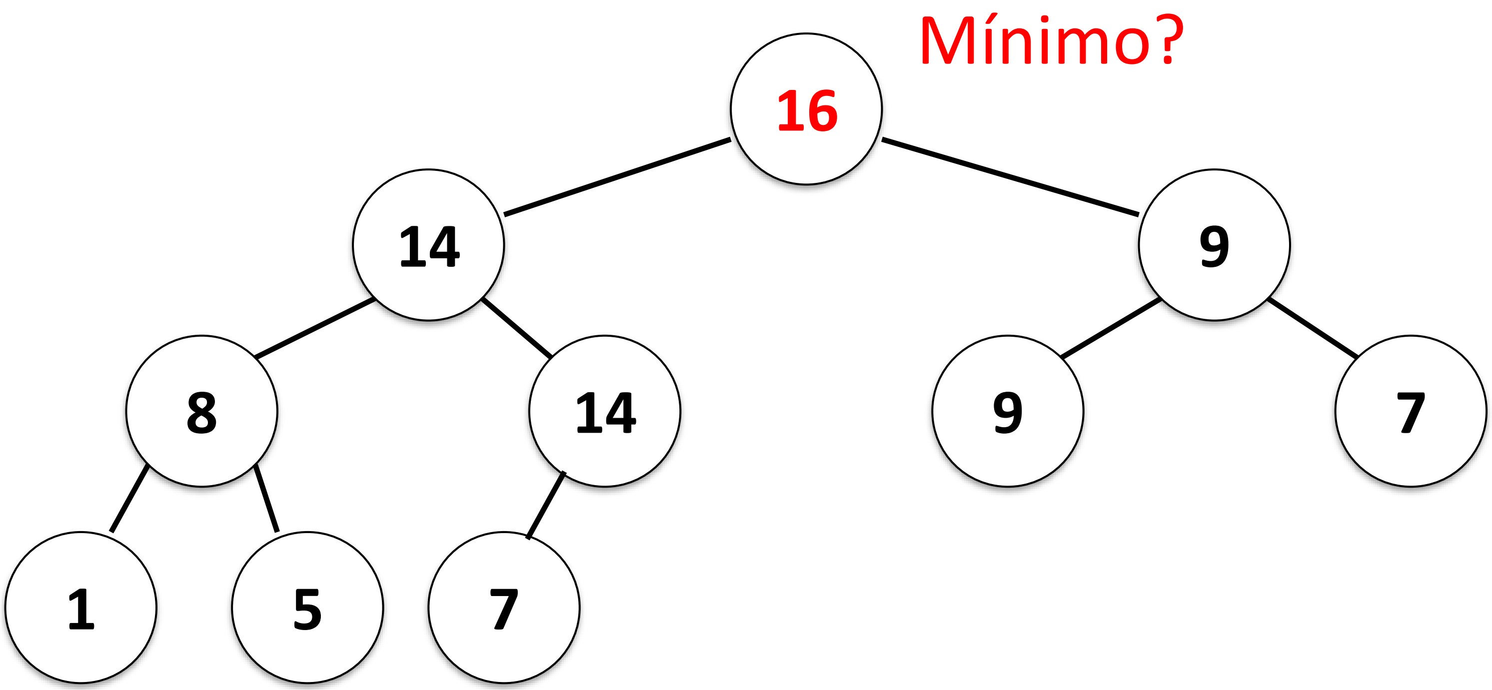


Exemplo: construir um Heap?

0	1	2	3	4	5	6	7	8	9
16	14	9	8	14	9	7	1	5	7

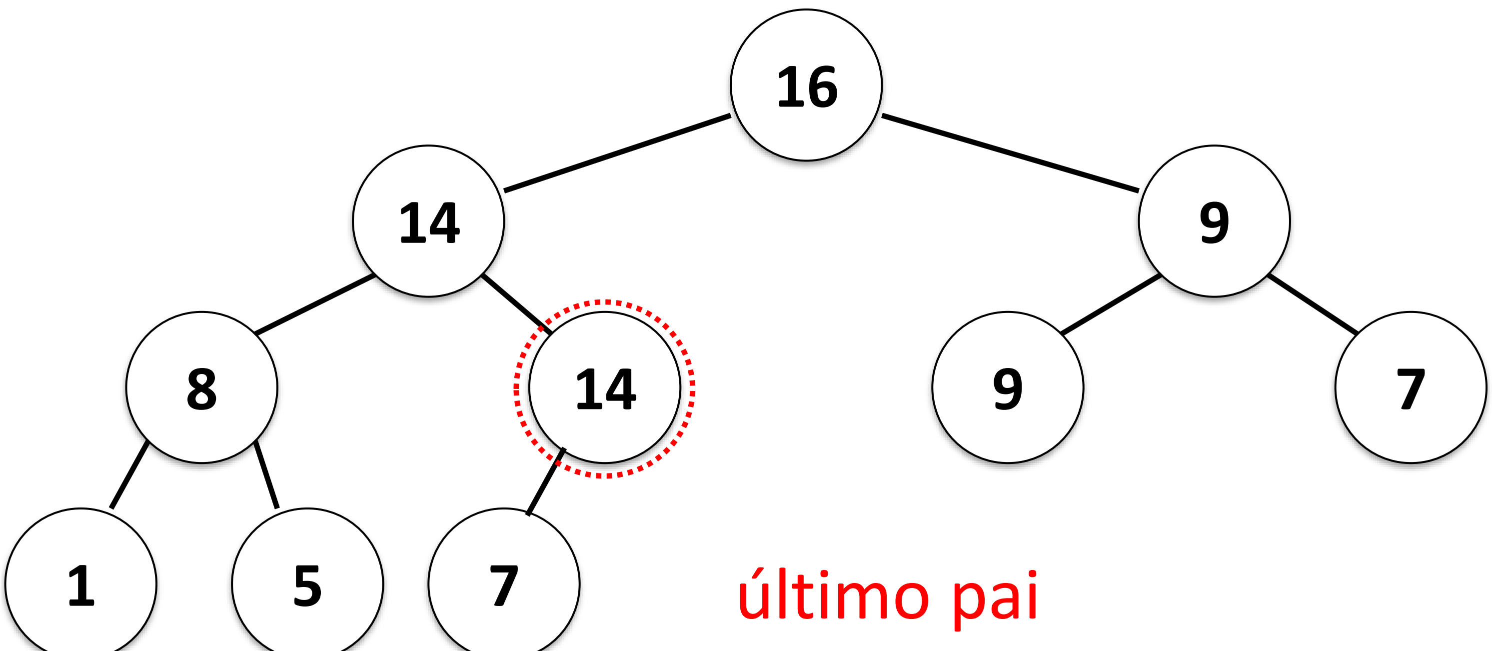
Exemplo: construir minHeap?

0	1	2	3	4	5	6	7	8	9
16	14	9	8	14	9	7	1	5	7



Exemplo: construir minHeap?

- Propriedade
  - Chave do **pai** é menor ou igual a dos seus **filhos**



último pai

Exemplo: construir minHeap?

- Propriedade
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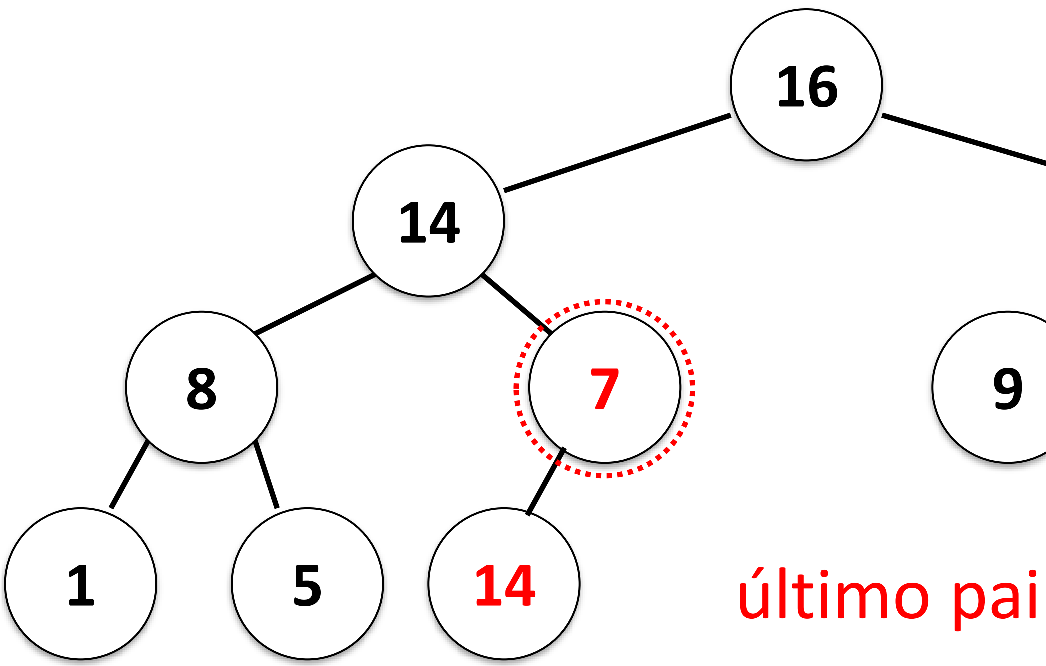

Exemplo: construir minHeap?

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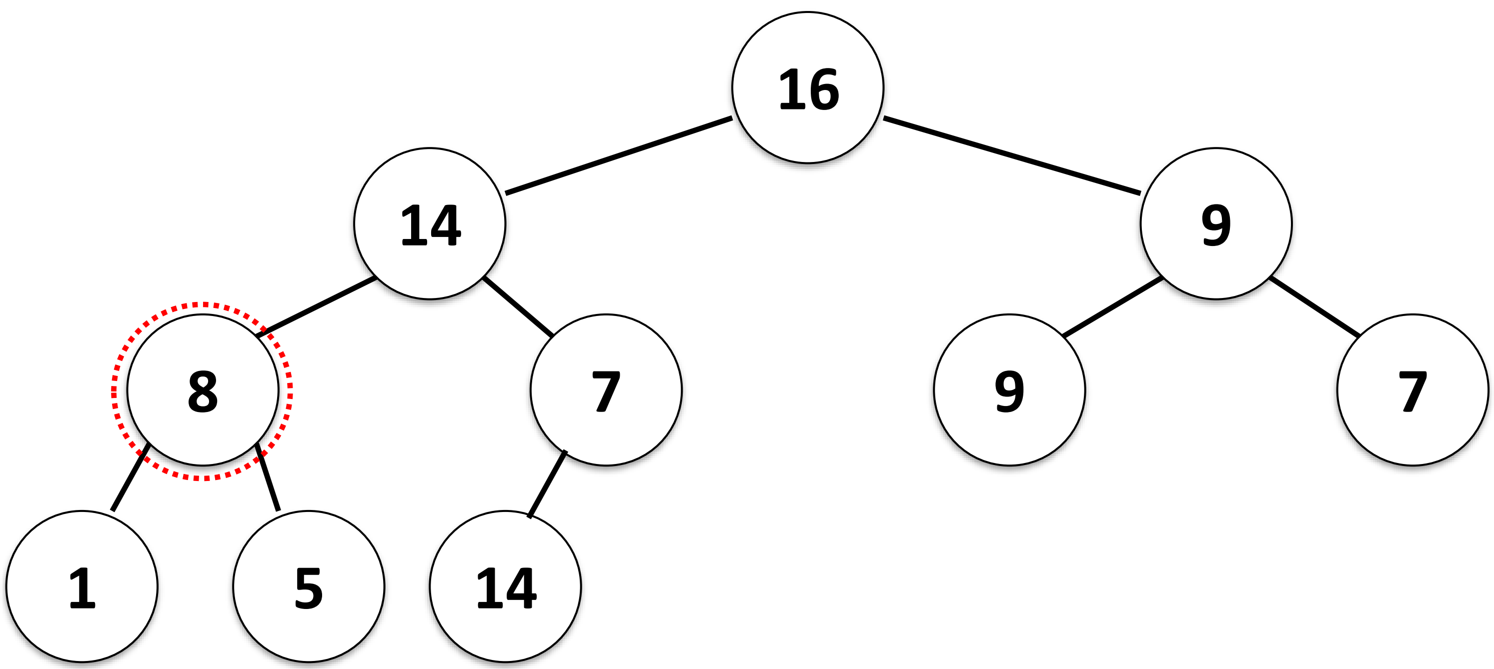

Exemplo: construir minHeap?

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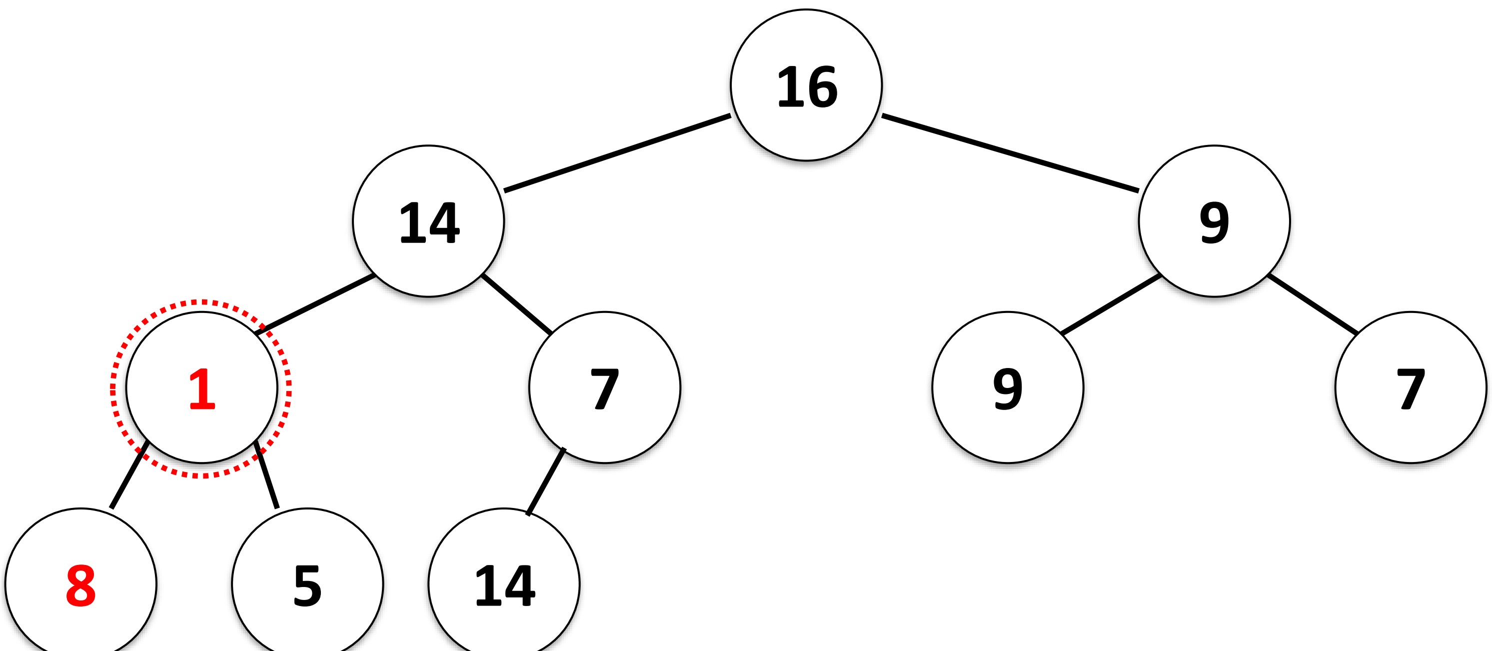

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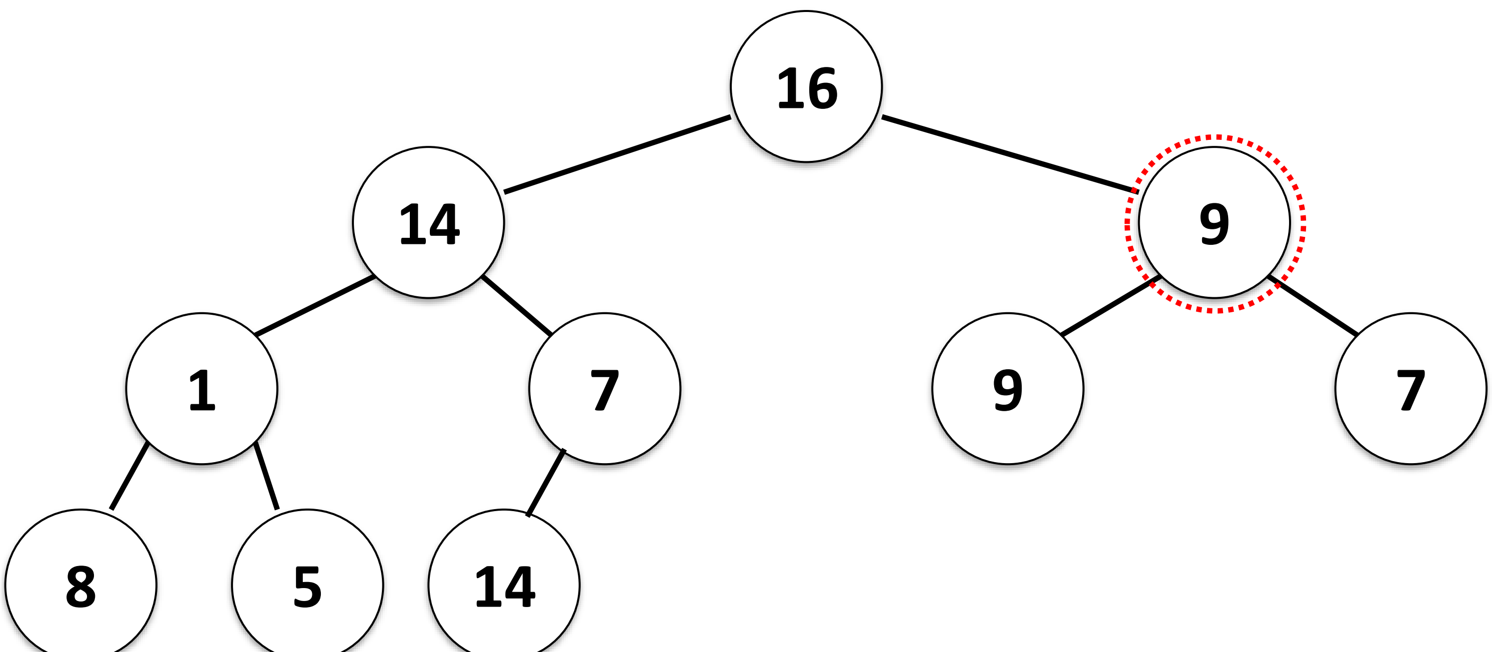

Exemplo: construir minHeap?

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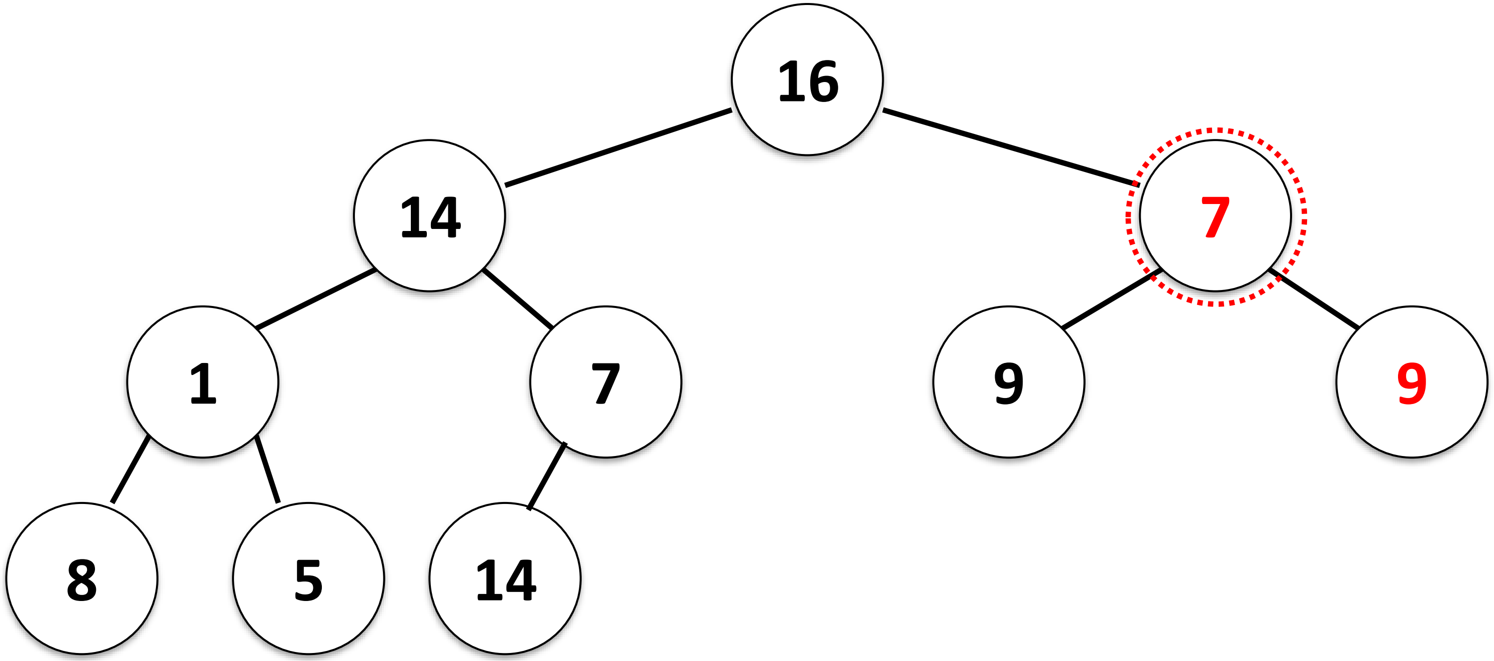

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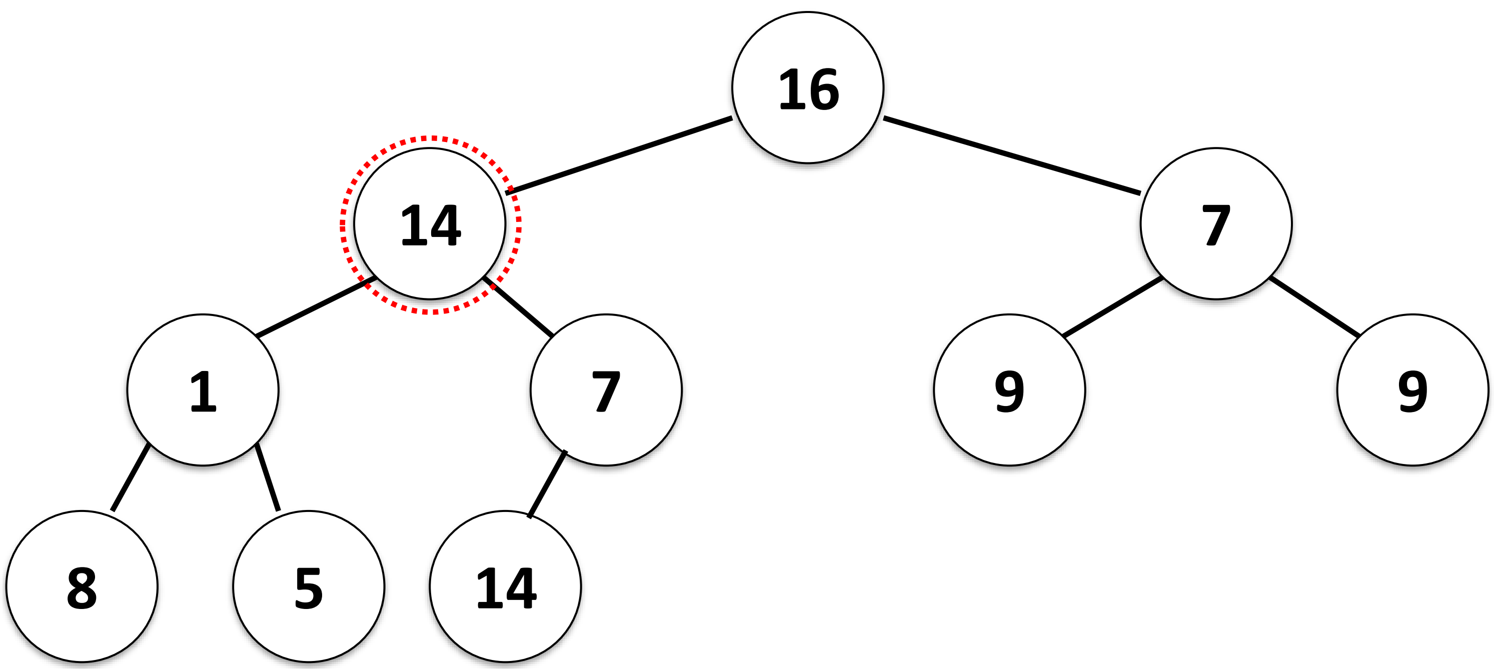

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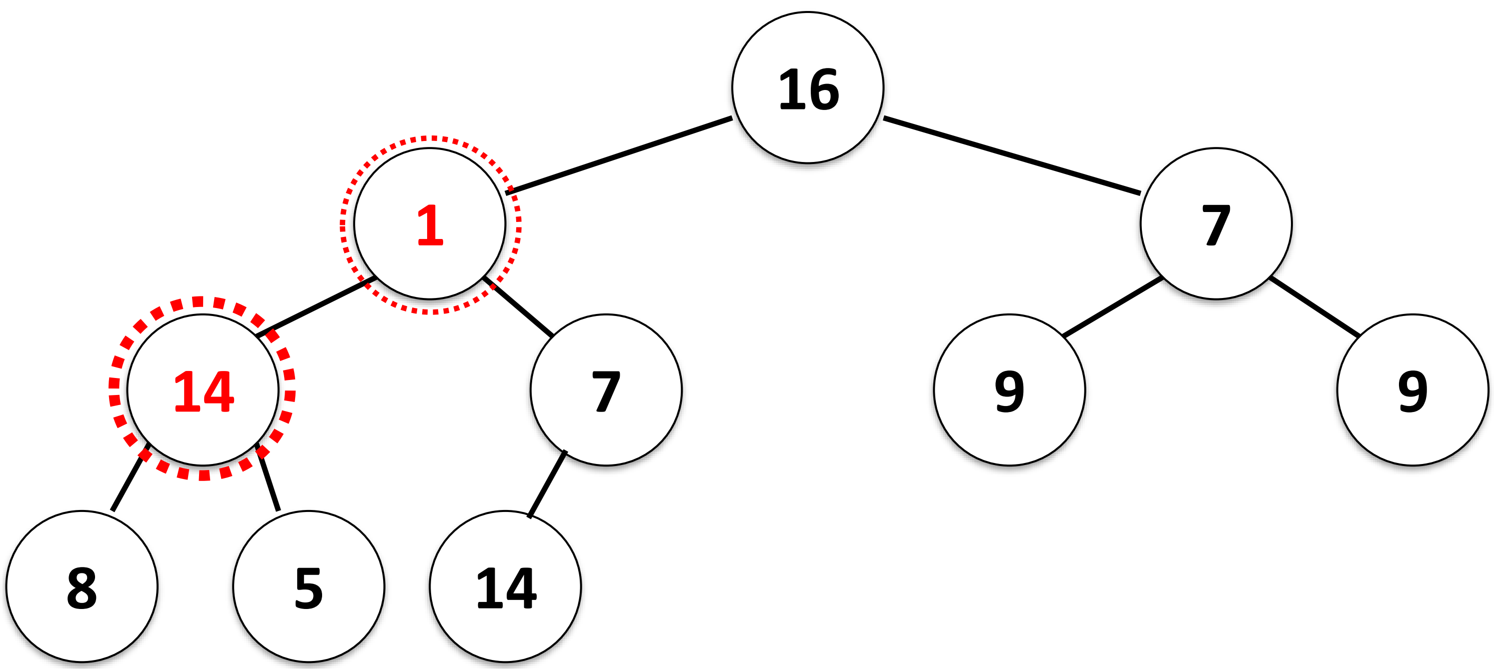

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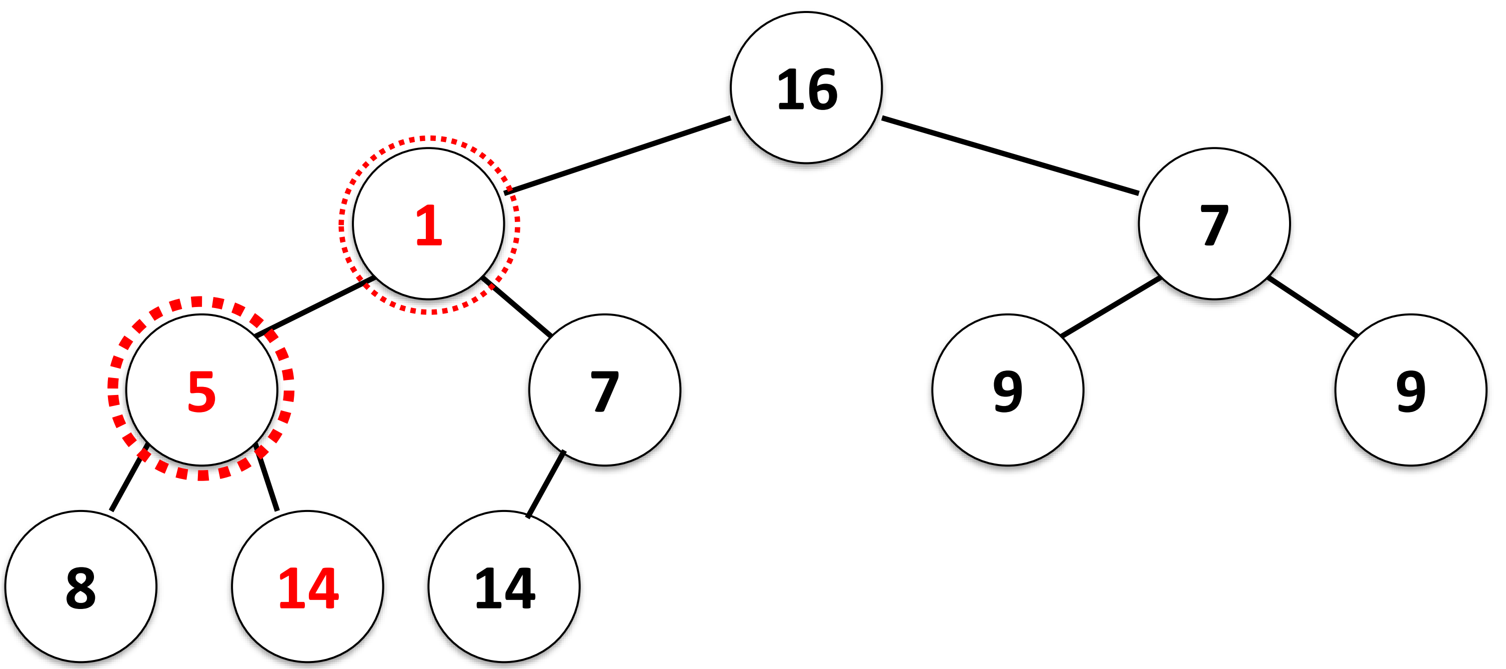

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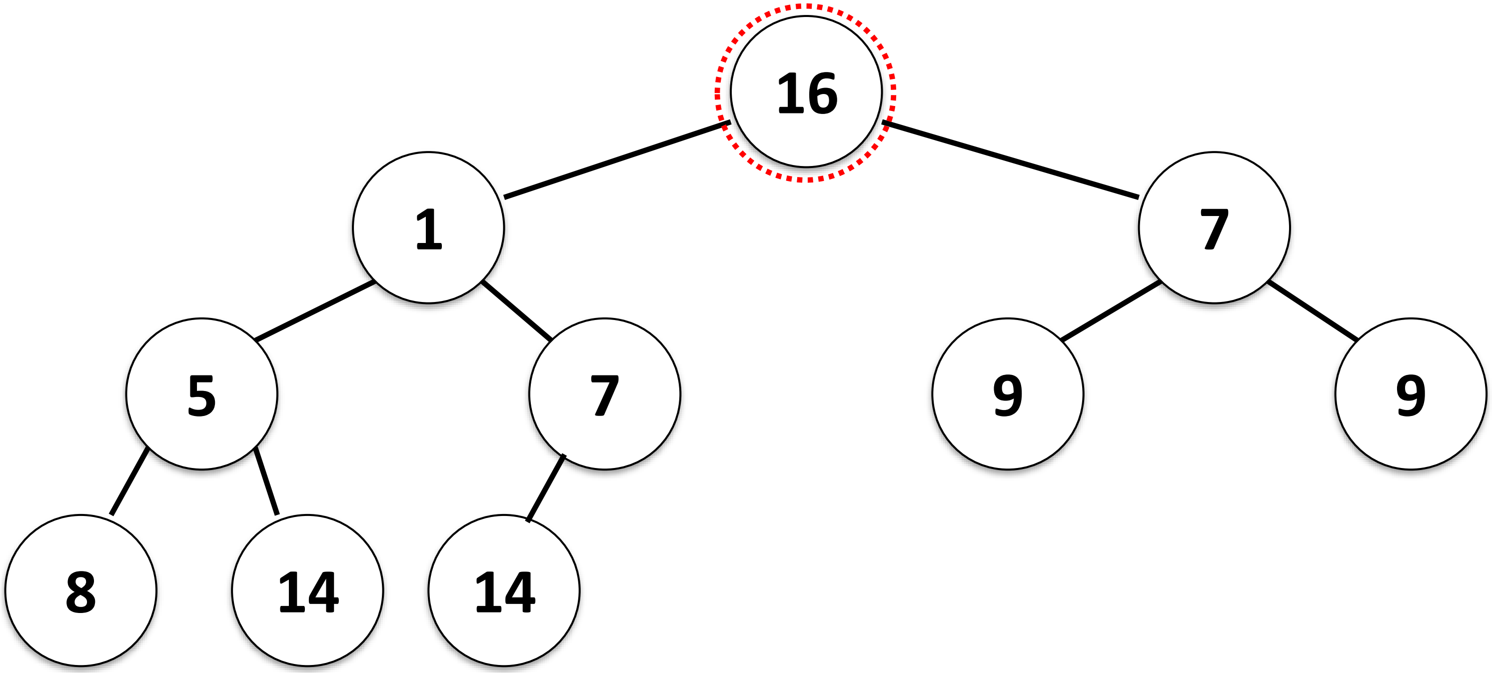

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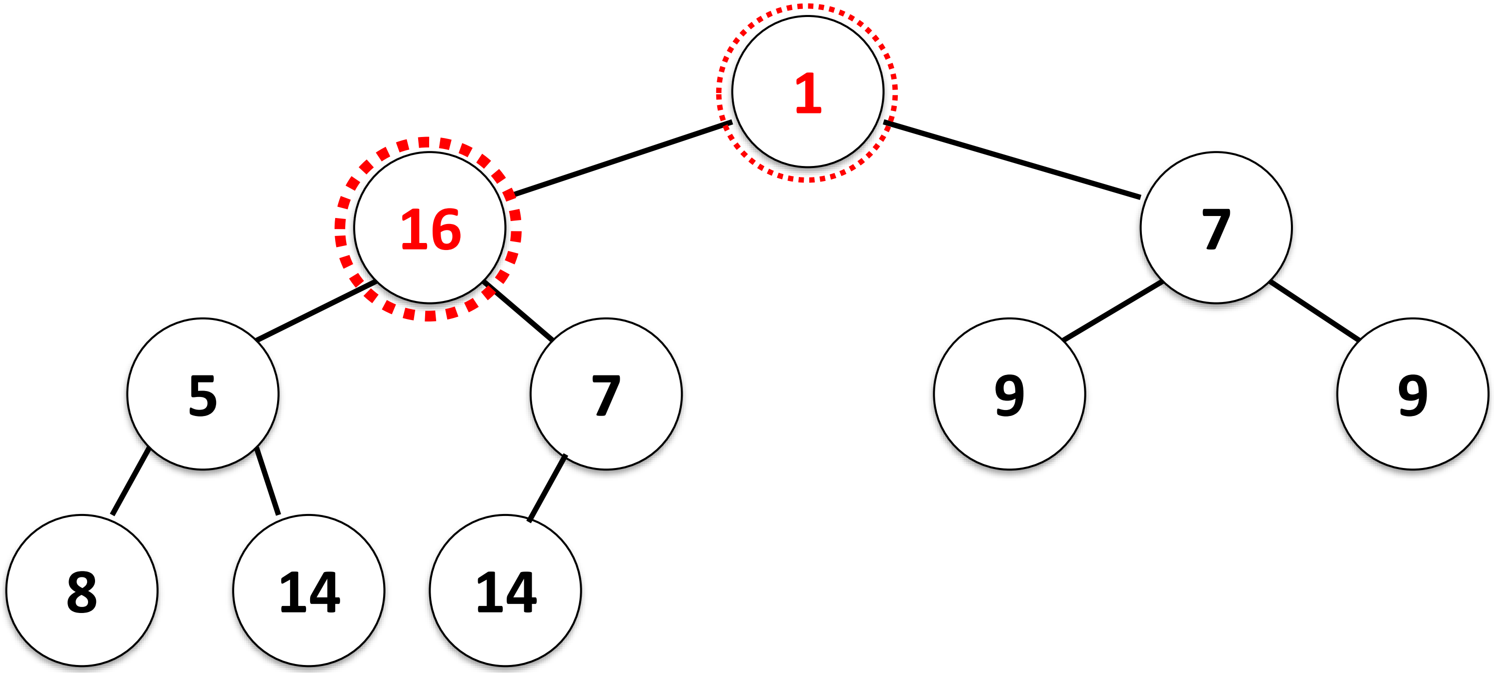

Exemplo: construir minHeap?

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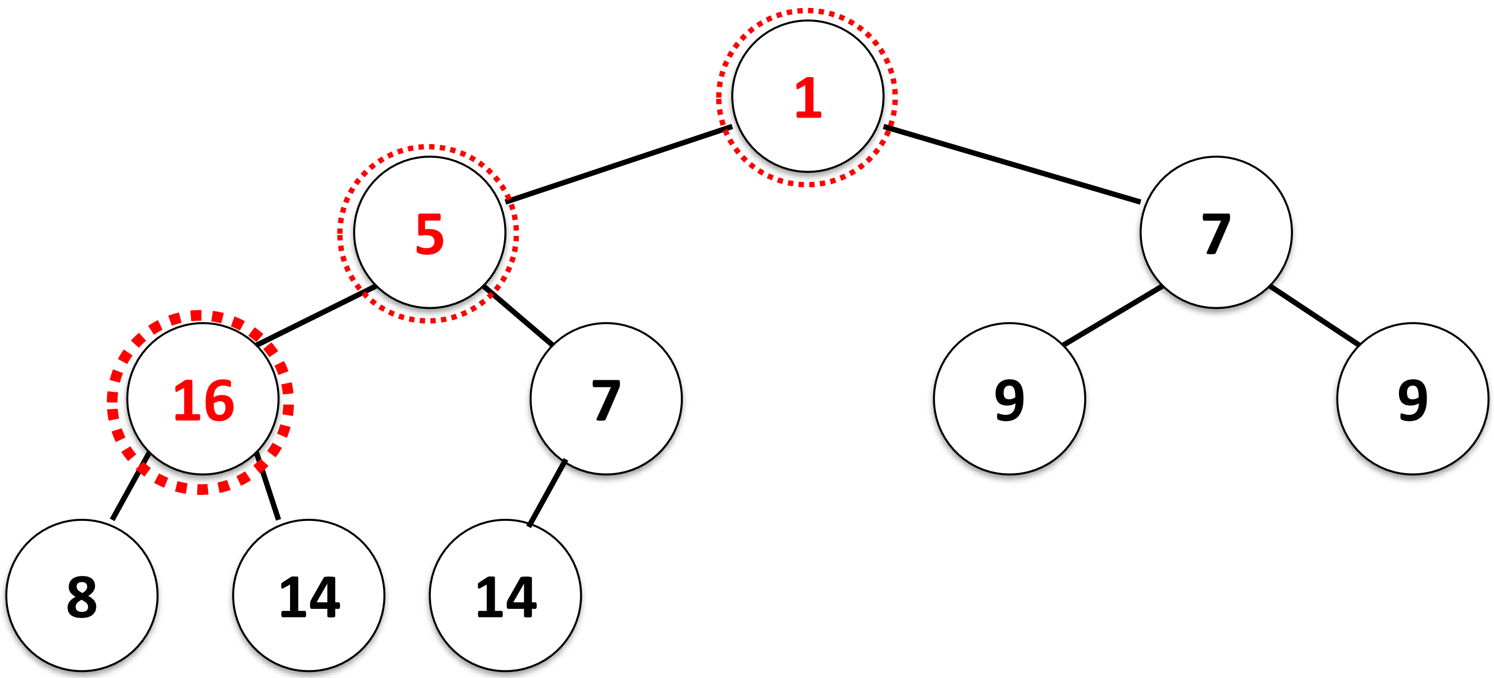
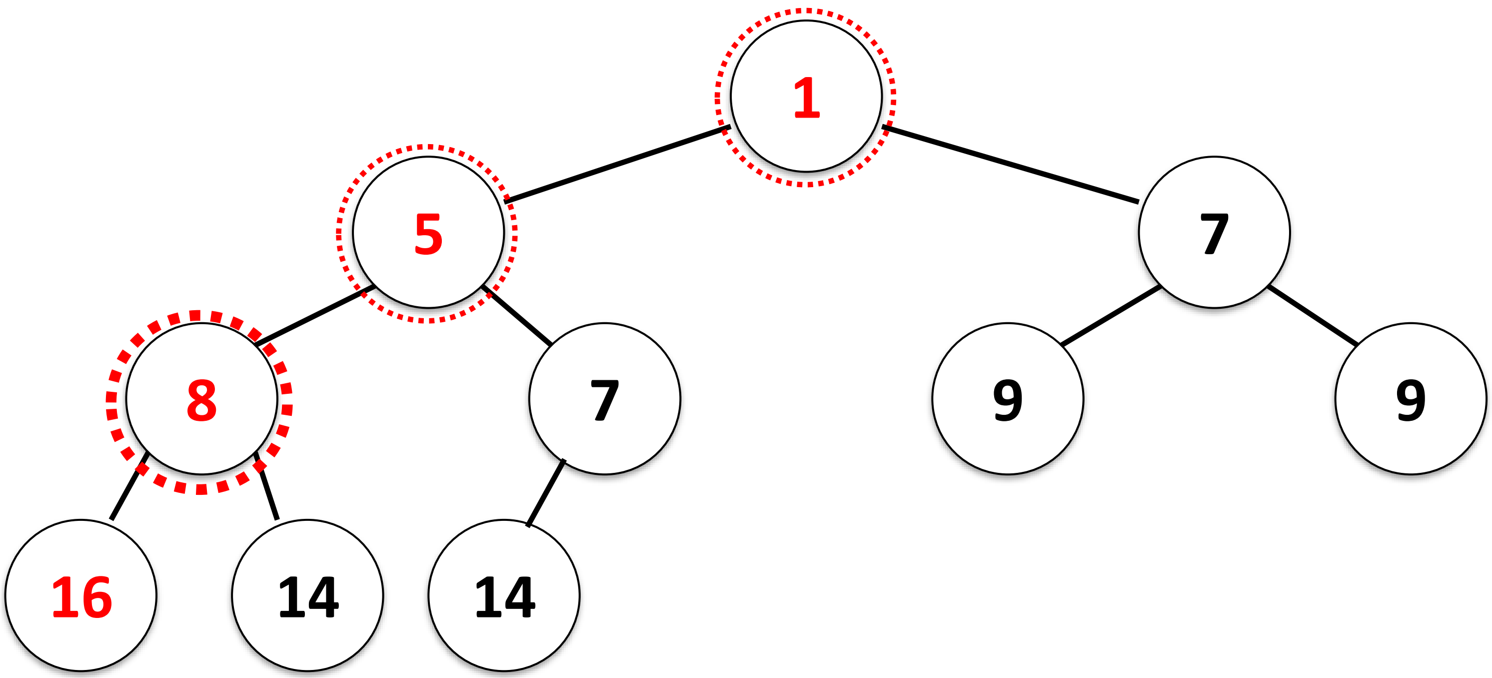


Exemplo: construir minHeap?

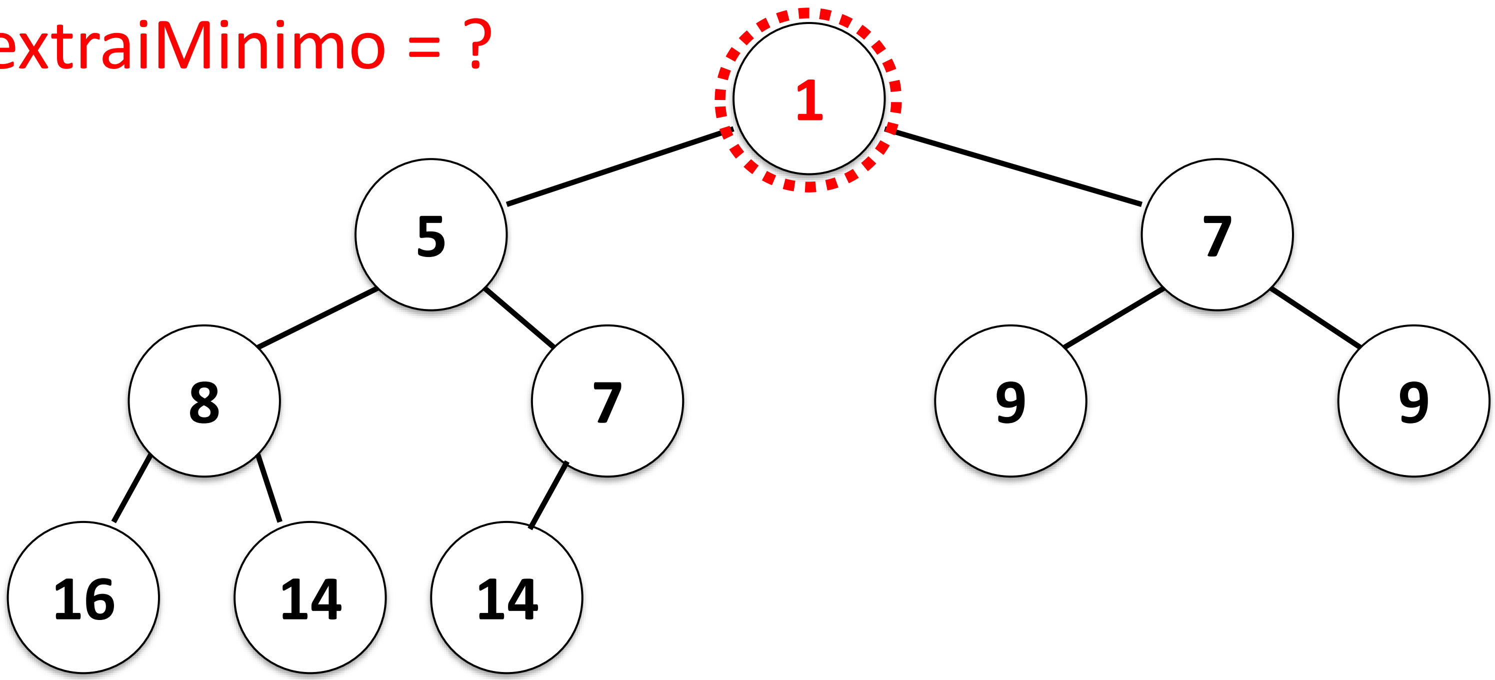
0	1	2	3	4	5	6	7	8	9
1	5	7	8	7	9	9	16	14	14



Exemplo: extraiMinimo?

0	1	2	3	4	5	6	7	8	9
1	5	7	8	7	9	9	16	14	14

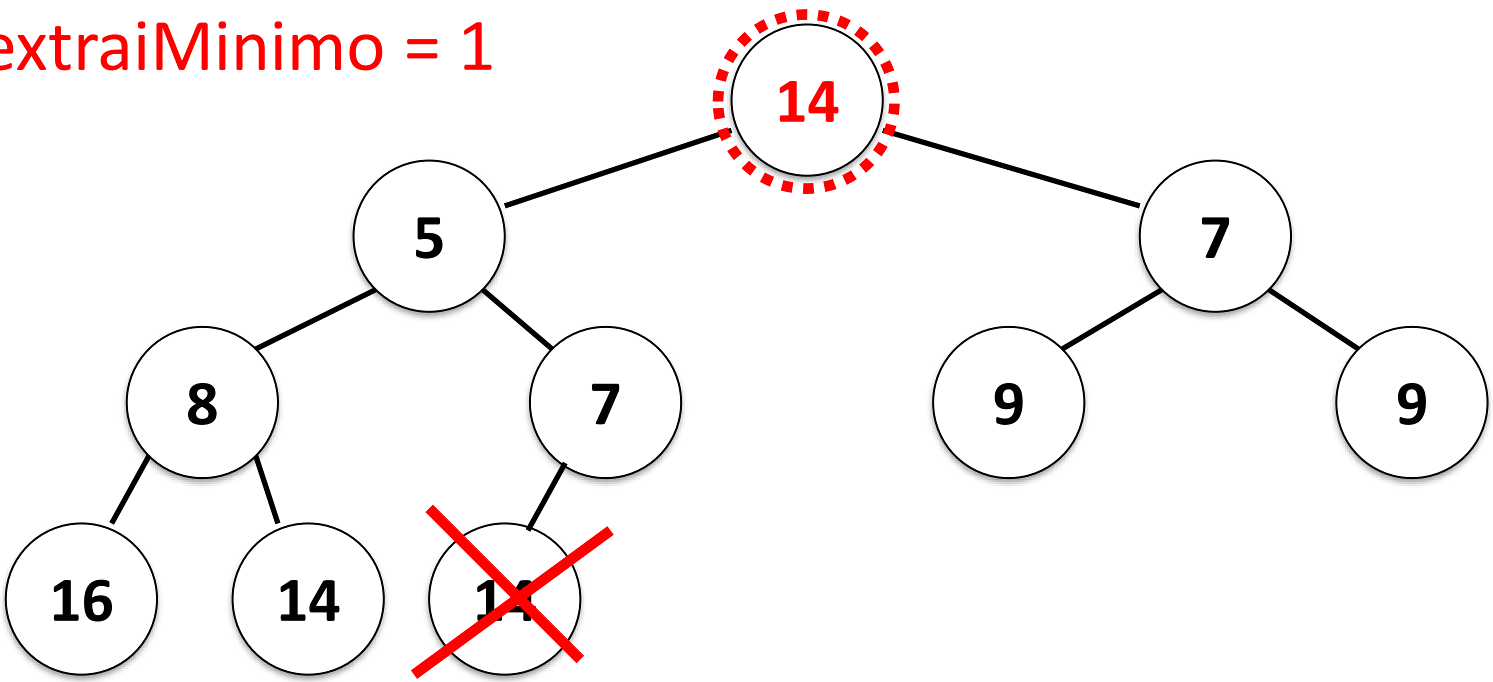
extraiMinimo = ?



Exemplo: extraiMinimo?

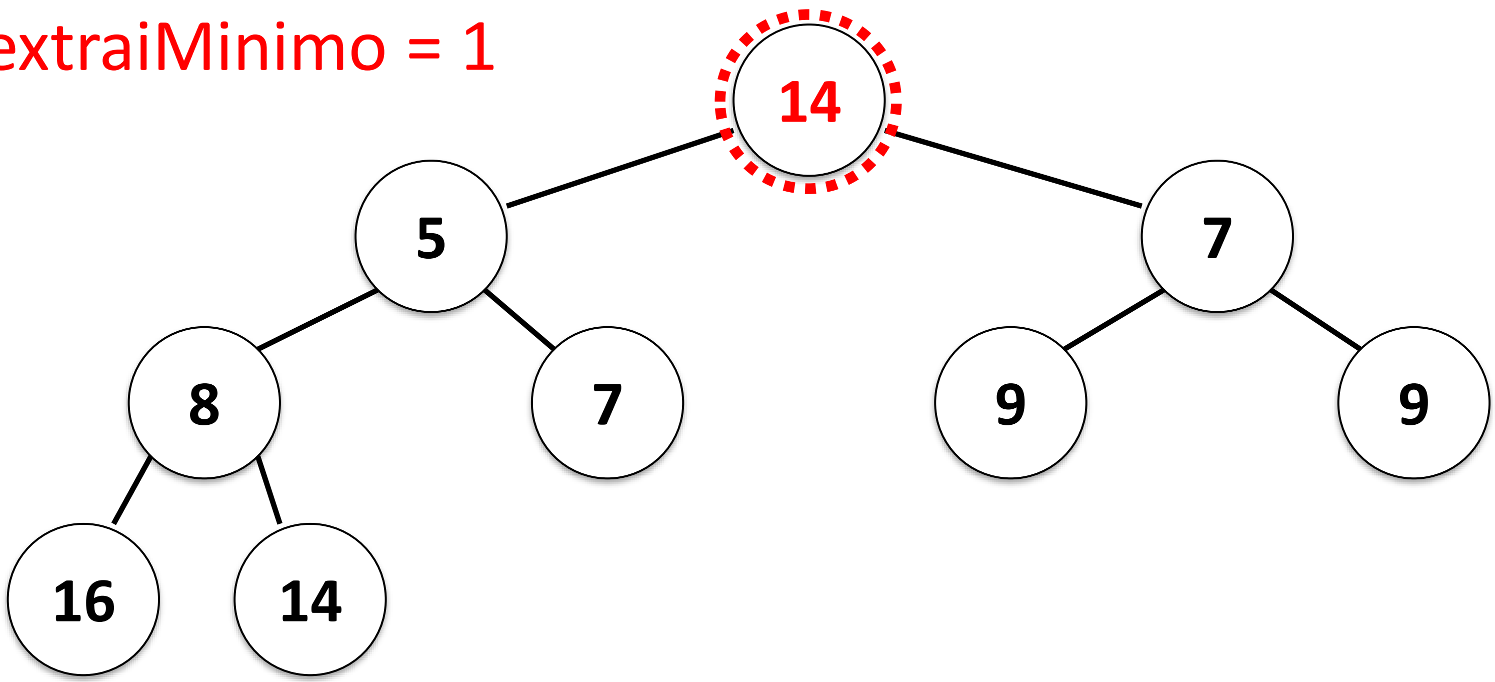
0	1	2	3	4	5	6	7	8	9
14	5	7	8	7	9	9	16	14	~~14~~

extraiMinimo = 1



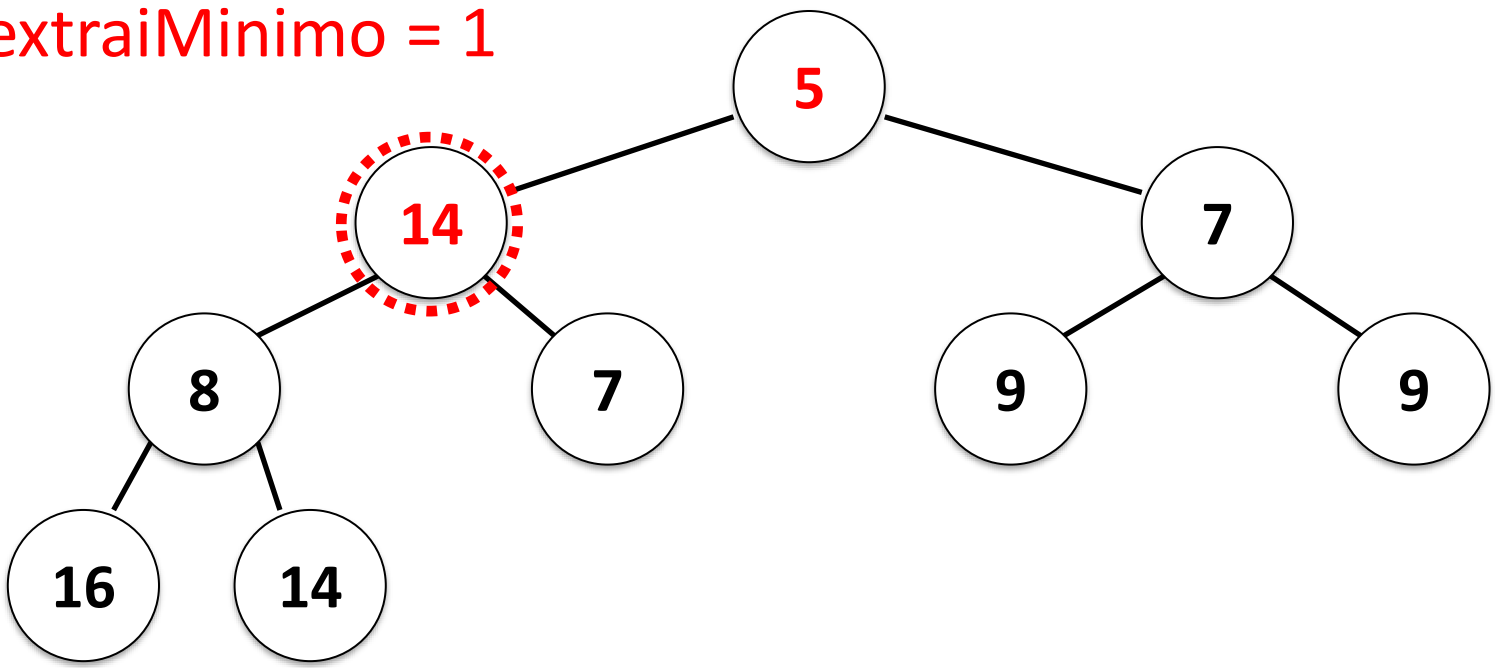
Desce chave

extraiMinimo = 1



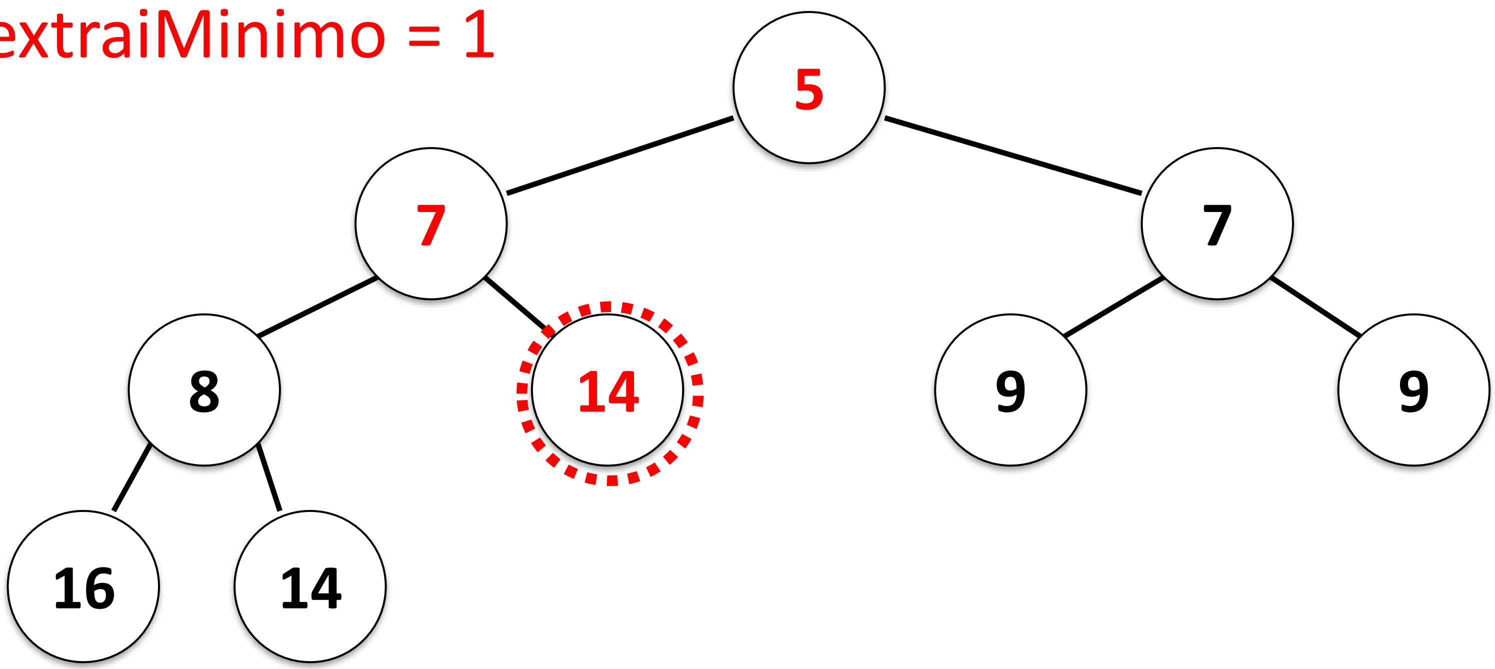
Desce chave

extraiMinimo = 1



Desce chave

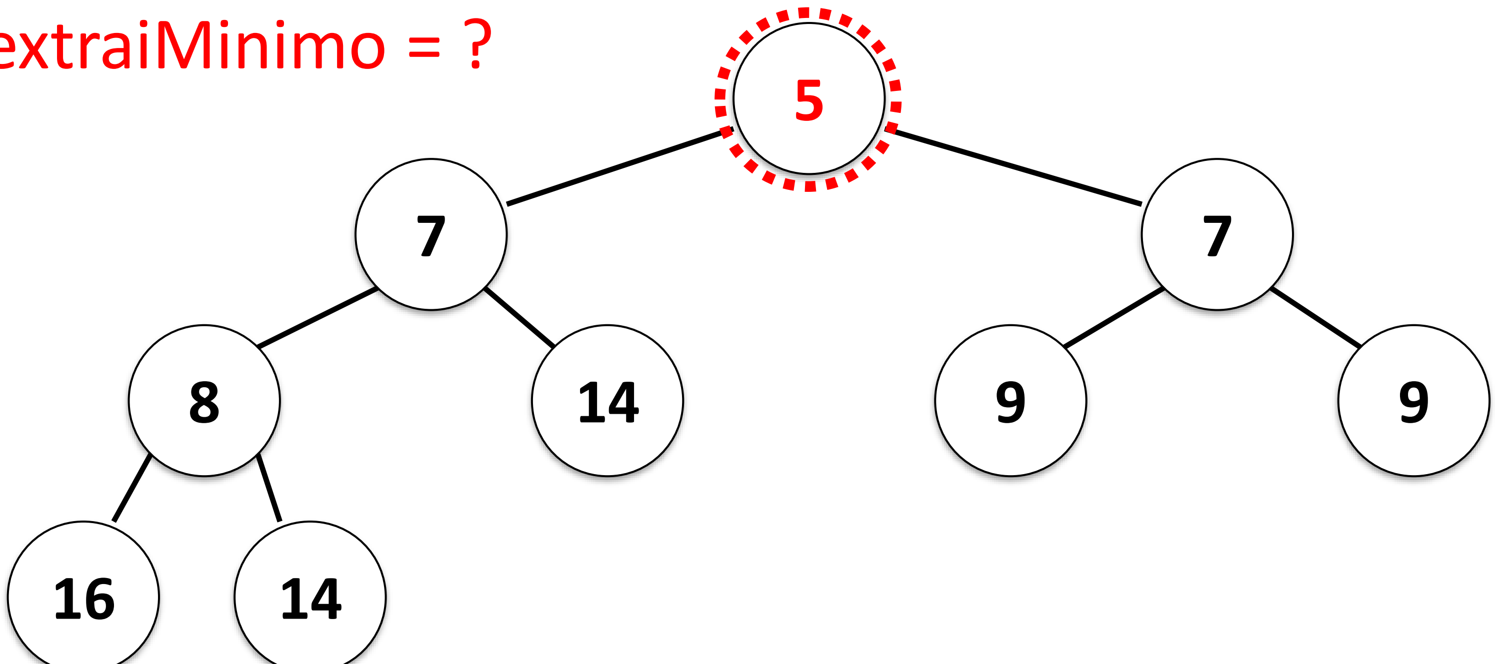
extraiMinimo = 1



Exemplo: extraiMinimo?

0	1	2	3	4	5	6	7	8	9
5	7	7	8	14	9	9	16	14	

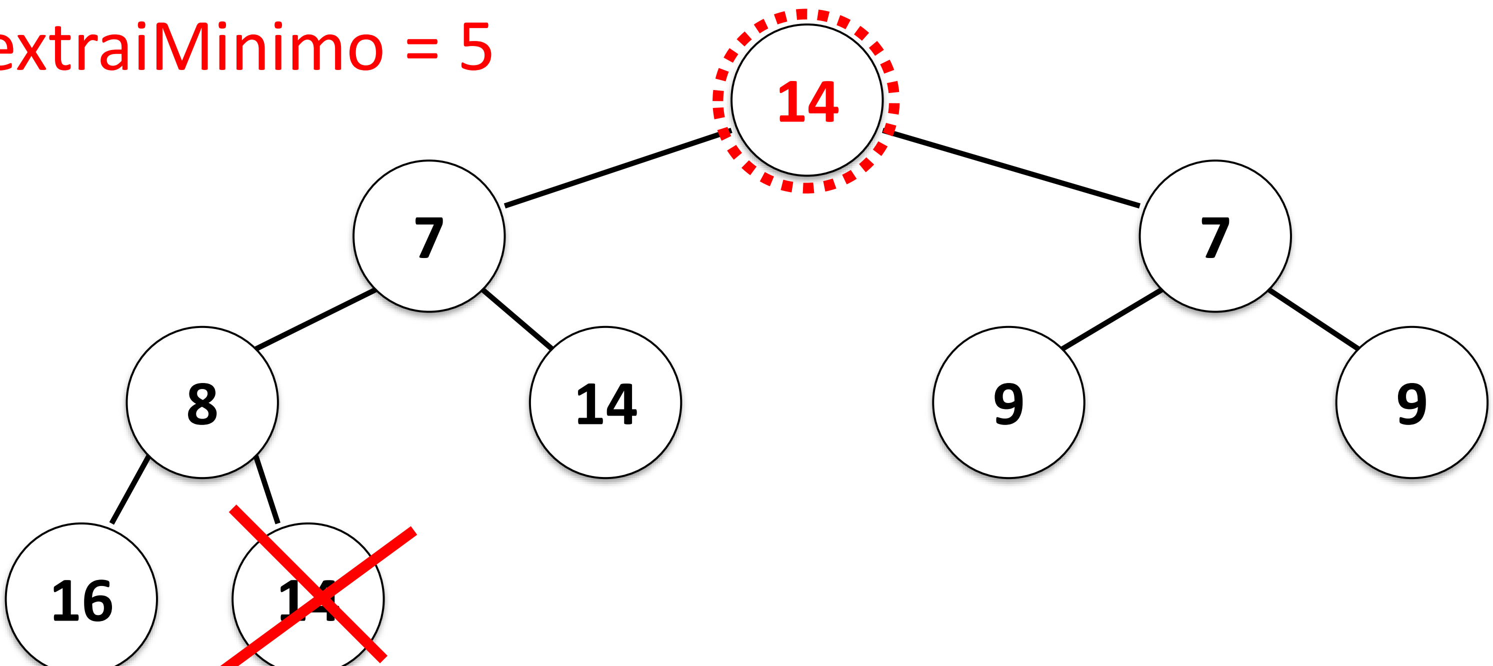
extraiMinimo = ?



Exemplo: extraiMinimo?

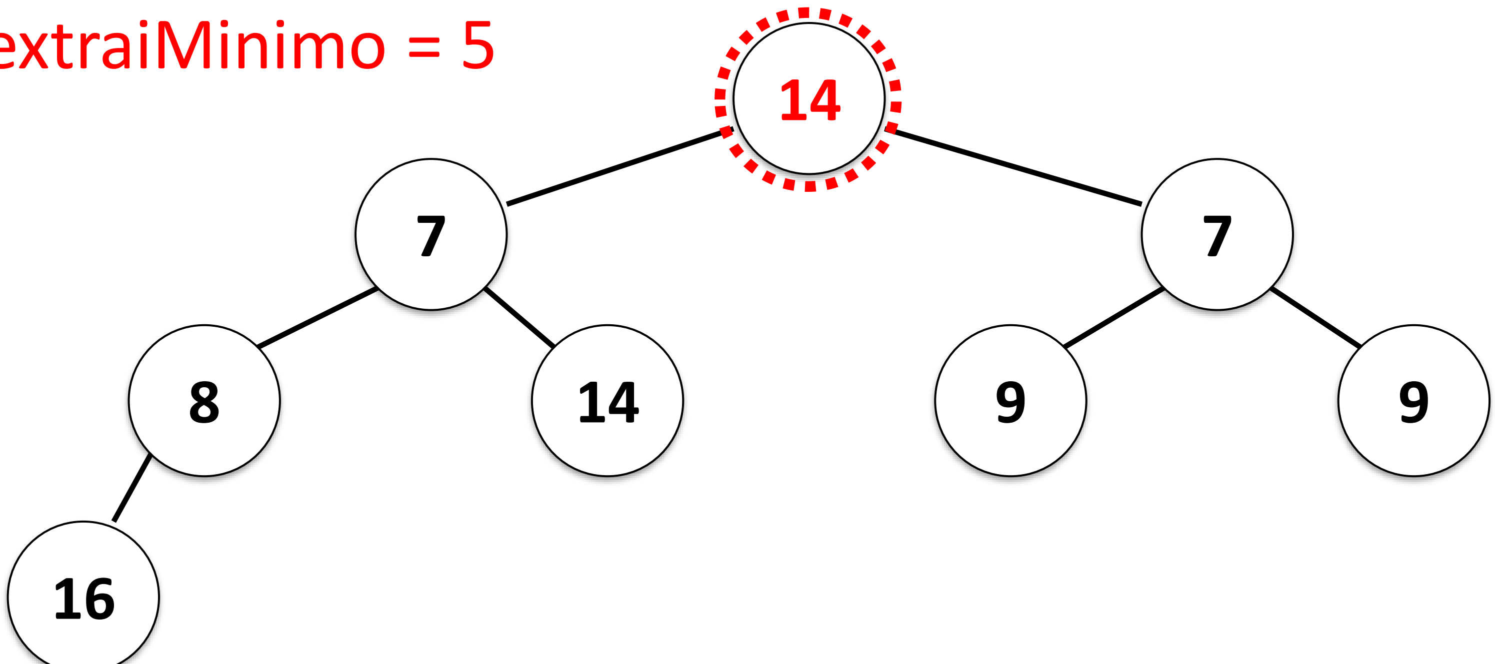
0	1	2	3	4	5	6	7	8	9
14	7	7	8	14	9	9	16	~~14~~	

extraiMinimo = 5



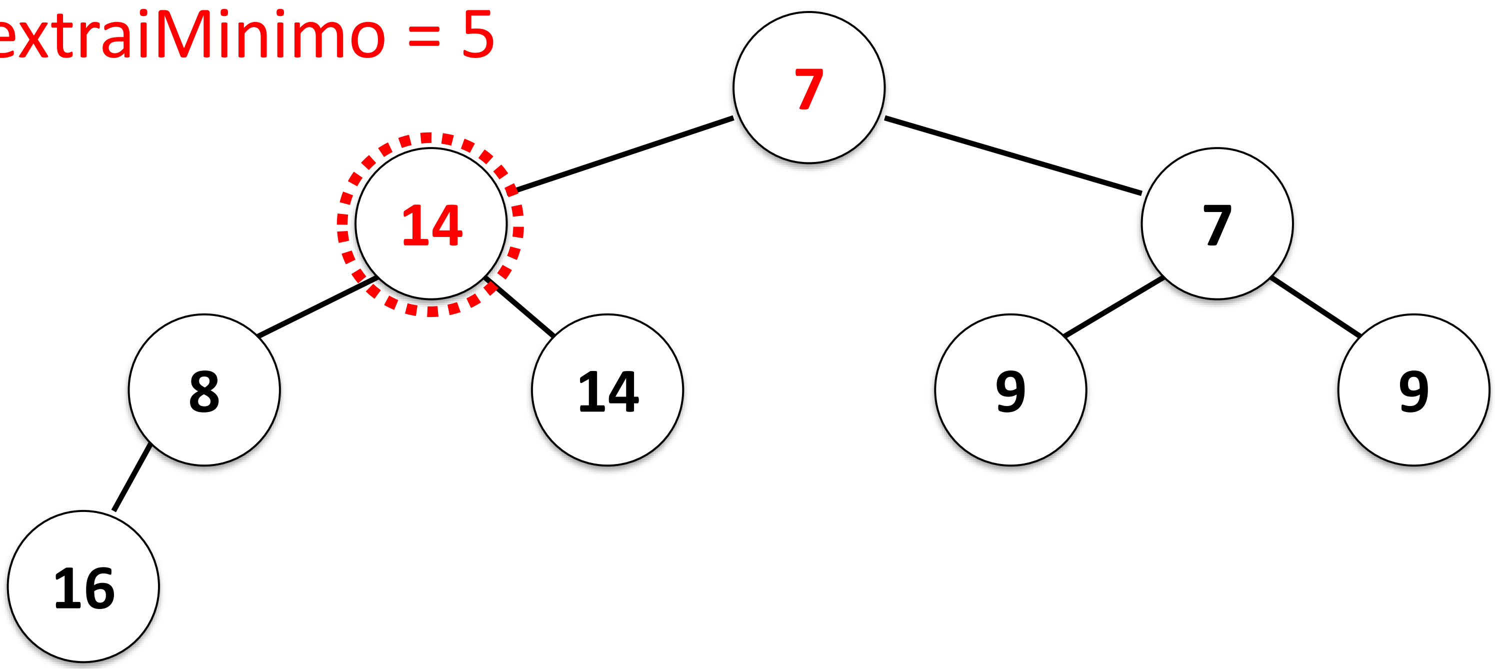
Desce chave

extraiMinimo = 5



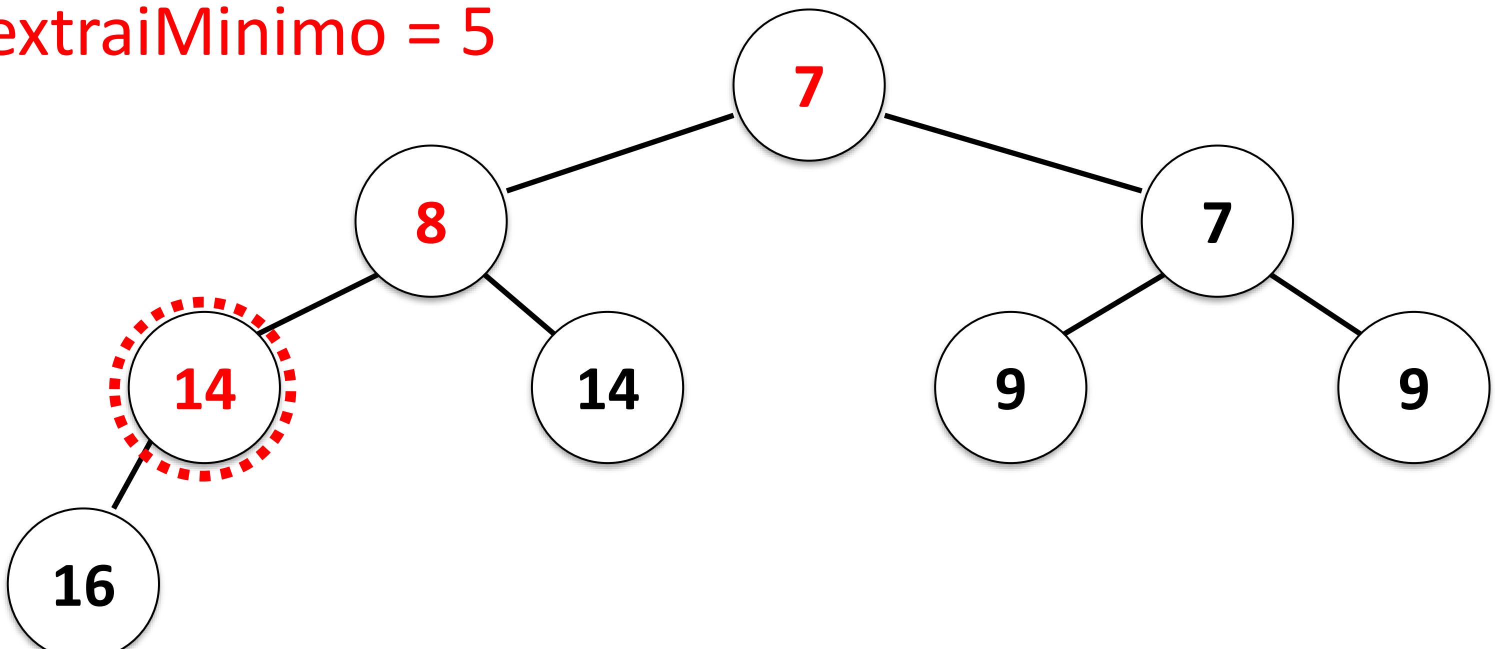
Desce chave

extraiMinimo = 5



Desce chave

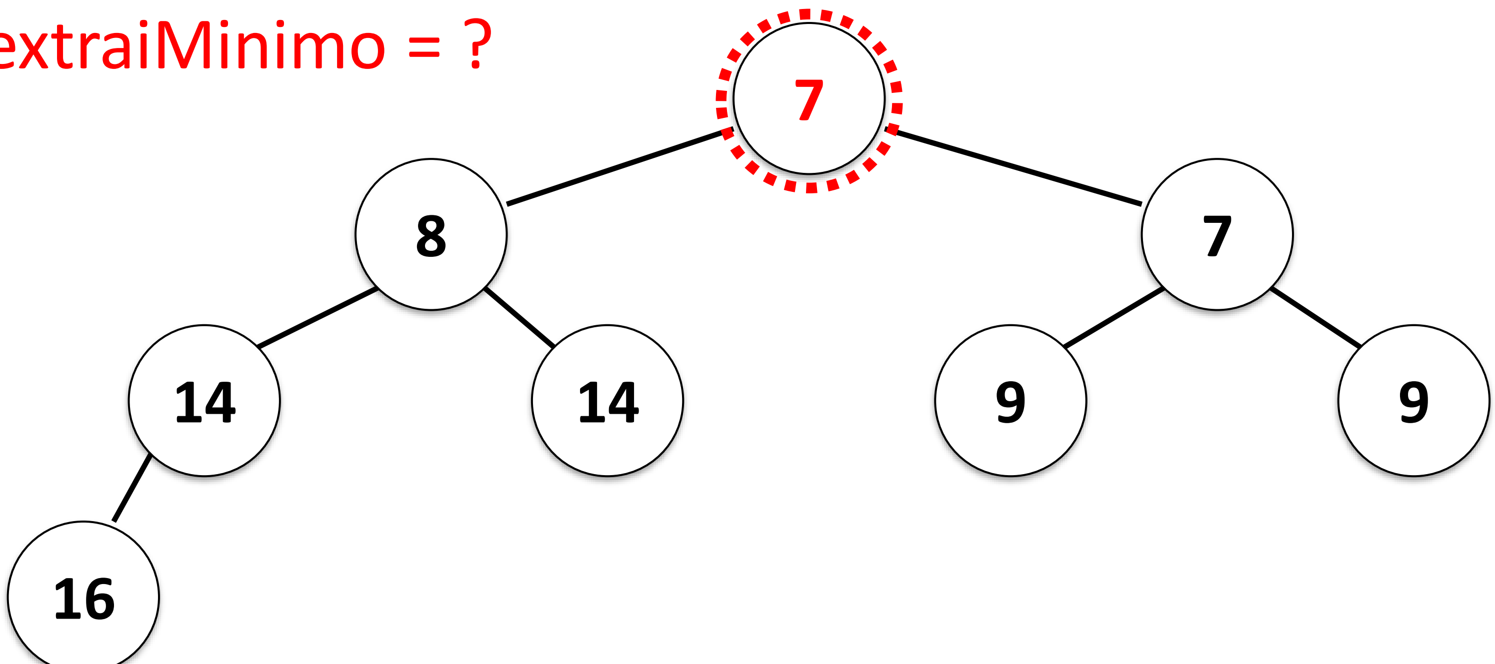
extraiMinimo = 5



Exemplo: extraiMinimo?

0	1	2	3	4	5	6	7	8	9
7	8	7	14	14	9	9	16		

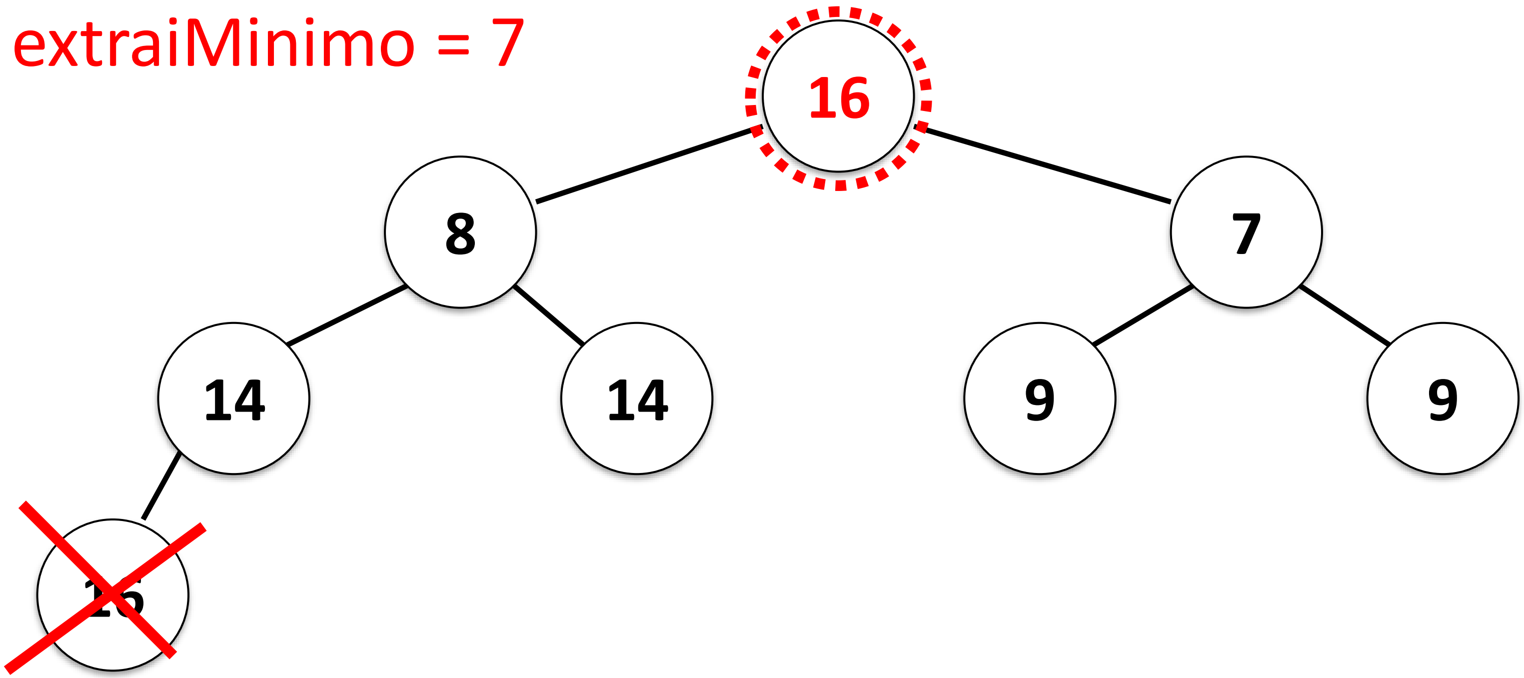
extraiMinimo = ?



Exemplo: extraiMinimo?

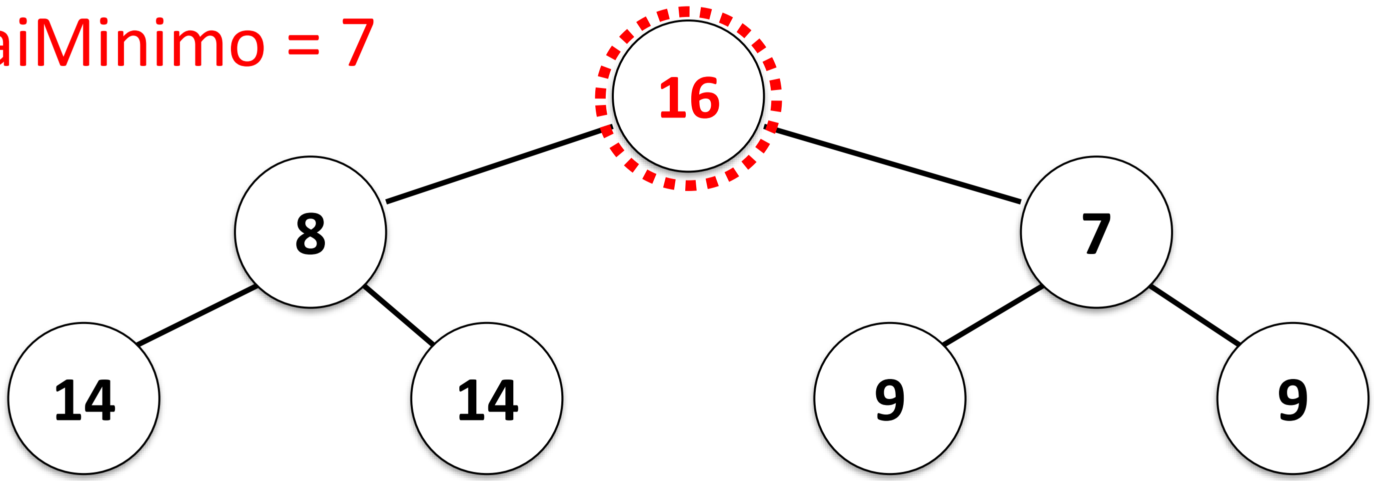
0	1	2	3	4	5	6	7	8	9
7	8	7	14	14	9	9	~~16~~		

extraiMinimo = 7



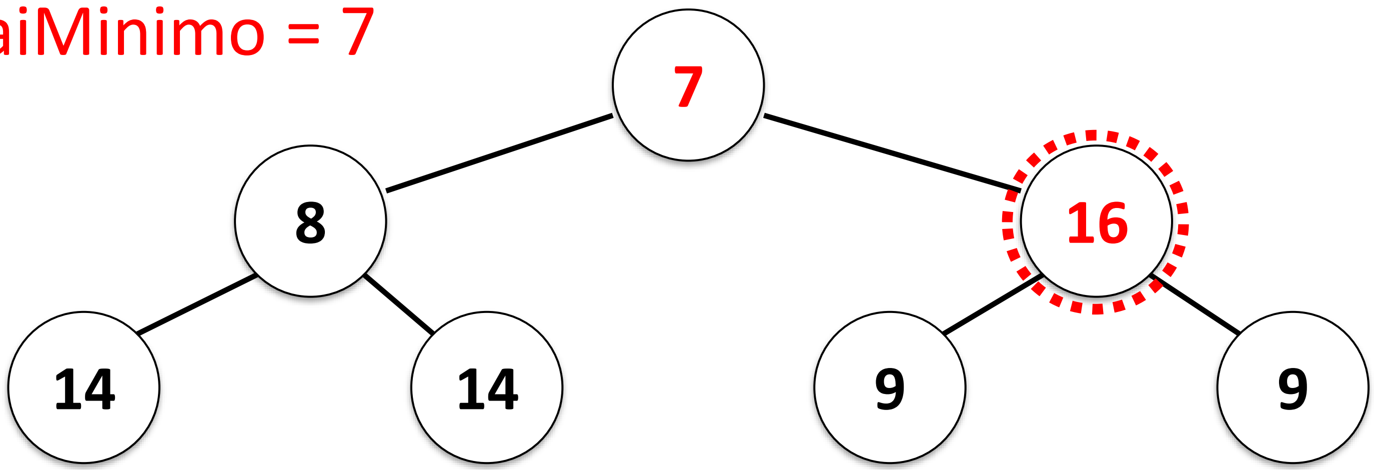
Desce chave

extraiMinimo = 7



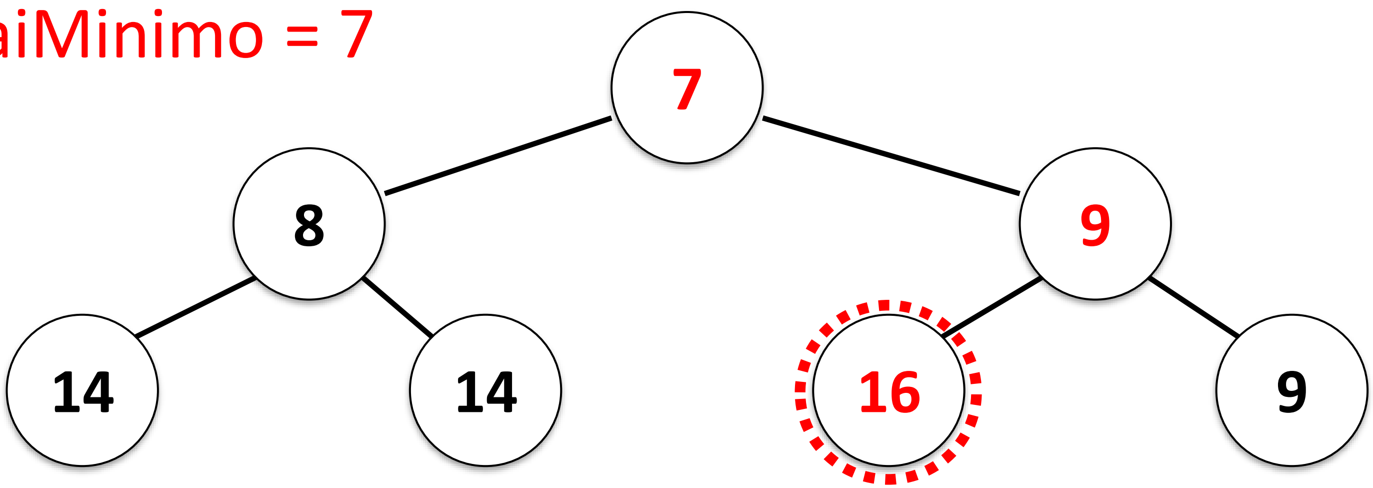
Desce chave

extraiMinimo = 7



Desce chave

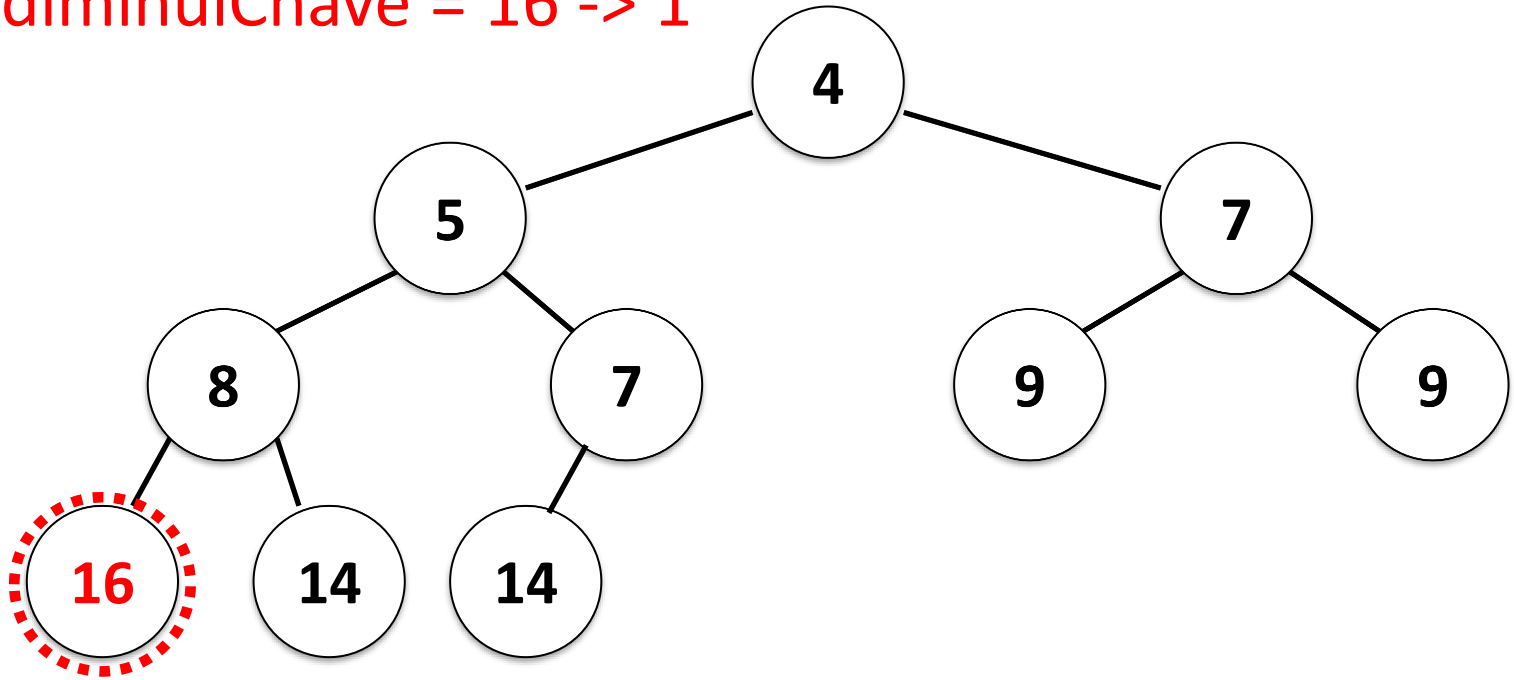
extraiMinimo = 7



Exemplo: diminuiChave?

0	1	2	3	4	5	6	7	8	9
1	5	7	8	7	9	9	16	14	14

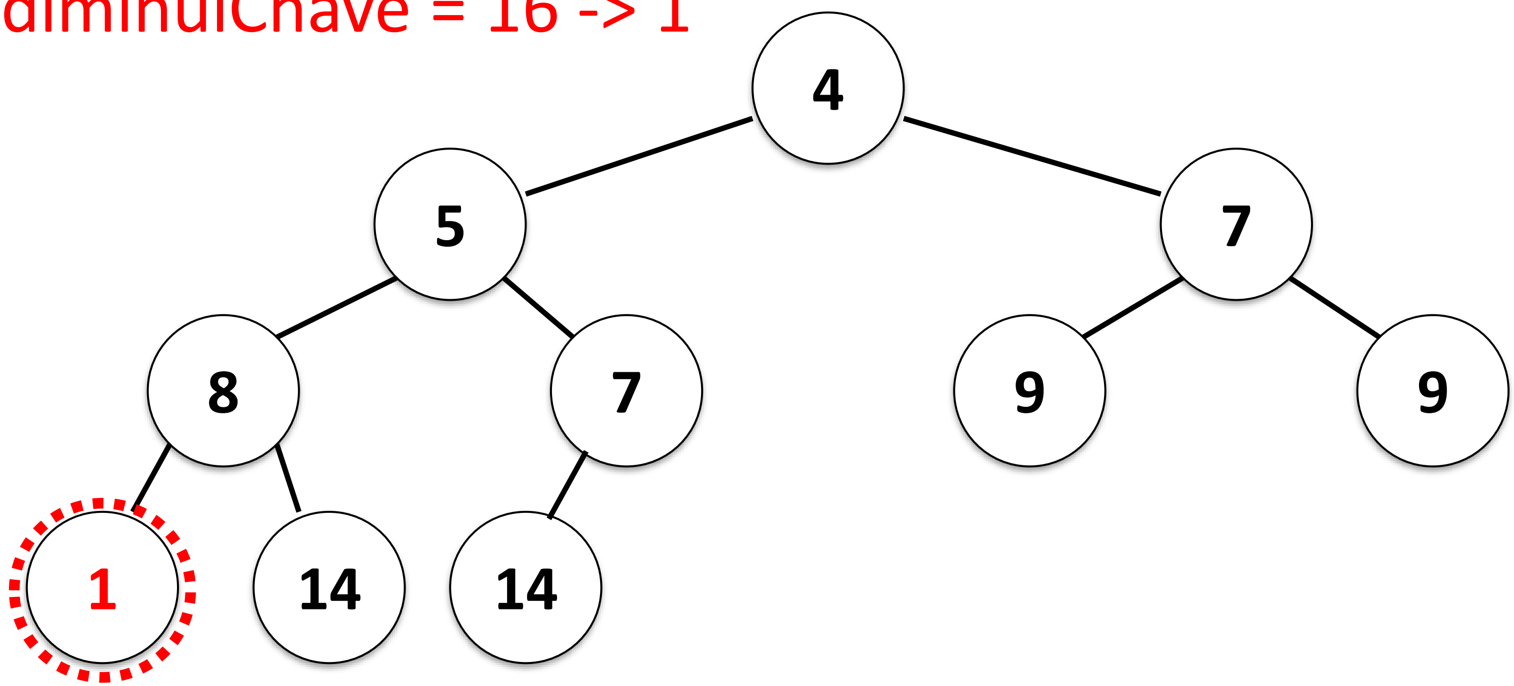
diminuiChave = 16 -> 1



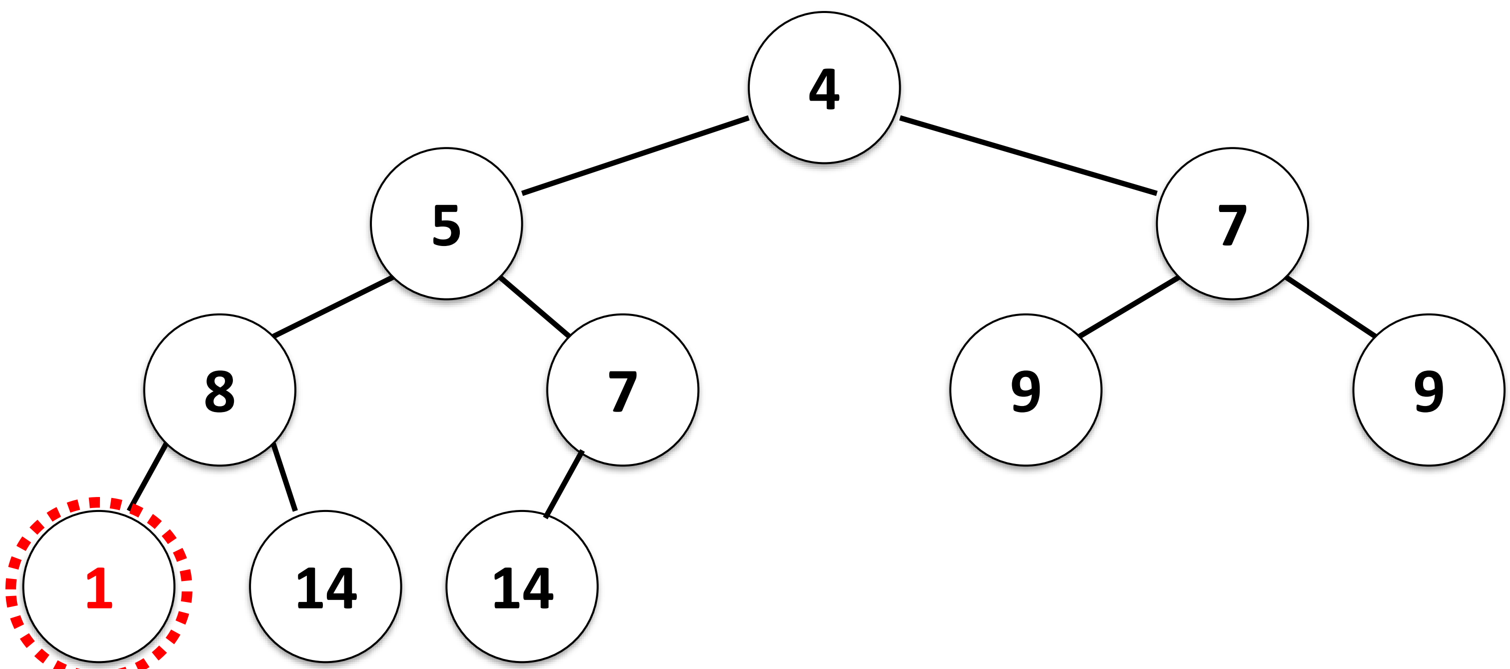
Exemplo: diminuiChave?

0	1	2	3	4	5	6	7	8	9
1	5	7	8	7	9	9	1	14	14

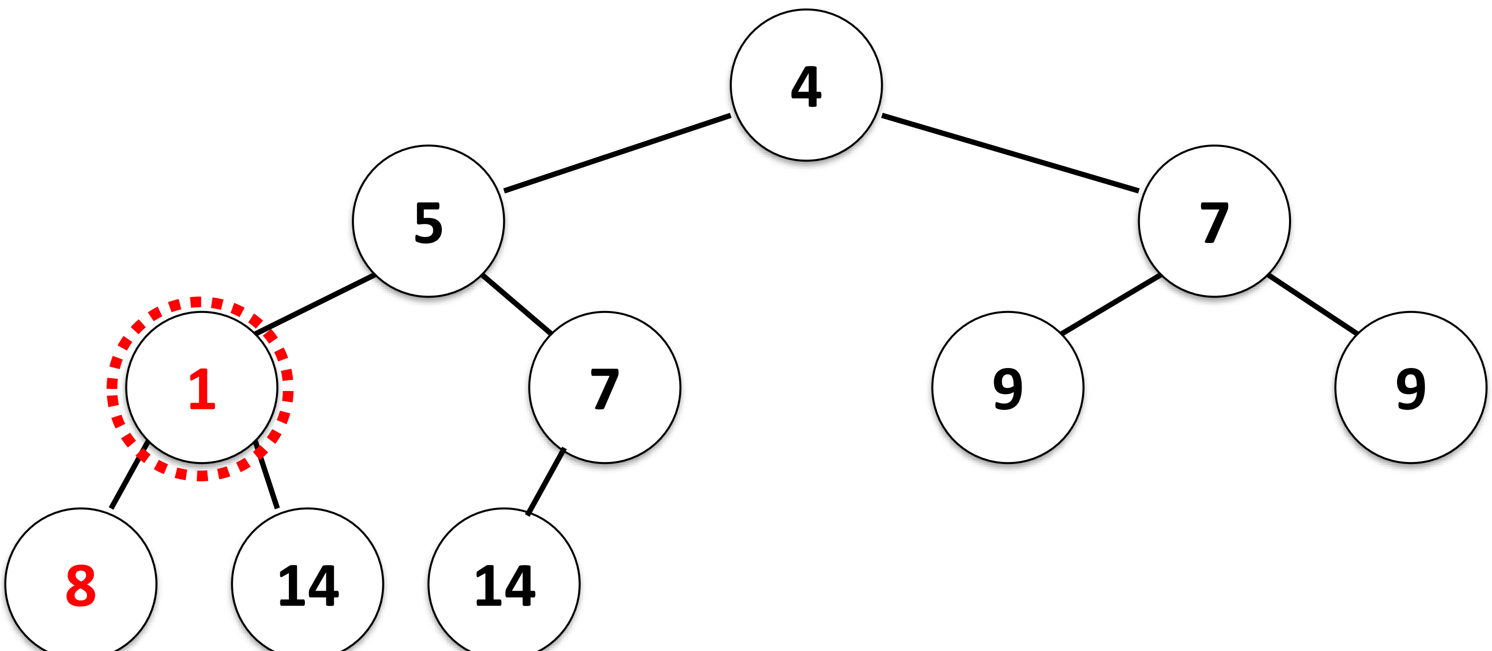
diminuiChave = 16 -> 1



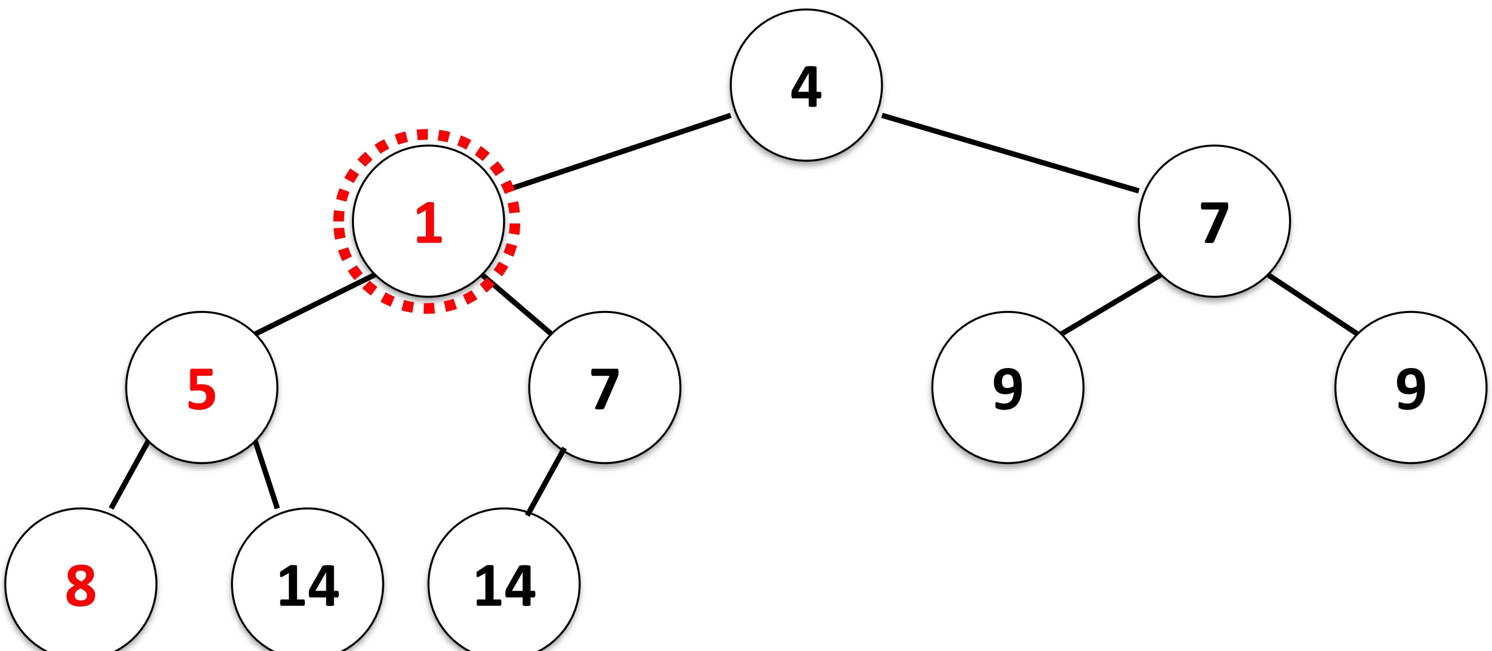
Sobe chave



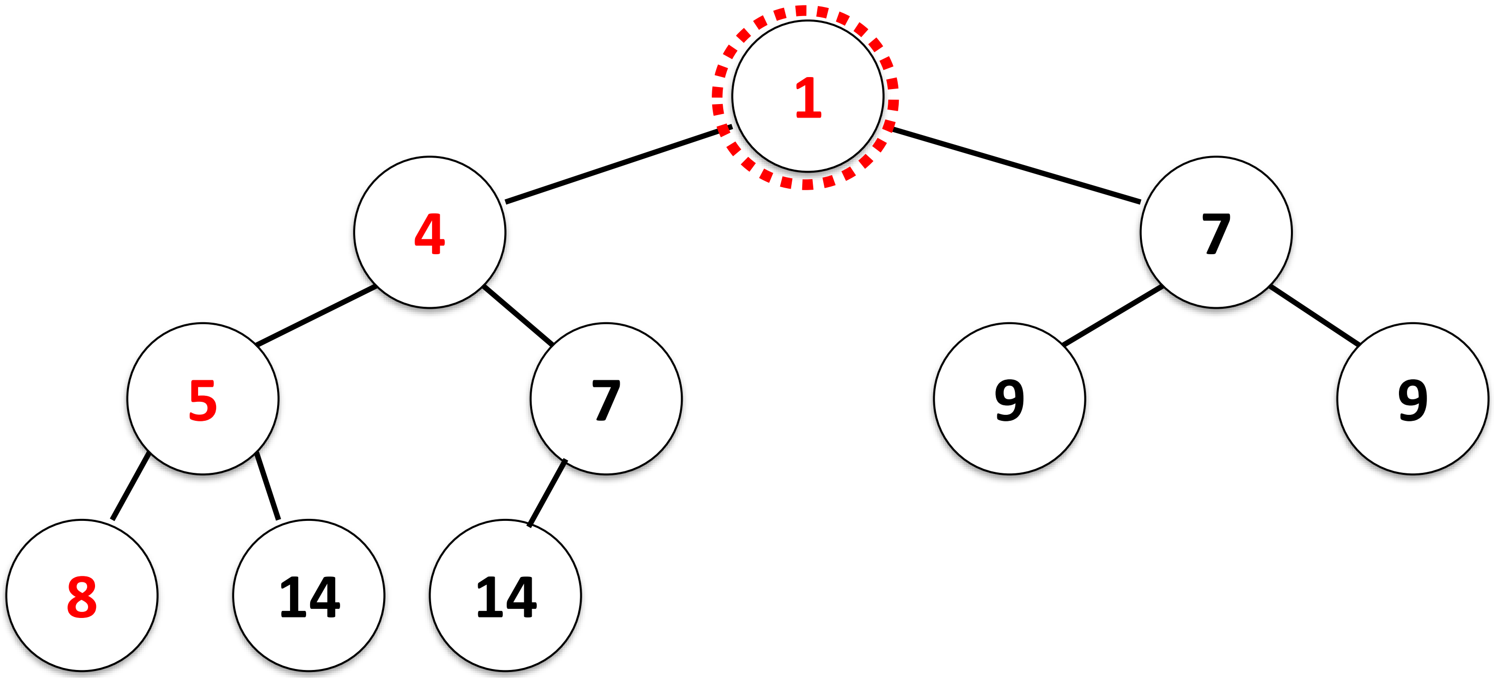
Sobe chave



Sobe chave



Sobe chave



Consumo de Tempo

- constroiMinHeap( ) consome $O(n)$ unidades de tempo.

- extraiMinimo( ) consome $O(\log n)$ unidades de tempo.

- diminuiChave( ) consome $O(\log n)$ unidades de tempo.

Dijkstra(**G**, **w**, **s**):	**Consumo de tempo:**
para cada vértice v_i em G.V faça	**O(n)**
v_i.**d** = INFINITO	**O(n)**
v_i.**p** = NIL	**O(n)**
s.d = 0	**O(1)**
Q = G.V	**O(n)**
enquanto **Q** != VAZIO faça	**O(n)**
u_i = **ExtraiMinimo**(**Q**)	**O(n)*O(log n)**
para cada v_i em G.**Adj**[u_i] faça	**O(m)**
se u_i.**d** + w(u,v) < v_i.**d**	**O(m)**
entao v_i.**d** = u_i.**d** + w(u,v)	**O(m)**
DiminuiChave(**Q**, v_i)	**O(m)*O(log n)**
v_i.**p** = u	**O(m)**

Total: T(n,m) = 5*O(n) + O(n log n) + 4*O(m) + O(m log n)
= O(n log n + m log n)

= O((n + m) * log n)

Hoje

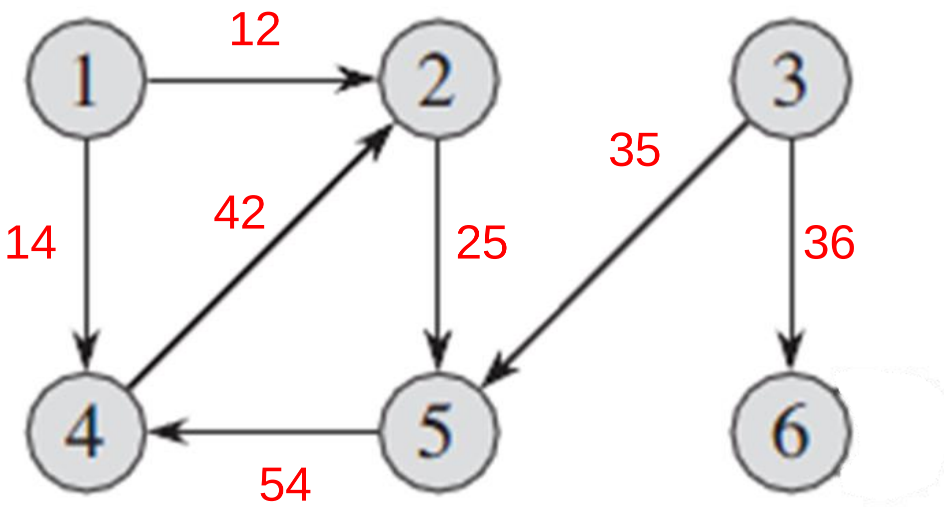
- Algoritmo de Floyd-Warshall
  – Cálculo da matriz de distâncias

- Pré-requisitos:
  – Grafo ponderado
  – Matriz **W**

Hoje

- Algoritmo de Floyd-Warshall
  - Cálculo da matriz de distâncias

- Grafo ponderado

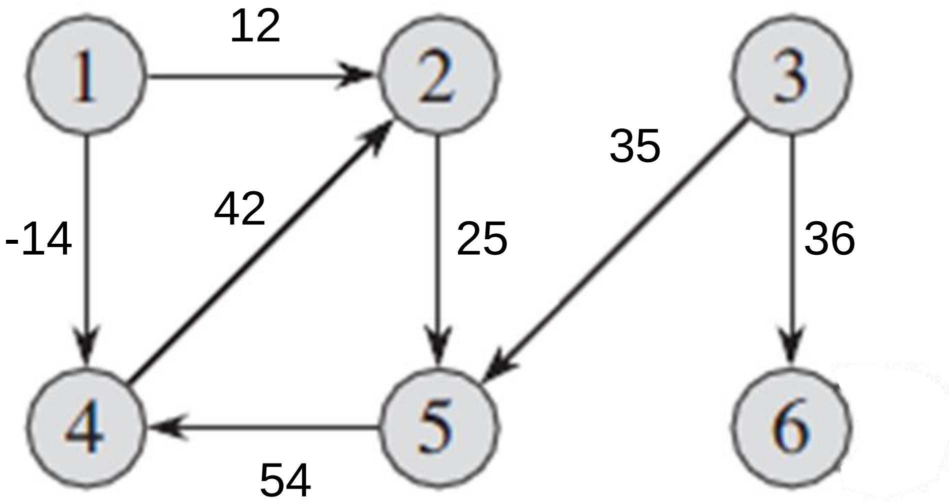


	1	2	3	4	5	6
1	0	12	0	14	0	0
2	0	0	0	0	25	0
3	0	0	0	0	35	36
4	0	42	0	0	0	0
5	0	0	0	54	0	0
6	0	0	0	0	0	0

Hoje

- Algoritmo de Floyd-Warshall
  - Cálculo da matriz de distâncias

- Matriz **W**



	1	2	3	4	5	6
1	0	12	∞	-14	∞	∞
2	∞	0	∞	∞	25	∞
3	∞	∞	0	∞	35	36
4	∞	42	∞	0	∞	∞
5	∞	∞	∞	54	0	∞
6	∞	∞	∞	∞	∞	0

Entrada: matriz **W**

Saída: matriz de distâncias

FLOYD-WARSHALL(W)

$n = W.rows$
$D^{(0)} = W$
for $k = 1$ **to** n
    let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix
    for $i = 1$ **to** n
        for $j = 1$ **to** n
            $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$
return $D^{(n)}$

Floyd-Warshall

- Matrizes

$$para\ k := 1\ at\acute{e}\ n$$

$$D^{(0)} \rightarrow D^{(1)} \rightarrow D^{(2)} \rightarrow D^{(3)} \rightarrow \ldots \rightarrow D^{(n)}$$

(W)

Matriz de distâncias

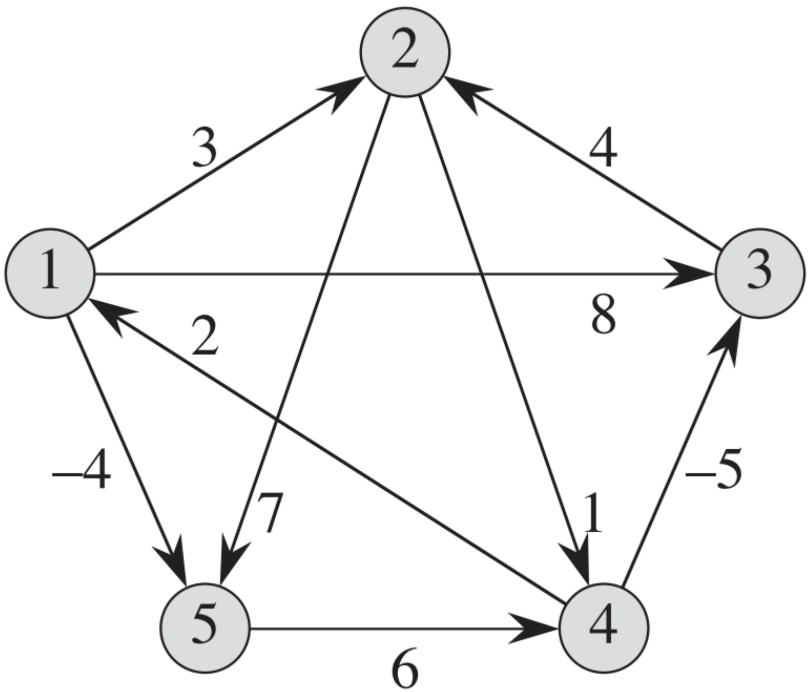
Entrada: matriz **W**

Saída: matriz de distâncias

FLOYD-WARSHALL(W)

$n = W.rows$
$D^{(0)} = W$
for $k = 1$ **to** n
    let $D^{(k)} = \left(d_{ij}^{(k)}\right)$ be a new $n \times n$ matrix
    for $i = 1$ **to** n
        for $j = 1$ **to** n
            $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$
return $D^{(n)}$

Exemplo



$D^0 =$

	1	2	3	4	5
1	**0**	3	8	∞	-4
2	∞	**0**	∞	1	7
3	∞	4	**0**	∞	∞
4	2	∞	-5	**0**	∞
5	∞	∞	∞	6	**0**

Entrada: matriz **W**

Saída: matriz de distâncias

FLOYD-WARSHALL(W)

$n = W.rows$
$D^{(0)} = W$
for $k = 1$ **to** n
    let $D^{(k)} = \left(d_{ij}^{(k)}\right)$ be a new $n \times n$ matrix
    for $i = 1$ **to** n
        for $j = 1$ **to** n
            $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$
return $D^{(n)}$

FLOYD-WARSHALL(W)

$n = W.rows$

$D^{(0)} = W$

for $k = 1$ **to** n

  let $D^{(k)} = \left(d_{ij}^{(k)}\right)$ be a new $n \times n$ matrix

  for $i = 1$ **to** n

    for $j = 1$ **to** n

      $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$

return $D^{(n)}$

Exemplo

$D^{(0)}$:

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

$D^{(1)}$:

	1	**2**	3	4	**5**
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	**5**	-5	0	**-2**
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

Exemplo

$$D^{(0)}: \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 3 & 8 & \infty & -4 \\ 2 & \infty & 0 & \infty & 1 & 7 \\ 3 & \infty & 4 & 0 & \infty & \infty \\ 4 & 2 & \infty & -5 & 0 & \infty \\ 5 & \infty & \infty & \infty & 6 & 0 \end{array} \qquad D^{(1)}: \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 3 & 8 & \infty & -4 \\ 2 & \infty & 0 & \infty & 1 & 7 \\ 3 & \infty & 4 & 0 & \infty & \infty \\ 4 & 2 & 5 & -5 & 0 & -2 \\ 5 & \infty & \infty & \infty & 6 & 0 \end{array}$$

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$$k = 1 \qquad d_{42}^{(1)} = min\left(d_{42}^{(0)}, d_{41}^{(0)} + d_{12}^{(0)}\right)$$

$$\infty \quad ? \; + \; ?$$

Exemplo

$D^{(0)}$:

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

$D^{(1)}$:

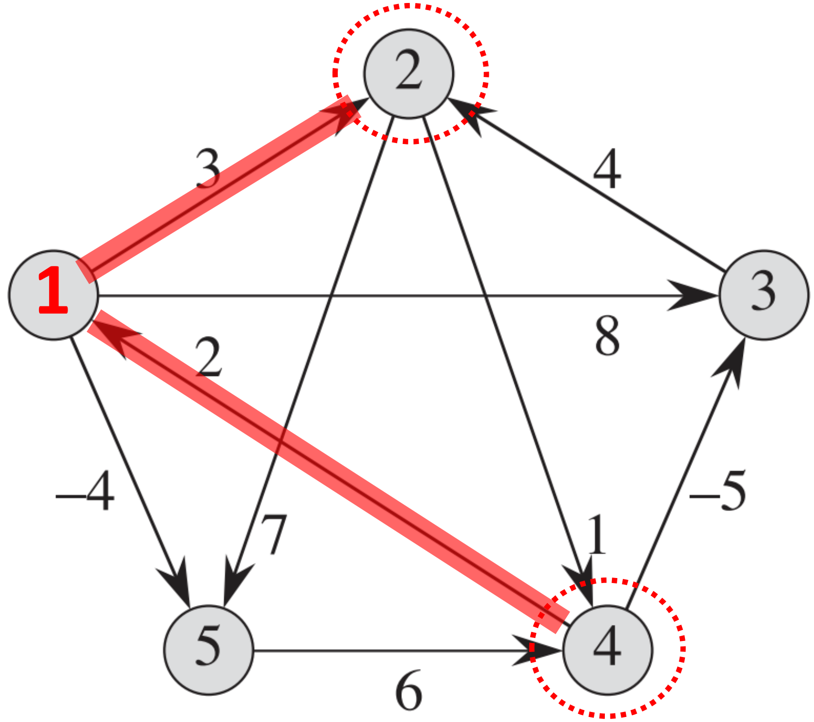
	1	**2**	3	4	**5**
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	**5**	-5	0	**-2**
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 1$

$$d_{42}^{(1)} = min\left(d_{42}^{(0)}, d_{41}^{(0)} + d_{12}^{(0)}\right)$$

$$\infty \quad 2 + 3$$



$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 1$

$$d_{42}^{(1)} = min\left(d_{42}^{(0)}, d_{41}^{(0)} + d_{12}^{(0)}\right)$$

$$\infty \qquad 2 \; + \; 3$$

Exemplo

$D^{(0)}$:

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

$D^{(1)}$:

	1	**2**	3	4	**5**
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	**5**	-5	0	**-2**
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 1$

$$d_{45}^{(1)} = min\left(d_{45}^{(0)}, d_{41}^{(0)} + d_{15}^{(0)}\right)$$

$$\infty \quad ? \quad + \quad ?$$

Exemplo

$D^{(0)}$:

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

$D^{(1)}$:

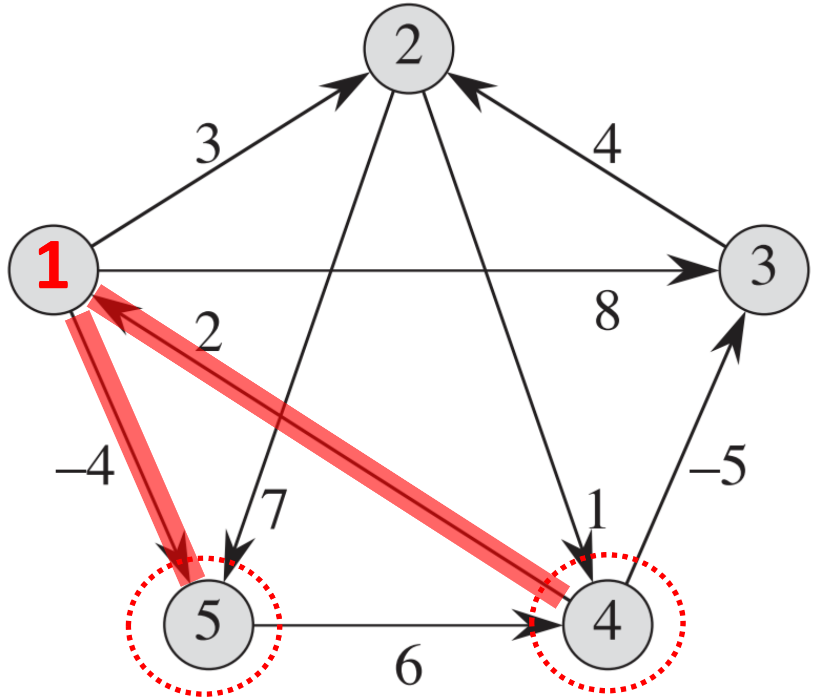
	1	**2**	3	4	**5**
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	**5**	-5	0	**-2**
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 1$

$$d_{45}^{(1)} = min\left(d_{45}^{(0)}, d_{41}^{(0)} + d_{15}^{(0)}\right)$$

$$\infty \quad 2 \quad + \quad -4$$



$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 1$ $$d_{45}^{(1)} = min\left(d_{45}^{(0)}, d_{41}^{(0)} + d_{15}^{(0)}\right)$$

$$\infty \quad 2 \quad + \quad -4$$

Floyd-Warshall

- Matrizes

$$D^{(0)} \rightarrow \boxed{D^{(1)} \rightarrow D^{(2)}} \rightarrow D^{(3)} \rightarrow D^{(4)} \rightarrow D^{(5)}$$

(W)

Exemplo

$D^{(1)}$:

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$D^{(2)}$:

	1	2	3	4	5
1	0	3	8	**4**	-4
2	∞	0	∞	1	7
3	∞	4	0	**5**	**11**
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

Exemplo

$D^{(1)}$:

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$D^{(2)}$:

	1	2	3	4	5
1	0	3	8	**4**	-4
2	∞	0	∞	1	7
3	∞	4	0	**5**	**11**
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 2$

$$d_{14}^{(2)} = min\left(d_{14}^{(1)}, d_{12}^{(1)} + d_{24}^{(1)}\right)$$

$$\infty \quad ? \quad + \quad ?$$

Exemplo

$D^{(1)}$:

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$D^{(2)}$:

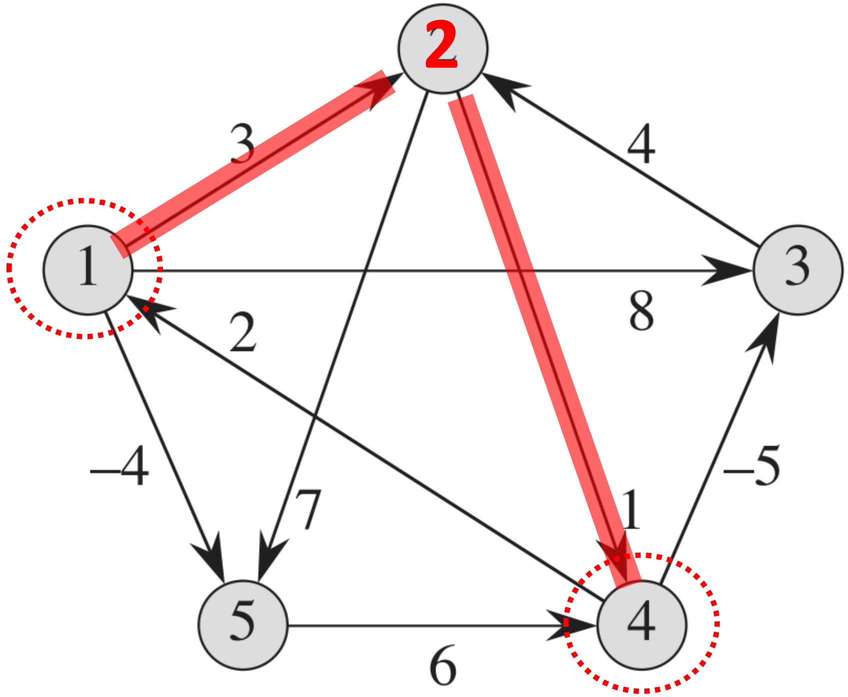
	1	2	3	4	5
1	0	3	8	**4**	-4
2	∞	0	∞	1	7
3	∞	4	0	**5**	**11**
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 2$

$$d_{14}^{(2)} = min\left(d_{14}^{(1)}, d_{12}^{(1)} + d_{24}^{(1)}\right)$$

$$\infty \quad 3 \quad + \quad 1$$



$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 2$ $$d_{14}^{(2)} = min\left(d_{14}^{(1)}, d_{12}^{(1)} + d_{24}^{(1)}\right)$$

$$\infty \quad 3 \quad + \quad 1$$

Exemplo

$D^{(1)}$:

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$D^{(2)}$:

	1	2	3	4	5
1	0	3	8	**4**	-4
2	∞	0	∞	1	7
3	∞	4	0	**5**	**11**
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 2$

$$d_{34}^{(2)} = min\left(d_{34}^{(1)}, d_{32}^{(1)} + d_{24}^{(1)}\right)$$

$$\infty \quad ? \quad + \quad ?$$

Exemplo

$D^{(1)}$:

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$D^{(2)}$:

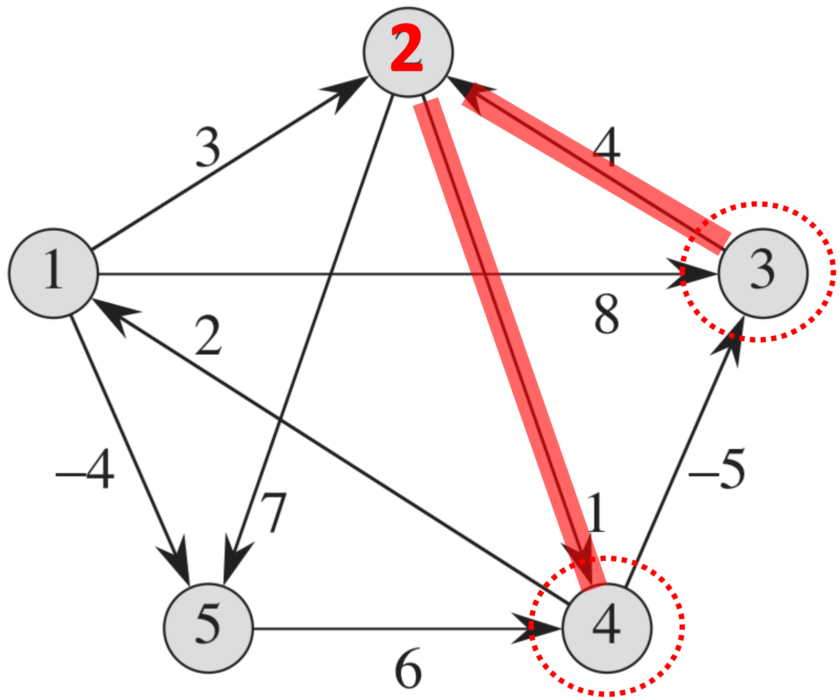
	1	2	3	4	5
1	0	3	8	**4**	-4
2	∞	0	∞	1	7
3	∞	4	0	**5**	**11**
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 2$

$$d_{34}^{(2)} = min\left(d_{34}^{(1)}, d_{32}^{(1)} + d_{24}^{(1)}\right)$$

$$\infty \quad 4 \quad + \quad 1$$



$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 2$ $$d_{34}^{(2)} = min\left(d_{34}^{(1)}, d_{32}^{(1)} + d_{24}^{(1)}\right)$$

$$\infty \quad 4 \quad + \quad 1$$

Exemplo

$D^{(1)}$:

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$D^{(2)}$:

	1	2	3	4	5
1	0	3	8	**4**	-4
2	∞	0	∞	1	7
3	∞	4	0	**5**	**11**
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 2$

$$d_{35}^{(2)} = min\left(d_{35}^{(1)}, d_{32}^{(1)} + d_{25}^{(1)}\right)$$

$$\infty \quad ? \quad + \quad ?$$

Exemplo

$D^{(1)}$:

	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$D^{(2)}$:

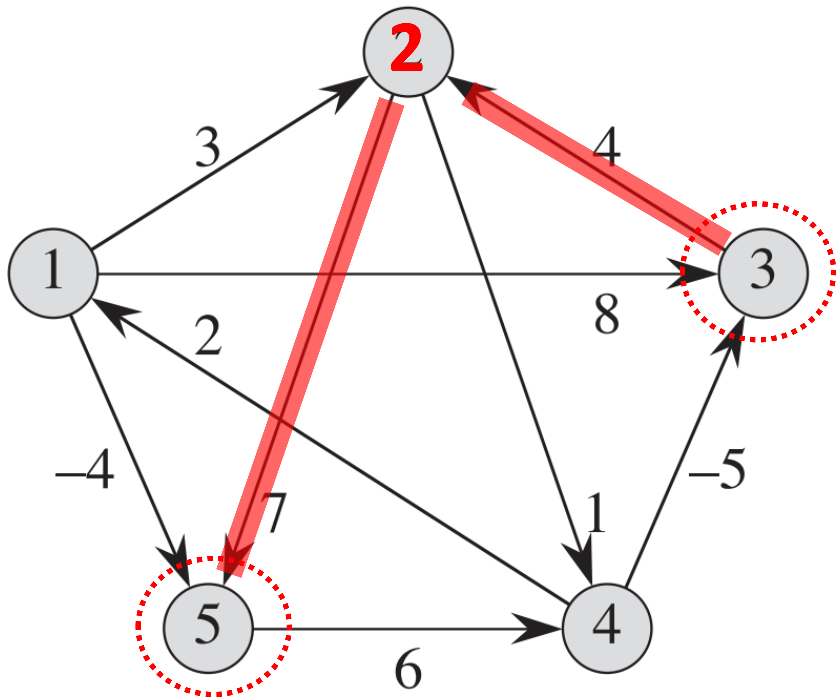
	1	2	3	4	5
1	0	3	8	**4**	-4
2	∞	0	∞	1	7
3	∞	4	0	**5**	**11**
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 2$

$$d_{35}^{(2)} = min\left(d_{35}^{(1)}, d_{32}^{(1)} + d_{25}^{(1)}\right)$$

$$\infty \quad 4 \quad + \quad 7$$



$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 2$

$$d_{35}^{(2)} = min\left(d_{35}^{(1)}, d_{32}^{(1)} + d_{25}^{(1)}\right)$$

$$\infty \quad 4 \quad + \quad 7$$

Floyd-Warshall

- Matrizes

$$D^{(0)} \rightarrow D^{(1)} \rightarrow \boxed{D^{(2)} \rightarrow D^{(3)}} \rightarrow D^{(4)} \rightarrow D^{(5)}$$

$$(W)$$

Exemplo

$D^{(2)}$:

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$D^{(3)}$:

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	**-1**	-5	0	-2
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

Exemplo

$D^{(2)}$:

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$D^{(3)}$:

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	**-1**	-5	0	-2
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 3$

$$d_{42}^{(3)} = min\left(d_{42}^{(2)}, d_{43}^{(2)} + d_{32}^{(2)}\right)$$

$$5 \quad ? \quad + \quad ?$$

Exemplo

$D^{(2)}$:

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	5	-5	0	-2
5	∞	∞	∞	6	0

$D^{(3)}$:

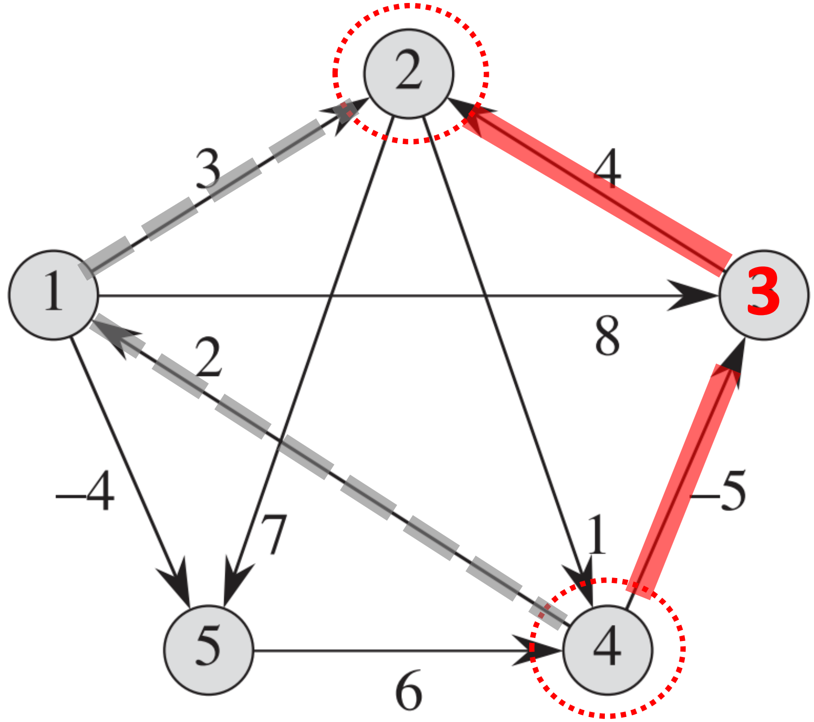
	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	**-1**	-5	0	-2
5	∞	∞	∞	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 3$

$$d_{42}^{(3)} = min\left(d_{42}^{(2)}, d_{43}^{(2)} + d_{32}^{(2)}\right)$$

$$5 \quad -5 + 4$$



$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 3$

$$d_{42}^{(3)} = min\left(d_{42}^{(2)}, d_{43}^{(2)} + d_{32}^{(2)}\right)$$

$$5 \qquad -5 + 4$$

Floyd-Warshall

- Matrizes

$$D^{(0)} \rightarrow D^{(1)} \rightarrow D^{(2)} \rightarrow \boxed{D^{(3)} \rightarrow D^{(4)}} \rightarrow D^{(5)}$$

$$(W)$$

Exemplo

$D^{(3)}$:

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	-1	-5	0	-2
5	∞	∞	∞	6	0

$D^{(4)}$:

	1	2	3	4	5
1	0	3	**-1**	4	-4
2	**3**	0	**-4**	1	**-1**
3	**7**	4	0	5	**3**
4	2	-1	-5	0	-2
5	**8**	**5**	**1**	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

Exemplo

$D^{(3)}$:

	1	2	3	4	5
1	0	3	8	4	-4
2	∞	0	∞	1	7
3	∞	4	0	5	11
4	2	-1	-5	0	-2
5	∞	∞	∞	6	0

$D^{(4)}$:

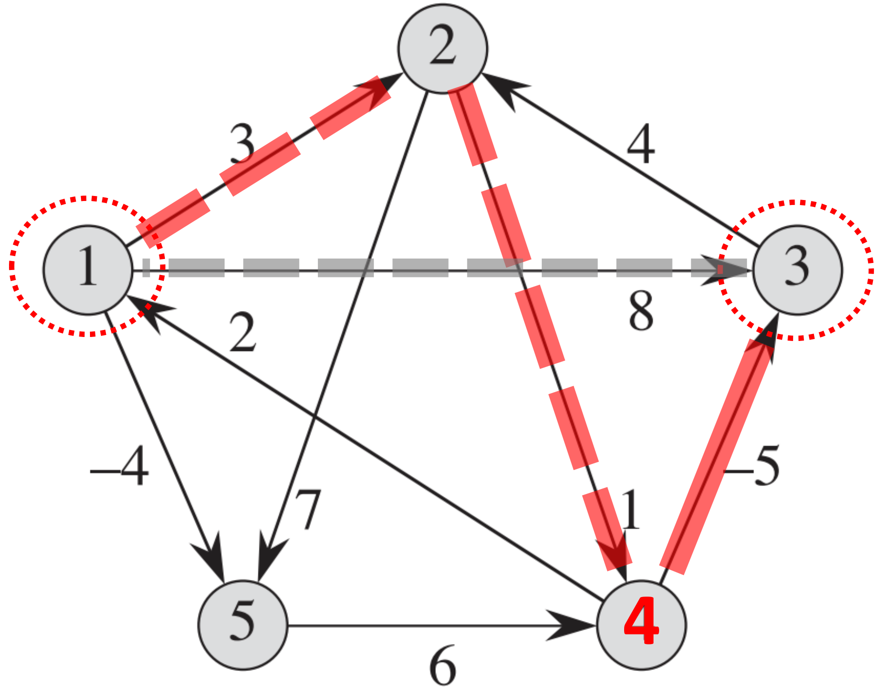
	1	2	3	4	5
1	0	3	**-1**	4	-4
2	**3**	0	**-4**	1	**-1**
3	**7**	4	0	5	**3**
4	2	-1	-5	0	-2
5	**8**	**5**	**1**	6	0

$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 4$

$$d_{13}^{(4)} = min\left(d_{13}^{(3)}, d_{14}^{(3)} + d_{43}^{(3)}\right)$$

$$8 \quad 4 \; + -5$$



$$d_{ij}^{(k)} = min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$k = 4$ $$d_{13}^{(4)} = min\left(d_{13}^{(3)}, d_{14}^{(3)} + d_{43}^{(3)}\right)$$

8 4 + −5

Floyd-Warshall

- Matriz de caminhos mínimos:

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)} , \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} . \end{cases}$$

Floyd-Warshall

$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(0)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(1)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

Floyd-Warshall

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \boxed{\infty} & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \boxed{\infty} & \boxed{\infty} \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(1)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \boxed{\text{NIL}} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \boxed{\text{NIL}} & \boxed{\text{NIL}} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & \boxed{4} & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \boxed{5} & \boxed{11} \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(2)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \boxed{2} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \boxed{2} & \boxed{2} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

Floyd-Warshall

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & \boxed{5} & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(2)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & \boxed{1} & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & \boxed{-1} & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & \boxed{3} & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

Floyd-Warshall

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(4)} = \begin{pmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

Exercício Programa

- 11-floydWarshall.py

Exercício

- Simule o algoritmo de Floyd-Warshall: