Special Topics in Logic and Security 1

Dataflow Analysis

Paul Irofti and Ioana Leuștean

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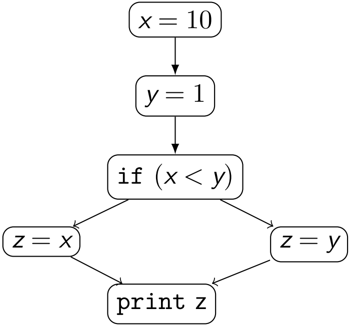
What is Static Analysis?

Static analysis involves the analysis of program properties without executing them.

Properties examples: type analysis, null pointers, unused assignments, code vulnerabilities (e.g. buffer overflow), etc.

Control Flow Graphs (CFG)

```
x = 10;
y = 1;
if  (x < y) z = x;
else        z = y;
print z
```



Control Flow Graphs (CFG)

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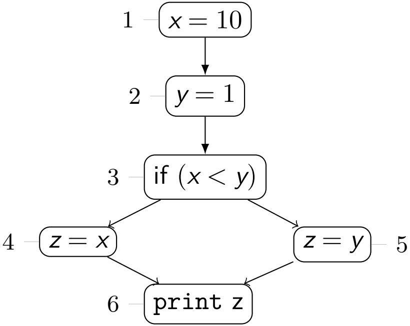
y = 1;
if  (x < y) z = x;
else        z = y;
print z
```

We label each of the graph's nodes.



Example: determining live variables

https://cs420.epfl.ch/c/06_dataflow-analysis.html

A variable is active (live) at a given point in the program's execution if it is possible for the current variable value to be read before it is overwritten.

For a node n from the CFG we will denote with v_n the set of variables that are live before the execution of n.

With that

$$v_n = \left(\bigcup\{v_i \mid i \text{ succesor of } n\} \setminus \mathit{Written}(n)\right) \cup \mathit{Read}(n)$$

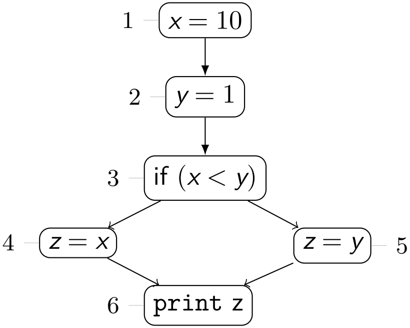
where

- $\mathit{Written}(n)$ is the set of variables that are redefined in n,
- $\mathit{Read}(n)$ is the set of variables that are read in n.

Example: determining live variables

Taking into account the above identified condition, in each node we obtain a set of constraints:

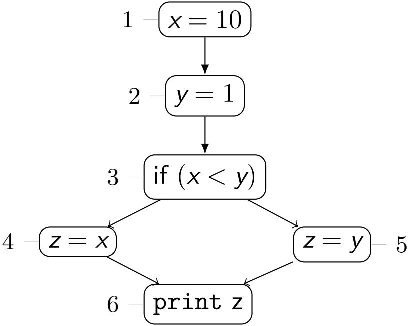
$$
\begin{array}{rcl}
v_1 & = & v_2 \setminus \{x\} \\
v_2 & = & v_3 \setminus \{y\} \\
v_3 & = & v_4 \cup v_5 \cup \{x, y\} \\
v_4 & = & (v_6 \setminus \{z\}) \cup \{x\} \\
v_5 & = & (v_6 \setminus \{z\}) \cup \{y\} \\
v_6 & = & \{z\}
\end{array}
$$



Example: determining live variables

Solving the system we obtain the solution:

$$\begin{array}{lcl} v_1 & = & \emptyset \\ v_2 & = & \{x\} \\ v_3 & = & \{x, y\} \\ v_4 & = & \{x\} \\ v_5 & = & \{y\} \\ v_6 & = & \{z\} \end{array}$$



CFG - remarks

- In the problem analysis from above, we associated a set of variables to each of the node from the control flow graph that represent the solution of a system of equations from $\mathcal{P}(Var)$, where Var is the set of variables from the entire program.
- In general, the analysis of problems from this class leads to a set of equations of latice L. Is there a generic method for solving such a system of equations?

POSET

Definition (partial order)

Let A be a set. A binary relation $R \subseteq A \times A$ is a **partial order** if:

- reflexive: $(x, x) \in R$ for any $x \in A$
- antisymmetric: $(x, y) \in R$ and $(y, x) \in R \implies x = y, \ \forall x, y \in A$
- transitive: $(x, y) \in R$ and $(y, z) \in R \implies (x, z) \in R, \ \forall x, y, z \in A$

Definition (POSET)

A partially ordered set (**poset**) is a pair (A, R), where A is a set and R is a partial order on A.

Key poset elements

Definition (infinimum and supremum)

Let A be a poset and $X \subseteq A$. The element $a \in A$ is called:

- lower bound for X if $a \leq x, \ \forall x \in X$
- upper bound for X if $x \leq a, \ \forall x \in X$
- **infimum** for X if a is the greatest lower bound; we denote $a = \inf X$
- **supremum** for X if a is the lowest upper bound; we denote $a = \sup X$

Definition (top and bottom)

Let $(A, \leq)$ be a poset. An element $e \in A$ is called:

- minimal element if $a \leq e \implies a = e$ (more than one)
- the least (**bottom**, minimum) element if $e \leq a \ \ \forall a \in A$ (unique)
- maximal element if $e \leq a \implies a = e$ (more than one)
- the greatest (**top**, maximum) element if $a \leq e \ \ \forall a \in A$ (unique)

Complete Partial Order (CPO)

Definition (chain)

A partial order is called total if any two elements are comparable. A totally ordered set (or a **chain**) is a pair $(A, \leq)$ where A is a set and $\leq$ is a total order on A.

Definition (CPO)

A complete partial order (CPO) is a poset $(C, \leq, \perp)$ such that C has a bottom element $\perp$ and $\sup X$ exists for any chain $X \subseteq C$.

Lattice

Definition

A poset $(L, \leq)$ is a **lattice** if $\exists \sup\{x_1, x_2\}, \ \inf\{x_1, x_2\}, \ \forall x_1, x_2 \in L$.
The infimum and the supremum become operations on L:

$$\vee : L \times L \to L, \quad x_1 \vee x_2 := \sup\{x_1, x_2\}$$
$$\wedge : L \times L \to L, \quad x_1 \wedge x_2 := \inf\{x_1, x_2\}$$

Proposition

The following identities hold:

- *associativity:* $(x \vee y) \vee z = x \vee (y \vee z)$, $\quad (x \wedge y) \wedge z = x \wedge (y \wedge z)$,
- *commutativity:* $x \vee y = y \vee x$, $\quad x \wedge y = y \wedge x$,
- *absorption:* $x \vee (x \wedge y) = x$, $\quad x \wedge (x \vee y) = x$.

Definition (complete)

A lattice $(L, \leq, \vee, \wedge)$ is complete if $\inf X$ and $\sup X$ exist for any $X \subseteq L$.

Monotony

Definition (monotone)

Given two posets $(A, \leq_A)$ and $(B, \leq_B)$, we say that a function $f \colon A \to B$ is monotone (increasing) if $a_1 \leq_A a_2 \implies f(a_1) \leq_B f(a_2), \forall a_1, a_2 \in A$.

Definition (continous)

Given two CPOs $(A, \leq_A)$ and $(B, \leq_B)$ we say that a function $f \colon A \to B$ is continuous if $f(\sup\{a_n \mid n \in \mathbb{N}\}) = \sup\{f(a_n) \mid n \in \mathbb{N}\}$ for any chain $\{a_n \mid n \in \mathbb{N}\}$ from A.

Remark

Continuity implies monotony.

Fixed Point

Definition (fixed point)

An element $a \in A$ is a fixed point of $f\colon A \to A$ if $f(a) = a$.

Theorem (Fixed point theorem for complete lattices (Knaster-Tarski))

If $(L, \leq)$ is a complete lattice and $F : L \to L$ is a monotone (increasing) function, then $a = \inf\{x \in L | F(x) \leq x\}$ is the least fixed point of F.

Theorem (Fixed point theorem for CPOs (Kleene))

If $(C, \leq, \perp)$ is a CPO and $F : C \to C$ is a continuous function, then $a = \sup\{F^n(\perp) | n \in N\}$ is the least fixed point of F.

Note that a exists since the sequence $F^0(\perp) = \perp \leq F(\perp) \leq F^2(\perp) \leq \cdots \leq F^n(\perp)$ is a chain.

Kleene Proof

Theorem (Fixed point theorem for CPOs (Kleene))

If $(C, \leq, \perp)$ *is a CPO and* $F : C \to C$ *is a continuous function, then* $a = \sup\{F^n(\perp) | n \in \mathbb{N}\}$ *is the least fixed point of* F*.*

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i) a is a fixed point

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i) a is a fixed point

$$
\begin{aligned}
F(a) &= F(\sup\{F^n(\perp) \mid n \in \mathbb{N}\}) \\
&= \sup\{F(F^n(\perp)) \mid n \in \mathbb{N}\} && \text{(continuity)} \\
&= \sup\{F^{n+1}(\perp) \mid n \in \mathbb{N}\} \\
&= \sup\{F^n(\perp) \mid n \in \mathbb{N}\} = a
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1. Assume b is another fixed point, i.e. $F(b) = b$.

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1. Assume b is another fixed point, i.e. $F(b) = b$.
2. We prove by induction on $n \geq 1$ that $F^n(\perp) \leq b$.

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2. We prove by induction on $n \geq 1$ that $F^n(\perp) \leq b$.
3. For $n = 0$, $F^0(\perp) = \perp \leq b$ because $\perp$ is the first element.

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1. Assume b is another fixed point, i.e. $F(b) = b$.
2. We prove by induction on $n \geq 1$ that $F^n(\perp) \leq b$.
3. For $n = 0$, $F^0(\perp) = \perp \leq b$ because $\perp$ is the first element.
4. If $F^n(\perp) \leq b$, then $F^{n+1}(\perp) \leq F(b)$ because F is increasing. Since $F(b) = b$ we get $F^{n+1}(\perp) \leq b$.

Knaster-Tarski Proof

Theorem

If $(L, \leq)$ is a complete lattice and $F : L \to L$ is a monotone (increasing) function, then $a = \inf\{x \in L | F(x) \leq x\}$ is the least fixed point of F.

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1. Let $X := \{x \in L | F(x) \leq x\} \implies a \leq x, \ \forall x \in X \implies F(a) \leq F(x) \leq x$

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2. Consequently $F(a)$ is a lower bound for $X \implies F(a) \leq a$

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3. It follows that $F(a) \in X \implies a \leq F(a)$
4. We proved that $F(a) \leq a$ and $a \leq F(a)$, so $F(a) = a$.

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ii) a is the least fixed point

1. assume b is another fixed point, i.e. $F(b) = b$

Knaster-Tarski Proof

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If $(L, \leq)$ is a complete lattice and $F : L \to L$ is a monotone (increasing) function, then $a = \inf\{x \in L | F(x) \leq x\}$ is the least fixed point of F.

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1. Let $X := \{x \in L | F(x) \leq x\} \implies a \leq x, \ \forall x \in X \implies F(a) \leq F(x) \leq x$
2. Consequently $F(a)$ is a lower bound for $X \implies F(a) \leq a$
3. It follows that $F(a) \in X \implies a \leq F(a)$
4. We proved that $F(a) \leq a$ and $a \leq F(a)$, so $F(a) = a$.

ii) a is the least fixed point

1. assume b is another fixed point, i.e. $F(b) = b$
2. then $b \in X$, so $a \leq b$.

The Product Lattice

Definition

We define the pointwise order relationship:

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \iff x_i \leq y_i,\ \forall i \in \{1, \ldots, n\}$$

Theorem

If $L_1, L_2, \ldots, L_n$ are lattices then so is the product

$$L_1 \times L_2 \times \cdots \times L_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in L_i\}$$

where the lattice order $\leq$ is defined pointwise.

Remark

Let L be a lattice and $n \geq 1$, note that $(L^n, \leq)$ is a lattice. More so, if L is a complete lattice then L^n is also a complete lattice.

Equation Systems in Complete Lattices

Let L be a complete lattice and $n \geq 1$ and $F_1, \ldots, F_n : L^n \to L$ a set of monotone functions. We want to solve the equation system:

$$\begin{array}{lcl} x_1 & = & F_1(x_1, \ldots, x_n) \\ x_2 & = & F_2(x_1, \ldots, x_n) \\ & \ldots & \\ x_n & = & F_n(x_1, \ldots, x_n) \end{array}$$

with $x_1, \ldots, x_n \in L$.

Definition

Denote the function $F : L^n \to L^n$ to be

$$F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), F_2(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$$

and observe that the system can be rewritten as

$$F(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$$

The equation system can be solved using the Fixed Point Theorem!

Determining live variables

It results that, in order to solve the system:

$$
\begin{array}{lcl}
v_1 & = & v_2 \setminus \{x\} \\
v_2 & = & v_3 \setminus \{y\} \\
v_3 & = & v_4 \cup v_5 \cup \{x, y\} \\
v_4 & = & (v_6 \setminus \{z\}) \cup \{x\} \\
v_5 & = & (v_6 \setminus \{z\}) \cup \{y\} \\
v_6 & = & \{z\}
\end{array}
$$

we need to find the solution to the equation (where v_i is denoted as x_i):

$$
F(x_1, x_2, x_3, x_4, x_5, x_6) = \\
(x_2 \setminus \{x\}\,,\ x_3 \setminus \{y\}\,,\ x_4 \cup x_5 \cup \{x, y\}\,,\ (x_6 \setminus \{z\}) \cup \{x\}\,,\ (x_6 \setminus \{z\}) \cup \{y\}\,,\ \{z\})
$$

in the complete lattice $(\mathcal{P}(Var), \subseteq, \emptyset, Var)$.

Fixed Point Theorem Solution

$F(x_1, x_2, x_3, x_4, x_5, x_6) =$
$(x_2 \setminus \{x\}\,,\; x_3 \setminus \{y\}\,,\; x_4 \cup x_5 \cup \{x, y\}\,,\; (x_6 \setminus \{z\}) \cup \{x\}\,,\; (x_6 \setminus \{z\}) \cup \{y\}\,,\; \{z\})$

Determining the least fixed-point:

	x_1	x_2	x_3	x_4	x_5	x_6
$\perp$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$F(\perp)$	$\emptyset$	$\emptyset$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$
$F^2(\perp)$	$\emptyset$	$\{x\}$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$
$F^3(\perp)$	$\emptyset$	$\{x\}$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$

where $\perp = (\emptyset, \ldots, \emptyset)$

IMP and live variables

Let the set of the variables successor to the node n be:

$$Join(v_n) = \bigcup \{ v_i \mid i \text{ succesor of } n \}$$

Then, the general formulation for live variables in node k is:

$$v_n = Join(v_n) \setminus Written(n) \cup Read(n)$$

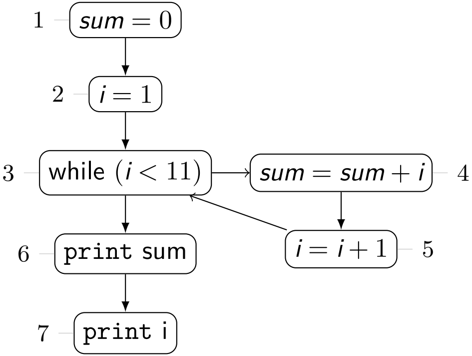
Particular formulations for IMP instructions:

- Expressions (E) — E — `x + 3, (x > 7)`
- Instruction blocks
  - Assignment — `x = E;` — $v_n = Join(v_n) \setminus \{x\} \cup Var(E)$
  - Conditional — `if (E)` — $v_n = Join(v_n) \cup Var(E)$
  - Loops — `while (E)` — $v_n = Join(v_n) \cup Var(E)$
  - Output — `print E;` — $v_n = Join(v_n) \cup Var(E)$

where $Var(E)$ represents the set of variables present in expression E.

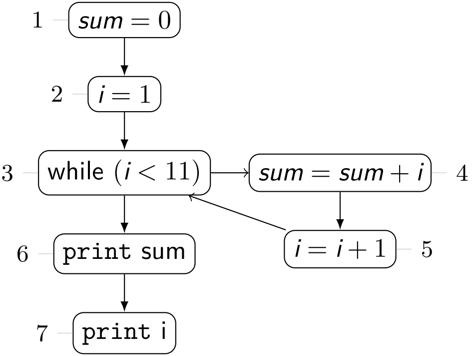
Control Flow Graphs (CFG)

```
sum = 0;
i = 1;
while (i < 11) {
    sum = sum + i;
    i = i + 1;
}
print sum
print i
```



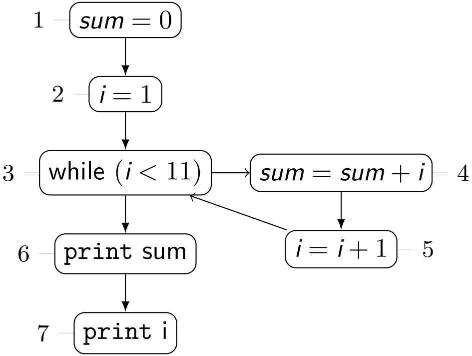
Control Flow Graphs (CFG)

$$
\begin{aligned}
v_1 &= v_2 \setminus \{sum\} \\
v_2 &= v_3 \setminus \{i\} \\
v_3 &= v_4 \cup v_6 \cup \{i\} \\
v_4 &= (v_5 \setminus \{sum\}) \cup \{i, sum\} \\
v_5 &= (v_3 \setminus \{i\}) \cup \{i\} \\
v_6 &= v_7 \cup \{sum\} \\
v_7 &= \{i\}
\end{aligned}
$$



Control Flow Graphs (CFG)

$$
\begin{aligned}
v_1 &= \emptyset \\
v_2 &= \{sum\} \\
v_3 &= \{i, sum\} \\
v_4 &= \{i, sum\} \\
v_5 &= \{i, sum\} \\
v_6 &= \{i, sum\} \\
v_7 &= \{i\}
\end{aligned}
$$



Solving with the Fixed Point Theorem

$F(x_1, x_2, x_3, x_4, x_5, x_6, x_7) =$
$(x_2 \setminus \{sum\}, x_3 \setminus \{i\}, x_4 \cup x_6 \cup \{i\}, (x_5 \setminus \{sum\}) \cup \{i, sum\}, (x_3 \setminus \{i\}) \cup \{i\}, x_7 \cup \{sum\}, \{i\})$

Determining the least fixed-point:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$\perp$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$F(\perp)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{sum\}$	$\{i\}$
$F^2(\perp)$	$\emptyset$	$\emptyset$	$\{sum, i\}$	$\{sum, i\}$	$\emptyset$	$\{sum\}$	$\{i\}$
$F^3(\perp)$	$\emptyset$	$\{sum\}$	$\{sum, i\}$	$\{sum, i\}$	$\{sum, i\}$	$\{sum\}$	$\{i\}$

where $\perp = (\emptyset, \ldots, \emptyset)$

Exercise

Perform liveness analysis for the following program:

```
x = 42;
while (x > 1) {
    y = x / 2;
    if (y > 3)
        x = x - y;
    z = x - 4;
    if (z > 0)
        x = x / 2;
    z = z - 1;
}
print x
```

Algorithms for determining the *lfp*

In order to obtain the least-fixed point (*lfp*) we use the following chain until we converge:

$$\mathbf{F}^0(\perp) = \perp \leq \mathbf{F}(\perp) \leq \mathbf{F}^2(\perp) \leq \cdots \leq \mathbf{F}^n(\perp) \leq \cdots$$

Which can be formalized in pseudo-code as:

Algorithm 1: NaiveLFP

```
x = (⊥, ⊥, ..., ⊥);
while x ≠ F(x) do
    x = F(x);
return x;
```

Complexity: depends on the height of lattice L^n and the evaluation cost of $\boldsymbol{F}(\boldsymbol{x})$.

Convergence: requires n iterations.

Algorithms for determining the *lfp*

Algorithm 1 does not profit from the lattice structure:

- recomputes the inputs that were not changed in the former iteration
- does not take into consideration the inputs that were already computed in the current iteration

In the first example we had:

$F(x_1, x_2, x_3, x_4, x_5, x_6) =$
$(x_2 \setminus \{x\}\,,\ x_3 \setminus \{y\}\,,\ x_4 \cup x_5 \cup \{x, y\}\,,\ (x_6 \setminus \{z\}) \cup \{x\}\,,\ (x_6 \setminus \{z\}) \cup \{y\}\,,\ \{z\})$

and the fixed-point is computed in 6 iterations

	x_1	x_2	x_3	x_4	x_5	x_6
$\perp$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$F(\perp)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{z\}$
$F^2(\perp)$	$\emptyset$	$\emptyset$	$\emptyset$	$\{x\}$	$\{y\}$	$\{z\}$
$F^3(\perp)$	$\emptyset$	$\emptyset$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$
$F^4(\perp)$	$\emptyset$	$\{x\}$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$
$F^5(\perp)$	$\emptyset$	$\{x\}$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$

Comparisson of *lfp* algorithms

$$F(\boldsymbol{x}) = (x_2 \setminus \{x\}\,,\; x_3 \setminus \{y\}\,,\; x_4 \cup x_5 \cup \{x, y\}\,,\; (x_6 \setminus \{z\}) \cup \{x\}\,,\; (x_6 \setminus \{z\}) \cup \{y\}\,,\; \{z\})$$

Ideally, an efficient algorithm would reduce the number of iterations:

	x_1	x_2	x_3	x_4	x_5	x_6
$\bot$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$F(\bot)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{z\}$
$F^2(\bot)$	$\emptyset$	$\emptyset$	$\emptyset$	$\{x\}$	$\{y\}$	$\{z\}$
$F^3(\bot)$	$\emptyset$	$\emptyset$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$
$F^4(\bot)$	$\emptyset$	$\{x\}$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$
$F^5(\bot)$	$\emptyset$	$\{x\}$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$

versus

	x_1	x_2	x_3	x_4	x_5	x_6
$\bot$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$F(\bot)$	$\emptyset$	$\emptyset$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$
$F^2(\bot)$	$\emptyset$	$\{x\}$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$
$F^3(\bot)$	$\emptyset$	$\{x\}$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{z\}$

Round Robin *lfp*

Gauss-Seidel-like component-wise updates:

- computes x_i based on $x_{j<i}$ from the current iteration
- *lfp* is attained even though the outer iteration may provide a different output than the outer iteration in Alg. 1

Algorithm 2: RoundRobinLFP

$(x_1, x_2, \ldots, x_n) = (\bot, \bot, \ldots, \bot)$;
while $(x_1, x_2, \ldots, x_n) \neq \boldsymbol{F}(x_1, x_2, \ldots, x_n)$ **do**
    for $i \in 1 \rightarrow n$ **do**
        $x_i = \boldsymbol{F}_i(x_1, x_2, \ldots, x_n)$;
return $(x_1, x_2, \ldots, x_n)$;

Complexity: depends on the height of lattice L^n and the cost of evaluating $\boldsymbol{F}_i(\boldsymbol{x})$.

Convergence: requires at most n iterations.

Chaotic *lfp*

Stochastic component wise updates:

- the order of operations is not important in Alg. 2
- computes x_i based on the x_j's from the current iteration; in no particular order
- Alg. 2 still recomputes the unchanged inputs from the former iteration
- the constraints need to be applied until convergence

Algorithm 3: ChaoticLFP

$(x_1, x_2, \ldots, x_n) = (\perp, \perp, \ldots, \perp)$;
while $\underline{(x_1, x_2, \ldots, x_n) \neq \boldsymbol{F}(x_1, x_2, \ldots, x_n)}$ **do**
  $i = \mathit{random}(1 \to n)$;
  $x_i = \boldsymbol{F}_i(x_1, x_2, \ldots, x_n)$;
return $(x_1, x_2, \ldots, x_n)$;

Complexity: first, it depends on how i is chosen at each iteartion, then it depends on the height of lattice L^n and then on the cost of evaluating $\boldsymbol{F}_i(\boldsymbol{x})$.

Convergence: the algorithm is not guaranteed to stop, but if it does the result is correct.

SimpleWorkList *lfp*

Define the map $dep : Nodes \to 2^{Nodes}$, where $dep(v)$ is the set of nodes whose information depends on the information of node v.

Algorithm 4: SimpleWorkListLFP

$(x_1, x_2, \ldots, x_n) = (\bot, \bot, \ldots, \bot)$;
$W = \{v_1, v_2, \ldots, v_n\}$;
while $\underline{W \neq \emptyset}$ **do**
    $v_i = W.removeNext()$;
    $y = \boldsymbol{F}_i(x_1, x_2, \ldots, x_n)$;
    if $\underline{y \neq x_i}$ **then**
        $x_i = y$;
        for $\underline{v_j \in dep(v_i)}$ **do**
            W.add(v_j);
return $(x_1, x_2, \ldots, x_n)$;

The functions *removeNext*() and *add*() choose a random node from W and add a node in W, respectively.

SimpleWorkList *lfp*

Properties

- computes only the particular constraints for the current node
- does not recompute the inputs unchanged during the former iteration
- computes x_i based on the x_j's from the current iteration; in no particular order
- constraints are applied until convergence
- each iteration has the same effect as the Alg. 3 iteration

Complexity: depends on the number of nodes n, the lattice height h, and the cost of evaluating $\boldsymbol{F}_i(\boldsymbol{x})$.

Convergence: a single iteration either decreases the work volume in W or climbs up the lattice; the algorithm stops in finite time because the lattice height is finite and the iterations stop when W is void.