TD 1 : Equations différentielles

https://s3-us-west-2.amazonaws.com/secure.notion-static.com/462a7fe0-c9d4-41ee-b145-b2873d34d5e5/S2_2023_TD2_EquaDiff_Etudiants.pdf

Exercice 1 : Equations linéaires du premier ordre

▼ 1. $xy' - 2y = 0$ sur $\mathbb{R}^{+*}$

$(E) : xy' - 2y = 0$

$I = \mathbb{R}^{+*}$

Rq : (E) *est homogene donc pas d'etape 2.*

Etape 1 : Resolution de $(E) : xy' - 2y = 0$

$y_0(x) = ke^{-\int \frac{b(x)}{a(x)} dx} = ke^{-\int \frac{-2}{x} dx} = ke^{ln(|x|)} = kx^2$

donc $y_0(x) = kx^2$, $k \in \mathbb{R}$

Etape 3 : Conclusion

$$S = S_0 = \left\{ \begin{array}{l} \mathbb{R}^{+*} \longrightarrow \mathbb{R} \\ x \longmapsto kx^2 \end{array} ; k \in \mathbb{R} \right\}$$

▼ 2. $(x^2 + 1)y' - y = 1$ sur $\mathbb{R}$

$(E) : (x^2 + 1)y' - y = 1$

$I = \mathbb{R}$

Etape 1 : Resolution de (E_0) : $(x^2+1)y' - y = 1$

$y_0(x) = ke^{-\int \frac{b(x)}{a(x)}dx} = ke^{-\int \frac{-1}{x^2+1}dx} = ke^{Arctan(x)}$

donc $y_0(x) = ke^{Arctan(x)}$, $k \in \mathbb{R}$

Etape 2 : Solution particuliere de (E)

$y_p = -1$ solution evidente de (E)

Etape 3 : Conclusion

$$S = \left\{ \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto ke^{Arctan(x)} - 1 \end{array} ; k \in \mathbb{R} \right\}$$

▼ 3. $x\ ln(x)y' - y = 4$ sur $]-1, +\infty[$

(E) : $x\ ln(x)y' - y = 4$

$I =]-1, +\infty[$

Etape 1 : Resolution de (E_0) : (E) : $x\ ln(x)y' - y = 4$

$y_0(x) = ke^{-\int \frac{b(x)}{a(x)}dx} = ke^{-\int \frac{-1}{x\ ln(x)}dx} = ke^{ln(|ln(x)|)} = k|ln(x)| = \alpha\ ln(x)$

donc $y_0(x) = \alpha\ ln(x)$, $\alpha \in \mathbb{R}$

Etape 2 : Solution particuliere de (E)

$y_p = -4$ solution evidente de (E)

Etape 3 : Conclusion

$$S = \left\{ \begin{array}{l}]-1,+\infty[\longrightarrow \mathbb{R} \\ x \longmapsto \alpha\, ln(x) - 4 \end{array} ; \alpha \in \mathbb{R} \right\}$$

▼ 4. $y' + y = e^x - 1$ sur $\mathbb{R}$

$(E) : y' + y = e^x - 1$

$I = \mathbb{R}$

Etape 1 : Resolution de $(E_0) : y' + y = 0$

$$y_0(x) = ke^{-\int \frac{b(x)}{a(x)}dx} = ke^{-\int \frac{1}{1}dx} = ke^{-x}$$

donc $y_0(x) = ke^{-x}$, $k \in \mathbb{R}$

Etape 2 : Solution particulière de (E)

On cherche une solution particulière (SP) de (E). On cherche y_p de la forme :

$$y_p(x) = k(x)e^{-x}$$
$$y_p'(x) = k'(x)e^{-x} - k(x)e^{-x}$$

y_p sol de (E)

$$\iff y_p'(x) + y_p(x) = e^x - 1$$
$$\iff k'(x)e^{-x} - k(x)e^{-x} + k(x)e^{-x} = e^x - 1$$
$$\iff k'(x)e^{-x} = e^x - 1$$
$$\iff k'(x) = \frac{e^x - 1}{e^{-x}}$$
$$\iff k'(x) = (e^x - 1)e^x$$
$$\iff k'(x) = e^{2x} - e^x$$

Prenons $k(x) = \int e^{2x} - e^x dx = \frac{1}{2}e^{2x} - e^x$

Ainsi $y_p(x) = (\frac{1}{2}e^{2x} - e^x)e^{-x} = \frac{e^x}{2} - 1$

Etape 3 : Conclusion

$$S = \left\{ \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto ke^{-x} + \frac{e^x}{2} - 1 \end{array} ; k \in \mathbb{R} \right\}$$

▼ 5. $y' - 2xy = (1-2x)e^x$ sur $\mathbb{R}$

(E) : $y' - 2xy = (1-2x)e^x$

$I = \mathbb{R}$

Etape 1 : Resolution de (E_0) : $y' - 2xy = 0$

$y_0(x) = ke^{-\int \frac{b(x)}{a(x)}dx} = ke^{-\int \frac{2x}{1}dx} = ke^{x^2}$

donc $y_0(x) = ke^{x^2}$, $k \in \mathbb{R}$

Etape 2 : Solution particulière de (E)

On cherche une solution particulière (SP) de (E). On cherche y_p de la forme :

$y_p(x) = k(x)e^{x^2}$

$y_p'(x) = k'(x)e^{x^2} + k(x)2xe^{x^2}$

y_p sol de (E)

$\iff y_p'(x) - 2xy_p(x) = (1-2x)e^x$

$\iff k'(x)e^{x^2} + 2xk(x)e^{x^2} - 2xk(x)e^{x^2} = (1-2x)e^x$

$\iff k'(x)e^{x^2} = (1-2x)e^x$

$\iff k'(x) = \frac{(1-2x)e^x}{e^{x^2}}$

$\iff k'(x) = (1-2x)e^{x-x^2}$

Prenons $k(x) = \int (1-2x)e^{x-x^2}dx = e^{x-x^2}$

Ainsi $y_p(x) = e^{x-x^2} \times e^{x^2} = e^x$

Etape 3 : Conclusion

$$S = \left\{ \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto ke^{x^2} + e^x \end{array} ; k \in \mathbb{R} \right\}$$

▼ 6. $y' - \frac{2y}{x+1} = (x+1)^3$ sur $]-1;+\infty[$

(E) : $y' - \frac{2y}{x+1} = (x+1)^3$

$I =]-1;+\infty[$

Etape 1 : Resolution de (E_0) : $y' - \frac{2y}{x+1} = 0$

$y_0(x) = ke^{-\int \frac{b(x)}{a(x)}dx} = ke^{-\int \frac{-2}{x+1}dx} = ke^{2\int \frac{1}{x+1}dx} = ke^{2\ln(x+1)} = k(x+1)^2$

donc $y_0(x) = k(x+1)^2$, $k \in \mathbb{R}$

Etape 2 : Solution particulière de (E)

On cherche une solution particulière (SP) de (E). On cherche y_p de la forme :

$y_p(x) = k(x) \times (x+1)^2$

$y_p'(x) = k'(x) \times (x+1)^2 + k(x) \times (2x+2)$

y_p sol de (E)

$\iff y'(x) - \frac{2y(x)}{x+1} = (x+1)^3$

$\iff k'(x) \times (x+1)^2 + k(x) \times (2x+2) - \frac{2k(x)\times(x+1)^2}{x+1} = (x+1)^3$

$\iff k'(x) \times (x+1)^2 + k(x) \times (2x+2) - k(x) \times (2x+2) = (x+1)^3$

$\iff k'(x) \times (x+1)^2 = (x+1)^3$

$\iff k'(x) = (x+1)$

Prenons $k(x) = \int x+1 \; dx = \frac{1}{2}x^2 + x$

Ainsi $y_p(x) = (\frac{1}{2}x^2 + x)(x+1)^2$

<u>Etape 3 :</u> Conclusion

$$S = \left\{ \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto (\frac{1}{2}x^2 + x)(x+1)^2 + k(x+1)^2 \end{array} ; k \in \mathbb{R} \right\}$$

▼ 7. $(1+x^2)y' + xy = 3x^3 + 3x$ sur $\mathbb{R}$

(E) : $(1+x^2)y' + xy = 3x^3 + 3x$

$I = \mathbb{R}$

<u>Etape 1 :</u> Resolution de (E_0) : $(1+x^2)y' + xy = 0$

$y_0(x) = ke^{-\int \frac{b(x)}{a(x)}dx} = ke^{-\int \frac{x}{1+x^2}dx} = ke^{-\frac{1}{2}\int \frac{2x}{x^2+1}dx} = ke^{-\frac{1}{2}\ln(x^2+1)} = k\frac{1}{\sqrt{x^2+1}}$

donc $y_0(x) = k\frac{1}{\sqrt{x^2+1}}$, $k \in \mathbb{R}$

<u>Etape 2 :</u> (Spéciale) Solution particulière de (E)

On cherche une solution particulière (SP) de (E). On cherche y_p de la forme :

$y_p(x) = \alpha x^2 + \beta x + \gamma \qquad (\alpha, \beta, \gamma) \in \mathbb{R}^3$

$y_p'(x) = 2\alpha x + \beta$

y_p sol de (E)

$$\iff (1+x^2)y'(x)+xy(x)=3x^3+3x$$
$$\iff (1+x^2)(2\alpha x+\beta)+x(\alpha x^2+\beta x+\gamma)=3x^3+3x$$
$$\iff 2\alpha x+\beta+2\alpha x^3+\beta x^2+\alpha x^3+\beta x^2+\gamma x=3x^3+3x$$
$$\iff x^3(2\alpha+\alpha)+x^2(\beta+\beta)+x(2\alpha+\gamma)+\beta=3x^3+3x$$
$$\iff x^3(3\alpha)+x^2(2\beta)+x(2\alpha+\gamma)+\beta=3x^3+3x$$

$$\iff \begin{cases} 3\alpha=3 \\ 2\beta=0 \\ 2\alpha+\gamma=0 \\ \beta=0 \end{cases} \iff \begin{cases} \alpha=1 \\ \beta=0 \\ 2\times 1+\gamma=3 \end{cases} \iff \begin{cases} \alpha=1 \\ \beta=0 \\ \gamma=1 \end{cases}$$

Ainsi $y_p(x)=x^2+1$

Etape 3 : Conclusion

$$S=\left\{ \begin{array}{l} \mathbb{R}\longrightarrow\mathbb{R} \\ x\longmapsto k\frac{1}{\sqrt{x^2+1}}+x^2+1 \end{array} ;k\in\mathbb{R}\right\}$$

▼ 8. $cos(t)y'-sin(t)=1$ sur $]-\frac{\pi}{2};+\frac{\pi}{2}[$

(E) : $cos(t)y'-sin(t)=1$

$I=]-\frac{\pi}{2};+\frac{\pi}{2}[$

Etape 1 : Resolution de (E_0) : $cos(t)y'-sin(t)=0$

$$y_0(x)=ke^{-\int\frac{b(x)}{a(x)}dx}=ke^{-\int\frac{\sin(t)}{\cos(t)}dx}=ke^{-ln(|\cos t|)}=\frac{k}{\cos t}$$

donc $y_0(x)=\frac{k}{\cos t}$, $k\in\mathbb{R}$

Etape 2 : Solution particulière de (E)

On cherche une solution particulière (SP) de (E). On cherche y_p de la forme :

$$y_p(x) = \frac{k}{\cos t}$$

$$y_p'(x) = \frac{k'(t)\cos t + k(t)\sin t}{\cos^2 t}$$

y_p sol de (E)

$$\iff cos(t)(y'(t)) - sin(t)y(t) = 1$$

$$\iff cos(t)\frac{k'(t)\cos t + k(t)\sin t}{\cos^2 t} - sin(t)\frac{k}{\cos t} = 1$$

$$\iff k'(t) + \frac{k(t)\sin t}{\cos t} - sin(t)\frac{k}{\cos t} = 1$$

$$\iff k'(t) = 1$$

Prenons $k(t) = \int 1 = t$

Ainsi $y_p(x) = \frac{k(t)}{\cos t}$

<u>Etape 3 :</u> Conclusion

$$S = \left\{ \begin{array}{l} \left]-\frac{\pi}{2};+\frac{\pi}{2}\right[\longrightarrow \mathbb{R} \\ t \longmapsto \frac{k}{\cos t} + \frac{t}{\cos t} \end{array} ; k \in \mathbb{R} \right\}$$

Or il faut aussi $y_0 = 2$,

or $y(0) = \frac{k}{\cos 0} + \frac{0}{\cos 0} = k$, donc $k = 2$.

Donc :

$$S = \left\{ \begin{array}{l} \left]-\frac{\pi}{2};+\frac{\pi}{2}\right[\longrightarrow \mathbb{R} \\ t \longmapsto \frac{2}{\cos t} + \frac{t}{\cos t} \end{array} \right\}$$

Exercice 3 : Equations linéaires du second ordre

▼ 1. $y'' - y' - 2y = -x^2 - 3x$

(E) : $y'' - y' - 2y = -x^2 - 3x$

$I = \mathbb{R}$

Etape 1 : Resolution de (E_0) : $y'' - y' - 2y = 0$

(C) : $r^2 - r - 2 = 0$

$r_1 = -1$ et $r_2 = 2$ sont solutions évidentes de (C)

donc $y_0(x) = k_1 e^{2x} + k_2 e^{-x}$, $(k_1, k_2) \in \mathbb{R}^2$

Etape 2 : SP de (E)

On cherche une solution sous la forme $y_p(x) = \alpha x^2 + \beta x + \gamma$, $(\alpha, \beta, \gamma) \in \mathbb{R}^3$

$y'_p(x) = 2\alpha x + \beta$

$y''p(x) = 2\alpha$

y_p sol de (E)

$$\iff y''(x) - y'(x) - 2y = -x^2 - 3x$$
$$\iff 2\alpha - 2\alpha x - \beta - 2\alpha x^2 - 2\beta x - 2\gamma = -x^2 - 3x$$
$$\iff x^2(-2\alpha) + x(-2\alpha - 2\beta) + (2\alpha - \beta - 2\gamma) = -x^2 - 3x$$

$$\iff \begin{cases} -2\alpha = -1 \\ -2\alpha - 2\beta = -3 \\ 2\alpha - \beta - 2\gamma = 0 \end{cases} \iff \begin{cases} \alpha = \frac{1}{2} \\ -2\alpha - 2\beta = -3 \\ 2\alpha - \beta - 2\gamma = 0 \end{cases}$$
$$\iff \begin{cases} \alpha = \frac{1}{2} \\ 2\beta = -2 \\ -\beta - 2\gamma = -1 \end{cases} \iff \begin{cases} \alpha = \frac{1}{2} \\ \beta = 1 \\ -2\gamma = 0 \end{cases} \iff \begin{cases} \alpha = \frac{1}{2} \\ \beta = 1 \\ \gamma = 0 \end{cases}$$

donc $y_p(x) = \frac{1}{2}x^2 + x$,

<u>Etape 3 :</u> Conclusion

$$S = \left\{ \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto k_1 e^{2x} + k_2 e^{-x} + \frac{1}{2}x^2 + x \end{array} ; (k_1, k_2) \in \mathbb{R}^2 \right\}$$

▼ 2. $y'' - 5y' + 6y = e^{2x}$

$(E) : y'' - 5y' + 6y = e^{2x}$

$I = \mathbb{R}$

<u>Etape 1 :</u> Resolution de $(E_0) : y'' - 5y' + 6y = 0$

$(C) : r^2 - 5r + 6 = 0$

$\Delta = 1$, $r_1 = 2$ et $r_2 = 3$

donc $y_0(x) = k_1 e^{2x} + k_2 e^{3x}$, $(k_1, k_2) \in \mathbb{R}^2$

<u>Etape 2 :</u> SP de (E)

On cherche une solution sous la forme $y_p(x) = Q(x)e^{2x}$

$y_p'(x) = (Q'(x) + 2Q(x))e^{2x}$

$y_p''(x) = (Q''(x) + 4Q'(x) + 4Q(x))e^{2x}$

y_p sol de (E)

$$\begin{aligned}
&\iff y_p'' - 5y_p' + 6y_p = e^{2x} \\
&\iff e^{2x}(Q'' + 4Q' + 4Q) - 5e^{2x}(Q' - 2Q) + 6e^{2x}(Q) = e^{2x} \\
&\iff e^{2x}(Q'' + 4Q' + 4Q - 5Q' - 10Q + 6Q) = e^{2x} \\
&\iff e^{2x}(Q'' - Q') = e^{2x} \\
&\iff Q'' - Q' = 1
\end{aligned}$$

Prenons $Q(x) = \alpha x + \beta$. On a $-\alpha = 1 \iff \alpha = -1$

Ainsi en prenant par exemple $\beta = 0$, $Q(x) = -x$.

d'où $y_p(x) = -xe^{2x}$.

Etape 3 : Conclusion

$$S = \left\{ \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto k_1 e^{2x} + k_2 e^{3x} - xe^{2x} \end{array} ; (k_1, k_2) \in \mathbb{R}^2 \right\}$$

▼ 3. $y'' - 4y' + 4y = xe^{2x}$

$(E) : y'' - 4y' + 4y = xe^{2x}$

$I = \mathbb{R}$

Etape 1 : Resolution de $(E_0) : y'' - 4y' + 4y = 0$

$(C) : r^2 - 4r + 4 = 0$

$\Delta = 0$, $r_1 = 2$

donc $y_0(x) = (k_1 x + k_2)e^{2x}$, $(k_1, k_2) \in \mathbb{R}^2$

Etape 2 : SP de (E)

On cherche une solution sous la forme $y_p(x) = Q(x)e^{2x}$

$y_p'(x) = (Q'(x) + 2Q(x))e^{2x}$

$y_p''(x) = (Q''(x) + 4Q'(x) + 4Q(x))e^{2x}$

y_p sol de (E)

$\iff y_p'' - 4y_p' + 4y_p = xe^{2x}$

$\iff e^{2x}(Q'' + 4Q' + 4Q - 4Q' - 8Q + 4Q) = xe^{2x}$

$\iff Q'' = x$

$\iff Q' = \frac{x^2}{2}$

$\iff Q = \frac{x^6}{6}$

d'où $y_p(x) = \frac{x^6}{6} e^{2x}$.

Etape 3 : Conclusion

$$S = \left\{ \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto (k_1 x + k_2)e^{2x} + \frac{x^6}{6}e^{2x} \end{array} ; (k_1, k_2) \in \mathbb{R}^2 \right\}$$

▼ 4. $y'' + y = e^x$

$(E) : y'' + y = e^x$

$I = \mathbb{R}$

Etape 1 : Resolution de (E_0) : $y'' + y = 0$

$(C) : r^2 + 1 = 0$

$\Delta = -4$, $r_1 = i$ et $r_2 = -i$

donc $y_0(x) = k_1 \cos(x) + k_2 \sin(x)$, $(k_1, k_2) \in \mathbb{R}^2$

Etape 2 : SP de (E)

On cherche une solution sous la forme $y_p(x) = Q(x)e^x$

$y_p'(x) = e^x(Q' + Q)$

$y_p''(x) = e^x(Q'' + 2Q' + Q)$

y_p sol de (E)

$\iff y'' + y = e^x$

$\iff e^x(Q'' + 2Q' + 2Q) = e^x$

$\iff Q'' + 2Q' + 2Q = 1$

Prenons $Q(x) = \alpha$. On a donc $\alpha = \frac{1}{2}$.

d'où $y_p(x) = \frac{1}{2}e^x$.

Etape 3 : Conclusion

$$S = \left\{ \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto k_1 \cos(x) + k_2 \sin(x) + \frac{1}{2} e^x \end{array} ; (k_1, k_2) \in \mathbb{R}^2 \right\}$$

▼ 5. $y'' + 2y' + 5y = xe^x$

$(E) : y'' + 2y' + 5y = xe^x$

$I = \mathbb{R}$

<u>Etape 1 :</u> Resolution de $(E_0) : y'' + 2y' + 5y = 0$

$(C) : r^2 + 2r + 5 = 0$

$\Delta = -16$, $r_1 = -1 + 2i$ et $r_2 = -1 - 2i$

donc $y_0(x) = k_1 \cos(2x) + k_2 \sin(2x)$, $(k_1, k_2) \in \mathbb{R}^2$

<u>Etape 2 :</u> SP de (E)

On cherche une solution sous la forme $y_p(x) = Q(x)e^x$

$y_p'(x) = e^x(Q' + Q)$

$y_p''(x) = e^x(Q'' + 2Q' + Q)$

y_p sol de (E)

$\iff y'' + 2y' + 5y = xe^x$

$\iff e^x(Q'' + 2Q' + Q + 2Q' + 2Q + 5Q) = xe^x$

$\iff Q'' + 2Q' + Q + 2Q' + 2Q + 5Q = x$

$\iff Q'' + 4Q' + 8Q = x$

Prenons $Q(x) = \alpha x + \beta$, $Q'(x) = \alpha$, $Q''(x) = 0$.

$Q'' + 4Q' + 8Q = x$

$\iff 4\alpha + 8\alpha x + 8\beta = x$

$$\iff \begin{cases} 8\alpha = 1 \\ 4\alpha - 2\beta = 0 \end{cases} \iff \begin{cases} \alpha = \frac{1}{8} \\ \beta = -\frac{1}{16} \end{cases}$$

d'où $y_p(x) = (\frac{1}{8}x - \frac{1}{16})e^x$.

<u>Etape 3 :</u> Conclusion

$$S = \left\{ \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto e^{-x}(k_1\cos(2x) + k_2\sin(2x)) + e^x(\frac{1}{8}x - \frac{1}{16}) \end{array} ; (k_1, k_2) \in \mathbb{R}^2 \right\}$$