Working Paper: Heteroskedasticity and Clustered Covariances from a Bayesian Perspective

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Abstract

We show that $\sqrt{n}$-consistent heteroskedasticity-robust and cluster-robust regression estimators and confidence intervals can be derived from fully Bayesian models of population sampling, addressing what we argue are deficiencies in current Bayesian regression models, which we review. In our model, the vexed question of how and when to "cluster" is answered by the sampling design encoded in the model: simple random sampling implies a heteroskedasticity-robust Bayesian estimator, and clustered sampling implies a cluster-robust Bayesian estimator, providing a Bayesian parallel to the work of Abadie et al. (2017). Our model is based on the Finite Dirichlet Process (FDP), a well-studied population sampling process that apparently originates with R.A. Fisher, and our findings may not be surprising to readers familiar with the frequentist properties of the closely related Bayesian Bootstrap, Dirichlet Process, and Efron "pairs" or "block" bootstraps. However, our application of FDP to robust regression is novel, and it fills a gap concerning Bayesian cluster-robust regression. Our approach has several advantages over related methods: we present a full probability model with clear assumptions about a sampling design, one that does not assume that all possible data-values have been observed (unlike many bootstrap procedures); and our posterior estimates and credible intervals can be regularized toward reasonable prior values in small samples, while achieving the desirable frequency properties of a bootstrap in moderate and large samples. However, our model also illustrates some limitations of "robust" procedures.

Perhaps the most widely used method in econometrics is the ordinary least squares (OLS) coefficient estimator with heteroskedasticity-robust or cluster-robust standard errors, which we will call "robust regression".[1] This method is frequentist, in the sense that the procedure is guaranteed to capture the truth with a certain frequency: the method produces confidence intervals which contain the true regression parameters at the nominal rate (e.g. 95% of the time). The method is also frequentist in the more technical sense that it does not require the user to specify a probability model of the entire data-generating process. [2] In particular, the outcome-variable's conditional variance may change as an arbitrary and unspecified function of the regressors ("heteroskedasticity") or may be correlated in some arbitrary and unspecified way within groups of observations ("clustered covariances"), and

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[1] Not to be confused with outlier-robust regression

[2] When applied to repeated instantiations of data — provided that an asymptotic approximation holds, which it does in many applications, even in moderate samples. The crucial assumption is that fourth mixed moments of the variables are finite.

the guarantee still holds. Arguably, such situations comprise most real-world economic settings, and such a general guarantee is highly desirable.

Frequentist procedures are often contrasted with Bayesian inference, which seeks to quantify one's rational beliefs given observed data, using a probabilistic model of the entire data-generating process, together with a prior probability distribution over the unknown parameters. There are good reasons to take this claim of rationality seriously[3], but also to hope that Bayesian methods could have some additional frequentist guarantees of reliability; in many cases, they do. Unfortunately, prominent and knowledgeable authors have concluded that Bayesian methods cannot straightforwardly accommodate robust regression, because the former relies on a full probability model and the latter does not (Sims, 2010). Certainly, no fully Bayesian counterpart to robust regression is widely used, and existing papers (of which we are aware) that offer Bayesian interpretations of robust regression either do not prove a precise and explicit correspondence to the robust estimators, or substantially depart from the standard Bayesian framework (depart from a full probability model, proper priors, Bayesian posterior distribution).

This apparent divergence between Bayesian and frequentist methods ought to concern "frequentist Bayesians" who would like their 95% credible intervals to contain the true parameter-values roughly 95% of the time, or at all (as many seem to expect); but also "Bayesian frequentists" who would like to interpret their confidence intervals as rational beliefs or inferences given the observed data (as virtually all do). It especially ought to concern applied researchers who are impatient with the above Bayes-frequentist quibbling, and who would like their conclusions to be independent of their philosophical foundations.

In the first part of this paper, we discuss how conventional Bayesian approaches to heteroskedasticity in linear regression either effectively assume that it does not exist, or they impose assumptions about the functional form of heteroskedasticity that we often have few ways to verify and good reasons to doubt. Evaluating these Bayesian approaches from a frequentist perspective using simulations and theory, we show that incorrect modeling assumptions about heteroskedasticity can lead to inferences about the regression parameters that are not only misleading (biased and asymptotically inconsistent), but also dogmatic (with low coverage rates), and often worsen as data increases.

In the second part of this paper, we address some of the above concerns by constructing fully Bayesian regression models, based on simple population sampling, whose point estimates converge to ordinary least squares (OLS) estimates and whose credible intervals converge to valid heteroskedasticity-robust or cluster-robust confidence intervals as data increases, assuring good frequentist performance for a broad range of priors and patterns of heteroskedasticity or clustered correlations, under relatively mild regularity conditions on the data-generating process.

Here is a summary of our main result. We have the following convergence in posterior distribution, almost surely under the true data-generating process as $n \longrightarrow \infty$:

$$\sqrt{n}\left(\beta_n - \hat{\beta}_n\right)|\mathbf{y}_n, \mathbf{X}_n \xrightarrow{d} \mathcal{N}\left(0, \lim_{n\to\infty} n\hat{\mathbb{V}}_n\right) \tag{1}$$

where

$$\hat{\mathbb{V}}_n = \left(\sum_{i=1}^{n} X_i^\top X_i\right)^{-1}\left[\sum_{i=1}^{n} X_i^\top \hat{\epsilon}_i \hat{\epsilon}_i^\top X_i\right]\left(\sum_{i=1}^{n} X_i^\top X_i\right)^{-1} \tag{2}$$

[3] For proofs that confidence intervals display strange pathologies only if they are non-Bayesian, see Casella (1992) and Müller and Norets (2016)

Informally, the above expression says that the Bayesian regression coefficients β_n given data $(\mathbf{y}_n, \mathbf{X}_n)$ are asymptotically distributed as a Gaussian random variable whose mean is the OLS coefficients $\hat{\beta}_n$, and whose covariance is a standard robust covariance estimator constructed from OLS residuals $\hat{\epsilon}$ and regressors X_i. Convergence in distribution does not imply convergence of moments, but under further regularity assumptions much like those in White (1980b), this can be strengthened into consistency of the posterior mean and posterior covariance of β_n.

Our model is based on the Finite Dirichlet Process (FDP), which was first described in a modern Bayesian context by Pitman (1996), who attributes the model to R.A. Fisher; important work was subsequently done by Ishwaran and Zarepour (2002) and Muliere and Secchi (2003), but not in a regression context and without reference to heteroskedasticity. Our theoretical results may not be surprising to readers familiar with the asymptotic frequency properties of the Dirichlet Process (introduced by Ferguson, 1973) and the Bayes Bootstrap (due to Rubin, 1981), to which the FDP is closely related and converges. It has been known since at least the work of Hjort (1994) and ALBERT Y Lo (1987) that both the Bayes Bootstrap and Dirichlet Process converge in various senses to the Efron (1979) "pairs" bootstrap, which is known to have heteroskedasticity robustness properties (Freedman, 1981), although as far as we are aware, no authors have used these general results from empirical process theory to explicitly connect Bayesian regression methods to heteroskedasticity and cluster-robust frequentist covariance estimators.

Our work is particularly indebted to the relatively small literature in which the Bayesian bootstrap and related methods are specifically considered for heteroskedasticity robust regression (Aitkin, 2008; Chamberlain and Imbens, 2003; Karabatsos, 2016; Lancaster, 2006; Poirier, 2011; Szpiro et al., 2010). Karabatsos (2016) is our nearest precedent, but his paper uses a mixture-of-Dirichlet-Processes model, and when discussing robustness it focuses on an improper prior in which the model becomes equivalent to the Bayes Bootstrap, and does not discuss asymptotics.

Our novel contribution is to relate a fully Bayesian population sampling model (which the Bayesian Bootstrap is not, and which the Dirichlet Process is not often described as) to both heteroskedasticity-robust and cluster-robust regression estimators (the above literature does not cover cluster-robustness) using asymptotic theory. This allows us to provide a relatively straightforward explanation (that does not require empirical process theory), using a finite-dimensional model that we can directly sample from (which the Dirichlet Process model is not), of how one can address both heteroskedasticity and clustered correlations from a Bayesian perspective. In comparison to existing frequentist methods, an advantage of the Bayesian approach is that the need to build an explicit probability model for the entire data-generating process serves to ground the discussion about "clustering"[4] in factually resolvable questions about how a population was actually sampled.

Since the Bayesian Bootstrap and the Dirichlet Process are limiting cases of our model, they are explicitly covered by our results.

0.1 Notation and Aims

In this section we clarify what we are trying to accomplish and so must introduce some notation. We will omit measure-theoretic details. Throughout this paper, we will focus on the simplest kinds of linear regressions, where observations $z_i = (y_i, X_i)$, with $i = 1, 2, \ldots, n$, are drawn independently and identically from some (true) data-generating probability distribution P^* with expectations

[4] Compare the differing non-Bayesian perspectives on clustering in Cameron and Miller (2015), Abadie et al. (2017), and MacKinnon and Webb (2019).

$\mathbb{E}^*$ that we may think of as an unknown population about which we are seeking inferences; the distribution of a sequence of datasets is given by an infinite product measure $\mathbb{P}^{*(\infty)}$ whose properties we will stipulate when we need them. To simplify discussion, we will sometimes refer to a generic observation $(y_i, X_i) = (y, X)$.

0.1.1 Notation for heteroskedasticity and clustering

In the heteroskedasticity scenario, we have $y_i \in \mathbb{R}$ and $X_i \in \mathbb{R}^k$ (as a row-vector), so that (y_i, X_i) can be considered a "row" of data. In the cluster-robustness scenario, each y_i is a column-vector with some dimension N_i , and each X_i is an $N_i \times k$ matrix, so that (y_i, X_i) is a "block" of data.[5] We will sometimes stack n observations in the usual way, writing $\mathbf{y} := (y_1, \ldots, y_n)^\top$, and $\mathbf{X} := (X_1^\top, \ldots X_n^\top)^\top$.

0.1.2 Standard robust regression estimators

In both the clustering and the heteroskedasticity case, we hope to draw inferences about a squared-error-minimizing linear approximation:

$$\beta^* := \arg\min_\beta \mathbb{E}^* \left[(y - X\beta)^\top (y - X\beta)\right] = \mathbb{E}^*[X^\top X]^{-1}\mathbb{E}^*[Xy] \tag{3}$$

Throughout, we will assume that $\mathbb{E}^*[X^\top X]$ is indeed invertible. In regular cases, an equivalent definition is $\beta^* := \arg\min_\beta \mathbb{E}^* \left[(\mathbb{E}^*[y|X] - X\beta)^\top (\mathbb{E}^*[y|X] - X\beta)\right]$, clarifying that we are approximating the conditional expectation function $\mathbb{E}^*[y|X]$.

If the true conditional expectation does happen to be linear, then $\mathbb{E}^*[y|X] = X\beta^*$, and so β^* quantifies the "slope" of the conditional expectation, making β^* clearly a useful estimand in this case. However, we need not (and our model does not) assume linearity in order to fit a linear regression. This perspective is advocated in an recent paper by Buja et al. (2019) and is implicit in a pivotal paper on the Bayes Bootstrap by Chamberlain and Imbens (2003). This view of regression perhaps originates in the seminal work of White (1980b), in which heteroskedastic-robust estimators are presented as generally robust to model misspecification.

In the frequentist literature, a key attraction of choosing β^* for one's estimand is that β^* is relatively easy to draw inferences about, at least when k remains much smaller than n. As is well-known, the sample OLS coefficients $\hat{\beta} := (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}$ are consistent, in the sense that $\hat{\beta} \longrightarrow \beta^*$ in probability as $n \longrightarrow \infty$, under relatively mild regularity conditions. In particular, consistency does not require that the conditional distribution of y given X be Gaussian or homoskedastic.

Under slightly stronger regularity conditions, but still without assuming that the conditional mean or conditional variance functions have any particular functional form, one can construct consistent confidence intervals for $\hat{\beta}$: that is, nominal $(1 - \alpha)\%$ intervals around $\hat{\beta}$ that (under hypothetically repeated sampling) capture the true β^* at a rate rapidly approaching $(1 - \alpha)\%$ as $n \longrightarrow \infty$. To make such confidence intervals, one uses an estimator of the covariance of $\hat{\beta}$ such as

[5] If N_i may vary in the population from 1 to some maximum size $\bar{N}$, then the marginal draws of y_i take values in $\mathcal{Y} = \mathbb{R}^1\cup, \ldots, \cup\mathbb{R}^{\bar{N}}$, and X_i in $\mathcal{X} = \mathbb{R}^{1\times k}\cup, \ldots, \cup\mathbb{R}^{\bar{N}\times k}$; and jointly, (y_i, X_i) takes values in $\mathcal{Y}\times\mathcal{X}$. The true conditional expectation $\mathbb{E}^*[y_i|X_i]$ can be viewed as mapping an $N_i \times k$ matrix $X_i \in \mathcal{X}$, to a vector in $\mathbb{R}^{N_i} \subset \mathcal{Y}$; the true conditional variance function $\mathbb{V}^*(y_i|X_i)$ maps X_i to a positive semidefinite matrix in $\mathbb{R}^{N_i\times N_i} \subset \mathcal{Y}\times\mathcal{Y}$.

that attributed to Eicker (1967), Huber (1967), and White (1980b):

$$\hat{\mathbb{V}}^{HC0}(\hat{\beta}) = (\mathbf{X}^\top \mathbf{X})^{-1} \left[\sum_{i=1}^{n} X_i^\top \hat{\epsilon}_i^2 X_i \right] (\mathbf{X}^\top \mathbf{X})^{-1} \tag{4}$$

where $\hat{\epsilon}_i = y_i - X_i \hat{\beta}$ are the OLS residuals. In the case of clustered covariances with $c = 1, \dots, C$ clusters, using our notation the Liang-Zeger covariance estimator (Liang and Zeger, 1986) is nearly identical:

$$\hat{\mathbb{V}}^{LZ}(\hat{\beta}) = (\mathbf{X}^\top \mathbf{X})^{-1} \left[\sum_{c=1}^{C} X_c^\top \hat{\epsilon}_c \hat{\epsilon}_c^\top X_c \right] (\mathbf{X}^\top \mathbf{X})^{-1} \tag{5}$$

where the N_c-vector $\hat{\epsilon}_c$ and the $N_c \times k$ matrix X_c correspond to the N_c rows of observations in cluster c. In our discussion, we will sometimes use $\hat{\mathbb{V}}$ to generically denote a robust covariance estimator. A large literature is devoted to variants of the above covariance estimators, with various "finite-sample corrections" and "degree-of-freedom" corrections; for a review see Cameron and Miller (2015).

0.1.3 When heteroskedasticity matters

To gain a concrete understanding of how heteroksedasticity can cause problems, we briefly consider the bivariate regression case where $X_i = (1, w_i)$ and w is univariate. Focusing on the slope coefficient $\hat{\beta}_2$ (usually the parameter of interest),[6]

$$\begin{aligned}
\mathbb{V}^{\hat{HC0}}(\hat{\beta}_2) &= \frac{1}{n} \frac{\frac{1}{n}\sum_i \hat{\epsilon}_i^2 \frac{(w_i - \bar{w})^2}{\frac{1}{n}\sum_l (w_l - \bar{w})^2}}{\left(\frac{1}{n}\sum_i (w_i - \bar{w})^2\right)^2} \\
&= \frac{1}{n} \frac{\frac{1}{n}\sum_i \left(\hat{\epsilon}_i^2 - \frac{1}{n}\sum_i \hat{\epsilon}_i^2\right)\left((w_i - \bar{w})^2 - \frac{1}{n}\sum_l (w_l - \bar{w})^2\right)}{\left(\frac{1}{n}\sum_i (w_i - \bar{x})^2\right)^2} + \frac{1}{n} \frac{\frac{1}{n}\left(\sum_i \hat{\epsilon}_i^2\right)\left(\frac{1}{n}\sum_i (w_i - \bar{w})^2\right)}{\left(\frac{1}{n}\sum_i (w_i - \bar{w})^2\right)^2} \\
&= \frac{1}{n} \frac{\hat{\text{Cov}}(\hat{\epsilon}^2, (w - \bar{w})^2)}{\hat{\text{Var}}(x)^2} + \hat{\mathbb{V}}^{OLS}(\hat{\beta}_2) \qquad (6)
\end{aligned}$$

We recognize the second term as the classical (homoskedastic) OLS variance estimator, written $\hat{\mathbb{V}}^{OLS}(\hat{\beta}_2)$. Since $\hat{\mathbb{V}}_{HC}(\hat{\beta})$ is (usually) consistent, we can conclude that $\hat{\mathbb{V}}^{OLS}(\hat{\beta}_2)$ is too small whenever the first term is positive: that is, when the squared residuals are positively correlated with $(w - \bar{w})^2$, the squared distance of the regressor from its mean. In other words, if the linear model's fit worsens (either due to increasing variance or nonlinearity) as X gets farther from its mean, nominal 95% confidence intervals based on the classical $\hat{\mathbb{V}}^{OLS}(\hat{\beta}_2)$ will tend to be overconfident or anti-conservative. As we shall see, the same is true for a close Bayesian equivalent to classical OLS.

0.1.4 Bayesian inference

A Bayesian model can be built from 1) a "sampling model" $\mathbb{P}(d\mathbf{z}_n|\lambda) = \prod_{i=1}^{n} P(dz_i|\lambda)$, in which the n datapoints $\mathbf{z}_n$ are independently and identically distributed $P(\cdot|\lambda)$ with unknown parameters

[6] See Buja et al. (2019, p.26) for the first equality.

λ in some set Θ, and 2) a prior distribution $\mathbb{P}(d\lambda)$over the parameters. Together, these produce a joint distribution $\mathbb{P}(d\mathbf{z}_n, d\lambda)$, and hence a posterior distribution of the parameters λ given the data $\mathbf{z}_n$. [7].

Within the Bayesian modeling framework, we will write probabilities $\mathbb{P}$, expectations $\mathbb{E}$, and variances $\mathbb{V}$ without asterisks, taking care not to confuse these with their true (data-generating) counterparts. We take the view that a Bayesian probability $\mathbb{P}$ represents beliefs or inferences about ("or models") some true unknown P^*, setting aside philosophical concerns about the well-foundedness of P^* or $\mathbb{P}$ for the purposes of this paper.

We will define our Bayesian linear regression coefficients j as frequentist econometricians define their estimand, with

$$\boldsymbol{\beta}_n := \arg\min_{\beta} \mathbb{E}\left[(y - X\beta)^\top (y - X\beta) \,|\lambda\right] \tag{7}$$

, where y and X are distributed according to a distribution $P(dy, dX|\lambda)$ that we will describe in greater detail in subsequent sections.

0.1.5 Evaluating Bayesian inferences from a frequentist perspective

When we evaluate a Bayesian model from a frequentist perspective, we will not assume that the model is correct; there need not exist any $\lambda^* \in \Theta$ such that $P(dz_i|\lambda^*) = P^*$.

We seek Bayesian models where $\boldsymbol{\beta}_n$ have a posterior distribution with the following robustness properties:

- $\sqrt{n}\,(\boldsymbol{\beta}_n - \beta^*)\,|\mathbf{z}_n \xrightarrow{d} \mathcal{N}\left(0, \lim_{n\to\infty} n\hat{\mathbb{V}}\right)$, $\mathbb{P}_*^\infty$-almost surely.

where $\hat{\mathbb{V}}(\hat{\boldsymbol{\beta}}_n)$ is a consistent covariance estimator: $n\hat{\mathbb{V}}(\hat{\boldsymbol{\beta}}_n) - n\mathbb{V}^*(\hat{\boldsymbol{\beta}}_n) \longrightarrow 0$, $\mathbb{P}_*^\infty$-almost surely. And more strongly, we may seek:

- $\mathbb{E}[\boldsymbol{\beta}_n|\mathbf{z}_n] - \hat{\boldsymbol{\beta}}_n \longrightarrow 0$, $\mathbb{P}_*^\infty$-almost surely.
- $n\mathbb{V}(\boldsymbol{\beta}_n|\mathbf{z}_n) - n\hat{\mathbb{V}}(\hat{\boldsymbol{\beta}}_n) \longrightarrow 0$ $\mathbb{P}_*^\infty$-almost surely.

1 Review of current Bayesian thinking about heteroskedasticity

In current Bayesian thinking about heteroskedasticity, two views appear to be prevalent: first, that it is usually safe to assume homoskedasticity, since heteroskedasticity usually does not matter in practice; and second, that if heteroskedasticity does matter in practice, then the true conditional variance $\mathbb{V}^*(y|X)$ must be modeled directly (using some model $\mathbb{V}(y|X, \lambda)$), as part of modeling the entire conditional distribution $\mathbb{P}^*(dy|X)$ (using some model $\mathbb{P}(dy|X, \lambda)$). In both approaches, the distribution of X is usually considered ancillary and therefore is not modeled. Our model fits within a third, smaller Bayesian literature in which we account for heteroskedasticity, but as a byproduct of modeling the joint distribution of y and X.

We will show using simulations and some theory that assuming homoskedasticity when heteroskedasticity is present can lead to credible intervals that are spuriously narrow — professing

[7] This holds even when Bayes' Theorem does not apply. (Polpo et al., 2015, Ch 1: What About the Posterior Distributions When the Model is Non-dominated?)

greater precision or certainty than a frequentist model not due to a better model or judicious priors, but rather due to an incorrect model that ignores the uncertainty coming from heteroskedasticity. We are more sympathetic to the view that the conditional variance function must be modeled (we will call this the "conditional modeling" view), but note that existing Bayesian conditional models make assumptions about heteroskedasticity that can be difficult to defend, and that can have the same consequences as ignoring heteroskedasticity if incorrect, as we show. Admittedly, we do not consider ways in which these models might be patched if their deficiencies became apparent to the researcher.

For the rest of this section we write $X_i^\top = \mathbf{x}_i$ as a vector, to clarify that we refer to the heteroskedastic case. Since clustered covariances usually include heteroskedasticity, our criticisms below also apply to them, but clustered covariances also present complexities of their own that we will not consider in this section for lack of space.

1.1 Ignoring or downplaying heteroskedasticity

According to a popular regression textbook with a Bayesian perspective, heteroskedasticity "does not affect what is typically the most important aspect of a regression model, which is the information that goes into the predictors and how they are combined..." and consequently heteroskedasticity is generally a minor issue. (Gelman, Hill, et al., 2021, p.154)

To examine this common view, let us consider a deceptively simple Bayesian model of heteroskedasticity to which we will return in later sections. For now, assume the variance function is fixed, $\mathbb{V}(y_i|\mathbf{x}_i) = \sigma^2(\mathbf{x}_i)$, and observations $(y_i, \mathbf{x}_i)$ are independently Normally distributed:

$$[y_i|\mathbf{X}, \boldsymbol{\beta}, \boldsymbol{\gamma}] \sim \mathcal{N}\left(\mathbf{x}_i^\top \boldsymbol{\beta}, \sigma^2(\mathbf{x}_i|\boldsymbol{\gamma})\right) \tag{8}$$

For illustrative purposes, we use an improper uniform prior on the regression coefficients $\boldsymbol{\beta}$. This impropriety will not affect our point; proper priors will yield increasingly similar conclusions as data increases.[8] Collecting the variances into a diagonal matrix $\Omega = \text{diag}\{\sigma^2(\mathbf{x}_i)\}$, standard manipulations (George E.P. Box and Tiao, 1973) show that the posterior distribution of $\boldsymbol{\beta}$ given $\mathbf{y}, X, \Omega$ is a Normal distribution with mean

$$\hat{\beta}_{GLS} := \left(\mathbf{X}^\top \Omega^{-1} \mathbf{X}\right)^{-1} \mathbf{X}^\top \Omega^{-1} \mathbf{y} \tag{9}$$

and covariance $\hat{V}_{GLS} := \left(\mathbf{X}^\top \Omega^{-1} \mathbf{X}\right)^{-1}$; that is, the posterior distribution is centered on the (frequentist) Generalized Least Squares estimator, which happens to be asymptotically efficient. [9]

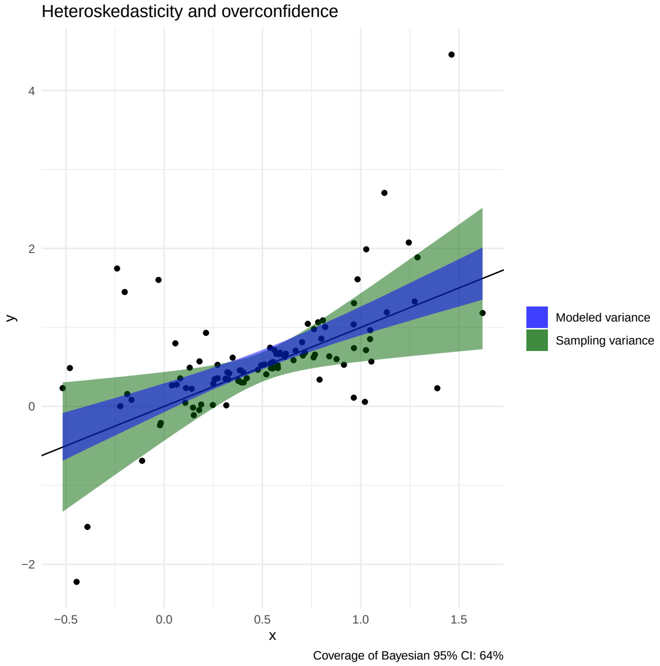
Crucially, we see that in this model, both the posterior mean and the posterior variance of $\boldsymbol{\beta}$ depend on Ω, which encodes the heteroskedasticity. Heteroskedasticity therefore affects not only the width of one's credible intervals, but also one's point estimates. As emphasized by Leamer (2010), this is not a minor effect, and it runs counter to Gelman and Hill's claim that heteroskedasticity does not affect important aspects of the model. Indeed, as we shall see in the next section, in cases where Ω is not known and must be modeled using a prior distribution, a poorly-chosen model for Ω can cause serious problems.

[8] As usual, the distribution of $\mathbf{x}$ is assumed to be ancillary and is not modeled. This is not the same as assuming that $\mathbf{x}$ is non-random.

[9] As Norets (2015) discusses, this model's Normality assumption often leads to excellent frequency properties even when the data are not in fact normally distributed.

Still, in the above model, we also see that assuming homoskedasticity, so that $\Omega = \sigma^2 I_n$ with some variance σ^2, results in the classical OLS coefficients $\hat{\beta} := (\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{y}$ and classical covariance $\hat{\mathbb{V}}_{OLS} := \sigma^2 \left(\mathbf{X}^\top \mathbf{X}\right)^{-1}$; for our argument, little changes if σ^2 is given a prior and estimated rather than known. Since $\hat{\boldsymbol{\beta}}$ is usually consistent for $\boldsymbol{\beta}^*$ even when heteroskedasticity exists, a pragmatic Bayesian might still insist that little is to be lost from ignoring known heteroskedasticity, except perhaps efficiency. This is not the case: one also risks overconfidence.

As we have shown in (6), classical confidence intervals that use $\hat{\mathbb{V}}_{OLS}$ can be spuriously narrow in the presence of heteroskedasticity and clustered correlations. To illustrate an extreme case, we show the results of a Bayesian OLS (homoskedastic) regression with a diffuse prior when data are in fact strongly heteroskedastic.



Above, we see that the Bayesian 95% probability region for $\mathbb{E}[y|\mathbf{x}, \beta] = \mathbf{x}^\top \beta$, (blue) is much narrower than the frequentist 95% sampling band for the Bayes estimator $\mathbf{x}^\top \hat{\beta}$ (green), and it happens that the Bayesian 95% posterior credible intervals capture the true slope coefficient only 64% of the time.[10] Here, Bayesian estimates that ignore heteroskedasticity are asymptotically consistent, but Bayesian inferences have much greater inferential certainty than a frequentist would think is warranted. There may be times when such certainty from a Bayesian model is justified, but this is not one of them, we argue. Here, the Bayesian model's extra certainty comes not from a better model or judicious use of prior information, but rather from simply ignoring an important source of uncertainty: heteroskedasticity.

[10] We estimated this with 10,000 simulated regressions. Currently, the coverage rate is calculated for a frequentist OLS model, which we'd expect to behave extremely similarly to a Bayesian OLS model with a diffuse prior, but is much faster to simulate. Different priors could be contemplated, but the usual recommended informative prior centered at a zero slope will not improve the coverage rate here, where the true regression slope is positive.

1.2 Conditional modeling of heteroskedasticity

It is clear that if a Bayesian knows there is heteroskedasticity, then her model must incorporate it somehow. In this subsection, we discuss recent work in which which $\mathbb{V}^*(y|\mathbf{x})$ is modeled explicitly, $\mathbb{E}^*[y_i|\mathbf{x}_i]$ is assumed to be linear in $\mathbf{x}$, and the distribution of $\mathbf{x}$ is considered ancillary and not modeled; we call this the "conditional modeling" approach, following (Gelman, Carlin, et al., 2020).

Bayesian conditional models for heteroskedasticity are numerous and we do not attempt to review or categorize them all here. [11] However, we can arrange many Bayesian conditional models along a specrum of how "wiggly"[12] $\mathbb{V}(y|\mathbf{x})$ is. We take readers on a brief tour of three kinds of heteroskedastic models: one that is not wiggly enough, some cutting-edge models that can be made just wiggly enough, and one that is altogether too wiggly.

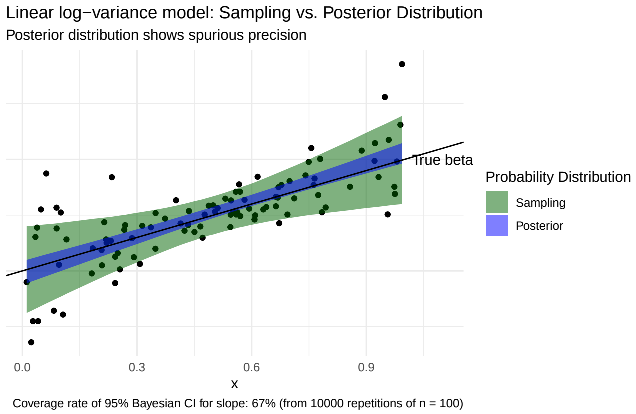
We will show in simulations that Bayesian models that assume too little or too much wiggliness can behave poorly indeed from a frequentist perspective, noting that it can be difficult or impossible in most real-world cases to verify whether one's wiggliness assumptions are correct.

1.2.1 Too little wiggliness

When Ω is not precisely known, but one suspects that heteroskedasticity might be lurking, perhaps the simplest way to extend (8) is with something like the following:

$$\sigma^2(\mathbf{x}_i) = \exp\{\mathbf{x}_i^\top \boldsymbol{\gamma}\} \tag{10}$$

giving the unknown parameters $\boldsymbol{\gamma}$ some prior. We choose this particular model because it appears to be the most common starting-point for modeling heteroskedasticity in the Bayesian (and frequentist) literature, and it is the default in Stan[13], one of the most widely-used Bayesian MCMC engines and modeling platforms. Eliding the details, we can estimate the posterior distribution of this model using Stan's default settings.



Unfortunately, the result hardly looks better than in the previous subsection. The posterior distribution is far narrower than the frequentist sampling distribution, and the Bayesian's 95%

[11] An extensive Bayesian literature is devoted to modeling heteroskedasticity, particularly in financial time-series, where the conditional variance is known as "volatility," and is essential to asset-pricing models such as the Black-Scholes model. We do not attempt to review this literature here.

[12] Perhaps measured in terms of integrated squared derivatives, as in the spline literature, where "wiggliness" is in fact a techical term.(Wood, 2017)

[13] Stan. (2022). Stan Development Team, Modeling.

credible intervals for the slope coefficients contain the true slope only 67% of the time, whereas the frequentist HC0 intervals have 92% coverage. So at least by the frequentist standard, the Bayesian is still overconfident, although there has been a slight improvement over the case where the Bayesian ignored heteroskedasticity entirely.

The origin of this disagreement is simple: the Bayesian model is wrong about the underlying process. By choosing $\sigma^2(\mathbf{x}_i) = \exp\{\mathbf{x}_i^\top \boldsymbol{\gamma}\}$, we have put all prior (and posterior) probability on a space of variance functions that does not contain the true variance function, which here happens to be $\sigma^{2(*)}(x) = (0.1 + 2(x - 0.5)^2)^2$. Since this is a problem with the likelihood, not the priors, more carefully chosen priors and more data will not generally fix the problem.

1.2.2 Just enough wiggliness

If unsatisfied with the above model, a Bayesian can turn to highly flexible models that use infinitely many parameters (naturally, these are called "nonparametric" models) to describe the conditional variance, in the hope that this much larger model-space will contain the true model, or a sufficiently good approximation of it. To take some leading recent examples of non-parametric or semiparametric models, Norets (2015), Pelenis (2014), and Zhao (2015) essentially extend model (8) with the unknown $\sigma^2(\mathbf{x}_i)$ modeled as a (perhaps transformed) linear combination of infinitely many basis functions $\psi_j(\mathbf{x}_i)$ with coefficients γ_j:

$$\sigma^2(\mathbf{x}_i) = \sum_{j=1}^{\infty} \gamma_j \psi_j(\mathbf{x}_i) \tag{11}$$

For example, in Norets, ψ_j are Bernstein polynomials. The above authors prove that their models produce consistent estimates and asymptotically correct confidence intervals under a very wide range of data-generating processes, even when the true conditional distribution is not at all Gaussian. As Norets explains, the key to such general success is that in large samples, Bayesian posterior distributions tend to concentrate around models that are "close" to the data-generating process in terms of Kullback-Liebler (KL) divergence[14]; and for Gaussian models like (8), the KL divergence is minimized by matching the conditional mean and variance in the model to the conditional mean and variance of the data-generating process — for any data-generating process. So if one is interested in robust Bayesian models of the mean and variance, Gaussian models tend to perform well (Kleijn and van der Vaart, 2006).

There are some difficulties with the above modeling approach. The desirable frequency properties rest heavily on smoothness assumptions about the true $\sigma^{(*)2}(\mathbf{x})$ that are difficult to verify (or even intuit) in most applications and are too technical to state precisely here, although they involve placing stochastic bounds on partial deriviatives of $\sigma^2(\mathbf{x})$ (eg. see Norets, 2015, p.410) in a way that matches the partial derivatives of the unknown true $\sigma^{(*)2}(\mathbf{x})$. These assumptions are far stronger than the regularity conditions that underpin frequentist robust models (compare to White (1980b), where no assumptions about derivatives are required).

Compounding this problem, when $\mathbf{x}_i$ has more than three dimensions or so, the nonparametrically modeled $\sigma^2(\mathbf{x}_i)$ may converge to the truth relatively slowly and be extremely computationally demanding, particularly if smoothness parameters are unknown and must also be given priors.[15] Clustered correlations appear nearly impossible to handle as nonparametric functions in this way,

[14] See Shalizi (2009)

[15] This is part of why we do not present MCMC simulations of these procedures here.

since in this case $\mathbb{V}(\mathbf{y}_c|X_c)$ is a matrix function whose dimension and form may change between clusters c.

1.2.3 Too much wiggliness

Next, we will show how someone who is dissatisfied with the above models and who is searching for arbitrarily flexible models of heteroskedasticity might construct a homoskedastic model as a limiting case. Indeed, the use of homoskedastic Student's-t models for heteroskedasticity (e.g., Geweke, 1993) appears to date back at least to Jeffreys' seminal *Theory of Probability* (Robert et al., 2009) and remains popular; it is currently a first-page Google search result for "Bayesian Heteroskedastic regression."[16][17]. Unfortunately, we will see that in simulations, the Student's t model can perform extremely poorly when faced with realistic patterns of heteroskedasticity. Ultimately, the Student's-t model illustrates the practical dangers not only of using homoskedastic models when the truth is heteroskedastic, but also of making ill-founded (non-)smoothness assumptions concerning $\sigma^2(\mathbf{x}_i)$.

Suppose that as above, each "error term" $\epsilon_i := y_i - \mathbf{x}_i^\top \beta$ is distributed independently as $\epsilon_i \sim N(0, \sigma^2(\mathbf{x}_i))$, and that to model $\sigma^2(\mathbf{x}_i)$ we use a linear combination of basis functions ψ_j with unknown coefficients $\gamma_j > 0$, similar to the semiparametric models in the previous section:

$$\sigma^2(\mathbf{x}_i) = \tau \sum_{j=1}^{J} \gamma_j \psi_j(\mathbf{x}_i) \tag{12}$$

For reasons that will become apparent, we include $\tau > 0$ as a scale factor; for now we will treat τ as fixed. Suppose that our basis consists of indicator functions $\psi_j(\mathbf{x}_i) = 1(\mathbf{x}_i \in A_j)$ for some partition $A_1, \ldots, A_J$ of the $\mathbf{x}$-space. This is the simplest form of B-spline. Suppose we use independent Inverse-Gamma priors on the coefficients, $\gamma_j \sim \text{InvGamma}(\frac{\nu}{2}, \frac{\nu}{2})$ for some fixed $\nu > 0$ (this enforces $\sigma^2(\mathbf{x}_i) > 0$), and because we want to accommodate nearly arbitrarily wiggly forms of heteroskedasticity, we make J much, much larger than any dataset we're likely to see. There is nothing intrinsically wrong with more parameters than data-points in a Bayesian analysis, and an extremely large J might even seem necessary if we want a fine partition and $\mathbf{x}$ has a few dozen dimensions.[18]

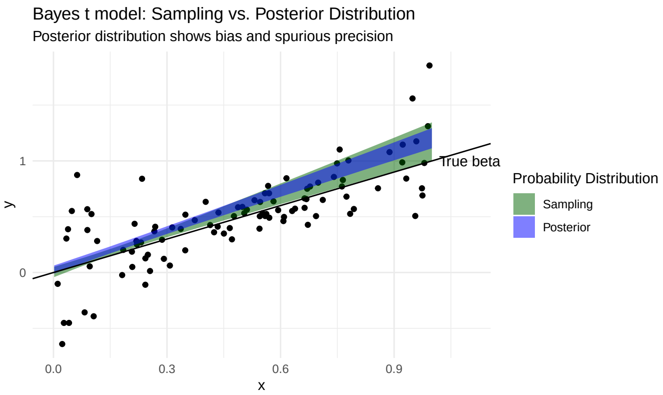
Consider what happens if data arrives and each observed $\mathbf{x}_i$ falls in its own member of the partition, with no member of the partition containing multiple observations (our main point would hold even with multiple observations per member, though the notation and qualifications would proliferate). In this case, each error term ϵ_i for $i = 1, \ldots, n$ is being modeled as $\epsilon_i \sim N(0, \sigma_i^2)$ independently, with each observation seeming to get its own variance $\sigma_i^2 = \tau\gamma_{j_i}$, with $\gamma_{j_i} \sim \text{InvGamma}(\frac{\nu}{2}, \frac{\nu}{2})$ independently.

Interestingly, integrating out each σ_i^2 while holdng the other parameters fixed, one sees that $\epsilon_i \sim t_\nu(\mathbf{x}^\top \beta, \tau^2)$, a Student's t distribution with ν degrees of freedom and a scale parameter equal to τ^2. This means that in this model, which is supposed to be almost arbitrarily heteroskedastic, each each error-term is effectively being modeled as *homoskedastic* for the given dataset. In many cases, this homoskedasticity leads to problems.

[16] As of Feb 2022. Stable link: https://web.archive.org/web/20220214205338/https://jrnold.github.io/bayesian_notes/heteroskedasticity.html

[17] This model is also sometimes recommended for outlier-robust regression, a different kind of robustness than we are concerned with (see West (1984))

[18] For example, if each vector $\mathbf{x}$ comprises 20 variables, and we wish to partition each variable's values into 10 subcategories, then there are $J = 10^{20}$ elements of the partition in total.



Coverage rate of 95% Bayesian CI for slope: 32% (from 10000 repetitions of n = 300)

In 10,000 simulations like the one pictured above, we find that 95% credible intervals based on the Student's t model capture the true slope (pictured in black) only 32 % of the time. Meanwhile, the frequentist HC-robust CIs (not pictured) capture the true value 94% of the time. In the picture, we see that the Bayesian estimate is strongly biased upwards, and the posterior distribution is also narrower than the sampling distribution of the Bayes estimate. This bias comes not from the prior (which is very diffuse in this case, although proper), but rather from the poorly-chosen Student's t likelihood, which does not actually model the heteroskedasticity in our chosen error-distribution. A maximum-likelihood estimator based on the same Student's t model would have the same pathologies. Because the problem is with the likelihood, not the prior, it only gets worse with more data.

A short explanation for this particular problem is that the Student's t model is akin to a weighted least squares model, where observations get less weight if they are farther from the regression line; more precisely, in Gibbs sampling the observations are iteratively re-weighted according to their inverse squared distance from the previous iteration's fitted regression line (see West (1984) for details). We made the true error distribution asymmetric as well as heteroskedastic, with a longer tail in the negative direction than in the positive direction, and with both tails of the error distribution getting longer as x increases (keeping the error term mean-zero). With this kind of error distribution, as x increases, the model systematically gives less and less weight to observations in the lengthening tail below the fitted regression line, biasing the regression line upwards. It may be somewhat unfair to test the Student's t model with asymmetric data, but the world is often unfairly asymmetrical; and as far as we are aware, no warnings about asymmetry appear in any of the standard Bayesian texts on the Student's t model.

What we have described is not peculiar to the Student's t model. When faced with heteroskedasticity, one would expect similar pathologies from any homoskedastic model, not just the Student's t.

1.3 Conclusion of review

This ladder of models, from one that ignores heteroskedasticity, to one with insufficient wiggliness of the conditional variance, to one with just enough wiggliness, to one with too much wiggliness, can be seen as a stylized story of how a researcher might expand a model to better fit the data. We have shown what happens when this story goes wrong at various stages: Bayesian 95% credible intervals may capture the true value much less often than 95%, being both biased and professing much greater inferential certainty than a frequentist would think is warranted — not because the Bayesian model makes better use of information, but simply because the Bayesian model incorrectly accounts for heteroskedasticity.

This leads us to search for other Bayesian models that do not hinge so crucially on assumptions about the wiggliness of conditional variance function.

2 Our model: Finite Dirichlet Process

In this section we examine the Finite Dirichlet Process model, which we believe answers many of the concerns that we raised in previous sections. However, one need not agree with our criticisms of the above models, or even agree with the premise that frequentist properties should matter to Bayesians, to consider our model to be a useful and descriptively plausible one for many kinds of economic data.

2.1 Sampling model

Suppose $\mathring{\mathbf{z}} = (\mathring{z}_1, \ldots, \mathring{z}_M) = ((\mathring{y}_1, \mathring{X}_1), \ldots, (\mathring{y}_M, \mathring{X}_M))$ is a set of $M < \infty$ distinct unknown values with respective unknown proportions $\boldsymbol{\theta} = \theta_1, \ldots, \theta_M$ in some population about which we seek inferences. Under simple random sampling with replacement, each observation z_i is drawn independently and identically from a categorical distribution on the given points $\mathring{\mathbf{z}} = \mathring{z}_1, \ldots, \mathring{z}_M$ with given probabilities $\boldsymbol{\theta} = \theta_1, \ldots, \theta_M$, which is described by the following probability transition kernel:

$$P(dz_i|\mathring{\mathbf{z}}, \boldsymbol{\theta}) := \sum_{j=1}^{M} \theta_j \delta_{\mathring{z}_j}(dz_i) \tag{13}$$

Recall that the Dirac measure $\delta_{\mathring{z}}(dz)$ is defined by $\int \delta_{\mathring{z}}(dz) f(z) = f(\mathring{z})$ for any measurable f.

Here, P represents the unknown population. One can think of each $\mathring{z}_j$ as a collection of numerical quantities associated with a distinct category j of sampling unit in the population, and $\theta_1, \ldots, \theta_M$ as proportions of the total population in each category. Unlike basic categorical models, the values $(\mathring{z}_1, \ldots, \mathring{z}_M)$ are not all known before sampling occurs.

In the standard heteroskedasticity scenario, each $\mathring{y}_j$ takes values in $\mathcal{Y} = \mathbb{R}$ and each $\mathring{X}_j$ take values in $\mathcal{X} = \mathbb{R}^k$, so $\mathring{z}_j$ in $\mathcal{Y} \times \mathcal{X}$ is essentially a row of regression-variables that would be sampled as a single unit, perhaps associated with an individual in a population. Naturally, observations z_i also occur in $\mathcal{Y} \times \mathcal{X}$.

In the cluster-robustness scenario, things are similar, except that each $\mathring{y}_j$ is a vector of N_j values, and each $\mathring{X}_j$ is a matrix of $\mathring{N}_j \times k$ values, so that each $(\mathring{y}_j, \mathring{X}_j)$ is a "cluster" of rows of regression-variables that would be sampled from the population as a single unit. To be more precise, if $\mathring{N}_j$ can be any integer from 1 to some maximum cluster size $\bar{N}$, then each $\mathring{y}_j$ takes values in $\mathcal{Y} = \mathbb{R}^1 \cup, \ldots, \cup \mathbb{R}^{\bar{N}}$, and $\mathring{X}_j$ in $\mathcal{X} = \mathbb{R}^{1 \times k} \cup, \ldots, \cup \mathbb{R}^{\bar{N} \times k}$.[19]

In this model, "clustering" is determined by our population of interest and how it is being sampled. If we are conducting a simple random cross-sectional sample of people from a population,

[19] Probability distributions on $\mathcal{Y} \times \mathcal{X}$ are not hard to construct from standard probability distributions: for example, first draw $\mathring{N}_j$ from some probability distribution on $1, \ldots \bar{N}$, then draw the $\mathring{N}_j$ "rows" of $(\mathring{y}_j, \mathring{X}_j)$ independently from some standard $1 + k$-dimensional distribution. Since these are finite unions of standard Borel spaces, they do not raise any concern about measurability.

then each $(\mathring{y}_j, \mathring{X}_j)$ is a single "row" of data for person j. If are randomly sampling people from the population but make multiple observations of each variable per person (over time, say), then it might be appropriate to consider each$(\mathring{y}_j, \mathring{X}_j)$ to be a "cluster" of $\mathring{N}_j$ observations for each person j in the population.

Data-generating processes like the above seem to describe a wide variety of social-science scenarios where we are sampling individuals or groups of individuals from a finite population. We conjecture that our model may also approximate sampling without replacement in large-population cases, similar to the closely related models by A. Lo (1987) and Aitkin (2008), an extension that we may consider in a later paper.

2.2 Regression functional

The parameters $\mathring{\mathbf{z}}, \boldsymbol{\theta}$ themselves are not necessarily of great interest, but they can be used to define a wide range of quantities of interest. Here, we focus on the usual least squares regression functional, assuming throughout that $\sum_{j=1}^{M} \theta_j \mathring{X}_j^\top \mathring{X}_j$ is (almost surely) invertible:

$$\beta(\mathring{\mathbf{z}}, \boldsymbol{\theta}) := \arg\min_{\beta} \mathbb{E}\left[(y - X\beta)^\top (y - X\beta) \,|\mathring{\mathbf{z}}, \boldsymbol{\theta}\right] = \left[\sum_{j=1}^{M} \theta_j \mathring{X}_j^\top \mathring{X}_j\right]^{-1} \sum_{j=1}^{M} \theta_j \mathring{X}_j^\top \mathring{y}_j \tag{14}$$

The prior and posterior distributions of $\mathring{\mathbf{z}}, \boldsymbol{\theta}$ induce prior and posterior distributions of β. The exact distribution of β is hard to characterize in closed form, but easy to draw samples from, since both the prior and the posterior $\mathring{\mathbf{z}}, \boldsymbol{\theta}$ are easy to sample from.

Other parameters could be derived by minimizing other objective functions, in a Bayesian parallel to frequentist "M-estimators" (see Chamberlain and Imbens 2003). We would expect such parameters to have similar robustness properties to this one.

2.3 Priors

In Bayesian reasoning, all unknowns need prior probability distributions. We assign the vector of unknown population proportions $\boldsymbol{\theta}$ the usual conjugate Dirichlet distribution on the $M-1$-dimensional simplex:

$$\text{Dir}(d\boldsymbol{\theta}|\alpha_1, \ldots, \alpha_M) = \frac{\Gamma(\sum_k \alpha_k)}{\prod_k \Gamma(\alpha_k)} \prod_{k=1}^{M} \theta_k^{\alpha_k - 1} d\boldsymbol{\theta} \tag{15}$$

with the Dirichlet distribution parameters $\alpha_j > 0$ all fixed at $\alpha_j = {}^{\alpha}\!/_{M}$; here,$\Gamma$ represents the Gamma function.

The unknown population-values are given independent and identical priors:

$$\mathring{z}_j \overset{iid}{\sim} F \tag{16}$$

for some distribution F on $\mathcal{Y} \times \mathcal{X}$ that has no atoms and has positive density everywhere.[20] Concretely, to draw a single $\mathring{z}_j$ from F, the analyst might start with a fixed prior "best guess" of the

[20] In the cluster case, I have in mind that every cluster-size N_j up to some $\bar{N}$ is given positive probability, and for any given cluster size the observations $(\mathring{y}_j, \mathring{X}_j)$ are drawn iid from a standard distribution with a (Lebesgue-dominated) density that is positive everywhere.

regression slope $\mathring{\beta}_0$, then draw $\mathring{N}_j$ from some prior (fix $\mathring{N}_j = 1$ in the heteroskedasticity case), then draw the $\mathring{N}_j$ rows of $\mathring{X}_j$ independently from some suitable k-dimensional distribution with full support, then draw $\mathring{\epsilon}_j$ from some distribution (independently of X_j, for simplicity) with full support on $\mathbb{R}^{N_j}$ and mean zero, then define the $\mathring{N}_j$ elements of $\mathring{y}_j$ as $\mathring{y}_j = X_j\beta_0 + \mathring{\epsilon}_j$. Other schemes could be devised; the main requirement is that they have full support on $\mathcal{Y} \times \mathcal{X}$.

Putting it all together, we will write the prior distribution as follows:

$$\Pi_0(d\boldsymbol{\theta}, d\mathring{\mathbf{z}}) = \mathrm{Dir}(d\boldsymbol{\theta}|\alpha_1, \ldots, \alpha_M) \prod_{k=1}^{M} F(d\mathring{z}_k) \tag{17}$$

These priors bear some explanation. As we will see from the posterior distribution, the parameters α_j can be interpreted as "counts" of (hypothetical) observations of the unknown population-values $\mathring{z}_j$. It makes sense to set $\alpha_j = \frac{\alpha}{M}$, since this is symmetric and keeps the sum of our prior counts $\sum_j \alpha_j$ invariant with respect to the number of latent points M; we do not want our "total prior data" to automatically increase with M.

The fixed parameter α essentially governs the "concentration" of the population; the smaller it is, the greater prior certainty we have that the population is concentrated at fewer values (a few θ_j are relatively large while the rest are small).

The distribution F describes one's prior expectations about how the population values are distributed, as we will see. The non-atomicity of F is simply a logical requirement: if $\mathring{z}_j$ are independently and identically distributed as F but we also want them to be distinct (almost surely), then their distribution can have no atoms.

The postulated number of distinct values M could be endowed with a prior, but for simplicity we consider it fixed at some sufficiently large value in any given analysis. We will discuss this more in our section on asymptotics, showing that the precise value of M matters little.

2.4 Prior Predictive Distribution

The prior predictive distribution is helpful for understanding the prior and is essential for defining the posterior distribution. We will derive it here.

For a single predicted value z, the prior predictive distribution is defined (Mazzi and Spizzichino in Polpo et al. (chapter by Mazzi and Spizzichino, 2015)) as any probability measure $P_0(dz)$ that satisfies

$$\int P_0(dz)h(z) = \int \Pi_0(d\boldsymbol{\theta}, d\mathring{\mathbf{z}}) \int P(dz|\mathring{\mathbf{z}}, \boldsymbol{\theta})h(z) \tag{18}$$

for any arbitrary measurable function $h(z)$ taking values in $\mathbb{R}$. Here,

$$\int P_0(dz)h(z) = \int \mathrm{Dir}(d\boldsymbol{\theta}|\alpha_1, \ldots, \alpha_M) \int \prod_{k=1}^{M} F(d\mathring{z}_k) \sum_j \theta_j \int \delta_{\mathring{z}_j}(dz)h(z) \tag{19}$$

$$\sum_j \frac{\alpha_j}{\sum_k \alpha_k} \int F(d\mathring{z}_j)h(\mathring{z}_j) \tag{20}$$

$$= \int F(d\mathring{z}_j)h(\mathring{z}_j) \tag{21}$$

using the fact that $\alpha_j = \frac{\alpha}{M}$.

Thus, the prior predictive distribution is simply F. This is why we say that F represents one's prior "best guess" about the population distribution.

2.5 Posterior Distribution

Given observed data $\mathbf{z}_n = z_1, \ldots, z_n$ consisting of m distinct values $\tilde{z}_1, \ldots, \tilde{z}_m$ with respective multiplicities $n_1, \ldots, n_m$ such that $\sum_{j=1}^{m} n_j = n$ and $m \leq M$, we learn that the "first" m population values $\mathring{z}_1, \ldots, \mathring{z}_m$ are fixed at the observed values $\tilde{z}_1, \ldots, \tilde{z}_m$, respectively, which we represent using Dirac delta distributions $\delta_{\tilde{z}_1}(d\mathring{z}_1), \ldots, \delta_{\tilde{z}_m}(d\mathring{z}_m)$. We learn nothing about the remaining unobserved population values $\mathring{z}_{m+1}, \ldots, \mathring{z}_M$, each of which retains the distribution F, independently and identically. Given the data $\mathbf{z}_n$ and $\mathring{\mathbf{z}}$, the conditional posterior distribution of $\boldsymbol{\theta}$ is Dirichlet, with the observed counts added to the "prior counts," much as in a standard Multinomial-Dirichlet categorical model.

Putting it all together, we may write the posterior distribution as

$$\Pi_n(d\boldsymbol{\theta}, d\mathring{\mathbf{z}}|\mathbf{z}_n) = \mathrm{Dir}(d\boldsymbol{\theta}|\alpha/M + n_1, \ldots \alpha/M + n_m, \alpha/M, \ldots, \alpha/M) \prod_{j=1}^{m} \delta_{\tilde{z}_j}(d\mathring{z}_j) \prod_{k=m+1}^{M} F(d\mathring{z}_k) \quad (22)$$

In (arbitrarily) picking the "first" m population-values specifically, we have rather freely used a property of our model called *exchangeability* to make the above expression as simple as possible, and arguably simpler. The true expression contains expressions like the above, but within a sum of all possible ways to assign population values to observed points. All of these assignments are equivalent in a precise sense, allowing us to pick one assignment arbitrarily. In the technical appendix, we discuss exchangeability and show that our simplification does not affect any meaningful posterior inferences.

A more formal argument

As Ishwaran and Zarepour (2002) write, Corollary 20 in Pitman (1996) states that for the Fisher model that we use, given distinct observed values $\tilde{z}_1, \ldots, \tilde{z}_m$ with multiplicities $n_1, \ldots, n_m$, the kernel $P(dz)$ can be represented in the following way:

$$P|\mathbf{z}_n = \sum_{j=1}^{m} \pi_j \delta_{\tilde{z}_j}(dz) + \pi_{m+1} \mathring{P}(dz) \quad (23)$$

where

$$(\pi_1, \ldots, \pi_m, \pi_{m+1}) \sim \mathrm{Dir}(\alpha/M + n_1, \ldots, \alpha/M + n_m, (M-m)\,\alpha/M) \quad (24)$$

independently of

$$\mathring{P}(dz) := \sum_{k=1}^{M-m} \lambda_k \delta_{\mathring{z}_j}(\cdot) \quad (25)$$

with iid $\mathring{z}_k \sim F(\cdot)$ and $(\lambda_1, \ldots, \lambda_{M-m}) \sim \mathrm{Dir}(\alpha/M, \ldots, \alpha/M)$.

Picking up where Isharan and Zarepour leave off, we observe that $\lambda_1, \ldots, \lambda_{M-m}$ are essentially proportions that subdivide or "decimate" the variable π_{m+1}. Since both $\lambda_1, \ldots, \lambda_{M-m}$ and $(\pi_1, \ldots, \pi_m, \pi_{m+1})$ are jointly Dirichlet, and moreover π_{m+1} corresponds to parameter $(M-m)\,\alpha/M =$

$\sum_{k=1}^{M-m} \alpha/M$, we may apply the "decimation" or "expansion" rule for Dirichlet distributions, (see Zhang (2008)), yielding:

$$(\pi_1, \ldots, \pi_m, \pi_{m+1}\lambda_1, \ldots, \pi_{m+1}\lambda_{M-m}) \sim \mathrm{Dir}(\alpha/M + n_1, \ldots, \alpha/M + n_m, \alpha/M, \ldots, \alpha/M) \tag{26}$$

So in the posterior distribution,

$$P(dz|\boldsymbol{\theta}, \mathring{\mathbf{z}}) = \sum_{j=1}^{m} \theta_j \delta_{\tilde{z}_j}(dz) + \sum_{k=m+1}^{M} \theta_k \delta_{\mathring{z}_j}(dz) \tag{27}$$

with

$$\boldsymbol{\theta} \sim \mathrm{Dir}(\alpha/M + n_1, \ldots \alpha/M + n_m, \alpha/M, \ldots, \alpha/M) \tag{28}$$

and iid $\mathring{z}_k \sim F(\cdot)$ for $k = m+1, \ldots, M$.

2.6 Posterior Predictive distribution

The posterior predictive distribution is helpful for understanding the posterior distribution, and also the effect (and therefore the meaning) of the prior distribution. It also allows us to draw connections to other closely-related models, and to begin considering our model's asymptotic properties.

Let h be any integrable real-valued function of a single prediction z. The posterior predictive distribution is fully characterized by the following expectation:

$$\mathbb{E}[h(z)|\mathbf{z}_n] \quad = \quad \int \Pi_n(d\boldsymbol{\theta}, d\mathring{\mathbf{z}}|\mathbf{z}_n) \int \sum_{l=1}^{M} \theta_l \delta_{\mathring{z}_l}(dz) h(z)$$

So

$$\mathbb{E}[h(z)|\mathbf{z}_n] = \int \mathrm{Dir}(d\boldsymbol{\theta}|\alpha/M{+}n_1, \ldots \alpha/M{+}n_m, \alpha/M, \ldots, \alpha/M) \left[\sum_{l=1}^{m} \theta_l h(\tilde{z}_j) + \sum_{l=m+1}^{M} \theta_l \int F(d\mathring{z}_l) h(\mathring{z}_l)\right] \tag{29}$$

Taking the expected value of the Dirichlet vector and rearranging,

$$\mathbb{E}[h(z)|\mathbf{z}_n] = \frac{n}{n+\alpha} \left[\frac{1}{n}\sum_{j=1}^{m} n_j h(\tilde{z}_j)\right] + \frac{\alpha}{n+\alpha}\frac{m}{M}\left[\frac{1}{m}\sum_{j=1}^{m} h(\tilde{z}_j)\right] + \frac{\alpha}{n+\alpha}\frac{M-m}{M}\mathbb{E}_0[h(\mathring{z})] \tag{30}$$

So the posterior expectation of $h(z)$ is a weighted average of, from left to right, 1) the sample mean, 2) the mean of the distinct observed values, and 3) the prior expectation, with weights that sum to one.

If the model "saturates" and $m = M$, then then the posterior expectation comprises the first two terms only:

$$\frac{n}{n+\alpha}\left[\frac{1}{n}\sum_{j=1}^{m} n_j h(\tilde{z}_j)\right] + \frac{\alpha}{n+\alpha}\left[\frac{1}{m}\sum_{j=1}^{m} h(\tilde{z}_j)\right] \tag{31}$$

This is the posterior predictive mean of a basic multinomial model supported on M known points with a symmetric Dirichlet prior with parameter α/M, to which the model reduces in this case. In comparison to the basic multinomial, the posterior mean of the FDP generally takes $\frac{M-m}{M}$ away from the weight of the mean of the population points and gives it to the prior mean, reflecting the lack of prior certainty about the unknown points.

The mean of distinct values, $\frac{1}{m}\sum_{j=1}^{m} h(\tilde{z}_j)$ in the second term, introduces a bias toward "more uncertain" (i.e. higher-entropy) distributions on the observed points. Desirable or not, this bias is usually small. As m approaches n, the mean of distinct values approaches the simple mean, so the term introduces less bias as its weight $\frac{\alpha}{n+\alpha}\frac{m}{M}$ increases. Furthermore, in many applications $M >> m$, making the term's contribution quite small. In fact, taking $M \longrightarrow \infty$, the second term of (30) vanishes and we have the posterior predictive mean of a Dirichlet Process (DP):

$$\frac{n}{n+\alpha}\left[\frac{1}{n}\sum_{j=1}^{m} n_j h(\tilde{z}_j)\right] + \frac{\alpha}{n+\alpha}\mathbb{E}_0[h(\mathring{z})] \tag{32}$$

Because the prior predictive distribution of the FDP is simply F, also as in a DP, and furthermore the DP is the *only* distribution with these prior and posterior predictive distributions (Albert Y. Lo, 1991), this functions as a simple demonstration that the FDP converges weakly (see Muliere and Secchi, 2003)to the DP as $M \longrightarrow \infty$. Indeed, this gives us license to consider $M \longrightarrow \infty$ as a well-defined Bayesian model. In comparison to the DP, the posterior mean of the FDP takes $\frac{m}{M}$ from the weight of the prior mean and adds it to weight of the mean of the distinct observed values, making the FDP slightly more responsive to the data than the DP.

If $\alpha \longrightarrow 0$ (an improper prior), then the posterior distribution reduces to a Bayes Bootstrap, with a posterior mean equal to the sample mean.

2.7 Posterior Distribution of the Regression Functional

Recall that our main quantity of interest is the regression functional

$$\beta(\mathring{\mathbf{z}}, \boldsymbol{\theta}) = \left[\sum_{j=1}^{M} \theta_j \mathring{X}_j^\top \mathring{X}_j\right]^{-1} \sum_{j=1}^{M} \theta_j \mathring{X}_j^\top \mathring{y}_j \tag{33}$$

, and that in the posterior distribution,

$$(\theta_1, \ldots, \theta_m, \theta_{m+1}, \ldots, \theta_M) \sim \mathrm{Dir}({}^{\alpha}/_{\mathrm{M}} + \mathrm{n}_1, \ldots, {}^{\alpha}/_{\mathrm{M}} + \mathrm{n}_\mathrm{m}, {}^{\alpha}/_{\mathrm{M}}, \ldots, {}^{\alpha}/_{\mathrm{M}}) \tag{34}$$

while the variables $(\mathring{X}_j, \mathring{y}_j)$ for $1 < j \leq m$ are fixed at the observed values $(\tilde{X}_j, \tilde{y}_j)$, and the remaining $(\mathring{X}_k, \mathring{y}_k) \overset{iid}{\sim} F(d\mathring{X}_k, d\mathring{y}_k)$ for $m < k \leq M$.

Like Lancaster (2006), we will use the well-known fact that any Dirichlet vector $(\theta_1, \ldots, \theta_M) \sim \mathrm{Dir}(\alpha_1, \ldots, \alpha_\mathrm{M})$ can be equivalently defined as $\theta_j = \frac{g_j}{\sum_{k=1}^{M} g_k}$ using independent Gamma-distributed random variables $g_k \sim \text{Gamma}(\alpha_k, 1)$[21]. We divide out the denominators $\sum_{k=1}^{M} g_k$ in the regression functional, which will render the terms of the sums independent and simplify our analysis considerably. For compact notation, we can collect $g_j \sim \text{Gamma}(n_j + \alpha/M, 1)$ for $1 \leq j \leq m$ into the diagonal matrix $\tilde{G}$, and collect $g_k \sim \text{Gamma}(\alpha/M, 1)$ for $m < k \leq M$ into the diagonal matrix $\mathring{G}$.

Then the posterior regression functional is:

$$\beta = \left[\sum_{j=1}^{m} g_j \tilde{X}_j^\top \tilde{X}_j + \sum_{k=m+1}^{M} g_k \mathring{X}_k^\top \mathring{X}_k\right]^{-1} \left[\sum_{j=1}^{m} g_j \tilde{X}_j^\top \tilde{y}_j + \sum_{k=m+1}^{M} g_k \mathring{X}_k \mathring{y}_k\right] \tag{35}$$

with $g_j \sim \text{Gamma}(n_j + \alpha/M, 1)$ for $1 \leq j \leq m$ and $g_k \sim \text{Gamma}(\alpha/M, 1)$ for $m < k \leq M$, with $(\mathring{X}_k, \mathring{y}_k) \overset{iid}{\sim} F(d\mathring{X}_k, d\mathring{y}_k)$, and with $\tilde{X}, \tilde{y}$ fixed.

Writing $\tilde{\mathbf{X}} = (\tilde{X}_1^\top, \ldots, \tilde{X}_m^\top)^\top$, $\tilde{\mathbf{y}} = (\tilde{y}_1, \ldots, \tilde{y}_m)^\top$, and similar for the $\mathring{X}_k, \mathring{y}_k$ variables, we have a more compact expression:

$$\beta = \left[\tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{X}} + \mathring{\mathbf{X}}^\top \mathring{G} \mathring{\mathbf{X}}\right]^{-1} \left[\tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{y}} + \mathring{\mathbf{X}}^\top \mathring{G} \mathring{\mathbf{y}}\right] \tag{36}$$

The following quantity behaves much like a Bayes Bootstrap OLS estimator (see Lancaster, 2006) and ultimately carries the desirable asymptotic properties of our model:

$$\tilde{\beta} := \left[\tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{X}}\right]^{-1} \tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{y}} \tag{37}$$

Let us show how β, $\tilde{\beta}$ and a prior regression parameter $\mathring{\beta}_0$ are related. Without loss of generality, we may suppose that the prior is of the form we suggested earlier, with $\mathring{y}_j = \mathring{X}_j \mathring{\beta}_0 + \mathring{\epsilon}_j$ for some fixed "prior best guess" slope parameter $\mathring{\beta}_0$ and some error term $\mathring{\epsilon}_j$ such that $\mathbb{E}[\mathring{\epsilon}_j | \mathring{\mathbf{X}}, \mathring{G}] = 0$. Then

$$\begin{aligned}
\beta \quad = \quad & \left[\frac{1}{n}\tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{X}} + \frac{1}{n}\mathring{\mathbf{X}}^\top \mathring{G} \mathring{\mathbf{X}}\right]^{-1} \left(\frac{1}{n}\tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{X}}\right) \tilde{\beta} + & (38)\\
& \left[\frac{1}{n}\tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{X}} + \frac{1}{n}\mathring{\mathbf{X}}^\top \mathring{G} \mathring{\mathbf{X}}\right]^{-1} \left(\frac{1}{n}\mathring{\mathbf{X}}^\top \mathring{G} \mathring{\mathbf{X}}\right) \mathring{\beta}_0 + & (39)\\
& \left[\frac{1}{n}\tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{X}} + \frac{1}{n}\mathring{\mathbf{X}}^\top \mathring{G} \mathring{\mathbf{X}}\right]^{-1} \frac{1}{n}\mathring{\mathbf{X}}^\top \mathring{G} \mathring{\boldsymbol{\epsilon}} & (40)
\end{aligned}$$

We see that the posterior β is a weighted average of 1) the bootstrap-like $\tilde{\beta}$ which contains the contribution of the data, 2) the prior expectation $\mathring{\beta}_0$, and 3) a mean-zero error term $\mathring{\mathbf{X}}^\top \mathring{G} \mathring{\boldsymbol{\epsilon}}$. The weights $\tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{X}}$ and $\mathring{\mathbf{X}}^\top \mathring{G} \mathring{\mathbf{X}}$ have the form of the classical least squares precision, and so can be interpreted naturally as describing the informativess of the data and the prior, respectively.

2.7.1 Computation

Equation (35) suggests a straightforward weighted bootstrap procedure for simulating from the posterior distribution. Repeat the following $s = 1, \ldots, S$ times:

[21] We use the "shape-scale" parametrization, so that $\mathbb{E}[g_k] = \mathbb{V}(g_k) = \alpha_k$.

1. Draw $g_j^{(s)} \sim \mathrm{Gamma}(n_j + {}^{\alpha}\!/\!{}_{M}, 1)$ for $1 \leq j \leq m$, and $g_k^{(s)} \sim \mathrm{Gamma}({}^{\alpha}\!/\!{}_{M}, 1)$ for $m < k \leq M$, all independently.
2. Draw $(\mathring{X}_k, \mathring{y}_k)^{(s)} \overset{iid}{\sim} F(d\mathring{X}_k, d\mathring{y}_k)$ for $m < k \leq M$, for some user-specified F.
3. Construct $\beta^{(s)}$ as in 35. This is a simple weighted least squares computation, where the weights are given by the Gamma variables g_i.

To approximate the posterior expectation $\int h(\beta)\Pi_n(d\beta|\mathbf{z}_n)$ for an arbitrary function h, one can use the Monte Carlo estimate $\frac{1}{S}\sum_{s=1}^{S} h(\beta^{(s)})$.

3 Asymptotics

We will consider infinite sequences of datasets $(\mathbf{z}_n)$ of n observations, with each observation drawn independently and identically from some unknown distribution P^*. The sequence of datasets is drawn from some true product distribution $\mathbb{P}^{*\infty}$. For each dataset $\mathbf{z}_n$, we consider a single posterior distribution $\Pi_n(d\boldsymbol{\theta}_n, d\mathring{\mathbf{z}}_n|\mathbf{z}_n)$ of the parameters $\boldsymbol{\theta}_n, \mathring{\mathbf{z}}_n$, as defined above, inducing a posterior distribution of the least squares functional $\boldsymbol{\beta}_n = \boldsymbol{\beta}_n(\boldsymbol{\theta}_n, \mathring{\mathbf{z}}_n)$. Our general strategy is as follows:

1. Observe that under plausible regularity conditions such as those in (White, 1980b), the sequence of datasets$(\mathbf{z}_n)$ has certain limit properties $\mathbb{P}^{*\infty}$-almost surely, such as that the OLS coefficients $\hat{\beta}_n$ and the (scaled) robust covariance estimator $n\hat{\mathbb{V}}_n$ are well-defined and consistent estimators.
2. Show that for any given sequence $(\mathbf{z}_n)$ with the above limiting properties, the sequence of posterior distributions converges: $\sqrt{n}\left(\beta_n - \hat{\beta}_n\right)|\mathbf{z}_n \xrightarrow{d} \mathcal{N}(0, \lim_{n\to\infty} n\hat{\mathbb{V}}_n)$, under some mild regularity assumptions on the prior. So the regression coefficients β_n converge in distribution to a Gaussian random variable with mean $\hat{\beta}_n$ and covariance $\hat{\mathbb{V}}_n$.
3. [Not yet done] We would like to formally conclude that the posterior mean $\mathbb{E}\left[\beta_n|\mathbf{z}_n\right]$ is a $\sqrt{n}$-consistent estimator. As described in van der Vaart (1998, p.17), asymptotic uniform integrability of $\sqrt{n}\beta_n|\mathbf{z}_n$[22] is the necessary and sufficient condition to ensure that the expectation of $\sqrt{n}\beta_n$ also converges, $\sqrt{n}\mathbb{E}\left[\beta_n|\mathbf{z}_n\right] - \sqrt{n}\hat{\beta}_n \longrightarrow 0$. Since $\hat{\beta}_n$ is $\sqrt{n}$-consistent, we can conclude that $\sqrt{n}\mathbb{E}\left[\beta_n|\mathbf{z}_n\right] - \sqrt{n}\beta^* \longrightarrow 0$.
4. [Not yet done] As above, under further regularity conditions the sequence $\left(n\left(\beta_n - \hat{\beta}_n\right)\left(\beta_n - \hat{\beta}_n\right)^\top\right)$ is asymptotically uniformly integrable, ensuring that that $n\mathbb{V}\left[\beta_n|\mathbf{z}_n\right] - n\hat{\mathbb{V}}_n \longrightarrow 0$. This allows us to conclude that the posterior covariance does converge as suggested above.

[22] A sequence (W_n) of random variables is asymptotically uniformly integrable if $\lim_{b\to\infty}\limsup_{n\to\infty}\mathbb{E}\left[|W_n|\,\mathbf{1}\left[|W_n| > b\right]\right] < \infty$. This essentially requires that the tails of the distribution of W_n contribute little to its mean for sufficiently large n. This is not especially difficult to obtain in realistic settings; a sufficient condition for uniform integrability (and hence asymptotic uniform integrability) is that the sequence be (uniformly) L_p-bounded for $p > 1$ (Cinlar, pp. 72); that is, $\mathbb{E}[|W_n|^p] < \infty$ for all n.

3.1 Asymptotic regimes

We will consider several asymptotic regimes in which data within each dataset are drawn independently and identically from some unknown distribution P^*, for the usual "triangular array" of data with increasing n. Recall that our model supposes that observations can take up to $M < \infty$ distinct values; m_n represents the number of distinct values in the sample.

- **Regime 0**: As n increases, the number of distinct observations in the sample, m_n, eventually exceeds our model's assumed total number of distinct values M and cause the model to fail.
- **Regime 1**: As n increases, the nondecreasing m_n eventually reaches some true number of distinct values $M^* \leq M$ in the population and stops increasing.
- **Regime 2**: As n increases, the nondecreasing m_n continues increasing without bound; however we also allow M_n to increase with n, setting $M_n = bn$ for some $b \geq 1$.
- **Regime 3**: Informally, one could consider first taking $M_n \longrightarrow \infty$, yielding a Dirichlet Process (Muliere and Secchi, 2003), then allowing n to increase, with m_n nondecreasing however we like.

Regime 0 is unlikely to pose a problem in practice. If, as is often the case in economics, observations are sampled from a population that is large but finite, then it always possible in principle to set M large enough that it will never be exceeded, particularly for sample sizes that economists will actually see. Consequently, we will set Regime 0 aside.

Regime 1 is not problematic for our model. However, a non-increasing m_n may not be particularly relevant to many economic studies, in which the true number of possible data-values M^* is much larger than n, and so m_n increases with n. Here, the problem is similar to that described in Abadie et al. (2017), who also consider increasingly large samples in a finite population. They argue that in most social science applications, the relevant asymptotic regime is not the one where the entire population is sampled.

Regime 2 is similar to that of Abadie et al. (2017), who consider a sequence of increasingly large populations as n increases. Regime 2 need not be taken as an un-Bayesian procedure to increase the prior parameter M as n increases in any given analysis; rather, it is a way of considering a sequence of analyses of increasingly large datasets, none of which has yet exhausted the set of distinct values in the model. Another interpretation of Regime 2 is that it approximates the Dirichlet Process model, a fully Bayesian model to which our model converges as M increases without bound (Muliere and Secchi, 2003), indicating that this approach is valid. Since the Dirichlet Process model is necessarily simulated using finite models like ours as approximations, our analysis will give us a concrete idea of how the simulated regression functional depends on the size (M) of our approximating model and sample size.

Regime 3 considers the case where we essentially use the Dirichlet Process itself (Muliere and Secchi, 2003). The following discussion applies to this case as well.

3.1.1 We can focus on the bootstrap-like $\tilde{\beta}$ in any asymptotic regime

Here, we will show that $\left(\frac{1}{n}\mathring{\mathbf{X}}^\top \mathring{G}\mathring{\mathbf{X}}\right)$ and $\frac{1}{\sqrt{n}}\mathring{X}^\top \mathring{G}\mathring{\boldsymbol{\epsilon}}$ in (39) and (40) vanish asymptotically, for any data-generating distribution and any asymptotic regime. This allows us to focus attention on $\tilde{\beta}$.

Consider $\frac{1}{\sqrt{n}}\mathring{X}^\top \mathring{G}\mathring{X} = \frac{1}{\sqrt{n}}\sum_{k=m_n+1}^{M_n} g_k \mathring{X}_k^\top \mathring{X}_k$, where $g_k \overset{iid}{\sim}$ Gamma$(\alpha/M_n, 1)$ and $(\mathring{y}_k, \mathring{X}_k) \overset{iid}{\sim} F$. All of the following expectations and variances are with respect to the posterior distribution. Notice that $\mathring{X}_k^\top \mathring{X}_k$ is always a $k \times k$ matrix, even in the cluster case. Suppose $\mathbb{E}[\mathring{X}_k^\top \mathring{X}_k]$ and $\mathbb{V}\left(\left(\mathring{X}_k^\top \mathring{X}_k\right)_{i,j}\right)$ are finite and constant across k for $m_n < k \leq M_n$. We will explicitly subscript m_n and M_n to emphasize that in some regimes they change with n. Then

$$\mathbb{E}[\frac{1}{\sqrt{n}}\sum_{k=m_n+1}^{M_n} g_k \mathring{X}_k^\top \mathring{X}_k] = \frac{1}{\sqrt{n}}(M_n - m_n)\frac{\alpha}{M_n}\mathbb{E}[\mathring{X}_k^\top \mathring{X}_k] \tag{41}$$

Above, we see that the mean diminishes to zero as n increases in Regimes 1, 2, and 3.

For the variance, we consider the i, j^{th} element of the matrix $\mathring{X}_k^\top \mathring{X}_k$:

$$\begin{aligned}\mathbb{V}\left(\frac{1}{\sqrt{n}}\sum_{k=m_n+1}^{M_n} g_k \left(\mathring{X}_k^\top \mathring{X}_k\right)_{i,j}\right) &= \frac{1}{n}\mathbb{V}\left(\mathbb{E}\left[\sum_{k=m_n+1}^{M_n} g_k \left(\mathring{X}_k^\top \mathring{X}_k\right)_{i,j} |\mathring{\mathbf{X}}\right]\right) \\ &+\frac{1}{n}\mathbb{E}\left[\mathbb{V}\left(\sum_{k=m_n+1}^{M_n} g_k \left(\mathring{X}_k^\top \mathring{X}_k\right)_{i,j} |\mathring{\mathbf{X}}\right)\right]\end{aligned}$$

$$\frac{1}{n}\mathbb{V}\left(\sum_{k=m_n+1}^{M_n} \frac{\alpha}{M} \left(\mathring{X}_k^\top \mathring{X}_k\right)_{i,j}\right) + \frac{1}{n}\mathbb{E}\left[\sum_{k=m_n+1}^{M_n} \frac{\alpha}{M} \left(\mathring{X}_k^\top \mathring{X}_k\right)_{i,j}^2\right] \tag{42}$$

$$\frac{1}{n}\left(\frac{\alpha}{M_n}\right)^2 (M_n - m_n)\mathbb{V}\left(\left(\mathring{X}_k^\top \mathring{X}_k\right)_{i,j}\right) + \frac{1}{n}\frac{\alpha}{M_n}(M_n - m_n)\mathbb{E}\left[\left(\mathring{X}_k^\top \mathring{X}_k\right)_{i,j}^2\right] \tag{43}$$

Similar to above, we see that the covariance matrix converges elementwise in probability to zero as n increases, in Regimes 1, 2, and 3.

By Chebyshev's inequality, $\frac{1}{\sqrt{n}}\mathring{X}^\top \mathring{G}\mathring{X} \longrightarrow \mathbf{0}$ in posterior probability.

The same reasoning applies to $\frac{1}{\sqrt{n}}\mathring{X}^\top \mathring{G}\mathring{\boldsymbol{\epsilon}}$.

By the continuous mapping theorem, we have rapid convergence:

$$\sqrt{n}\left(\beta_n - \tilde{\beta}_n\right) \xrightarrow{p} 0 \tag{44}$$

in posterior probability, for *any* data sequence with $(\mathbf{z}_n)$ with m_n and M_n behaving as specified in Regimes 1,2, or 3. This will allow us to derive asymptotic results for $\tilde{\beta}_n$ that we can transfer to β_n.

3.1.2 A Bernstein von Mises theorem

Here, we derive a central limit theorem for the Bayesian posterior distribution given a fixed sequence of datasets $\left((\mathbf{y}, \mathbf{X})_n\right)$. For conciseness, we use $\mathbb{E}[\cdot]$ and $\mathbb{V}(\cdot)$ to denote expectations and variances under the posterior distribution, leaving it implicit that we condition on $\mathbf{y}, \mathbf{X}$.

We begin by relating the Bayesian $\tilde{\beta} := \left[\tilde{\mathbf{X}}^\top \tilde{G}\tilde{\mathbf{X}}\right]^{-1} \tilde{\mathbf{X}}^\top \tilde{G}\tilde{\mathbf{y}}$ to the OLS $\hat{\beta}$. First define the residuals of the distinct observed values. $\hat{\tilde{\boldsymbol{\epsilon}}} := \tilde{\mathbf{y}} - \tilde{\mathbf{X}}\hat{\beta}$. Then $\tilde{\beta} = \hat{\beta} + \left(\tilde{\mathbf{X}}^\top \tilde{G}\tilde{\mathbf{X}}\right)^{-1} \tilde{\mathbf{X}}^\top \tilde{G}\hat{\tilde{\boldsymbol{\epsilon}}}$, so we can construct the variable that will have a normal limiting distribution:

$$\sqrt{n}\left(\tilde{\beta}-\hat{\beta}\right)=\left(\frac{1}{n}\tilde{\mathbf{X}}^\top\tilde{G}\tilde{\mathbf{X}}\right)^{-1}\frac{1}{\sqrt{n}}\tilde{\mathbf{X}}^\top\tilde{G}\hat{\tilde{\boldsymbol{\epsilon}}} \tag{45}$$

Taking this apart, we begin by considering sequences of $\frac{1}{\sqrt{n}}\tilde{\mathbf{X}}^\top\tilde{G}\hat{\tilde{\boldsymbol{\epsilon}}} = \frac{1}{\sqrt{n}}\sum_{j=1}^{m_n}\tilde{g}_j\tilde{X}_j^\top\hat{\tilde{\epsilon}}_j$, with $\tilde{g}_j \sim \text{Gamma}(n_j+\alpha/M_n,1)$. For a central limit theorem, we prefer to work with sums of n terms, so we use the fact that each Gamma-distributed $\tilde{g}_j$ is equal in distribution to the sum $\mathring{g}_j+\sum_{i=1}^{n_j}g_{ji}$ for independent $\mathring{g}_j \sim \text{Gamma}(\alpha/M_n,1)$ and $g_{ji}\sim\text{Gamma}(1,1)$, so with a slight abuse of notation,

$$\frac{1}{\sqrt{n}}\tilde{\mathbf{X}}^\top\tilde{G}\hat{\tilde{\boldsymbol{\epsilon}}}=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}g_jX_j^\top\hat{\epsilon}_j+\frac{1}{\sqrt{n}}\sum_{j=1}^{m_n}\mathring{g}_j\tilde{X}_j^\top\hat{\tilde{\epsilon}}_j \tag{46}$$

Keeping the data fixed, this has following posterior mean:

$$\bar{\mu}_n=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}X_j^\top\hat{\epsilon}_j+\frac{\alpha}{M_n}\frac{1}{n}\sum_{i=1}^{m_n}\tilde{X}_i^\top\hat{\tilde{\epsilon}}_i=\frac{\alpha}{M_n}\frac{1}{n}\sum_{i=1}^{m_n}\tilde{X}_i^\top\hat{\tilde{\epsilon}}_i \tag{47}$$

where the first term vanishes due to the orthogonality of X and the residuals. The posterior variance is:

$$\bar{Q}_n=\frac{1}{n}\sum_{i=1}^{n}X_i^\top\hat{\epsilon}_i\hat{\epsilon}_i^\top X_i+\frac{\alpha}{M_n}\frac{1}{n}\sum_{i=1}^{m_n}\tilde{X}_i^\top\hat{\tilde{\epsilon}}_i\hat{\tilde{\epsilon}}_i^\top\tilde{X}_i \tag{48}$$

Notice that the first term of (48) is the center of the robust covariance estimator.

We assume that we are Regimes 1, 2, or 3 , and that the following assumptions hold:

1. Non-colinearity: $\frac{1}{n}\sum_{i=1}^{n}X_i^\top X_i \longrightarrow \mathbb{E}^*[X^\top X]$, where $\mathbb{E}^*[X^\top X]$ is positive definite.

2. Bounded mixed fourth moments of X: $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(X_i^\top X_i\right)_{j,k}^2<\infty$ for all j , k.

3. Second moments of distinct residuals vanish when multiplied by $\frac{\alpha}{M_n}\frac{m_n}{n}$:

$$\left(\frac{\alpha}{M_n}\frac{m_n}{n}\right)\left(\frac{1}{m_n}\sum_{i=1}^{m_n}\tilde{X}_i^\top\hat{\tilde{\epsilon}}_i\right)\longrightarrow 0 \tag{49}$$

4. Fourth mixed moments of distinct residuals vanish when multiplied by $\frac{\alpha}{M_n}\frac{m_n}{n}$:

$$\left(\frac{\alpha}{M_n}\frac{m_n}{n}\right)\left(\frac{1}{m_n}\sum_{i=1}^{m_n}\tilde{X}_i^\top\hat{\tilde{\epsilon}}_i\hat{\tilde{\epsilon}}_i^\top\tilde{X}_i\right)\longrightarrow 0 \tag{50}$$

5. Asymptotic variance is well defined: $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i^\top\hat{\epsilon}_i\hat{\epsilon}_i^\top X_i=Q$, where Q is positive definite and finite.

6. Levy condition: For all i, $\lim_{n\to\infty}\left(n\bar{Q}_n\right)^{-1}\mathbb{V}(w_i)=\mathbf{0}$

Assumption 3) implies $\bar{\mu}_n \longrightarrow 0$. Assumption 4) implies $\bar{Q}_n \longrightarrow Q$. Then Assumptions 3-6 provide the conditions for the multivariate Lindberg-Levy Central Limit Theorem (Greene, 2012), so that as $n \longrightarrow \infty$,

$$\frac{1}{\sqrt{n}} \tilde{\mathbf{X}}^\top \tilde{G} \hat{\tilde{\boldsymbol{\epsilon}}} | \mathbf{y}_n, \mathbf{X}_n \xrightarrow{d} \mathcal{N}(0, Q) \tag{51}$$

Now we turn to the other constituent of $\tilde{\beta}$ and show that it converges in posterior probability. Assumption 1) implies $\mathbb{E}[\frac{1}{n}\tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{X}}] \longrightarrow \mathbb{E}^*[X^\top X]$. Since $\mathbb{V}\left(\sum_{i=1}^m \left(\frac{1}{n}\tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{X}}\right)_{j,k}\right) = \frac{1}{n^2}\sum_{i=1}^n \left(X_i^\top X_i\right)_{j,k}^2 + \frac{1}{n^2}\frac{\alpha}{M_n}\sum_{i=1}^{m_n}\left(\tilde{X}_i^\top \tilde{X}_i\right)_{j,k}^2$, and moreover $0 \leq \sum_{i=1}^{m_n}\left(\tilde{X}_i^\top \tilde{X}_i\right)_{j,k}^2 \leq \sum_{i=1}^{m_n} n_j \left(\tilde{X}_i^\top \tilde{X}_i\right)_{j,k}^2 = \sum_{i=1}^n \left(X_i^\top X_i\right)_{j,k}^2$, Assumption 2) implies $\mathbb{V}\left(\sum_{i=1}^m \left(\frac{1}{n}\tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{X}}\right)_{j,k}\right) \longrightarrow 0$. Then by Chebyshev's inequality, 1) and 2) imply the following convergence in posterior probability as $n \longrightarrow \infty$:

$$\frac{1}{n} \tilde{\mathbf{X}}^\top \tilde{G} \tilde{\mathbf{X}} \xrightarrow{p} \mathbb{E}^*[X^\top X] \tag{52}$$

By Slutsky's Lemma and the continuous mapping theorem, (51) and (52) imply:

$$\sqrt{n}\left(\tilde{\beta} - \hat{\beta}\right) | \mathbf{y}_n, \mathbf{X}_n \xrightarrow{d} \mathcal{N}\left(0, \mathbb{E}^*[X^\top X]^{-1} Q \mathbb{E}^*[X^\top X]^{-1}\right) \tag{53}$$

By (44), we can substitute β for $\tilde{\beta}$, above.

Discussion of the conditions

Assumption 1), which ensures non-collinearity, Assumption 2, which assumes that the fourth mixed moments of the X distribution are finite, and Assumption 5), which ensures a well-defined asymptotic variance, are all standard assumptions for frequentist robust regression, holding $\mathbb{P}_*^\infty$-almost surely under regularity conditions used by White (1980b). Assumptions 3) and 4) are not standard, but they clearly hold $\mathbb{P}_*^\infty$-almost surely under the White (1980b) regularity conditions in Regime 1 (where m_n eventually reaches an upper bound less than M), Regime 2 (where $M_n = bn$ where $b \geq 0$) and Regime 3 (the Dirichlet Process case). Given assumptions 3) and 4), Assumption 6 is simply the standard Levy condition that each variance is vanishingly small compared to the sum of variances, also holding $\mathbb{P}_*^\infty$-almost surely under standard assumptions. This allows us to conclude that (53) holds $\mathbb{P}_*^\infty$-almost surely.

4 Simulations

4.1 Heteroskedasticity

In very small samples, using our method with a well-informed (i.e. fairly accurate concerning the true slope) prior may substantially improve the coverage rate of Bayesian CIs in comparison to White's HC0 intervals. A poorly-informed prior would probably worsen the coverage rate.

In small-to-moderate samples, there may be worse coverage, perhaps due to higher-order asymptotics.

Table 1: Coverage Rates: Bayes vs. HC0 - Small sample, good prior

Coeff	Coverage Bayes Boot	Coverage Hc0
1	1	0.839
2	1	0.833

Using 10000 simulated datasets of size n = 10. Prior slope = 0.9; true slope = 1

Table 2: Coverage Rates: FDM vs. HC0 - Small sample, good prior

Coeff	Coverage Bayes Boot	Coverage Hc0
1	0.907	0.931
2	0.905	0.927

Using 10000 simulated datasets of size n = 100. Prior slope = 0.9; true slope = 1

In moderate-to-large samples, even poorly informed priors are washed out and the posterior coefficient estimates and standard errors are numerically very close to OLS with HC0, which has good coverage rates.

Table 3: Bayes vs. HC0 - one moderate sample

Coeff	OLS Estimate	Bayes Estimate	HC0 SE	Bayes SE
1	0.009	0.011	0.045	0.044
2	1.014	1.009	0.072	0.072

Using 1 simulated datasets of size n = 500. Prior slope = 0; true slope = 1

5 Discussion

Traditional Bayesian X-conditional modeling tells a story about how each y_i is generated given each x_i. We have shown that when this story is wrong about the conditional variance of y_i given x_i, the model may be highly misleading and overconfident about the true regression coefficients, at least from a frequentist perspective. In this paper, we tell a different story, about how each y_i and x_i are jointly sampled from a finite population, which we believe to be plausible on its own merits as a Bayesian description of many data-generating processes and our uncertainty about them. We also obtain some relatively broad but not universal guarantees that at least with sufficient data, the regression coefficients' credible intervals will tend to capture the true regression coefficients at the nominal rate, acheiving the correct amount of (frequentist) confidence. Both stories rely on assumptions, and as is usually the case, neither set of assumptions can be ranked as uniformly weaker than the other.[23]

[23] In X-conditional modeling, X need not be *iid* provided that it is ancillary to $y|X$.(see, e.g., Norets, 2015) This is weaker than the iid assumption we use in our joint modeling approach. On the other hand, our joint model effectively

While Buja et al. (2019) characterize OLS with robust standard errors as “assumption-lean,” we prefer instead to emphasize the usefulness of our assumptions. The discreteness and finiteness of our model may be viewed as disadvantageous, but they are simple consequences of the assumption that we are sampling from a finite population, and hence a discrete process, which is usually the case in the social sciences. Our assumed sampling process tells us how to handle clustering: simple random sampling implies a heteroskedasticity-robust Bayesian estimator, and clustered sampling implies a cluster-robust Bayesian estimator. This provides a plausible phyiscal explanation of why one would have independent error terms (in the former case), or error terms that are correlated within clusters but independent between clusters (in the latter case). We also assume that we know very little about the shape of the conditional variance function and even the conditional mean function, which is in many cases more plausible than assuming linearity and homoskedasticity.

Many other sampling designs could be contemplated in a framework similar to this one, and we may pursue some of these in a later paper. Most notable among these is sampling without replacement, which we have strong reason to believe will yield similar results to the above model, owing to the well-known convergence of the hypergeometric to the multinomial as the number of categories increases (A. Lo, 1987).

Authors with Bayesian and other model-based perspectives have raised thoughtful objections to the use of robust standard errors. King and Roberts (2015) have argued that “Robust Standard Errors Expose Methodological Problems They Do Not Fix,” to quote their paper’s title (see also Freedman, 2006; Leamer, 2010; Meng and Xie, 2014; Pelenis, 2012; Sims, 2010). These authors point out various ways that robust standard errors, whenever they substantially differ from classical standard errors (and traditional Bayesian credible intervals), provide evidence that one’s model is inefficient at best, and possibly misspecified to the point of being grossly misleading about the scientific questions at stake. We agree, as did White (1980a, p.828) in the paper that introduced robust standard errors to economics. Like White, we argue that robust standard errors and, we add, their Bayesian equivalents, can be part of a larger workflow in which the model is critiqued and amended. The Bayesian framework we use here is particularly amenable to this perspective, since we may decide that the vector of squared-error-minimizing linear coefficients β^* is not a useful estimand, or that its Bayesian counterpart β is not a useful summary of the posterior distribution, without changing the underlying Finite Dirichlet Process model, which remains weakly consistent for the true P^* under relatively broad conditions (Muliere and Secchi, 2003). Indeed, the decomposition (6) shows us that if $\mathbb{V}^{\hat{HC0}}(\hat{\beta}_2) > \hat{\mathbb{V}}^{OLS}(\hat{\beta}_2)$, then the linear model’s fit worsens specifically in the tails of the X-distribution, either due to nonlinearity or to heteroskedasticity (or both): if we seek ways to improve on the linear model, then robust standard errors (Bayesian or otherwise) show us a place to start.

However, not every model can be a stepping-stone to another model; at some point, one must stop and draw inferences. In many cases, a fully Bayesian model which provides $\sqrt{n}$-consistent heteroskedasticity-robust and cluster-robust regression estimators and confidence intervals, even if it is inefficient, is a good place to stop.

assumes less about the conditional distribution of y given X.

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