Models of Peano Arithmetic

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Preface

Nonstandard models of (first-order) Peano arithmetic were first constructed by Skolem (1934), and (implicitly) by Gödel (1930, 1931) using his incompleteness and completeness theorems. The subject can be said to have come of age with Harvey Friedman's work on *initial segments* of nonstandard models, when he realized the importance of the standard system of a model, rediscovering and extending Scott's work (Scott 1962) on what are today called Scott sets in the process. The most celebrated advance in the subject is the Paris–Harrington theorem (Paris and Harrington 1977), arising from work by Paris and Kirby on *indicators*, which in turn was inspired by Friedman's work. At roughly the same time, we have a line of research into extensions of models of Peano arithmetic and types over such models, starting with the MacDowell–Specker theorem (MacDowell and Specker 1961) and followed by the work of Gaifman (his *splitting theorem* (1972) and his construction of a *definable type* (1976)) and others. In yet another line of attack, initiated by Abraham Robinson, and continued by Krajewski, Kotlarski, Lachlan, and others, we have the notion of *satisfaction classes* over a model of Peano arithmetic, and this links the subject up with the study of *recursively saturated* and *resplendent* models of Barwise, Schlipf, and Ressayre.

This book is about these developments in the model theory of Peano arithmetic, and in particular the interplay between the first-order theory, recursion-theoretic aspects of the language, and the structural properties of models. I have tried to make the presentation as straightforward as possible, with a first-year postgraduate reader in mind; however, much of the material here is at 'undergraduate level' and, on the other hand, there are also more difficult exercises (usually with hints) and—later on—more advanced material to keep other readers happy too.

I have kept to standard or self-explanatory notation where possible, and have given the most elementary proofs available, even if this means repeating ideas and arguments occasionally. Sometimes shorter or slicker proofs are possible, but often these obscure the essential ideas at hand. The advantages of this 'straightforward' approach are firstly, that (I hope) the material will be more accessible for a reader new to the subject (for whom a certain amount of repetition of similar ideas in different situations will be no bad thing), and secondly, that a more advanced reader will be able to dip into the book at almost any point. This last advantage is especially important for the chapter on recursive saturation (Chapter 15), a subject which arises naturally out of consideration of nonstandard models but which has applications throughout model theory. Having said this, though, a reader who takes the book in its proper sequence will find a beautiful story emerging that links model theory and recursion theory in many surprising ways.

The prerequisites for reading this book are set out in Chapter 0, and will probably be fewer than might at first be imagined. On the model-theoretic side, a basic knowledge of completeness, compactness, and elementary chains of models for first-order languages suffices. In particular the omitting-types theorem is *not* assumed, although many of the proofs here will use a similar kind of argument to that in the proof of omitting types. On the recursion-theoretic side even less background is needed: a basic knowledge of the notions 'primitive recursive', 'recursive', 'r.e.', and these notions relativized to an oracle suffices.

There are two main points where recursion theory comes into the discussion of non-standard models. The first is concerned with the language of arithmetic, $\mathscr{L}_A$, and initial segments of models. It turns out that the natural numbers $\mathbb{N}$ form an initial segment of each non-standard model of arithmetic, and a natural class of formulas of $\mathscr{L}_A$, the Σ_1 formulas, coincides with the r.e. sets on $\mathbb{N}$. Moreover, the truth of Σ_1 formulas is 'preserved' when one passes from validity in an initial segment of a model to validity in the full model. The second point is concerned with the notion of proof in the predicate calculus. (The reader is assumed to have encountered some such notion, although the exact definition will not matter for our purposes.) The important fact about such notions is that they are recursively 'checkable' and 'enumerable'. Thus there is a suitable coding (or 'Gödel numbering') of formulas and proofs into integers in $\mathbb{N}$ so that a computer program could verify that a given $n \in \mathbb{N}$ is the code for a valid proof of a certain sentence, and a second computer program could be written that lists all such codes of valid proofs. The reader will see both these points at work throughout this book.

I learned most of the material presented below during 1984–7 whilst I was a research student in Manchester. Lecture notes from courses by Jeff Paris (on models of arithmetic) and George Wilmers (on recursive saturation) were extremely helpful in the preparation of this text, and the debt I owe these two people will be clear from my presentation below. There have been many other papers that I have consulted for specific parts of the text below, and I thank these authors too. I would also like to thank Angus Macintyre and Alex Wilkie for their interest and encouragement, and all the people who looked through preliminary drafts of this text and suggested improvements.

The quotations at the head of each chapter are intended to amuse the reader and lighten the text. They may only have a slight relevance to the chapter concerned, but I hope you will enjoy them!

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Contents

0

Background

'Consider it logically,' Valdez said. 'What is my problem? *To find Cathy*. What do I know about Cathy? Nothing.'

'That doesn't sound so good,' Marvin said.

'But it is only half of the problem. Granted that I know nothing about Cathy, what do I know about *Finding*?'

'What?' Marvin asked.

'It happens that I know everything about Finding,' Valdez said triumphantly, gesturing with his graceful terracotta hands. 'For it happens that I am an expert in the Theory of Searches!'

Robert Sheckley
Mindswap

Although the reader is not expected to know anything about nonstandard models of arithmetic, we *will* assume a little knowledge of model theory and arithmetic (or rather, recursion theory.) The purpose of this chapter is to review this background material. But first, a few general remarks.

In the text, 's.t.' is an abbreviation for 'such that', 'iff' is an abbreviation for 'if and only if', and □ denotes the end (or absence) of a proof. The exercises are part of the text, so should be read, and preferably attempted! An * against an exercise indicates that the result will be used later in the text, a + indicates that the exercise is slightly harder than usual. To prevent possible criticism on grounds of unfairness, these two categories of exercises usually have hints, either in the wording of the question or in brackets afterwards. Some exercises require some extra background knowledge (from algebra or set theory, for example). These are intended only for readers with this background.

Our metatheory can be taken to be ZFC throughout, and our set-theoretic notation is standard. The only set-theoretic results needed are elementary results of cardinal arithmetic, such as $\kappa^2=\kappa$ for an infinite cardinal κ. Cardinal numbers are thought of as initial ordinals in the usual way. We use the aleph notation $\aleph_0, \aleph_1, \ldots, \aleph_\alpha, \ldots$ for cardinals, or (very occasionally) the alternative notation $\omega_0, \omega_1, \ldots, \omega_\alpha, \ldots$ when we want to emphasize the ordinal corresponding to a given cardinal. Thus ω_α can be thought of as the least order-type of a well-ordering of cardinality $\aleph_\alpha$.

One final (non-mathematical) remark: in *Mindswap*, Valdez *does* eventually find Cathy!

0.1 BACKGROUND MODEL THEORY

We will assume the reader has some basic knowledge of model theory. In fact, Jane Bridge's book (Bridge 1977) is quite sufficient, although other texts are also suitable, such as books specializing in model theory (for example, Chang and Keisler 1973, Chapter 1 and parts of Chapters 2 and 3), or certain chapters from books about mathematical logic in general (for example, Barwise 1977, Chapters A1 and A2). The purpose of this section is to fix some notation and state the model-theoretic results that we shall require later.

Models are structures for a first-order language, and so consist of a non-empty set (called the *domain*) together with various *relations, functions*, and privileged elements (called *constants*). We shall denote models by the letters $M, N, K, L, \ldots$. If M is a model then the domain of M will also be denoted M. The *cardinality of* M, $\mathrm{card}(M)$, denotes the cardinality of the *domain* of M.

Finite sequences of elements of a model (popularly called *tuples*) are denoted $\bar{a}, \bar{b}, \ldots$. Thus '$\bar{a} \in M$' means '$\bar{a}$ is a sequence of elements of the domain of M of finite length'.

Traditionally, model theory is the study of the interplay between the mathematical structure of a model and its first-order language. All first-order languages in this book contain the following *logical symbols*:

Boolean operations ('and', 'or', and 'not'): $\wedge, \vee, \neg$;
equality: $=$;
quantifiers: $\exists, \forall$;
variables*: $\mathsf{v}_0\ \mathsf{v}_1\ \mathsf{v}_2 \ldots \mathsf{v}_i \ldots$;
brackets: ( );

together with a *nonlogical symbol* for each function, relation, and constant. (Note that equality is *always* present.) In practice we shall use $x, y, z, u, v, w, \ldots$ (etc.) as variables (which should be considered as suitable v_is) and occasionally different style brackets such as { } and [ ] in place of ( ) when writing first-order formulas and sentences.

The implication symbols, $\rightarrow$ and $\leftrightarrow$, and the 'there exists a unique' symbol, $\exists!$, are introduced in the usual way as the following shorthands:

$$\begin{array}{rll} \varphi \rightarrow \psi & \text{for} & \neg\varphi \vee \psi; \\ \varphi \leftrightarrow \psi & \text{for} & (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi); \\ \exists! x \psi(x) & \text{for} & \exists x(\psi(x) \wedge \forall y(\psi(y) \rightarrow y = x)). \end{array}$$

Brackets are often omitted with the conventions that $\neg$ is more binding than $\wedge$ or $\vee$, which in turn are more binding than $\rightarrow$ and $\leftrightarrow$. Notice too the use of $\psi(y)$ to mean 'the formula obtained by substituting y for all free

* Here we are thinking of each variable as a separate symbol of the language. An alternative approach is to have two symbols, v and $'$, and think of v_i as the string of symbols $\mathsf{v}'' \cdots '$ (i 's).

occurrences of x in $\psi(x)$'. We also employ the notations $\bigwedge_{i=1}^{n}\varphi_i$ and $\bigvee_{i=1}^{n}\psi_i$ for $(\varphi_1\wedge(\varphi_2\wedge\cdots\wedge(\varphi_n)\cdots))$ and $(\psi_1\vee(\psi_2\vee\cdots\vee(\psi_n)\cdots))$ respectively, and often write $t\neq s$ in place of $\neg(t=s)$.

If M is a structure for the first-order language $\mathscr{L}$, $\varphi(\bar{x})$ is an $\mathscr{L}$-formula with free-variables among $\bar{x}=(x_0, x_1, \ldots, x_n)$, and $\bar{a}=(a_0, a_1, \ldots, a_n)\in M$, then we write $M\models\varphi(\bar{a})$ to mean 'φ is true in M when each variable x_i is interpreted by a_i' using Tarski's well-known definition of truth. (NB, $=$ is *always* interpreted by equality in M.) Similarly if $\Phi(\bar{x})$ is a set of formulas in $\bar{x}$, $M\models\Phi(\bar{a})$ means $M\models\varphi(\bar{a})$ for each $\varphi(\bar{x})\in\Phi(\bar{x})$. If t is a closed term in $\mathscr{L}$ ('closed' $\doteq$ 'has no free-variables'), t^M denotes its interpretation (also called its *realization*) in M. Similarly, if s is a function, relation, or constant symbol of $\mathscr{L}$, then s^M denotes its interpretation in M. Usually, however, we will drop the superscript M and denote t^M or s^M by just t or s, unless confusion may arise from so doing.

If $\mathscr{L}$, $\mathscr{L}'$ are first-order languages, and every non-logical symbol of $\mathscr{L}$ is also a symbol of $\mathscr{L}'$, then an $\mathscr{L}'$-structure M may be converted into an $\mathscr{L}$-structure $M\restriction\mathscr{L}$ (called the *reduct* of M to $\mathscr{L}$) by 'forgetting' about the extra relation, function, or constant symbols in $\mathscr{L}'$. In the other direction, an $\mathscr{L}$-structure N may be converted to an $\mathscr{L}'$-structure by adding new relations, functions, and constants corresponding to the extra symbols in $\mathscr{L}'$. For example, if relations $R_1, \ldots, R_n$, functions $f_1, \ldots, f_k$, and constants $c_1, \ldots, c_l$ were added to N, the notation of the new structure would be $(N, R_1, \ldots, R_n, \ldots, f_1, \ldots, f_k, c_1, \ldots, c_l)$ and would be said to be an *expansion* of N. Of special importance is the structure $(N, a)_{a\in N}$ obtained by adding a constant for every $a\in N$.

A *theory* in a first-order language $\mathscr{L}$ is simply a set of sentences of $\mathscr{L}$. (These sentences are thought of as being the *axioms* of the theory.) We assume that the reader has encountered some notion of a *formal proof* in the predicate calculus, and if T is an $\mathscr{L}$-theory and σ is an $\mathscr{L}$-sentence we write $T\vdash\sigma$ to mean 'there is a proof of σ from sentences in T'. Similarly, if S is a set of sentences of $\mathscr{L}$ we write $T\vdash S$ to mean $T\vdash\sigma$ for each $\sigma\in S$. T is said to be *consistent* iff for no $\mathscr{L}$-sentence σ do we have both $T\vdash\sigma$ and $T\vdash\neg\sigma$. The two turnstile symbols, $\vdash$ and $\models$, are related by the following theorem of Gödel's.

THEOREM 0.1 (The completeness theorem). Let T be a theory in the first-order language $\mathscr{L}$ and let σ be an $\mathscr{L}$-sentence. Then $T\vdash\sigma$ iff for all models M for $\mathscr{L}$: $M\models T\Rightarrow M\models\sigma$. □

The most basic result in model theory is the *compactness theorem*.

THEOREM 0.2 (Compactness). If T is an $\mathscr{L}$ theory, then T has a model iff every finite subset $S\subseteq T$ has a model. □

If M, N are models for the same first-order language $\mathscr{L}$, then M is a *submodel* of N (or a *substructure* of N), $M\subseteq N$, iff the domain of M is a subset of the domain of N containing the constants of N and closed under the functions of N, and each non-logical symbol in $\mathscr{L}$ is interpreted in M according to the restriction of its interpretation in M. (Unless stated otherwise, $M \subseteq N$ will mean 'M is a submodel of N' rather than 'the domain of M is included in the domain of N'.) M is an *elementary* submodel of N, $M < N$, iff $M \subseteq N$, and for each formula $\varphi(\bar{x})$ and each $\bar{a} \in M$

$$M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\bar{a}).$$

In particular, if $M < N$ then M and N satisfy the same sentences; the converse of this statement need not be true, however.

We can now state the two Löwenheim–Skolem theorems.

THEOREM 0.3 (The downward Löwenheim–Skolem theorem). If the first-order language $\mathscr{L}$ has at most $\lambda \geqslant \aleph_0$ symbols and N is an $\mathscr{L}$-structure of cardinality $\kappa > \lambda$, then there is an elementary substructive $M < N$ with $\text{card}(M) = \lambda$. □

THEOREM 0.4 (The upward Löwenheim–Skolem theorem). If $\mathscr{L}$ has at most $\lambda \geqslant \aleph_0$ symbols, M is an $\mathscr{L}$-structure and κ a cardinal with $\kappa \geqslant \text{card}(M) \geqslant \lambda$, then there is a proper elementary extension $N > M$ of M with $\text{card}(N) = \kappa$. □

A few words about the machinery used to prove Theorems 0.3 and 0.4 are in order here. It is useful to have some sort of criterion for deciding when a model M is an elementary submodel of another model N. The following turns out to be useful.

THEOREM 0.5 (The Tarski–Vaught test for elementary extensions). Let $M \subseteq N$ be models for the same language $\mathscr{L}$. Then the following are equivalent:

(a) $M < N$;

(b) for each $\mathscr{L}$-formula $\varphi(\bar{x}, y)$ and for each $\bar{a} \in M$,

$$N \models \exists y \varphi(\bar{a}, y) \Rightarrow \text{there exists } b \in M \text{ s.t. } N \models \varphi(b, \bar{a}). \qquad \square$$

Theorem 0.3 follows by putting just enough into M to satisfy $\text{card}(M) = \lambda$ and condition (b) of Theorem 0.5.

The proof of the upward Löwenheim–Skolem theorem rests on the idea of the *complete diagram* of a model M, that is, the set of all sentences true in the structure $\bar{M} = (M, a)_{a \in M}$. If $\bar{N}$ satisfies the complete diagram of M, then there is a canonical embedding $f: \bar{M} \to \bar{N}$ given by sending each $a \in M$ to the element of $\bar{N}$ realizing the constant corresponding to a. If N is the reduct of $\bar{N}$ to the original language, then it is easy to see that f sends M

onto an elementary substructure of N. The proof of Theorem 0.4 now goes by showing that there is an $\bar{N}$ satisfying the complete diagram of M and having the appropriate cardinality, using the compactness theorem. To ensure that N is a *proper* elementary extension, we just make $\bar{N}$ satisfy the set of sentences $\{c \neq a | a \in M\}$ for another new constant symbol c.

We conclude this section with the lemma on *elementary chains* of models.

LEMMA 0.6. If $M_0 < M_1 < M_2 < \cdots < M_i < M_{i+1} \cdots$ are $\mathscr{L}$-structures, then the union of this chain, $M_\infty = \bigcup_i M_i$, is an $\mathscr{L}$-structure and is an elementary extension of each M_i. □

In most applications of Lemma 0.6 the indexing variable i will range over the non-negative integers, $\mathbb{N}$. Occasionally though, we will need i to range over ordinals less than some given ordinal α: in this case $M_\infty = \bigcup_{i<\alpha} M_i$ and the proof of the lemma is by transfinite induction on ordinals less than α.

0.2 BACKGROUND RECURSION THEORY

On occasions throughout this hook we shall be concerned whether a set $A \subseteq \mathbb{N}^k$ is *computable* (or, as we shall usually say, *recursive*). We are thinking here of an idealized computer rigorously carrying out a fixed (finite) set of instructions. The computer is finite and is only able to use a finite amount of memory at a given time, but the situation is idealized in the senses that, firstly, we suppose that it always has enough memory to complete the calculation in hand and, secondly, we are always able to wait as long as is needed for the computation to finish. The set A is then *computable* is there is such a computer which, on any input $\bar{n} \in \mathbb{N}^k$, eventually returns an answer, 'yes' or 'no', according to whether $\bar{n} \in A$ or not.

There are many ways to formalize this notion, each with its own advantages and disadvantages. (See Cutland (1980), Chapter 3, for a discussion of several of the machine models available.) For technical reasons the following definitions will be convenient.

DEFINITION. The class of *partial recursive functions*, $\mathscr{C}$, is the least class of functions $f: A \to \mathbb{N}$ for some $A \subseteq \mathbb{N}^k$ and some $k \geq 1$, such that:

(a) $\mathscr{C}$ contains the *zero* and *successor* functions,

$$0(x) = 0 \text{ for all } x \in \mathbb{N},$$

$$S(x) = x + 1 \text{ for all } x \in \mathbb{N},$$

and, for each $1 \leq i \leq n \in \mathbb{N}$, $\mathscr{C}$ contains the *projection* function

$$U_i^n(x_1, x_2, \ldots, x_n) = x_i;$$

(b) if $f(x_1, x_2, \ldots, x_k)$ and $g_1(\bar{y}), \ldots, g_k(\bar{y})$ are all in $\mathscr{C}$ then so is the *composition* of f and $g_1, \ldots, g_k$,

$$h(\bar{y}) = f(g_1(\bar{y}), g_2(\bar{y}), \ldots, g_k(\bar{y}))$$

with the convention that if any of $g_1(\bar{y}), \ldots, g_k(\bar{y})$ is undefined, or if $f(g_1(\bar{y}), \ldots, g_k(\bar{y}))$ is undefined, then so is $h(\bar{y})$;

(c) $\mathscr{C}$ is closed under *primitive recursion*, i.e. if $f(\bar{x})$ and $g(\bar{x}, y, z)$ are in $\mathscr{C}$ then so is $h(\bar{x}, y)$ defined by

$$h(\bar{x}, 0) = f(\bar{x})$$

$$h(\bar{x}, y+1) = \begin{cases} g(\bar{x}, y, h(\bar{x}, y)) \\ \text{undefined, if } h(\bar{x}, y) \text{ is undefined;} \end{cases}$$

(d) $\mathscr{C}$ is closed under *minimalization*, i.e., if $g(\bar{x}, y)$ is in $\mathscr{C}$ then so is $h(\bar{x}) = (\mu y)(g(\bar{x}, y) = 0)$ defined by

$$h(\bar{x}) = \text{the least } y \text{ such that}$$

$$g(\bar{x}, 0), (\bar{x}, 1), \ldots, g(\bar{x}, y) \text{ are all defined and } g(\bar{x}, y) = 0$$

$$h(\bar{x}) \text{ is undefined if there is no such } y.$$

DEFINITION. The class of *primitive recursive functions*, $\mathscr{PR}$, is the least class of functions containing 0, S and U_i^n for each $1 \leqslant i \leqslant n$, and closed under composition and primitive recursion.

It is immediate from these definitions that $\mathscr{PR}$ only contains *total* functions, whereas functions in $\mathscr{C}$ need not be defined on all arguments, and that $\mathscr{PR} \subseteq \mathscr{C}$.

The following three easy propositions will be of particular importance later on.

PROPOSITION 0.7. The following functions are in $\mathscr{PR}$ (and hence in $\mathscr{C}$):

(a) $x + y$;

(b) $x \cdot y$;

(c) $x \dot{-} 1 = \begin{cases} 0 & \text{if } x = 0 \\ x - 1 & \text{otherwise;} \end{cases}$

(d) $x \dot{-} y = \begin{cases} 0 & \text{if } y \geqslant x \\ x - y & \text{otherwise;} \end{cases}$

(e) $\max(x, y)$;

(f) $\min(x, y)$;

(g) $\mathrm{eq}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y; \end{cases}$

(h) $\mathrm{lt}(x, y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{if } x \geqslant y. \end{cases}$ □

PROPOSITION 0.8. Both $\mathcal{PR}$ and $\mathcal{C}$ are closed under *definition by cases*, i.e., if $g, f_0, f_1, \ldots, f_k$ are in $\mathcal{PR}$ (resp. $\mathcal{C}$) then so is

$$h(\bar{x}) = \begin{cases} f_0(\bar{x}) \text{ if } g(\bar{x}) = 0 \\ f_1(\bar{x}) \text{ if } g(\bar{x}) = 1 \\ \quad\vdots \\ f_{k-1}(\bar{x}) \text{ if } g(\bar{x}) = k-1 \\ f_k(\bar{x}) \text{ if } g(\bar{x}) \geqslant k \\ \text{undefined, if } g(\bar{x}) \text{ is undefined.} \end{cases}$$ □

PROPOSITION 0.9. $\mathcal{PR}$ and $\mathcal{C}$ are closed under *bounded minimalization*, i.e., if $g(\bar{x}, y) \in \mathcal{PR}$ (resp. $\mathcal{C}$) then so is

$$f(\bar{x}, z) = (\mu y \leqslant z)(g(\bar{x}, y) = 0)$$

defined by

$$f(\bar{x}, z) = \begin{cases} \text{the least } y \leqslant z \text{ such that } g(\bar{x}, 0), g(\bar{x}, 1), \ldots, \\ g(\bar{x}, y) \text{ are all defined and } g(\bar{x}, y) = 0, \\ \text{if such } y \text{ exists;} \\ z, \text{ if } g(\bar{x}, 0), g(\bar{x}, 1), \ldots, g(\bar{x}, z) \\ \text{are all defined and non-zero;} \\ \text{undefined, otherwise.} \end{cases}$$ □

DEFINITION. The *recursive functions* are the *total* functions $f: \mathbb{N}^k \to \mathbb{N}$ (for some $k > 0$) in $\mathcal{C}$. A set $A \subseteq \mathbb{N}^k$ is *recursive* iff its characteristic function

$$\chi_A(\bar{x}) = \begin{cases} 1 & \text{if } \bar{x} \in A \\ 0 & \text{if } \bar{x} \notin A \end{cases}$$

is recursive. Similarly a set A is *primitive recursive* iff $\chi_A \in \mathcal{PR}$.

It is a fact that the class $\mathcal{C}$ coincides exactly with all the other definitions of the notion of 'partial computable function' (based on the intuition described at the beginning of this section) that have been suggested at various times. This leads us to the following thesis.

CHURCH–TURING THESIS. A partial function $f: A \subseteq \mathbb{N}^k \to \mathbb{N}$ is in $\mathscr{C}$ iff there is an algorithm for some idealized computer that computes f.

The Church–Turing thesis cannot be proved mathematically: it says that $\mathscr{C}$ captures *exactly* our informal notion of 'computable'. We will often use this thesis to show that certain functions, f, are in $\mathscr{C}$, by informally describing an algorithm for f. In this way we gain in both brevity and clarity: the function f could have been given by its formal definition in $\mathscr{C}$, but the definition would be long and the various properties of f obscure.

DEFINITION. A set $A \subseteq \mathbb{N}^k$ is *recursively enumerable* (or *r.e.* for short) iff A is the domain of some partial recursive function f; that is

$$\text{for all } \bar{x} \in \mathbb{N}^k (f(\bar{x}) \text{is defined} \Leftrightarrow \bar{x} \in A).$$

Clearly, by composing with the zero function, we may assume the function $f(\bar{x})$ above has value 0 whenever it is defined.

Notice that every recursive set A is trivially r.e., since if χ_A is recursive then $f(\bar{x}) = (\mu y)(1 \dot{-} \chi_A(\bar{x}) = 0)$ is in $\mathscr{C}$. This observation proves one direction of the following theorem.

THEOREM 0.10. (See, e.g., Cutland (1980), p. 124). $A \subseteq \mathbb{N}^k$ is recursive iff both A and its complement $A^c = \mathbb{N}^k - A$ are r.e. □

The following useful fact explains the origins of the term 'recursively enumerable'.

THEOREM 0.11 (See Cutland (1980), p. 125). $A \subseteq \mathbb{N}$ is r.e. iff there is a recursive function $f: \mathbb{N} \to \mathbb{N}$ s.t. $A = \{f(0), f(1), \ldots\}$. □

Notice too that (if A is infinite) the enumeration of A in Theorem 0.11, $\{f(0), f(1), \ldots\}$, can be taken to be without repetitions, for if f is computable then f' recursively defined by

$$\begin{aligned} f'(0) &= f(0) \\ f'(x+1) &= \text{the first value in the sequence} \\ &\qquad f(0), f(1), f(2), \ldots \text{ not equal to any} \\ &\qquad \text{of } f'(0), f'(1), \ldots, f'(x). \end{aligned}$$

is also computable, by the Church–Turing thesis.

We should mention here that there are r.e. sets $A \subseteq \mathbb{N}$ that are not recursive. (Examples of such sets will be given later in the text.)

We finish off this section with a few words about *relative computability*. If A, B are subsets of $\mathbb{N}$ we will often want to consider whether or not B is *recursive relative to* A (usually we say: *recursive in* A) meaning, roughly, 'if A were recursive then so would B be'. This is achieved by the following definition.

DEFINITION. If $A \subseteq \mathbb{N}^k$ for some $k \geq 1$, then $\mathscr{C}^A$ is the least class of partial functions on $\mathbb{N}$ containing χ_A, the zero, successor and projection functions, and close under composition, primitive recursion, and minimalization. A set $B \subseteq \mathbb{N}^l$ is *recursive in* A iff $\chi_B \in \mathscr{C}^A$.

Similarly we can define $\mathscr{PR}^A$ and hence the notion of a set B being *primitive recursive in* A. In all these notions A is referred to as an *oracle*, terminology that originates from the idea of an idealized computer running a program that stops every so often to consult something like the Delphic oracle to determine whether or not a certain tuple of integers $\bar{n}$ is in A or not.

1

The standard model

We never know how high we are
Till we are called to rise
And then if we are true to plan
Our statures touch the skies—

Emily Dickinson

Throughout this book $\mathscr{L}_A$ will denote the *first-order language of arithmetic* whose nonlogical symbols consist of the following: the constant symbols, 0 and 1; the binary relation symbol, $<$; and the two binary function symbols, $+$ and $\cdot$. From one point of view this book is about a particular $\mathscr{L}_A$-structure, $\mathbb{N}$, and a particular axiom system by which theorems about $\mathbb{N}$ can be deduced.

The structure $\mathbb{N}$ (called the *standard model*) is the $\mathscr{L}_A$-structure whose domain is the set of non-negative integers, $\{0, 1, 2, 3, \ldots\}$, and where the symbols in $\mathscr{L}_A$ are given their obvious interpretation. The axiom system involved is the formal system of *Peano arithmetic (PA)* which consists of a small number of basic axioms together with the axiom scheme of *mathematical induction*. Much more will be said about *PA* later. This chapter is concerned with introducing the reader to some of the ideas that will come up again and again in the course of this book.

To discover more about $\mathbb{N}$ and the strengths and limitations of *PA* we will concentrate our attention on those $\mathscr{L}_A$-structures that look very much like $\mathbb{N}$ from the point of view of first-order logic, but are *not isomorphic* to $\mathbb{N}$. (Such structures are said to be *nonstandard*.)

It was Skolem who first showed that nonstandard models of all true first-order $\mathscr{L}_A$-sentences exist (Skolem, 1934). (Here and throughout the book the unqualified word 'true' always means 'true in $\mathbb{N}$'.) Nowadays Skolem's result is proved using the compactness theorem as follows.

Let $\mathrm{Th}(\mathbb{N})$ denote the complete $\mathscr{L}_A$-theory of the standard model, i.e. $\mathrm{Th}(\mathbb{N})$ is the collection of all true $\mathscr{L}_A$-sentences. For each $n \in \mathbb{N}$ we let $\underline{n}$ be the closed term $(\cdots(((1+1)+1)+1)+\cdots+1)$ (n 1s) of $\mathscr{L}_A$; $\underline{0}$ is just the constant symbol 0. We now expand our language $\mathscr{L}_A$ by adding to it a new constant symbol c, obtaining the new language $\mathscr{L}_c$, and consider the following $\mathscr{L}_c$-theory with axioms

$$\sigma \quad (\text{for each } \sigma \in \mathrm{Th}(\mathbb{N}))$$

and

$$c > \underline{n} \quad (\text{for each } n \in \mathbb{N}).$$

This theory is consistent, for each finite fragment of it is contained in

$$T_k = \mathrm{Th}(\mathbb{N}) \cup \{c > \underline{n} \mid n < k\}$$

for some $k \in \mathbb{N}$, and clearly the $\mathscr{L}_c$-structure $(\mathbb{N}, k)$ with domain $\mathbb{N}$, 0, 1, $+$, $\cdot$ and $<$ interpreted naturally, and c interpreted by the integer k, satisfies T_k. Thus by the compactness theore $\bigcup_{k \in \mathbb{N}} T_k$ is consistent and has a model M_c. The first thing to notice about M_c is that

$$M_c \vDash c > \underline{n}$$

for all $n \in \mathbb{N}$, and hence it contains an 'infinite' integer. In particular this means that the reduct, M, of M_c to our original language $\mathscr{L}_A$ is not isomorphic to the standard model $\mathbb{N}$, for any isomorphism $h: \mathbb{N} \to M$ would have to send each $n \in \mathbb{N}$ to $\underline{n}^M$ (the element realizing the closed term $\underline{n}$ in M) and

$$M \vDash \forall x, y(x > y \to \neg x = y),$$

since this sentence is true in $\mathbb{N}$. It follows that the element realizing c in M_c cannot be in the image of h, and so M is nonstandard. By the Löwenheim–Skolem theorems we can take M_c (and hence M) to be of any infinite cardinality we like; this shows that there are lots of non-isomorphic models of $\mathrm{Th}(\mathbb{N})$ around.

The map $h: \mathbb{N} \to M$ sending $n \in \mathbb{N}$ to $\underline{n}^M$ is in fact an embedding of $\mathscr{L}_A$-structures. To check that h is 1–1, notice that if $n, k \in \mathbb{N}$ with $n \neq k$ then

$$\mathbb{N} \vDash \neg(\underline{n} = \underline{k}),$$

so the sentence $\neg(\underline{n} \neq \underline{k})$ is in $\mathrm{Th}(\mathbb{N})$ and hence true in M. Similarly h preserves $<$, $+$, and $\cdot$, since for any $n, m, k \in \mathbb{N}$

$$\begin{aligned} n < m \quad &\text{iff} \quad \mathbb{N} \vDash \underline{n} < \underline{m} \quad &&\text{iff} \quad M \vDash \underline{n} < \underline{m}, \\ n + m = k \quad &\text{iff} \quad \mathbb{N} \vDash \underline{n} + \underline{m} = \underline{k} \quad &&\text{iff} \quad M \vDash \underline{n} + \underline{m} = \underline{k}, \end{aligned}$$

and

$$n \cdot m = k \quad \text{iff} \quad \mathbb{N} \vDash \underline{n} \cdot \underline{m} = \underline{k} \quad \text{iff} \quad M \vDash \underline{n} \cdot \underline{m} = \underline{k}.$$

For this reason we will always identify $\mathbb{N}$ with the image of h in M, and thus consider $\mathbb{N}$ as a substructure of every model $M \vDash \mathrm{Th}(\mathbb{N})$. Then M is nonstandard just in case it has elements a not equal to any standard integer $n \in \mathbb{N}$. (Such elements a are called *nonstandard elements* of M.)

The embedding $h: \mathbb{N} \to M$ (which, by our convention, we can now regard as an inclusion $\mathbb{N} \subseteq M$) has another important property relating to the

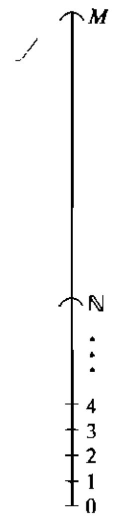

Fig. 1 A nonstandard model of arithmetic

order $<$. Notice that $<$ is a linear order on M with a least element 0 and no greatest element. (These properties are expressed by a first-order $\mathscr{L}_A$-sentence that is true in $\mathbb{N}$, and hence true in M.) Now for each $k \in \mathbb{N}$ we have

$$M \vDash \forall x(x < \underline{k} \rightarrow (x = \underline{0} \vee x = \underline{1} \vee x = \underline{2} \cdots \vee x = \underline{k-1}))$$

since this sentence is true in $\mathbb{N}$. It follows from this that $\mathbb{N}$ is an *initial segment* of M, and M is an *end-extension* of $\mathbb{N}$ (in symbols: $\mathbb{N} \subseteq_e M$); i.e., the inclusion $\mathbb{N} \subseteq M$ has the property that

$$\text{for all } n \in \mathbb{N} \text{ and for all } a \in M,$$

$$M \vDash a < \underline{n} \Rightarrow a \in \mathbb{N},$$

so that no new elements are added below any given $n \in \mathbb{N}$. We can draw a picture of this situation (see Fig. 1). (In Fig. 1 the order $<$ increases upwards. Remember: neither $\mathbb{N}$ nor M has a greatest element.) Thus, in particular $a \in M$ is nonstandard just in case that $M \vDash a > \underline{n}$ for each $n \in \mathbb{N}$. (We write this in shorthand thus: $a > \mathbb{N}$.) One of the main goals of this book is to investigate this picture in much greater detail. For example: are there any other interesting initial segments $I \subseteq_e M$ lying between $\mathbb{N}$ and M? (The answer turns out to be an emphatic 'yes'.)

How can such nonstandard models be of interest to anyone whose main concern is $\mathbb{N}$ itself? The key is in the existence of 'infinite' elements in a nonstandard M and the first-order properties they may possess. For example, suppose $\theta(x)$ is an $\mathscr{L}_A$-formula with only one free-variable x and suppose $M \models \theta(a)$ for some $a \in M$ where $M \models \mathrm{Th}(\mathbb{N})$ and a is nonstandard. Then $M \models \exists x\theta(x)$ and so $\mathbb{N} \models \exists x\theta(x)$ (for if $\mathbb{N} \models \forall x \neg\theta(x)$ then so would $M \models \forall x \neg\theta(x)$ since $M \models \mathrm{Th}(\mathbb{N})$). But even more is true: if $n \in \mathbb{N}$, then $M \models a > \underline{n}$ since a is non-standard, so

$$M \models \exists x(\theta(x) \wedge x > \underline{n});$$

hence $\mathbb{N} \models \exists x(\theta(x) \wedge x > \underline{n})$. It follows that there are *infinitely many* $k \in \mathbb{N}$ satisfying $\mathbb{N} \models \theta(\underline{k})$. Conversely, suppose there are infinitely many $k \in \mathbb{N}$ satisfying $\mathbb{N} \models \theta(\underline{k})$, and suppose $M \models \mathrm{Th}(\mathbb{N})$ is nonstandard. Then since $\mathbb{N} \models \forall x \exists y(y > x \wedge \theta(y))$, we have

$$M \models \forall x \exists y(y > x \wedge \theta(y)).$$

Thus for any $b \in M$, and in particular for any *nonstandard* $b \in M$, there is $a > b$ in M satisfying $M \models \theta(a)$. We have proved that any nonstandard $M \models \mathrm{Th}(\mathbb{N})$ has a nonstandard $a \in M \models \theta(a)$ iff there are infinitely many $k \in \mathbb{N}$ satisfying $\mathbb{N} \models \theta(\underline{k})$. This observation is the basis of many elegant 'non-standard' proofs of theorems about $\mathbb{N}$.

A second example that will concern us through this book is the idea of a nonstandard number $a \in \mathbb{N}$ *coding* a (possibly infinite) subset S of $\mathbb{N}$. Suppose $p_0 = 2, p_1 = 3, p_2 = 5, \ldots$ are the (standard) primes and let S be any subset of $\mathbb{N}$. We construct a nonstandard model M as before by expanding $\mathscr{L}_A$ to $\mathscr{L}_c$, but this time considering the theory T with axioms

$$\begin{aligned} &\mathrm{Th}(\mathbb{N}) \cup \{c > \underline{n} \mid n \in \mathbb{N}\} \\ &\cup \{\forall x \neg(\underline{p_k} \cdot x = c) \mid k \notin S\} \\ &\cup \{\exists x(\underline{p_k} \cdot x = c) \mid k \in S\}. \end{aligned}$$

Notice that every finite fragment of T is contained in one of the theories T_l given by axioms

$$\begin{aligned} &\mathrm{Th}(\mathbb{N}) \cup \{c > \underline{n} \mid n < l\} \\ &\cup \{\forall x \neg(\underline{p_k} \cdot x = c) \mid k \notin S \,\&\, k < l\} \\ &\cup \{\exists x(\underline{p_k} \cdot x = c) \mid k \in S \,\&\, k < l\} \end{aligned}$$

for some $l \in \mathbb{N}$. Let q be any prime in $\mathbb{N}$ with $q > l$, and let r be the integer

$$r = q \cdot \prod_{\substack{k < l \\ k \in S}} p_k.$$

It is easy to see that the $\mathscr{L}_c$-structure $(\mathbb{N}, r)$ with c interpreted by the integer r satisfies T_l. Thus each finite subset of T has a model and so T is consistent. Let $M_c \vDash T$, and let M be the reduct of M_c to the original language $\mathscr{L}_A$, and suppose $a \in M$ realizes the constant c in M_c. Then *a codes* the set S, i.e., we can recover our set S as follows:

$$S = \{n \in \mathbb{N} \mid M \vDash \exists x(\underline{p_n} \cdot x = a)\}.$$

Thus (since $S \subseteq \mathbb{N}$ was arbitrary) there are nonstandard models $M \vDash \mathrm{Th}(\mathbb{N})$ with elements $a > \mathbb{N}$ coding the set of primes, or the set of composites, or the set of powers of 2, . . . and so on. We conclude this chapter with a simple application of these ideas to the number of countable models of $\mathrm{Th}(\mathbb{N})$.

THEOREM 1.1. There are exactly $2^{\aleph_0}$ non-isomorphic countable models of $\mathrm{Th}(\mathbb{N})$.

Proof. Clearly $2^{\aleph_0}$ is the maximum number of models that could exist. We must show that this maximum is attained.

If $M \vDash \mathrm{Th}(\mathbb{N})$ is any countable model and $a \in M$, then the set *coded by a* is the set

$$S_a = \{n \in \mathbb{N} \mid M \vDash \exists x(\underline{p_n} \cdot x = a)\}.$$

So $S_a \subseteq \mathbb{N}$ and (since M is countable) at most countably many subsets of $\mathbb{N}$ are coded by some $a \in M$. We have shown, however, that any set $S \subseteq \mathbb{N}$ is coded by some a in some countable model $M \vDash \mathrm{Th}(\mathbb{N})$. If there were only $\kappa < 2^{\aleph_0}$ non-isomorphic countable models M_i of $\mathrm{Th}(M)$ (where i ranges over a set I of cardinality κ), then the number of subsets $S \subseteq \mathbb{N}$ coded by some a in some M_i is at most

$$\leq \sum_{i \in I} \mathrm{card}(M_i) = \kappa \cdot \aleph_0 = \max(\kappa, \aleph_0),$$

which is less than $2^{\aleph_0}$, a contradiction, since there are $2^{\aleph_0}$ subsets $S \subseteq \mathbb{N}$. □

Exercises for Chapter 1

1.1 Suppose that M is an $\mathscr{L}_A$-structure satisfying $\mathrm{Th}(\mathbb{N})$. Verify that the embedding $h: n \mapsto \underline{n}^M$ is an elementary embedding of $\mathbb{N}$ into M, i.e., h maps $\mathbb{N}$ onto an elementary substructure of M.

1.2 By considering the formulas

$$\exists x(\overbrace{x+x+\cdots+x}^{p}=y)$$

and

$$\forall x \neg(\overbrace{x+x+\cdots+x}^{p}=y),$$

show that there are $2^{\aleph_0}$ non-isomorphic countable structures satisfying all true sentences in the language whose only non-logical symbol is $+$.

1.3 Classify (up to isomorphism) all countable models of $\mathrm{Th}(\mathbb{N}, 0, 1, S)$ in the language consisting of constants 0, 1, and a unary function symbol, S, which is interpreted in $\mathbb{N}$ as the *successor* operation:

$$S(x)=x+1.$$

In particular, verify that, up to isomorphism, there are only $\aleph_0$ countable models of this theory.

2

Discretely ordered rings

What's done we partly may compute
But know not what's resisted.

Burns
Address to the Unco Guid

We start our study of models of arithmetic in earnest by writing down some simple axioms that are obviously true in $\mathbb{N}$. The resulting theory, PA^-, is described in Section 2.1 and is our 'base theory' to which we shall later add induction axioms.

PA^- has some very nice features which we explore in this chapter. In particular we will show in Section 2.2 that any model of PA^- may be regarded as an end-extension of the standard model, and we introduce the notion of a Σ_1 sentence, and show that PA^- is strong enough to prove all true Σ_1 sentences. (The dual question, whether PA^- or any reasonable extension of PA^- is strong enough to *disprove* all *false* Σ_1 sentences turns out to be of a rather different nature—as perhaps prophesied by Burns! This will be the subject of the next chapter.)

2.1 THE AXIOMS OF PA^-

Our first five axioms state that the binary functions $+$ and $\cdot$ are associative, commutative, and satisfy the distributive law.

Ax1: $\forall x, y, z((x+y)+z=x+(y+z))$

Ax2: $\forall x, y(x+y=y+x)$

Ax3: $\forall x, y, z((x\cdot y)\cdot z=x\cdot(y\cdot z))$

Ax4: $\forall x, y(x\cdot y=y\cdot x)$

Ax5: $\forall x, y, z(x\cdot(y+z)=x\cdot y+x\cdot z)$.

As will already be apparent, we write our $\mathscr{L}_A$-formulas in the 'natural' way, so for example we write $x+y$ instead of $+(x, y)$. We also employ all the usual abbreviations, omitting brackets where possible (with the convention that the multiplication operation is more binding than that of addition). Often we will omit the multiplication sign altogether. We also employ the notation of the last chapter for the closed terms $\underline{n}=1+1+\cdots+1$ (n 1s) representing the integer n. By Ax1 it does not matter how this is bracketed (but for sake of definiteness we may assume $1+1+\cdots+1$ is implicitly bracketed $((\cdots((1+1)+1)+\cdots)+1)$.) $\underline{0}$ and $\underline{1}$ are just the constant terms 0, 1 respectively. Similarly if $n\in\mathbb{N}$ and x is a variable, then nx denotes the term $(\cdots((x+x)+x)+\cdots+x)$ (n xs) if $n>0$, and denotes $\underline{0}$ if $n=0$; also x^n denotes $(\cdots((x\cdot x)\cdot x)\cdot\cdots\cdot x)$ (n xs) if $n>0$, and denotes $\underline{1}$ if $n=0$.

The next two axioms state that 0 is an identity for $+$ and a zero for $\cdot$, and that 1 is an identity for $\cdot$.

Ax6: $\forall x((x+0=x)\wedge(x\cdot 0=0))$

Ax7: $\forall x(x\cdot 1=x)$.

$\mathbb{N}$ is, of course, equipped with a linear order. This is expressed by the following axioms:

Ax8: $\forall x,y,z((x<y\wedge y<z)\rightarrow x<z)$

Ax9: $\forall x\neg x<x$

Ax10: $\forall x,y(x<y\vee x=y\vee y<x)$.

It follows from these axioms that

$$\forall x,y(x<y\rightarrow\neg y<x)$$

so the three possibilities in Ax10 are mutually exclusive. We use $x\leqslant y$ as an abbreviation for $x<y\vee x=y$, so by Ax8, Ax9, and Ax10 we have

$$\forall x,y(x\leqslant y\leftrightarrow\neg y<x).$$

Also: $x>y$ means $y<x$; $x\geqslant y$ means $y\leqslant x$; and $x<y<z$ means $x<y\wedge y<z$; etc. We define $\max(x_1,\ldots,x_n)$ to be the $<$-greatest element of $\{x_1,\ldots,x_n\}$, so

$$\forall\bar{x},y(\max(x_1,\ldots,x_n)=y\leftrightarrow(\bigvee_{i=1}^{n}y=x_i)\wedge(\bigwedge_{i=1}^{n}y\geqslant x_i))$$

and max is a well-defined function on structures satisfying Ax8, Ax9, and Ax10 for all finite $n\in\mathbb{N}$. Similarly $\min(x_1,\ldots,x_n)$ is the $<$-least element of $\{x_1,\ldots,x_n\}$.

Next are the axioms that say that $+$ and $\cdot$ respect the order:

Ax11: $\forall x,y,z(x<y\rightarrow x+z<y+z)$

Ax12: $\forall x,y,z(0<z\wedge x<y\rightarrow x\cdot z<y\cdot z)$

And, of course, if $x<y$ then we can subtract x from y:

Ax13: $\forall x,y(x<y\rightarrow\exists z\, x+z=y)$

Before we give the last two axioms let us note that Ax1–Ax13 imply converses to Ax11, Ax12 and show that the z in Ax13 is unique.

For the converse to Ax11 suppose $x+z<y+z$ and $\neg x<y$. Then $y \leqslant x$, and if $x=y$ then $x+z=y+z<y+z$ contradicting Ax9, and if $y<x$ then $y+z<x+z$ by Ax11, hence $y+z<x+z<y+z$, contradicting Ax8 and Ax9.

For the converse to Ax12 suppose $x \cdot z<y \cdot z$ and that $z \geqslant 0$. Then certainly $z \neq 0$ since $x \cdot 0=y \cdot 0=0$ by Ax6, and if $\neg x<y$ then $y \leqslant x$ so $y \cdot z \leqslant x \cdot z<y \cdot z$ by Ax12, contradicting Ax8 and Ax9 again.

To show that the z in Ax13 is unique, suppose $x<y$ and u, v satisfy $x+u=x+v=y$ and $u \neq v$. By Ax10 we may assume that $u<v$. But then $x+u<x+v$ by Ax11, Ax2, contradicting Ax9.

The last two axioms state that the order $<$ is *discrete* and that 0 is the least natural number.

Ax14: $\quad 0<1 \wedge \forall x(x>0 \rightarrow x \geqslant 1)$

Ax15: $\quad \forall x(x \geqslant 0)$

(The word *discrete* should not be confused with the word *discreet*, even though they both have the same Latin root!)

The axioms Ax1–Ax15 constitute the theory PA^-. PA^- is the 'base theory' that we shall assume holds in every $\mathscr{L}_A$-structure we consider. (Later on we will add induction axioms to PA^- to get the full PA or *Peano arithmetic*.) The standard model $\mathbb{N}$ clearly satisfies PA^-, as does any nonstandard model of $\mathrm{Th}(\mathbb{N})$ since Ax1–Ax15 are all first-order. Another example is given by the $\mathscr{L}_A$-structure $\mathbb{Z}[X]^+$ which is defined as follows:

$\mathbb{Z}[X]$ is the ring of polynomials in one variable X with coefficients from $\mathbb{Z}$. $\mathbb{Z}[X]$ is made into an $\mathscr{L}_A$-structure by defining an order $<$ making X 'infinitely large'. More specifically, if $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ with $a_n \neq 0$, then we define $a_0+a_1X+\cdots+a_nX^n>0$ iff $a_n>0$. Given any two polynomials $p, q \in \mathbb{Z}[X]$, we define $p>q$ iff $p-q>0$. $\mathbb{Z}[X]^+$ is the $\mathscr{L}_A$-structure of all *non-negative* polynomials in $\mathbb{Z}[X]$, i.e., $\mathbb{Z}[X]^+$ has domain

$$\{p \in \mathbb{Z}[X] \mid \mathbb{Z}[X] \vDash p \geqslant 0\}.$$

It is straightforward to check that $\mathbb{Z}[X]^+ \vDash PA^-$. $\mathbb{Z}[X]^+$ does not satisfy $\mathrm{Th}(\mathbb{N})$, however, since for example $\mathbb{N} \vDash \forall x \exists y(2y=x \vee 2y+1=x)$, whereas if $y=a_0+a_1X+\cdots+a_nX^n \in \mathbb{Z}[X]^+$ and $0 \leqslant 2y=2a_0+2a_1X+\cdots+2a_nX^n \leqslant X$, then $a_1=a_2=\cdots a_n=0$, $y=a_0$, and so both $2y<X$ and $2y+1<X$, so $\mathbb{Z}[X]^+ \vDash \forall y(2y \neq X \wedge 2y+1 \neq X)$.

There is a correspondence between models of PA^- and so-called *discretely ordered rings*, which we now describe. Suppose $M \vDash PA^-$. Define

$R = M^2/\sim$ where $\sim$ is the equivalence relation

$$(a, b) \sim (c, d) \text{ iff } a + d = b + c$$

on M^2. (Exercise: verify that this is indeed an equivalence.) We may denote the equivalence class of (a, b) by $[a, b]$ and define $+, \cdot, <$ on R by

$$[a, b] + [c, d] = [a + c, b + d]$$
$$[a, b] \cdot [c, d] = [ac + bd, bc + ad]$$
$$[a, b] < [c, d] \text{ iff } a + d < b + c.$$

It is easy to check that these are all well-defined, and that if 0, 1 are interpreted in R by $[0, 0]$ and $[1, 0]$ respectively, then the $\mathscr{L}_A$-structure satisfies Ax1–Ax14 together with

Ax16: $\quad \forall x \exists y (x + y = 0)$.

($\mathscr{L}_A$-structures satisfying Ax1–Ax14 together with Ax16 are called *discretely ordered rings*.) It is also easy to verify that the map

$$a \mapsto [a, 0]$$

is an embedding of $\mathscr{L}_A$-structures $M \to R$ sending M onto the collection of non-negative elements of R.

Conversely, if R is any discretely order ring and M is the $\mathscr{L}_A$-structure consisting of all non-negative elements of R, then $M \vDash PA^-$, and repeating the above construction we recover our original ring R. This justifies calling *PA^- the theory of the non-negative parts of discretely ordered rings*.

We conclude Section 2.1 by showing that for any $a \in M \vDash PA^-$ there are no elements between a and $a + 1$. The following definition will be useful.

Definition. Let $M \vDash PA^-$. The *modified subtraction operation*, $\dot{-}$, is defined on M by

$$a \dot{-} b = \begin{cases} 0 & \text{if } M \vDash a \leqslant b \\ \text{the unique } c \text{ in } M \text{ s.t.} & \\ M \vDash b + c = a & \text{if } M \vDash a > b. \end{cases}$$

We shall often write $a - b$ for $a \dot{-} b$ when we know that $a \geqslant b$.

Proposition 2.1. Let $M \vDash PA^-$. Then

$$M \vDash \forall x, y (y > x \to y \geqslant x + 1).$$

Proof. If $x, y \in M \models y > x$ let $z = y \dot{-} x$ so $y = x + z$ and hence $z > 0$ (for if $z \leqslant 0$ then $x + z = y \leqslant x$, contradiction). If follows from Ax14 that $z \geqslant 1$, hence $y = x + z \geqslant x + 1$ by Ax11. □

Exercises for Section 2.1

2.1 Show that, in PA^-, $\mathscr{L}_A$-terms are 'exactly' polynomials over $\mathbb{Z}$ with non-negative coefficients; i.e., for each such polynomial $p(\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n)$ define an $\mathscr{L}_A$-term $t_p(\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n)$ representing $p(\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n)$ and verify that the map $p \mapsto t_p$ induces a bijection between all such polynomials and the set of $\mathscr{L}_A$-terms under the equivalence

$$t(\bar{\mathsf{v}}) \sim s(\bar{\mathsf{v}}) \text{ iff } PA^- \vdash \forall \bar{\mathsf{v}}(t(\bar{\mathsf{v}}) = s(\bar{\mathsf{v}})).$$

2.2 Define an order $<$ on $\mathbb{N}[X]$ (the set of all polynomials in the variable X with coefficients from $\mathbb{N}$) making X 'infinitely large' and satisfying Ax8–Ax12. Show that the resulting $\mathscr{L}_A$-structure does not satisfy PA^-. Which axioms are not satisfied?

2.3 Let $M \models \mathrm{Th}(\mathbb{N})$ be nonstandard with $a \in M \setminus \mathbb{N}$. Define a map

$$f\colon \mathbb{Z}[X]^+ \to M$$

by

$$p(X) \mapsto p(a) \in M.$$

Show that f is well-defined and is an embedding of $\mathscr{L}_A$-structures. Conclude that $\mathbb{Z}[X]^+$ is isomorphic to a substructure of M.

Now let σ be the sentence $\forall \bar{x} \psi(\bar{x})$ of $\mathscr{L}_A$ *where* ψ *is quantifier-free*. Show that

$$\mathbb{N} \models \sigma \Rightarrow M \models \sigma \Rightarrow \mathbb{Z}[X]^+ \models \sigma$$

and hence $\mathbb{Z}[X]^+ \models \forall_1 - \mathrm{Th}(\mathbb{N})$, the set of all true $\mathscr{L}_A$-sentences of the form $\forall \bar{x} \psi(\bar{x})$ with $\psi(\bar{x})$ quantifier-free.

2.4 Verify that any *ordered ring R* (i.e., an $\mathscr{L}_A$-structure satisfying Ax1–Ax13 and Ax16) satisfies

$$\forall x, y (x < y \leftrightarrow y - x > 0);$$

thus to define an order on a ring R it is sufficient to specify the positive elements of R.

2.5 Let R be the ring $\mathbb{Z}[X, Y]/(X^2 - 2Y^2)$, i.e., R is the polynomial ring $\mathbb{Z}[X, Y]$ modulo the equivalence

$$p(X, Y) \sim q(X, Y) \quad \text{iff} \quad p - q = r \cdot (X^2 - 2Y^2) \text{ for some polynomial } r \in \mathbb{Z}[X, Y].$$

Show that R can be discretely ordered. Hence show that

$$PA^- \nvdash \forall x, y(x^2 \neq 2y^2).$$

2.6 Show that all orderd rings are integral domains, i.e., satisfy

$$\forall x, y(x \cdot y = 0 \rightarrow x = 0 \vee y = 0),$$

and have characteristic zero, i.e.,

$$\neg(\underline{n} = 0)$$

for each $n \in \mathbb{N}$. Give an example of an integral domain of characteristic zero that cannot be ordered.

2.7 Show that $\mathbb{Z}[X]/(2X^2 - 1)$ can be ordered, but not discretely ordered.

2.8 Show that $\mathbb{Z}[X, Y, Z]/(XZ - Y^2)$ can be discretely ordered with $0 < X < Y < Z$. Deduce that the notions 'prime' and 'irreducible' are not equivalent in PA^-, i.e., that

$$PA^- \nvdash \forall x(\text{prime}(x) \leftrightarrow \text{irred}(x)),$$

where prime(x) denotes

$$x > 1 \wedge \forall y, z, w(xw = yz \rightarrow \exists v(xv = y \vee xv = z))$$

and irred(x) denotes

$$x > 1 \wedge \forall u, v(uv = x \rightarrow u = x \vee v = x).$$

2.2 INITIAL SEGMENTS AND END-EXTENSIONS

The most important property of models of PA^- is Theorem 2.2 below (which generalizes an observation we made about models of Th($\mathbb{N}$) in the last chapter). Recall that if M and N are $\mathscr{L}_A$-structures with N a substructure of M, then N is an *initial segment* of M, M is an *end-extension* of N, or (in symbols) $N \subseteq_e M$ iff

$$\text{for all } x \in N, \text{ for all } y \in M\ (M \vDash y < x \Rightarrow y \in N).$$

N is a *proper* initial segment if, in addition, $N \neq M$.

THEOREM 2.2. Let $M \vDash PA^-$. Then the map $\mathbb{N} \rightarrow M$ given by $n \mapsto \underline{n}^M$ is an embedding of $\mathscr{L}_A$-structures sending $\mathbb{N}$ onto an initial segment of M.

The proof of Theorem 2.2 depends on the following four simple lemmas.

LEMMA 2.3. If $n, l, k \in \mathbb{N}$ and $n = l + k$, then $PA^- \vdash \underline{n} = \underline{l} + \underline{k}$.

Proof. If $k = 0$ this follows from Ax6. If $k > 0$ this follows from Ax1 and a simple induction on k. □

LEMMA 2.4. If $n, l, k \in \mathbb{N}$ and $n = l \cdot k$ then $PA^- \vdash \underline{n} = \underline{l} \cdot \underline{k}$.

Proof. We fix l and argue by inducation on k. For $k = 0$, $PA^- \vdash \underline{0} = \underline{l} \cdot \underline{0}$ by Ax6. If $PA^- \vdash \underline{n} = \underline{l} \cdot \underline{k}$, $k' = k + 1$ and $n' = n + l$, then $PA^- \vdash \underline{l} \cdot \underline{k}' = \underline{l} \cdot (\underline{k} + 1) = \underline{l} \cdot \underline{k} + \underline{l}$ by Ax5 and Ax7; hence $PA^- \vdash \underline{l} \cdot \underline{k}' = \underline{n} + \underline{l}$ by the induction hypothesis, hence $PA^- \vdash \underline{l} \cdot \underline{k}' = \underline{n}'$ by Lemma 2.3. □

LEMMA 2.5. If $n, k \in \mathbb{N}$ with $n < k$, then $PA^- \vdash \underline{n} < \underline{k}$.

Proof. By Ax14 $PA^- \vdash \underline{0} < \underline{1}$; and by Ax11 and Ax8, $PA^- \vdash 0 < \underline{k} \Rightarrow \mathrm{PA}^- \vdash 0 < \underline{k} + 1 = \underline{k+1}$. So $PA^- \vdash 0 < \underline{k}$ for all $k \in \mathbb{N} \setminus \{0\}$, by induction on k.

Now suppose $n, k \in \mathbb{N}$ are arbitrary s.t. $n < k$, and let $l = k - n$. Then $PA^- \vdash \underline{n} + \underline{l} = \underline{k}$ by lemma 2.3 and $PA^- \vdash \underline{l} > 0$ by the previous paragraph. Hence $PA^- \vdash \underline{n} + \underline{l} > \underline{n}$ by Ax11 so $PA^- \vdash \underline{k} > \underline{n}$ as required. □

LEMMA 2.6. For all $k \in \mathbb{N}$ $PA^- \vdash \forall x (x \leqslant \underline{k} \rightarrow x = \underline{0} \vee x = \underline{1} \vee \cdots \vee x = \underline{k})$.

Proof. By induction on k. For $k = 0$ this follows from Ax15. Suppose

$$PA^- \vdash \forall x (x \leqslant \underline{k} \rightarrow \mathrm{x} = \underline{0} \vee \mathrm{x} = \underline{1} \vee \cdots \vee \mathrm{x} = \underline{k});$$

then $PA^- \vdash \forall x (x \leqslant \underline{k+1} \rightarrow x \leqslant \underline{k} \vee x = \underline{k+1})$ by Proposition 2.1, hence

$$PA^- \vdash \forall x (x \leqslant k + 1 \rightarrow x = \underline{0} \vee x = \underline{1} \vee \cdots \vee \mathrm{x} = \underline{k} \vee \mathrm{x} = k + 1)$$

as required. □

Proof of Theorem 2.2. Lemmas 2.3, 2.4, and 2.5 show that the map $n \mapsto \underline{n}^M$ respects $+, \cdot, <$. Since $PA^- \vdash \forall x, y (x < y \rightarrow x \neq y)$, Lemma 2.5 also shows that this map is an embedding (i.e., it is also 1–1). Finally Lemma 2.6 shows that the image $N = \{\underline{n}^M | n \in \mathbb{N}\}$ is an initial segment of M. □

Because of Theorem 2.2 we will always identify $\mathbb{N}$ with the smallest initial segment $\{\underline{n}^M | n \in \mathbb{N}\}$ of a model of PA^-. This means that fig. 1 in Chapter 1 applies whenever $M \vDash PA^-$. In particular this implies that there will be no confusion if we denote the closed $\mathscr{L}_A$-term $\underline{n}$ simply by n; we shall employ this notation from now on.

Initial segments and end-extensions play a key role in the study of models of arithmetic, and it is worth while considering them in relation to models of PA^- first.

EXAMPLES

(a) The standard model $\mathbb{N}$ clearly has no proper initial segments.

(b) The only proper initial segment of $\mathbb{Z}[X]^+$ is $\mathbb{N}$. There are, however, many other proper *cuts* of $\mathbb{Z}[X]^+$ (i.e., proper subsets, S, of the domain of $\mathbb{Z}[X]^+$

satisfying $p<q\in S \Rightarrow p\in S$ and closed under the successor function $S(x)=x+1$) such as

$$I_k=\{p\in \mathbb{Z}[X]^+ \mid p \text{ has degree} \leq k\}$$

for each non-zero $k\in\mathbb{N}$ (these cuts are closed under $+$ but not under $\cdot$) and

$$J_p=\{q\in \mathbb{Z}[X]^+ \mid q\leq p+n \text{ for some } n\in\mathbb{N}\}$$

for each polynomial $p\in\mathbb{Z}[x]^+$of non-zero degree (these are not even closed under $+$).

(c) Any model $M\models PA^-$ has a proper end-extension satisfying PA^-, for given $M\models PA^-$ let R be the discretely ordered ring associated with M, and let $R[X]$ have order given by

$$a_0+a_1X+\cdots+a_nX^n>0 \quad \text{iff} \quad a_n>0$$

for each polynomial $a_0+a_1X+\cdots+a_nX^n\in R[X]$ with $a_0, a_1, \ldots, a_n\in R$ and $a_n\neq 0$. Then $R[X]^+$, the structure consisting of the non-negative elements of $R[X]$, is a model of PA^- and the composition of the natural embeddings

$$M\rightarrow R\rightarrow R[X]$$

is an embedding of M onto a proper initial segment of $R[X]^+$.

Definition

(a) If t is an $\mathscr{L}_A$-term then $\forall x<t(\ldots)$ is an abbreviation for $\forall x(x<t\rightarrow\ldots)$ and $\exists x<t(\ldots)$ is an abbreviation for $\exists x(x<t\wedge\ldots)$. Similarly $\forall x\leq t(\ldots)$ and $\exists x\leq t(\ldots)$ are shorthand for $\forall x(x\leq t\rightarrow\ldots)$ and $\exists x(x\leq t\wedge\ldots)$ respectively. These quantifiers are said to be *bounded*.

(b) A formula $\theta(\bar{x})$ of $\mathscr{L}_A$ is Δ_0 iff all quantifiers in θ are bounded. A formula $\psi(\bar{x})$ is Σ_1 iff it is of the form $\exists\bar{y}\theta(\bar{x},\bar{y})$ with $\theta(\bar{x},\bar{y})\in\Delta_0$ (so that the only unbounded quantifiers of ψ are existential and occur at the beginning of ψ). $\psi(\bar{x})$ is Π_1 iff it is of the form $\forall\bar{y}\theta(\bar{x},\bar{y})$ with $\theta(\bar{x},\bar{y})\in\Delta_0$.

The use of 'Π' and 'Σ' in this notation is a left-over from the days when the Boolean operations '$\wedge$' and '$\vee$' were denoted '$\cdot$' and '$+$' respectively. It was then natural to write $\forall x\psi(x)$ as $\Pi_x\psi(x)$ and $\exists x\psi(x)$ as $\Sigma_x\psi(x)$.

In the definitions of Σ_1 and Π_1 we allow the unbounded block of quantifiers '$\exists\bar{y}$' or '$\forall\bar{y}$' to be empty: in this way any Δ_0 formula is automatically both a Π_1 formula and a Σ_1 formula. Notice also that a conjunction or a disjunction of Σ_1 formulas is equivalent (in the preducate calculus) to a Σ_1 formula. For example $(\exists\bar{y}\theta_1(\bar{x},\bar{y}))\wedge(\exists\bar{y}\theta_2(\bar{x},\bar{y}))$ is equivalent to $\exists\bar{y},\bar{z}(\theta_1(\bar{x},\bar{y})\wedge\theta_2(\bar{x},\bar{z}))$ for some suitable choice of new variables $\bar{z}$. Similarly Π_1 is closed under $\wedge$ and $\vee$. Also, the negation of a Σ_1

formula is equivalent to a Π_1 formula and vice versa, since $\neg\exists\bar{y}\theta(\bar{x}, \bar{y})$ is equivalent to $\forall\bar{y}\neg\theta(\bar{x}, \bar{y})$ and $\neg\forall\bar{y}\theta(\bar{x}, \bar{y})$ is equivalent to $\exists\bar{y}\neg\theta(\bar{x}, \bar{y})$.

Many useful and interesting properties of numbers turn out to be Δ_0, such as:

$$\begin{array}{rl} \text{`}x-y\text{ exists'} & \exists z\leqslant x(z+y=x); \\ \text{`}x\text{ is irreducible'} & x>1\wedge\forall u<x\forall v<x\neg(u\cdot v=x); \end{array}$$

and so on. The importance of the class of Δ_0 formulas is that they are *absolute* with respect to end-extensions.

DEFINITION. If Γ is a class of $\mathscr{L}_A$-formulas and $M\subseteq N$ are $\mathscr{L}_A$-structures, then M is a Γ-*elementary substructure* of N, $M\prec_\Gamma N$, iff

$$\text{whenever } \bar{a}\in M \text{ and } \gamma(\bar{x})\in\Gamma \text{ then}$$
$$M\vDash\gamma(\bar{a})\Leftrightarrow N\vDash\gamma(\bar{a}).$$

If $M\prec_\Gamma N$, then the formulas in Γ are said to be *absolute* for the extension $M\subseteq N$.

THEOREM 2.7. Let $M\subseteq_e N$ both be $\mathscr{L}_A$-structures, with N an end-extension of M. Then $M\prec_{\Delta_0} N$.

Proof. The proof goes by induction on the complexity of Δ_0 formulas, which is (for the purposes of this proof) defined to be the number of $\wedge$, $\vee$, $\neg$, $\exists$, or $\forall$ symbols in the formula. The induction hypothesis is that

$$\left.\begin{array}{l} \text{For all } \theta(\bar{x})\in\Delta_0 \text{ with complexity} \leqslant n \text{ and} \\ \text{for all } \bar{a}\in M \text{ we have} \\ \quad M\vDash\theta(\bar{a})\Leftrightarrow N\vDash\theta(\bar{a}). \end{array}\right\} \qquad (*)$$

This is clearly true for $n=0$ since formulas of complexity 0 are atomic.

Assuming (*) holds, suppose $\theta(\bar{x})=\theta_1(\bar{x})\wedge\theta_2(\bar{x})$ is of complexity $n+1$ and $\bar{a}\in M$. Then

$$\begin{array}{llll} & M\vDash\theta(\bar{a}) & & \\ \Leftrightarrow & M\vDash\theta_1(\bar{a}) & \text{and} & M\vDash\theta_2(\bar{a}) \\ \Leftrightarrow & N\vDash\theta_1(\bar{a}) & \text{and} & N\vDash\theta_2(\bar{a}) \quad (\text{using } (*)) \\ \Leftrightarrow & N\vDash\theta(\bar{a}). & & \end{array}$$

Similar arguments show that $M\vDash\theta(\bar{a})\Leftrightarrow N\vDash\theta(\bar{a})$ if $\bar{a}\in M$ and $\theta(\bar{x})$ is either $\theta_1(\bar{x})\vee\theta_2(\bar{x})$ or $\neg\theta_1(\bar{x})$ and has complexity $n+1$.

If $\theta(\bar{x})$ is $\forall y < t(\bar{x})\theta_1(\bar{x}, y)$, is of complexity $n+1$, and $\bar{a} \in M$, then

$$\begin{aligned} &M \models \theta(\bar{a}) \\ &\Leftrightarrow \text{for all } b < t(\bar{a}) \quad \text{in} \quad M, M \models \theta_1(\bar{a}, b); \end{aligned}$$

however, if $\bar{a} \in M$ then $t(\bar{a}) \in M$ since M is closed under $+, \cdot$, and since $M \subseteq_e N$ we have

$$\{b \in N \mid N \models b < t(\bar{a})\} = \{b \in M \mid M \models b < t(\bar{a})\}$$

so, using (*), we deduce

$$\begin{aligned} &M \models \theta(\bar{a}) \\ &\Leftrightarrow \text{for all } b < t(a) \text{ in } N, N \models \theta_1(\bar{a}, b) \\ &\Leftrightarrow N \models \theta(\bar{a}). \end{aligned}$$

The case when $\theta(\bar{x})$ is $\exists y < t(\bar{x})\theta_1(\bar{x}, y)$ is proved in exactly the same way. □

COROLLARY 2.8. Suppose $M \subseteq_e N$ are $\mathscr{L}_A$-structures and $\varphi(\bar{x})$, $\psi(\bar{x})$ are $\mathscr{L}_A$-formulas with $\varphi(\bar{x}) \in \Sigma_1$ and $\psi(\bar{x}) \in \Pi_1$. Then for any $\bar{a} \in M$

(a) $$M \models \varphi(\bar{a}) \Rightarrow N \models \varphi(\bar{a})$$

(b) $$N \models \psi(\bar{a}) \Rightarrow M \models \psi(\bar{a}).$$

Proof. If $\varphi(\bar{x})$ is $\exists \bar{y}\theta(\bar{x}, \bar{y})$ with $\theta \in \Delta_0$ and $\bar{a}, \bar{b} \in M \models \theta(\bar{a}, \bar{b})$ then $N \models \theta(\bar{a}, \bar{b})$ by Theorem 2.7, so $N \models \exists \bar{y}\theta(\bar{a}, \bar{y})$, i.e., $N \models \varphi(\bar{a})$.

If $\psi(\bar{x})$ is $\forall \bar{y}\chi(\bar{x}, \bar{y})$ with $\chi \in \Delta_0$ and $\bar{a} \in M$ with $N \models \forall \bar{y}\chi(\bar{a}, \bar{y})$ then for all $\bar{b} \in M$ $N \models \chi(\bar{a}, \bar{b})$, so for all $\bar{b} \in M$ $M \models \chi(\bar{a}, \bar{b})$ by 2.7, i.e., $M \models \forall \bar{y}\chi(\bar{a}, \bar{y})$. □

COROLLARY 2.9. Let $\Sigma_1 - \mathrm{Th}(\mathbb{N})$ denote $\{\sigma \mid \sigma \in \Sigma_1, \sigma \text{ is an } \mathscr{L}_A\text{-sentence, and } \mathbb{N} \models \sigma\}$. Then $PA^- \vdash \Sigma_1 - \mathrm{Th}(\mathbb{N})$.

Proof. We show that any model $M \models PA^-$ satisfies $\Sigma_1 - \mathrm{Th}(\mathbb{N})$. The corollary then follows by the completeness theorem.

By Theorem 2.2 we may assume $\mathbb{N} \subseteq_e M$, so if $\sigma \in \Sigma_1 - \mathrm{Th}(\mathbb{N})$ then $\mathbb{N} \models \sigma$ and hence, by 2.8, $M \models \sigma$, and we are finished. □

Corollary 2.9 shows that PA^- can prove a large number of true sentences. We saw, however, that PA^- cannot prove *all* true sentences. For example, since

$$\mathbb{Z}[X]^+ \models PA^- + \neg\forall x \exists y \leqslant x(2y = x \vee 2y + 1 = x).$$

This does leave open the possibility that some 'nicely axiomatized' theory extending PA^- could prove all true sentences. We take up this question in the next chapter.

Exercises for 2.2

2.9 Show that, for any $\mathscr{L}_A$-formula $\theta(x, \bar{y})$ and for any $\mathscr{L}_A$-term $t(\bar{y})$,

$$\exists x < t(\bar{y})\theta(x, \bar{y})$$

and

$$\neg\forall x < t(\bar{y})\neg\theta(x, \bar{y})$$

are equivalent in the predicate calculus.

2.10 Find a closed $\mathscr{L}_A$-term t and an $\mathscr{L}_A$-formula $\theta(x)$ such that

$$\mathbb{N} \models \forall x < t\theta(x)$$

and

$$\mathbb{N} \models \forall x < t\neg\theta(x).$$

(This rather silly exercise is here as a warning against a potential source of mistakes!)

2.11 If $M \subseteq_e N$ are $\mathscr{L}_A$-structures and $N \models PA^-$ show that $M \models PA^-$.

2.12 Verify the following variation on the Tarski–Vaught test for elementary substructures:

If $M \subseteq N$ are substructures then $M \prec_{\Delta_0} N$ iff for all $\bar{a} \in M$, for all $\mathscr{L}_A$-formulas $\theta(x, \bar{y}) \in \Delta_0$ and for all $\mathscr{L}_A$-terms $t(\bar{y})$, $N \models \exists x < t(\bar{a})\theta(x, \bar{a}) \Rightarrow$ there exists $b \in M$ such that $N \models b < t(\bar{a}) \wedge \theta(b, \bar{a})$.

2.13 Show that there are exactly $\aleph_0$ proper cuts of $\mathbb{Z}[X]^+$.

(The next two exercises push Corollary 2.9 further, by considering subtheories of PA^-. The bracketing of the terms $\underline{n}$ is important.)

2.14 The following eight axioms give the $\mathscr{L}_A$-theory Q:

$$\begin{aligned}
&Q1: && \forall x, y(x+1=y+1 \rightarrow x=y)\\
&Q2: && \forall x(0 \neq x+1)\\
&Q3: && \forall x(x \neq 0 \rightarrow \exists y(x=y+1))\\
&Q4: && \forall x(x+0=x)\\
&Q5: && \forall x, y(x+(y+1)=(x+y)+1)\\
&Q6: && \forall x(x \cdot 0=0)\\
&Q7: && \forall x, y(x \cdot (y+1)=(x \cdot y)+x)\\
&Q8: && \forall x, y(x<y+1 \leftrightarrow \exists z((x+z)+1=y)).
\end{aligned}$$

(a) Exhibit a model of Q in which $<$ is not interpreted as a linear order.

(b) Given that $M \vDash Q$, verify that the map $n \mapsto \underline{n}^M$ is an embedding of $\mathscr{L}_A$-structures $\mathbb{N} \to M$ with image an initial segment of M. Hence deduce $Q \vdash \Sigma_1 - \mathrm{Th}(\mathbb{N})$.

2.15 The following axiom and axiom schemes define the $\mathscr{L}_A$-theory S:

$$
\begin{array}{ll}
S0: & 0+1=1 \\
S1_k: & \neg 0=\underline{k}+1 \\
S2_{k,l}: & \underline{k}+1=\underline{l}+1 \to \underline{k}=\underline{l} \\
S3_k: & \underline{k}+0=\underline{k} \\
S4_{k,l}: & \underline{k}+(\underline{l}+1)=(\underline{k}+\underline{l})+1 \\
S5_k: & \underline{k}\cdot 0=0 \\
S6_{k,l}: & \underline{k}\cdot(\underline{l}+1)=(\underline{k}\cdot\underline{l})+\underline{k} \\
S7_k: & \forall x(x<\underline{k} \leftrightarrow (x=\underline{0} \vee x=\underline{1} \vee \cdots \vee x=k-1),
\end{array}
$$

where k, l range over all non-negative integers.

(a) Show that for all $\mathscr{L}_A$-structures M:

$$
\begin{aligned}
M \vDash S \text{ iff } & \mathbb{N} \cong \{\underline{n}^M \mid n \in \mathbb{N}\} \subseteq_e M \\
\text{iff } & \mathbb{N} \cong \{\underline{n}^M \mid n \in \mathbb{N}\} \prec_{\Delta_0} M \\
\text{iff } & M \vDash \Sigma_1 - \mathrm{Th}(\mathbb{N}).
\end{aligned}
$$

(b) Show that S is not finitely axiomatized.

3

Gödel incompleteness

You said we've a wise creator
And I replied you're right, but look,
You claim he's timeless and nowhere.
Such terms, for all we know, could be
A secret language: which amounts
To saying we cannot think straight.

Abu al-Ala al-Ma'arri
Translated by G. B. H. Wightman and A. Y. al-Udhari

This chapter is devoted to the proof of Gödel's first incompleteness theorem for recursively axiomatized theories extending PA^-. Gödel's theorem is often considered because of its role in foundational issues in mathematics (in this context the first incompleteness theorem tells us that 'arithmetic truth is not axiomatizable'). We shall be also interested in it for a different reason: the incompleteness theorem, in conjunction with the completeness theorem, equips us with a rich supply of interesting models. The techniques we employ, such as the representability of recursive functions in PA^- and the diagonalization lemma, will also yield other well-known and interesting results in logic, such as Tarski's theorem on the undefinability of truth and Gödel's second incompleteness theorem. These results will not (except for a brief remark in Chapter 15) be needed in the sequel, and so are left as exercises with detailed hints.

The chapter is in two sections: in the first we explore some of the connections between expressibility in $\mathscr{L}_A$ and recursion theory, and in the second we give the proof of the first incompleteness theorem in (essentially) the form first given by Rosser (1936).

3.1 Σ_1 FORMULAS AND THE COMPUTABLE SETS

We aim to describe the connections between *expressibility* in $\mathscr{L}_A$ and the notions 'recursive', 'r.e.', etc. The key result here is the theorem characterizing $\mathscr{C}$ in terms of $\mathscr{L}_A$ (Theorem 3.3). This result, together with Theorems 2.2 and 2.7 of the last chapter, go a long way to explaining why recursion theory is relevant to the study of models of arithmetic.

Our first lemma shows that the truth of Δ_0 formulas is easily computed.

LEMMA 3.1. For all Δ_0 formulas $\theta(\bar{x})$ of $\mathscr{L}_A$ the function $\chi_\theta(\bar{x})$ defined by

$$\chi_\theta(\bar{x}) = \begin{cases} 1 & \text{if} \quad \mathbb{N} \models \theta(\bar{x}) \\ 0 & \text{if} \quad \mathbb{N} \models \neg\theta(\bar{x}) \end{cases}$$

is primitive recursive.

Proof. By induction on the complexity of Δ_0 formulas (where 'complexity' is defined in the same way as in the proof of Theorem 2.7).

By Proposition 0.7, $\mathscr{PR}$ contains the functions $x+1$, $x+y$, $x \cdot y$, eq, and lt. So, as $\mathscr{PR}$ is closed under composition, it contains a function for each term $t(\bar{x})$ of $\mathscr{L}_A$, and hence contains

$$\chi_{t=s}(\bar{x}) = \mathrm{eq}(t(\bar{x}), s(\bar{x}))$$

and

$$\chi_{t<s}(\bar{x}) = \mathrm{lt}(t(\bar{x}), s(\bar{x}))$$

for each pair of $\mathscr{L}_A$-terms $t(\bar{x})$, $s(\bar{x})$. Thus the lemma is true for formulas θ of complexity zero.

For the induction step, if $\theta_1(\bar{x})$ and $\theta_2(\bar{x})$ are Δ_0 and $\chi_{\theta_1}, \chi_{\theta_2} \in \mathscr{PR}$ then

$$\chi_{\theta_1 \wedge \theta_2}(\bar{x}) = \min(\chi_{\theta_1}(\bar{x}), \chi_{\theta_2}(\bar{x}))$$

$$\chi_{\theta_1 \vee \theta_2}(\bar{x}) = \max(\chi_{\theta_1}(\bar{x}), \chi_{\theta_2}(\bar{x}))$$

and

$$\chi_{\neg\theta_1}(\bar{x}) = 1 \dot{-} \chi_{\theta_1}(\bar{x})$$

are all in $\mathscr{PR}$ by Proposition 0.7 and composition again. Finally, if $\theta(\bar{x}, y)$ is Δ_0, $\chi_\theta(\bar{x}, y) \in \mathscr{PR}$, $t(\bar{x})$ is an $\mathscr{L}_A$-term and $\psi(\bar{x})$ is the Δ_0 formula $\exists y < t(\bar{x})\theta(\bar{x}, y)$, then

$$\chi_\psi(\bar{x}) = 1 \dot{-} \mathrm{eq}(t(\bar{x}), (\mu y \leqslant t(\bar{x}))(1 \dot{-} \chi_\theta(\bar{x}, y) = 0))$$

is in $\mathscr{PR}$, using Proposition 0.9 and composition; similarly if φ is $\forall y < t(\bar{x})\theta(\bar{x}, y)$ then

$$\chi_\varphi(\bar{x}) = \mathrm{eq}(t(\bar{x}), (\mu y \leqslant t(\bar{x}))(\chi_\theta(\bar{x}, y) = 0))$$

is also in $\mathscr{PR}$. □

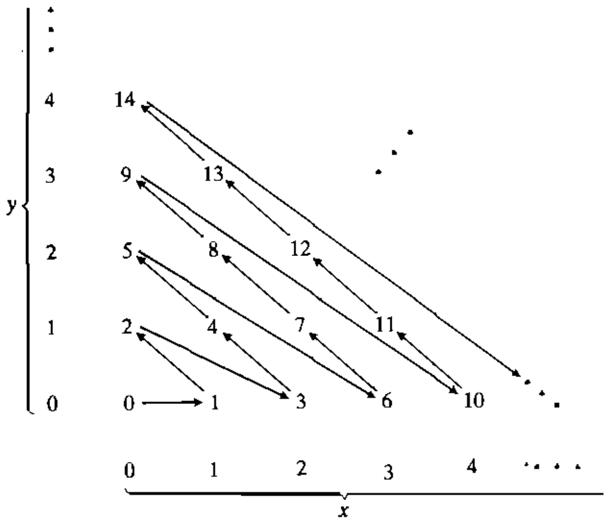

Fig. 1 The pairing function $(x+y)(x+y+1)/2+y$.

We shall extend our notation for functions defined by minimalization in a rather straightforward way: if $\theta(\bar{x}, y)$ is a Δ_0 formula, $(\mu y)(\theta(\bar{x}, y))$ denotes the least y such that $\mathbb{N} \models \theta(\bar{x}, y)$, if it exists, undefined otherwise. Similarly $(\mu y \leqslant z)(\theta(\bar{x}, y))$ is the least $y \leqslant z$ such that $\theta(\bar{x}, y)$, if this exists, and z otherwise. Lemma 3.1 shows that these functions are in $\mathscr{C}$ and $\mathscr{PR}$ respectively.

We should also mention that there is no converse to Lemma 3.1: there do in fact exist sets $A \subseteq \mathbb{N}$ with $\chi_A \in \mathscr{PR}$ but $\chi_A \neq \chi_\theta$ for all Δ_0 formulas $\theta(x)$. We will not prove this here; we will however give a characterization of the subsets of $\mathbb{N}$ that are defined by Σ_1 formulas. To do this we will need some way of coding tuples $\bar{x} \in \mathbb{N}$ by a single number in $\mathbb{N}$.

If $x, y \in \mathbb{N}$ we define $\langle x, y\rangle$ to be

$$\langle x, y\rangle = \frac{(x+y)(x+y+1)}{2} + y.$$

Since either $x+y$ or $x+y+1$ is even, $\langle x, y\rangle$ is always a non-negative integer, and $\langle,\rangle$ is a function $\mathbb{N}^2 \to \mathbb{N}$. It is easy to see that $\langle,\rangle$: $\mathbb{N}^2 \to \mathbb{N}$ is actually a bijection. (Figure 2 provides a hint); $\langle,\rangle$ is a primitive recursive function, since it may be obtained using composition from $+$ and the function $f(x) = x(x+1)/2$ given by $f(0) = 0$ and $f(x+1) = f(x) + x + 1$. It will also be important to observe that for any $x, y \in \mathbb{N}$, $x, y \leqslant \langle x, y\rangle < (x+y+1)^2$ and that the relation $\langle x, y\rangle = z$ is given by the formula $(x+y)(x+y+1) + 2y = 2z$ which, having no quantifiers at all is trivially Δ_0.

We extend the $\langle,\rangle$ notation to triples, quadruples, quintuples, etc. by inductively setting

$$\langle x_1, x_2, \ldots, x_k\rangle = \langle x_1, \langle x_2, \ldots, x_k\rangle\rangle$$

for each $k > 2$. The result is a family of bijections $\langle, \ldots, \rangle: \mathbb{N}^k \to \mathbb{N}$ with similar properties: each such function is in $\mathcal{PR}$ since it is obtained by composition from $\langle,\rangle$; each satisfies

$$x_1, x_2, \ldots, x_k \leqslant \langle x_1, x_2, \ldots, x_k\rangle < (x_1 + x_2 + \cdots + x_k + 1)^{2^k};$$

and each is given by a Δ_0 formula, for example $\langle x_1, \ldots, x_k\rangle = z$ is

$$\exists y \leqslant z(\langle x_1, y\rangle = z \wedge \langle x_2, \ldots, x_k\rangle = y),$$

where $\langle x_1, \ldots, x_{k-1}\rangle = z$ and $\langle x, y\rangle = z$ are Δ_0 formulas that have previously been defined.

The pairing function is very useful. We will however need slightly more: we will need to have a way of coding finite sequences in a way that is *uniform* in the length of the sequence. This is achieved in the following lemma of Gödel's.

Lemma 3.2 (Gödel's lemma). There is a Δ_0 formula $\theta(x, y, z)$ such that

$$\mathbb{N} \vDash \forall x, y \exists ! z \theta(x, y, z)$$

and, for all $k \in \mathbb{N}$ and all $z_0, z_1, \ldots, z_{k-1} \in \mathbb{N}$, there is $x \in \mathbb{N}$ such that

$$\text{for all } i < k \ \mathbb{N} \vDash \theta(x, i, z_i). \qquad \square$$

This lemma first appeared in Gödel's famous 1931 paper, and we shall give a proof of it later, in Chapter 5 below. (Indeed we shall prove rather more than is stated above: this is the reason for delaying the proof.) An alternative proof of Lemma 3.2 is indicated in the exercises at the end of this section.

Nowadays many suitable $\theta(x, y, z)$ satisfying Lemma 3.2 are known. For the rest of this section we shall fix one such formula and write it more suggestively as $(x)_y = z$ to indicate that x codes a sequence $(x)_0, (x)_1, \ldots$.

Definition. Let $f: \mathbb{N}^k \to \mathbb{N}$ be a partial function, i.e., $\mathrm{dom}(f) = A \subseteq \mathbb{N}^k$ for some set A. The *graph of* f, Γ_f, is the $(k+1)$-ary relation on $\mathbb{N}$ defined by

$$\Gamma_f(x_1, \ldots, x_k, y) \Leftrightarrow f(x_1, \ldots, x_k) = y.$$

We can now give the promised theorem characterizing $\mathscr{C}$.

THEOREM 3.3. Let $f\colon \mathbb{N}^k \to \mathbb{N}$ be a partial function on $\mathbb{N}$. Then $f \in \mathscr{C}$ iff Γ_f is equivalent to a Σ_1 $\mathscr{L}_A$-formula, i.e., iff there is a Σ_1 formula $\theta(x_1, \ldots, x_k, y)$ such that, for all $\bar{x}$, $y \in \mathbb{N}$, $\Gamma_f(\bar{x}, y) \Leftrightarrow \mathbb{N} \models \theta(\bar{x}, y)$.

Proof. To prove the '$\Leftarrow$' direction, we first define the function

$$\text{first}(x) = (\mu y \leqslant x)(\exists z \leqslant x(\langle y, z\rangle = x)).$$

first(x) is in $\mathscr{PR}$ by virtue of Lemma 3.1 and the fact that $x = \langle y, z\rangle$ is a Δ_0 formula. Now suppose the Σ_1 formula $\psi(x_1, \ldots, x_k, y)$ is equivalent to $\Gamma_f(x_1, \ldots, x_k, y)$, where f is a partial function $\mathbb{N}^k \to \mathbb{N}$, and suppose $\psi(\bar{x}, y)$ is $\exists \bar{z}\varphi(\bar{x}, y, \bar{z})$ where φ is Δ_0. Then using Lemma 3.1 again we may define $g \in \mathscr{C}$ by

$$g(\bar{x}) = (\mu z)(\exists v, \bar{w} \leqslant z(\langle v, \bar{w}\rangle = z \wedge \varphi(\bar{x}, v, \bar{w})))$$

using minimalization. Recall also that (if $\bar{w}$ is a tuple of length $\geqslant 2$) $\langle v, \bar{w}\rangle = \langle v, \langle \bar{w}\rangle\rangle$ so first$(\langle v, \bar{w}\rangle) = v$. This means that

$$\text{first}(g(\bar{x})) = \begin{cases} \text{the least } y \text{ s.t. } \mathbb{N} \models \psi(\bar{x}, y), \\ \text{if such } y \text{ exists} \\ \text{undefined, otherwise,} \end{cases}$$

since for each $\bar{x} \in \mathbb{N}$ there is at most one $y \in \mathbb{N}$ s.t. $\mathbb{N} \models \psi(\bar{x}, y)$. But $\psi(\bar{x}, y)$ is equivalent to Γ_f, so first$(g(\bar{x})) = f(\bar{x})$, where these are defined, hence $f \in \mathscr{C}$.

For the converse, let us say that a partial function $\mathbb{N}^k \to \mathbb{N}$ whose graph is equivalent to a Σ_1 formula *has Σ_1 graph*. We shall show that the class of all partial functions having Σ_1 graph contains the zero, successor, and projection functions, and is closed under composition, primitive recursion, and minimalization. It will follow that this class contains all of $\mathscr{C}$.

The graphs of the zero, successor and projection functions are clearly $y = 0$, $y = x + 1$, and $y = x_i$ respectively. If $f(x_1, \ldots, x_k)$, $g_1(\bar{z}), \ldots, g_k(\bar{z})$ have Σ_1 graphs $\Gamma_f(x, y)$, $\Gamma_{g_1}(\bar{z}, y), \ldots, \Gamma_{g_k}(\bar{z}, y)$ respectively, then the graph of $f(g_1(\bar{z}), \ldots, g_k(\bar{z}))$ is given by

$$\exists u_1, \ldots, u_k(\textstyle\bigwedge_{i=1}^{n} \Gamma_{g_i}(\bar{z}, u_i) \wedge \Gamma_f(u_1, \ldots, u_k, y))$$

which is equivalent to a Σ_1 formula since Σ_1 is closed under conjunctions and existential quantification. Hence the class of partial functions on $\mathbb{N}$ with Σ_1 graph is closed under composition.

If $f(\bar{x}, y)$ is obtained by primitive recursion as

$$f(\bar{x}, 0) = g(x)$$
$$f(\bar{x}, y+1) = h(\bar{x}, y, f(\bar{x}, y))$$

where both $g(\bar{x})$ and $h(\bar{x}, y, z)$ have Σ_1 graphs $\Gamma_g(\bar{x}, y)$ and $\Gamma_h(\bar{x}, y, z, w)$ then, using Gödel's lemma, the graph of f is

$$\Gamma_f(\bar{x}, y, z) \Leftrightarrow \exists u, v\{\Gamma_g(\bar{x}, v) \wedge (u)_0 = v \wedge (u)_y = z \wedge \forall i < y \exists r, s[(u)_i = r \wedge (u)_{i+1} = s \wedge \Gamma_h(\bar{x}, i, r, s)]\}$$

the idea being to code the sequence $f(\bar{x}, 0), f(\bar{x}, 1), \ldots, f(\bar{x}, y)$ by the single number u. But this formula is equivalent to a Σ_1 formula also, since $(x)_y = z$ is Δ_0 and if $\Gamma_h(\bar{x}, i, r, s)$ is $\exists \bar{t}\psi(\bar{x}, i, r, s, \bar{t})$ with $\psi \in \Delta_0$, then

$$\forall i < y \exists r, s[(u)_i = r \wedge (u)_{i+1} = s \wedge \Gamma_h(\bar{x}, i, r, s)]$$

is easily seen to be equivalent to

$$\exists w \forall i < y \exists r, s, \bar{t} \leqslant w[(u)_i = r \wedge (u)_{i+1} = s \wedge \psi(\bar{x}, i, r, s, \bar{t})]$$

by considering $w =$ the maximum of suitable r, s, $\bar{t}$ satisfying $(u)_i = r \wedge (u)_{i+1} = s \wedge \psi(\bar{x}, i, r, s, \bar{t})$ as i ranges over $\{0, 1, \ldots, y-1\}$. (This trick will be used many times in this book and, later on, will become the *collection axioms*.)

Finally, to show closure under minimalization, if $f(\bar{x}) = (\mu y)(g(\bar{x}, y) = 0)$ and the graph of $g(\bar{x}, y)$, $\Gamma_g(\bar{x}, y, z)$, is given by a Σ_1 formula, then $\Gamma_f(\bar{x}, y)$ is given by

$$\exists u\{(u)_y = 0 \wedge \forall i < y(\neg(u)_i = 0) \wedge \forall j \leqslant y \exists v(\Gamma_g(\bar{x}, j, v) \wedge (u)_j = v)\}$$

using Gödel's lemma to code the sequence $g(\bar{x}, 0), g(\bar{x}, 1), \ldots, g(\bar{x}, f(\bar{x}))$. But this too is equivalent to a Σ_1 formula, since if $\Gamma_g(\bar{x}, y, z)$ is $\exists \bar{w}\varphi(\bar{x}, y, z, \bar{w})$ with $\varphi \in \Delta_0$, then $\forall j \leqslant y \exists v(\Gamma_g(\bar{x}, j, v) \wedge (u)_j = v)$ is equivalent in $\mathbb{N}$ to

$$\exists s \forall j \leqslant y \exists v, \bar{w} \leqslant s(\varphi(\bar{x}, j, v, \bar{w}) \wedge (u)_j = v)$$

by a similar argument. □

We can now use this theorem to give characterizations of 'r.e.' and 'recursive' in terms of $\mathscr{L}_A$.

COROLLARY 3.4. $A \subseteq \mathbb{N}^k$ is r.e. iff there is a Σ_1 formula $\psi(x_1, \ldots, x_k)$ such that, for all $\bar{x} \in \mathbb{N}$, $\bar{x} \in A$ iff $\mathbb{N} \models \psi(\bar{x})$.

Proof. Suppose A is r.e., then for some $f(\bar{x}) \in \mathscr{C}$, $A = \mathrm{dom}(f)$. This means that for all $\bar{x} \in \mathbb{N}$, $\bar{x} \in A$ iff $\mathbb{N} \models \exists v \Gamma_f(\bar{x}, v)$, and $\exists v \Gamma_f(\bar{x}, v)$ is equivalent to a Σ_1 formula by Theorem 3.3.

Conversely if $A = \{\bar{x} \mid \mathbb{N} \models \psi(\bar{x})\}$ with $\psi \in \Sigma_1$ define $f(\bar{x}) = 0$ if $\mathbb{N} \models \psi(\bar{x})$, and $f(\bar{x})$ undefined otherwise. The graph of f, $\Gamma_f(\bar{x}, y)$, is then given by the Σ_1 formula

$$y = 0 \wedge \psi(\bar{x})$$

so $f \in \mathscr{C}$ by Theorem 3.3, and so A is r.e., since $A = \mathrm{dom}(f)$. □

COROLLARY 3.5. $A \subseteq \mathbb{N}^k$ is recursive iff there is a Σ_1 formula $\psi(x_1, \ldots, x_k)$ and a Π_1 formula $\varphi(x_1, \ldots, x_k)$ such that, for all $\bar{x} \in \mathbb{N}$, $\bar{x} \in A$ iff $\mathbb{N} \models \psi(\bar{x})$ iff $\mathbb{N} \models \varphi(\bar{x})$.

Proof. Recall that A is recursive iff both A and its complement A^c are r.e. This means that, by Corollary 3.4, A is recursive iff '$\bar{x} \in A$' and '$\bar{x} \in A^c$' are equivalent to Σ_1 formulas iff '$\bar{x} \in A$' is equivalent to both a Σ_1 formula and a Π_1 formula, since the negation of a Σ_1 formula is equivalent to a Π_1 formula. □

There are also useful versions of Theorem 3.3 and Corollaries 3.4 and 3.5 that work for the function classes $\mathscr{C}^A$ and the corresponding notions of being r.e. or recursive in the oracle A. Let $A \subseteq \mathbb{N}^k$ for some $k \geqslant 1$ and let $\mathscr{L}_A(A)$ be the first-order language obtained by adding to $\mathscr{L}_A$ a new k-ary relation symbol (which we shall also denote A). $\mathbb{N}$ can be considered as an $\mathscr{L}_A(A)$ structure by interpreting the new relation symbol by the set A. Thus: $\mathbb{N} \models A(\bar{n}) \Leftrightarrow \bar{n} \in A$. The formula classes Σ_1^A, Π_0^A and Δ_0^A of $\mathscr{L}_A(A)$ are defined in exactly the same way as the corresponding classes Σ_1, Π_1 and Δ_0 of $\mathscr{L}_A$ except that now the relation symbol A may occur in Σ_1, Π_1 or Δ_0 formulas. Notice that the characteristic function $\chi_A(\bar{x})$ of A has Δ_0^A graph given by

$$\chi_A(\bar{x}) = y \Leftrightarrow (y = 1 \wedge A(\bar{x})) \vee (y = 0 \wedge \neg A(\bar{x})).$$

We can now state the 'relativized' versions of Theorem 3.3 and Corollaries 3.4 and 3.5. (The proofs are exactly the same.)

THEOREM 3.3^A. If $A \subseteq \mathbb{N}^k$ with $k \geqslant 1$, and $\mathbb{N}$ is considered as an $\mathscr{L}_A(A)$-structure by interpreting $\mathbb{N} \models A(\bar{n}) \Leftrightarrow \bar{n} \in A$, then a partial function f on $\mathbb{N}$ is in $\mathscr{C}^A$ iff Γ_f is equivalent (in $\mathbb{N}$) to a Σ_1^A formula. □

COROLLARY 3.4^A. With $A \subseteq \mathbb{N}^k$ and $\mathbb{N}$ considered as an $\mathscr{L}_A(A)$-structure as before, a set $B \subseteq \mathbb{N}^l$ is r.e. in A iff the relation '$\bar{x} \in B$' is equivalent (in $\mathbb{N}$) to a Σ_1^A formula. □

COROLLARY 3.5^A. With $A \subseteq \mathbb{N}^k$ and $\mathbb{N}$ as before, a set $B \subseteq \mathbb{N}^l$ is recursive in A iff '$\bar{x} \in B$' is equivalent (in $\mathbb{N}$) to both a Σ_1^A formula and a Π_1^A formula. □

Exercises for Section 3.1

(These exercises provide a proof of Gödel's lemma, Lemma 3.2. The ideas used for this proof are from Smullyan (1961) and Quine (1946).)
3.1 Show that 'x is a power of 2' is equivalent in $\mathbb{N}$ to a Δ_0 formula.
(*Hint*: Use the facts that 'y divides x' and '2 divides y' are equivalent in $\mathbb{N}$ to Δ_0 formulas.)
3.2 A natural number $n \in \mathbb{N}$ can be represented by its *dyadic numeral* $\varepsilon_{k-1}\varepsilon_{k-2} \cdots \varepsilon_2\varepsilon_1\varepsilon_0$ where each ε_i is either the symbol 1 or the symbol 2, $k \in \mathbb{N}$ (possibly $k=0$) and $n = \Sigma_{i=0}^{k-1} \varepsilon_i 2^i$. Show that this representation gives a 1–1 correspondence between $\mathbb{N}$ and strings of the symbols 1, 2 of finite length.
3.3 (a) Show that 'y is represented by a string of 2s of the same length of x' is equivalent to a Δ_0 formula $\varphi(x, y)$.
(b) Hence deduce 'the dyadic numeral for z is the concatenation of those for x, y' is equivalent in $\mathbb{N}$ to a Δ_0 formula $z = x^\frown y$.
3.4 Define 'x is a substring of y' ($x \subseteq_p y$) on strings of the symbols 1, 2, and show this is represented by a Δ_0 formula in $\mathbb{N}$ via the correspondence in Exercise 3.2. Hence consider

$$\begin{aligned}\psi(x, y) \leftrightarrow \exists u, w \leqslant y(&\text{‘}u \text{ is a string of 2s'} \wedge u \neq 1 \wedge u \subseteq_p y \wedge \\ &\forall z \leqslant y(z \subseteq_p y \wedge \text{‘}z \text{ is a string of 2s'} \rightarrow z \leqslant u) \wedge \\ &w = u^\frown 1^\frown x^\frown 1^\frown u \wedge w \subseteq_p y \wedge \neg u \subseteq_p x)\end{aligned}$$

and

$$\begin{aligned}\theta(x, y, z) \leftrightarrow (&\psi(\langle z, y\rangle, x) \wedge \forall v < z \neg\psi(\langle v, y\rangle, x)) \vee \\ &(\forall u \leqslant x \neg\psi(\langle u, y\rangle, x) \wedge z = 0).\end{aligned}$$

Show that $\theta(x, y, z)$ satisfies the conclusion to Gödel's lemma.

3.2 INCOMPLETENESS

To apply the results of Section 3.1 to theories of arithmetic we need some notion of a function or set being *represented in a theory**. This achieved by the following definition:

* Some authors call the notion we define here that of being *strongly represented in a theory*.

DEFINITION: Let $f\colon \mathbb{N}^k \to \mathbb{N}$ be a total function and T an $\mathcal{L}_A$-theory extending PA^-. Say f is *represented in* T iff there is an $\mathcal{L}_A$-formula $\theta(x_1, \ldots, x_k, y)$ such that, for all $\bar{n} \in \mathbb{N}^k$,

(a) $$T \vdash \exists! y\theta(\bar{n}, y)$$

and

(b) $$\text{if } k = f(\bar{n}) \quad \text{then} \quad T \vdash \theta(\bar{n}, k).$$

(Notice that, since T extends PA^-, we can, without ambiguity, identify each $n \in \mathbb{N}$ with some closed term of $\mathcal{L}_A$ that has value n in $\mathbb{N}$.)

Similarly we say a set $S \subseteq \mathbb{N}^k$ is *represented in* T iff there is an $\mathcal{L}_A$-formula $\theta(x_1, \ldots, x_k)$ such that for all $\bar{n} \in \mathbb{N}^k$

(c) $$\bar{n} \in S \Rightarrow T \vdash \theta(\bar{n})$$

and

(d) $$\bar{n} \notin S \Rightarrow T \vdash \neg\theta(\bar{n}).$$

The function f, or the set S, is *Σ_1-represented in T* if the formula θ can be taken to be Σ_1.

LEMMA 3.6. Let $f\colon \mathbb{N}^k \to \mathbb{N}$ be recursive. Then f is Σ_1-represented in PA^-.

Proof. If $f\colon \mathbb{N}^k \to \mathbb{N}$ is recursive it is total and its graph, Γ_f, is given by a Σ_1 formula, by Theorem 3.3. Let this formula be $\exists \bar{z}\varphi(\bar{x}, y, \bar{z})$ with $\varphi \in \Delta_0$; by replacing this with $\exists z\, \exists \bar{w} < z\varphi(\bar{x}, y, \bar{w})$ if necessary we may assume that the tuple $\bar{z}$ is actually a *single variable*, z.

Now consider the Δ_0 formula $\psi(\bar{x}, y, z)$ given by

$$\varphi(\bar{x}, y, z) \wedge \forall u, v \leq y + z(u + v < y + z \rightarrow \neg\varphi(\bar{x}, u, v)).$$

We claim the Σ_1 formula $\exists z\psi(\bar{x}, y, z)$ satisfies conditions (a) and (b) in the definition above of 'f is represented in PA^-'.

We verify (b) first. Let $\bar{n} \in \mathbb{N}$ and $k = f(\bar{n})$. Then k is the *unique* element of $\mathbb{N}$ satisfying $\mathbb{N} \models \exists z\varphi(\bar{n}, k, z)$. Now let l be the least element of $\mathbb{N}$ satisfying $\mathbb{N} \models \varphi(\bar{n}, k, l)$. It is easy to check that $\mathbb{N} \models \psi(\bar{n}, k, l)$ also, and hence $\mathbb{N} \models \exists z\psi(\bar{n}, k, z)$. But then $\exists z\psi(\bar{n}, k, z)$ is a true Σ_1 sentence and so by Corollary 2.9, $PA^- \vdash \exists z\psi(\bar{n}, k, z)$ as required.

To verify (a), suppose $\bar{n} \in \mathbb{N}$, $k = f(\bar{n})$ and l is the least element of $\mathbb{N}$ satisfying $\mathbb{N} \models \varphi(\bar{n}, k, l)$ as before. Let $M \models PA^-$ be arbitrary and $a, b \in M \models \psi(\bar{n}, a, b)$. We will show that $a = k$, which suffices by the completeness theorem. Since $\varphi(\bar{n}, k, l)$ is Δ_0 and true in $\mathbb{N}$ we have $M \models \varphi(\bar{n}, k, l)$ and by consideration of ψ and the fact that $<$ is a linear order on M we deduce that $a, b \leq k + l$, and hence both a and b are in $\mathbb{N}$. But $\mathbb{N} \prec_{\Delta_0} M \models \psi(\bar{n}, a, b)$

by Theorems 2.7 and 2.2, so as ψ is Δ_0 we deduce $\mathbb{N}\models\psi(\bar{n}, a, b)$, and hence $\mathbb{N}\models\exists z\varphi(\bar{n}, a, z)$, so $a=k$, as required. □

COROLLARY 3.7. All recursive sets $A\subseteq\mathbb{N}^k$ are Σ_1-represented in PA^-.

Proof. If $A\subseteq\mathbb{N}^k$ is recursive, then its characteristic function χ_A is recursive. Let $\theta(\bar{x}, y)$ be a Σ_1 formula representing $\chi_A(\bar{x})=y$ in PA^-. Then $\theta(\bar{x}, 1)$ represents A in PA^-, since

$$PA^-\vdash\theta(\bar{n}, 1)\Leftrightarrow\chi_A(\bar{n})=1\Leftrightarrow\bar{n}\in A, \qquad \text{and}$$

$$PA^-\vdash\neg\theta(\bar{n}, 1)\Leftrightarrow PA^-\vdash\theta(\bar{n}, 0)\Leftrightarrow n\notin A$$

for each $\bar{n}\in\mathbb{N}$. □

The two key ideas in the proof of the incompleteness theorem are those of *Gödel-numbering* of formulas and proofs and the use of a *diagonal* argument. A Godel-numbering is an injection $\sigma\mapsto\ulcorner\sigma\urcorner$ taking formulas (considered as strings of symbols from some finite alphabet*) to natural numbers. We will require our Gödel-numbering to have the following properties:

(a) it is a computable function (we assume some natural notion of computability on strings of symbols over a finite alphabet);
(b) its image is a computable subset of $\mathbb{N}$;
(c) its inverse (where this inverse is defined) is also computable. Thus we may recalculate the formula σ from its Godel-number $\ulcorner\sigma\urcorner$.

There are many suitable Gödel-numberings to choose from and, for the purposes of this chapter, the choice does not matter. To give a specific example, however, if there are 26 symbols in our alphabet, say, we could associate with each symbol, s, a unique integer, n_s, from $\{0, 1, 2, \ldots, 25\}$ and define our Gödel-numbering by sending a string $s_0s_1s_2\ldots s_k$ to the integer $\ulcorner s_0s_1s_2\ldots s_k\urcorner=\Sigma_{i=0}^k 26^i\cdot(n_{s_i}+1)$. This Gödel-numbering may be extended or modified to code *proofs* in a rather natural way: for example if we regard proofs as certain kinds of finite sequences of formulas we may introduce a new symbol # to our alphabet and code the sequence of formulas $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_k$ as the Gödel-number of the string $\sigma_0\#\sigma_1\#\sigma_2\#\cdots\#\sigma_k$.

The second idea in the proof of the incompleteness theorem is the use of diagonalization; this part of the argument is summarized by the following very useful lemma.

LEMMA 3.8 (The diagonalization lemma). Let T be an $\mathscr{L}_A$-theory that Σ_1-represents all recursive functions, and let $\theta(x)$ be an $\mathscr{L}_A$-formula with one

* Here we can conveniently think of the variables of $\mathscr{L}_A$ being formed from two symbols v and ′ so that v_i is $\mathrm{v}''\cdots'$ (i primes).

free-variable x. Then there is a sentence G of $\mathcal{L}_A$ such that

$$T \vdash G \leftrightarrow \theta(\ulcorner G \urcorner).$$

Moreover if $\theta(x)$ is a Π_1 formula, then G may be taken to be equivalent to a Π_1 sentence also.

Proof. Let diag be the recursive function defined by

$$\mathrm{diag}(n) = \begin{cases} \ulcorner \forall y(y = \underline{n} \to \sigma(y)) \urcorner & \text{if } n = \ulcorner \sigma(x) \urcorner, \\ & \quad \text{the Gödel-number of an } \mathcal{L}_A\text{-formula} \\ 0, & \text{otherwise.} \end{cases}$$

(For the purposes of this proof we revert back to our notation $\underline{n}$ for the canonical closed term denoting the integer n.) The function diag is clearly recursive by our choice of Gödel-numbering function and by the Church–Turing thesis. Let $\delta(x, y)$ be a Σ_1 formula representing $\mathrm{diag}(x) = y$ in T, and let $\psi(x)$ be the formula

$$\forall z(\delta(x, z) \to \theta(z)).$$

Let $n = \ulcorner \psi(x) \urcorner$ and let G be the sentence

$$\forall y(y = \underline{n} \to \psi(y)),$$

i.e., G is the sentence whose Gödel-number is $\mathrm{diag}(\ulcorner \psi(x) \urcorner)$. Clearly if $\theta(x)$ is Π_1, then both $\psi(x)$ and G are equivalent to Π_1 formulas. Thus it suffices to prove $T \vdash G \leftrightarrow \theta(\ulcorner G \urcorner)$.

Now clearly $T \vdash G \leftrightarrow \psi(\underline{n})$, i.e. $T \vdash G \leftrightarrow \forall z(\delta(\underline{n}, z) \to \theta(z))$, and since $\delta(x, y)$ represents diag in T and $\ulcorner G \urcorner = \mathrm{diag}(n)$ we have

$$T \vdash \forall z(\delta(\underline{n}, z) \leftrightarrow z = \ulcorner G \urcorner)$$

so

$$T \vdash G \leftrightarrow \theta(\ulcorner G \urcorner)$$

as required. □

We can now prove the main theorems of this chapter: The Gödel–Rosser incompleteness theorems.

THEOREM 3.9 (A rudimentary form of the Gödel–Rosser theorem). Let T be a recursive set of Gödel-numbers of $\mathcal{L}_A$-sentences such that
(a) T is *simply consistent*, i.e. for no $\mathcal{L}_A$-sentence σ is it the case that both $\ulcorner \sigma \urcorner \in T$ and $\ulcorner \neg \sigma \urcorner \in T$;
(b) $T \supseteq \{\ulcorner \sigma \urcorner | \sigma \text{ is a } \Sigma_1 \text{ or } \Pi_1\ \mathcal{L}_A\text{-sentence s.t. } PA^- \vdash \sigma\}$.
Then T is incomplete, in the sense that there is a Π_1 sentence τ s.t. neither $\ulcorner \tau \urcorner$ nor $\ulcorner \neg \tau \urcorner$ is in T.

Proof. Since T is recursive, there is a Σ_1 formula $\theta(x)$ such that, for all $n \in \mathbb{N}$,

$$n \in T \Leftrightarrow PA^- \vdash \theta(n)$$

and

$$n \notin T \Leftrightarrow PA^- \vdash \neg\theta(n).$$

Using Lemma 3.8 there is a Π_1 sentence τ such that $PA^- \vdash \tau \leftrightarrow \neg\theta(\ulcorner\tau\urcorner)$. Then, using (a) and (b),

$$\ulcorner\tau\urcorner \in T \Rightarrow PA^- \vdash \theta(\ulcorner\tau\urcorner) \Rightarrow PA^- \vdash \neg\tau \Rightarrow \ulcorner\neg\tau\urcorner \in T$$

and

$$\ulcorner\neg\tau\urcorner \in T \Rightarrow \ulcorner\tau\urcorner \notin T \Rightarrow PA^- \vdash \neg\theta(\ulcorner\tau\urcorner) \Rightarrow PA^- \vdash \tau \Rightarrow \ulcorner\tau\urcorner \in T,$$

showing that neither $\ulcorner\tau\urcorner$ nor $\ulcorner\neg\tau\urcorner$ is in T. □

Corollary 3.10 (The Gödel–Rosser incompleteness theorem). Let T be a consistent recursively axiomatized $\mathscr{L}_A$-theory extending PA^-. Then there is a Π_1 sentence τ such that neither $T \vdash \tau$ nor $T \vdash \neg\tau$.

Proof. If T were consistent, recursive, and complete for Π_1 sentences, then the set

$$S = \left\{ \ulcorner\sigma\urcorner \,\middle|\, \begin{array}{l} \sigma \text{ is an } \mathscr{L}_A\text{-sentence, } \sigma \text{ is } \Pi_1 \text{ or } \Sigma_1 \\ \text{and } T \vdash \sigma \end{array} \right\}$$

would satisfy the hypothesis of Theorem 3.9. (S is recursive since by assumption $T \vdash \sigma$ iff $T \nvdash \neg\sigma$ for all Σ_1 or Π_1 sentences σ. Thus to decide if a Σ_1 or Π_1 sentence σ has a Gödel-number in S we can systematically search through all formal proofs from axioms in T until we come across either a proof of σ or a proof of $\neg\sigma$. Then $\ulcorner\sigma\urcorner \in S$ or $\ulcorner\sigma\urcorner \notin S$ respectively.)

But by Theorem 3.9 there is then a Π_1 sentence τ such that $\ulcorner\tau\urcorner \notin S$ and $\ulcorner\neg\tau\urcorner \notin S$. This is the required contradiction. □

The weaker form of Corollary 3.10, where 'T is consistent' is replaced by 'T is ω-consistent' (meaning: for all $\mathscr{L}_A$-formulas $\theta(x)$, if $T \vdash \theta(n)$ for all $n \in \mathbb{N}$ then $T + \forall x \theta(x)$ is consistent) is known as 'Gödel's first incompleteness theorem' and was first proved by Gödel in Gödel (1931). The improvement to Corollary 3.10 is due to Rosser (1936).

Corollary 3.11. Let $T \supseteq PA^-$ be a recursively axiomatized $\mathscr{L}_A$-theory. Then T has $2^{\aleph_0}$ complete extensions.

Proof. By Corollary 3.10, whenever U is a recursive consistent extension of PA^- there is a Π_1 sentence σ_U such that both $U+\sigma_U$ and $U+\neg\sigma_U$ are consistent. Define recursively-axiomatized thories T_s (where s is a (possibly empty) finite sequence of 0s and 1s) inductively by

$$T_\varnothing = T$$

$$T_{s0} = T_s + \sigma_{T_s}$$

$$T_{s1} = T_s + \neg\sigma_{T_s}.$$

(Notice that each T_s is a finite extension of T and so is clearly recursively axiomatized.) For each function $f\colon \mathbb{N}\to\{0,1\}$ let T_f be $\bigcup_{n\in\mathbb{N}} T_{f\restriction n}$, where $f\restriction n$ is the finite sequence $f(0), f(1), f(2), \ldots, f(n-1)$ of 0s and 1s. Clearly each T_f is consistent, and so has a complete consistent extension T'_f. We claim that if $f\neq g$ then $T'_f\neq T'_g$, which suffices since there are $2^{\aleph_0}$ functions $\mathbb{N}\to\{0,1\}$. But if $f\neq g$ then there is an n such that $f\restriction n=g\restriction n$ but $f(n)\neq g(n)$, and if U is the theory $T_{f\restriction n}=T_{g\restriction n}$ then, by the construction, T_f and T_g disagree about the sentence σ_U, so $T'_f\neq T'_g$. □

COROLLARY 3.12. Let $T\supseteq PA^-$ be a consistent $\mathscr{L}_A$-theory with a recursive axiomatization. Then there are $2^{\aleph_0}$ countable nonelementarily-equivalent models of T.

Proof. Each of the complete theories T'_f of Corollary 3.11 has a countable model, by the completeness theorem. □

Exercises for Section 3.2

3.5 Show that, for all functions $f\colon \mathbb{N}^k\to\mathbb{N}$ (where $k\geqslant 1$), f is represented in PA^- implies that f is recursive. Hence deduce that f is represented in PA^- iff f is Σ_1-represented in PA^-.

3.6 Let Q and S be the $\mathscr{L}_A$-theories of Exercises 2.14 and 2.15. The theory R is axiomatized by S together with the scheme

$$\forall x(x<\underline{n}\vee x=\underline{n}\vee x>\underline{n})$$

for all $n\in\mathbb{N}$. Prove that $Q\vdash R\vdash S$ but $S\nvdash R$ and $R\nvdash Q$, and that all recursive functions are Σ_1-represented in R.

3.7* (Tarski's theorem on the undefinability of truth). Let $M\vDash PA^-$ and let $\ulcorner\cdot\urcorner$ be a Gödel-numbering of $\mathscr{L}_A$-formulas satisfying (a), (b), and (c) on p. 37.

Show that there is no $\mathscr{L}_A$-formula $\theta(x)$ such that, for all $n\in\mathbb{N}$, $M\vDash\theta(n)$ iff n is the Gödel-number of an $\mathscr{L}_A$-sentence σ and $M\vDash\sigma$.

(*Hint:* If such a formula $\theta(x)$ exists consider the $\mathscr{L}_A$-sentence G satisfying

$$PA^-\vdash G\leftrightarrow\neg\theta(\ulcorner G\urcorner)$$

given by the diagonalization lemma.)

3.8* (Gödel's second incompleteness theorem). Gödel's second incompleteness theorem is often paraphrased by saying 'sufficiently strong theories T cannot prove their own consistency'. If this is to mean anything at all we must be able to express 'the sentence σ is provable in T' as a formula prov($\ulcorner\sigma\urcorner$) in the language of T. Let T be a consistent $\mathscr{L}_A$-theory extending PA^-, fix some Gödel-numbering $\ulcorner\cdot\urcorner$ of $\mathscr{L}_A$-formulas satisfying (a), (b), and (c) on p. 37, and suppose $\theta(x)$ is an $\mathscr{L}_A$-formula with one free-variable such that, for any sentences σ, τ of $\mathscr{L}_A$,

(a) $T \vdash \sigma \Rightarrow T \vdash \theta(\ulcorner\sigma\urcorner)$

(b) $T \vdash \theta(\ulcorner\sigma \rightarrow \tau\urcorner) \rightarrow (\theta(\ulcorner\sigma\urcorner) \rightarrow \theta(\ulcorner\tau\urcorner))$

(c) $T \vdash \theta(\ulcorner\sigma\urcorner) \rightarrow \theta(\ulcorner\theta(\ulcorner\sigma\urcorner)\urcorner)$.

($\theta(x)$ is said to be a *provability predicate for* T. Such T are typically rather stronger than PA^-. The theory PA, to be introduced in the next chapter, does have a provability predicate.) The *consistency* of T can now be expressed by saying '$0=1$ is not provable', i.e., by the sentence $\neg\theta(\ulcorner 0=1\urcorner)$.

Prove the second incompleteness theorem, that for T and θ as above,

$$T \nvdash \neg\theta(\ulcorner 0=1\urcorner).$$

(*Hint:* Let ψ be an $\mathscr{L}_A$-sentence satisfying $T \vdash \psi \leftrightarrow \neg\theta(\ulcorner\psi\urcorner)$. Using (a), show that $T \nvdash \psi$. Now by considering the sentence $\psi \rightarrow (\theta(\ulcorner\psi\urcorner) \rightarrow 0=1)$ and using (a), (b), and (c) show that $T \vdash \neg\theta(\ulcorner 0=1\urcorner) \rightarrow \psi$ and hence deduce that $T \nvdash \neg\theta(\ulcorner 0=1\urcorner)$.)

3.9 Let $f: \mathbb{N}^k \rightarrow \mathbb{N}$ be a (total) function, and T an $\mathscr{L}_A$-theory. Show that f is represented in T iff the graph of f is represented in T.

4

The axioms of Peano Arithmetic

$1 \varepsilon N.$
$a \varepsilon N . \supset . a+1 \varepsilon N.$
$a, b \varepsilon N . \supset : a=b . = . a+1=b+1.$
$a \varepsilon N . \supset . a+1 -= 1.$
$k \varepsilon K \therefore 1 \varepsilon k \therefore x \varepsilon N . x \varepsilon k : \supset_x x+1 \varepsilon k :: \supset . N \supset k.$

Giuseppe Peano, *Arithmetices principia nova methodo exposita*, Turin: Bocca, 1889.

The quotations at the head of this chapter is how Peano stated his famous five axioms of arithmetic. (His notation looks surprisingly modern. In particular his use of $\supset$ for implication and set-inclusion has become the modern-day $\supseteq$ and $\supset$, although the 'subset' relation symbol $\supset$ is now written the other way round, as $\subseteq$!) Essentially, Peano's axioms are those of PA^- together with the *second-order induction axiom*,

$$\forall X(0 \in X \wedge \forall x(x \in X \rightarrow x+1 \in X) \rightarrow \forall y(y \in X)) \qquad (*)$$

where X ranges over all subsets of the domain of the stucture under consideration, and x, y range over elements of this domain.

Peano (and, independently, Dedekind) realized that these axioms actually characterize the standard model $\mathbb{N}$, i.e. up to a canonical isomorphism there is only one such structure satisfying PA^- together with (*), namely $\mathbb{N}$, (see Exercise 4.4). This observation isolates the most important feature of $\mathbb{N}$, i.e., *induction*, and tells us a great deal about the nature of induction and recursive definitions. It also, however, begs many difficult philosophical and set-theoretic questions about the nature of subsets of $\mathbb{N}$.

The approach taken in the study of the theory that is variously known nowadays as 'first-order Peano arithmetic', 'Peano arithmetic' or simply 'PA' bypasses these questions by restricting the induction axiom (*) to subsets X that are defined in a rather concrete way, namely to those subsets X defined by a *first-order* $\mathscr{L}_A$-formula. The result is a weaker theory which no longer characterizes $\mathbb{N}$, but a theory that is extremely natural, and one on which much of our understanding of $\mathbb{N}$ is based. In fact, having a large supply of interesting models is actually a bonus here! If PA turned out to be rather weak, however, it would be uninteresting. Fortunately PA is surprisingly strong and encapsulates almost all of what are known today as 'elementary number theory' and 'finite combinatorics', as we shall see.

PA, then, is the first-order theory we get when we add to our base theory, PA^-, as much of the Peano–Dedekind induction axiom (*) as possible, but remain in the realm of the first-order language $\mathscr{L}_A$. To be specific, if $\varphi(x, \bar{y})$, is an $\mathscr{L}_A$-formula, *the axiom of induction on x in $\varphi(x, \bar{y})$*, $I_x\varphi$, is the sentence

$$\forall\bar{y}(\varphi(0,\bar{y})\wedge\forall x(\varphi(x,\bar{y})\rightarrow\varphi(x+1,\bar{y}))\rightarrow\forall x\varphi(x,\bar{y})).$$

Here x is the *induction variable* and $\bar{y}$ are the *parameters*. In most cases, however, it will be clear from the context which variable is the induction variable, and when this is the case we denote $I_x\varphi$ more simply as $I\varphi$. PA is the first-order theory axiomatized by PA^- together with induction axioms $I_x\varphi$ for all $\mathscr{L}_A$-formulas φ. More generally, if Γ is a class of $\mathscr{L}_A$-formulas then the theory $I\Gamma$ is the first-order theory axiomatized by PA^- together with all $I_x\varphi$ for $\varphi\in\Gamma$. For example this definition gives us theories $I\Delta_0$, $I\Sigma_1$, and $I\Pi_1$ when $\Gamma=\Delta_0$, Σ_1 or Π_1, respectively.

There are two important remarks to make at this point. The first is that, whilst the Peano–Dedekind second-order axiom of induction is given by a single sentence, PA is given by an axiom *scheme*. (We shall see later that no finite set of $\mathscr{L}_A$-sentences can axiomatize PA.) The second is that PA is *recursively* axiomatized (albeit not *finitely* axiomatized) that is, there is a recursive procedure to decide if a given sentence σ is an axiom of PA. It follows by Gödel's incompleteness theorem that PA is not complete. Indeed, by Corollary 3.11, PA has $2^{\aleph_2}$ complete extensions, only one of which is $\mathrm{Th}(\mathbb{N})$.

In the next chapter we intend to back up our claim that PA is sufficiently strong to prove most of elementary number theory, and indeed that there are natural ways of expressing number-theoretic and finite-combinatorial statements in PA. The rest of this chapter is devoted to preparing the ground, and to introducing the reader to the style of argument we shall use inside PA. One particularly pleasant conclusion from this chapter's discussion is that the other natural induction schemes (such as *total induction* and the *least-number principle*) turn out to be a consequence of PA as we have defined it above.

4.1 ALTERNATIVE INDUCTION SCHEMES

This section addresses the problem of developing techniques for proving theorems within PA. In particular we will be interested in altenative induction schemes that can be justified in PA. The first very simple (but very useful) example sets the scene nicely.

Suppose $n\in\mathbb{N}$, $\varphi(x)$ is a formula, and we wish to show $\mathbb{N}\models\forall x\leqslant n\varphi(x)$.

Then it is clearly sufficient to show that

$$\mathbb{N} \models \varphi(0) \wedge \forall x < n(\varphi(x) \rightarrow \varphi(x+1)),$$

and indeed the stronger statement, $\mathbb{N} \models \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1))$ might not be true. We can express this principle by the scheme

$$\forall \bar{y}, z(\varphi(0, \bar{y}) \wedge \forall x < z(\varphi(x, \bar{y}) \rightarrow \varphi(x+1, \bar{y})) \rightarrow \forall x \leqslant z \varphi(x, \bar{y})) \quad (+)$$

over all $\mathscr{L}_{\mathrm{A}}$-formulas $\varphi(x, \bar{y})$. Does PA prove all instances of $(+)$? Fortunately the answer is 'yes'.

Proposition 4.1. *PA* proves all instances of the scheme $(+)$ of 'induction up to z'.

Proof. Let M be an arbitrary model of PA, and let $\varphi(x, \bar{y})$ be any $\mathscr{L}_{\mathrm{A}}$-formula. We shall show that M satisfies $(+)$ and conclude from the completeness theorem that the sentence $(+)$ is provable in PA.

So suppose $\bar{a}$, $b \in M$ and

$$M \models \varphi(0, \overset{\bar{a}}{\cancel{b}}) \wedge \forall x < b(\varphi(x, \bar{a}) \rightarrow \varphi(x+1, \bar{a}));$$

we must show $M \models \forall x \leqslant b \varphi(x, \bar{a})$. Let $\psi(x, \bar{y}, z)$ be the $\mathscr{L}_{\mathrm{A}}$-formula $(x \leqslant z \wedge \varphi(x, \bar{y})) \vee (x > z)$. Then clearly $M \models \psi(x, \bar{a}, b)$ for all $x > b$, and

$$M \models \psi(0, \bar{a}, b) \wedge \forall x < b(\psi(x, \bar{a}, b) \rightarrow \psi(x+1, \bar{a}, b))$$

so

$$M \models \psi(0, \bar{a}, b) \wedge \forall x(\psi(x, \bar{a}, b) \rightarrow \psi(x+1, \bar{a}, b))$$

hence by $I\psi$ in M, $M \models \forall x \psi(x, \bar{a}, b)$, hence $M \models \forall x \leqslant b \varphi(x, \bar{a})$, as required. □

Notice in particular the use of the completeness theorem and our argument 'inside the model M'. This will be our usual method of showing that PA proves a given sentence σ, and from now on we shall omit the explicit reference to the completeness theorem. This method has two advantages: firstly, the notation is much more concise and the proof shorter; and secondly, the important idea or ideas in the proof are much more clearly expressed. It is an instructive exercise to write out a proof of $PA \vdash (+)$ in one's favourite formal proof-system for the predicate calculus!

The next result will be used a lot: it shows that PA proves the *least-number principle.* If $\varphi(x, \bar{y})$ is an $\mathscr{L}_{\mathrm{A}}$-formula, $L_x\varphi$ denotes the sentence

$$\forall \bar{y}(\exists x \varphi(x, \bar{y}) \rightarrow \exists z(\varphi(z, \bar{y}) \wedge \forall w < z \neg \varphi(w, \bar{y})))$$

and we drop the subscript 'x' if it is clear from the context.

PROPOSITION 4.2. *PA* proves all instances $L_x\varphi$ of the least number principle.

Proof. Let $M \vDash PA$ be arbitrary, let $\bar{a}, b \in M$, and let $\varphi(x, \bar{y})$ be an $\mathscr{L}_A$-formula such that $M \vDash \varphi(b, \bar{a})$. If the conclusion of the least-number principle is false in M, i.e., if

$$M \vDash \forall z(\varphi(z, \bar{a}) \to \exists w < z \varphi(w, \bar{a})) \qquad (\ddagger)$$

we shall deduce a contradiction. The formula $\theta(x, \bar{y})$ is defined to be

$$\forall z(z < x \to \neg\varphi(z, \bar{y})).$$

Notice that (since 0 is the smallest element of M) $M \vDash \theta(0, \bar{a})$. Now suppose $x \in M$ and $M \vDash \theta(x, \bar{a})$; we show that $M \vDash \theta(x+1, \bar{a})$.

If $z \in M \vDash z < x+1$ then *either* $z < x$ so $M \vDash \neg\varphi(z, \bar{a})$ (since $M \vDash \theta(x, \bar{a})$) *or* $z = x$, but then $M \vDash \forall w < x \neg\varphi(w, \bar{a})$, so by our supposition ($\ddagger$) we have $M \vDash \neg\varphi(z, \bar{a})$. Hence $M \vDash \theta(x+1, \bar{a})$, as required. Thus

$$M \vDash \theta(0, \bar{a}) \wedge \forall x(\theta(x, \bar{a}) \to \theta(x+1, \bar{a}))$$

so by $I\theta$ in M we have $M \vDash \forall x \theta(x, a)$, i.e., $M \vDash \forall z \neg\varphi(z, \bar{a})$, contradicting the existence of $b \in M \vDash \varphi(b, \bar{a})$. □

The other important induction scheme is the *principle of total induction*. If $\varphi(x, \bar{y})$is an $\mathscr{L}_A$-formula $T_x\varphi$ is the sentence

$$\forall \bar{y}(\forall x(\forall z < x \varphi(z, \bar{y}) \to \varphi(x, \bar{y})) \to \forall x \varphi(x, \bar{y})).$$

PROPOSITION 4.3. *PA* proves all instances $T_x\varphi$ of the principle of total induction.

Proof. Let $M \vDash PA$, $\bar{a} \in M$, and $\varphi(x, \bar{y})$ be an $\mathscr{L}_A$-formula. Suppose

$$M \vDash \forall x(\forall z < x \varphi(z, \bar{a}) \to \varphi(x, \bar{a})). \qquad (\bullet)$$

Then if $b \in M \vDash \neg\varphi(b, \bar{a})$, by $L\neg\varphi$ in M and Proposition 4.2 there is a *least* such $b \in M$ satisfying $\neg\varphi(b, \bar{a})$. But this contradicts ($\bullet$); hence $M \vDash \forall x \varphi(x, \bar{a})$, as required. □

In closing this section we remark that there are converses to Propositions 4.1, 4.2, and 4.3: over PA^- the axiom schemes of induction up to z, the least-number principle, the principle of total induction, and the ordinary scheme of induction are all equivalent; that is if a model $M \models PA^-$ satisfies one of these schemes it satisfies them all. (See Exercises for Section 4.1.)

Exercises for Section 4.1

4.1 Prove the remark in the last paragraph, by showing that, if $M \models PA^-$ satisfies *either* the principle of total induction, *or* the scheme of induction up to z, then $M \models PA$.

4.2 Recall the theory Q of Exercise 2.14, and let Ind denote $\{I_x\varphi | \varphi$ an $\mathscr{L}_A$-formula$\}$. Prove that PA^- may be replaced by Q in the definition of PA by proving that the following sentences hold in all models of Ind $+ Q$, and hence Ind $+ Q \vdash PA^-$.

$$\forall x, y, z((y+z)+x=y+(z+x))$$

$$\forall x(x+0=0+x)$$

$$\forall x(x+1=1+x)$$

$$\forall x, y(x+y=y+x)$$

$$\forall x, y, z((y+z)\cdot x=(y\cdot x)+(z\cdot x))$$

$$\forall x(0\cdot x=0)$$

$$\forall x(1\cdot x=x)$$

$$\forall x, y(x\cdot y=y\cdot x)$$

$$\forall x, y, z((z\cdot y)\cdot x=z\cdot(y\cdot x))$$

$$\forall x, y(1+y+x\neq x)$$

$$\forall x, y\exists z(x+1+z=y\vee x=y\vee y+1+z=x)$$

(*Hint:* In each case, use induction on x.)

4.3 Show that the axiom $Q3$ of Q follows from Ind together with the other axioms of Q.

Construct a model of the least-number principle, $LNP=\{L_x\varphi|\varphi$ an $\mathscr{L}_A$-formula$\}$, together with all axioms of Q except $Q3$.

(*Hint:* Let M be a non-standard model of Th($\mathbb{N}$), $a>\mathbb{N}$ in M and consider $K=\mathbb{N}\cup\{x\in M|M\models x\geqslant a\}$.)

4.4 Prove that any structure $(M, 0, S)$ (where 0 is a constant and S is a unary function on M) satisfying $\forall x, y(S(x)=S(y)\rightarrow x=y)$, $\forall x\neg S(x)=0$, and the second-order induction axiom (*) on p. 42 is isomorphic to $(\mathbb{N}, 0, S)$ (where $S(x)=x+1$); and that the isomorphism h: $(M, 0, S)\rightarrow(\mathbb{N}, 0, S)$ is unique.

(*Hint:* Consider the set $\{a\in M|M\models a=SS\cdots S(0)$ (n Ss) some $n\in\mathbb{N}\}$.)

4.5⁺ Find second-order formulas involving non-logical symbols $0, S$ only that define $+, \cdot$ in $(\mathbb{N}, 0, S)$. (You may use quantification over subsets $X\subseteq\mathbb{N}^k$ and the

relations '$(x_1, x_2, \ldots, x_k) \in X$' for various fixed $k \in \mathbb{N}$.) Show that $+, \cdot$ are not definable in $(\mathbb{N}, 0, S)$ by first-order formulas.
(*Hint:* For the second part, consider Exercises 1.2, 1.3.)

4.2 INTRODUCING NEW RELATIONS AND FUNCTIONS

Suppose that, working PA, we had proved

$$\forall \bar{x} \exists y \varphi(\bar{x}, y) \wedge \forall \bar{x}, y, z(\varphi(\bar{x}, y) \wedge \varphi(\bar{x}, z) \to y = z)$$

where $\varphi(\bar{x}, y)$ is an $\mathscr{L}_A$-formula with all free-variables amongst $\bar{x}, y$. Normal mathematical practice would then indicate that we could add a new function symbol $f(\bar{x})$ together with the *defining axiom*

$$\forall \bar{x} \varphi(\bar{x}, f(\bar{x})) \tag{1}$$

to PA, obtaining a new theory in an expanded language. Similarly, given a formula $\psi(\bar{x})$ we could introduce a new relation symbol R together with defining axiom

$$\forall \bar{x}(\psi(\bar{x}) \leftrightarrow R(\bar{x})). \tag{2}$$

In this section we will address the question of what the relationship is between these new theories and the old, and in particular the problem of justifying the scheme of induction in the expanded language.

Because we may want to iterate our results (so that we can consider expansions of $\mathscr{L}_A$ by *several* new function and relation symbols), we fix the following notation: $\mathscr{L}'_A$ will denote an expansion of $\mathscr{L}_A$ by function and relation symbols, and $PA(\mathscr{L}'_A)$ will denote an $\mathscr{L}'_A$ theory consisting of PA^-, suitable defining axioms for the new function and relation symbols (of the form (1) or (2) above), and the axiom scheme of induction for *all* $\mathscr{L}'_A$-formulas.

Now suppose $\mathscr{L}'_A$ and $PA(\mathscr{L}'_A)$ are as in the last paragraph, and that

$$PA(\mathscr{L}'_A) \vdash \forall \bar{x} \exists ! y \varphi(\bar{x}, y)$$

where $\varphi(\bar{x}, y)$ is an $\mathscr{L}'_A$-formula. Let f be a new function symbol (of arity equal to k = the length of $\bar{x}$.) We define $\mathscr{L}''_A$ to be the language $\mathscr{L}'_A \cup \{f\}$, and we define the following 'translation' of $\mathscr{L}''_A$-formulas $\theta(\bar{v})$ to $\mathscr{L}'_A$-formulas $\bar{\theta}(\bar{v})$ by induction on terms and formulas. (The reader should note that, despite the notational complexity, the idea behind the following definition is very simple, and is just that of replacing every formula $\theta(\ldots, f(\bar{t}), \ldots)$ by $\exists z(\varphi(\bar{t}, z) \wedge \theta(\ldots, z, \ldots))$ for a suitable new variable z.)

$(1)^f$ If $\theta(\bar{v})$ is an atomic $\mathscr{L}'_A$-formula (so does not contain any occurrence of f) then $\bar{\theta}(\bar{v})$ is just $\theta(\bar{v})$.

$(2)^f$ Suppose $\theta_i(\bar{v}, y)$ is $s_i(\bar{v})=y$ for $1 \leq i \leq k$, and $\psi(\bar{v}, y)$ is $t(\bar{v})=y$, where the $s_i(\bar{v})$ and $t(\bar{v})$ are $\mathscr{L}''_A$-terms and each of $\bar{\theta}_i(\bar{v}, y)\,(1 \leq i \leq k)$ and $\bar{\psi}(\bar{v}, y)$ have already been defined. Then:

(a) if $\theta(\bar{v})$ is $t(\bar{v})=f(s_1(\bar{v}), \ldots, s_k(\bar{v}))$ or $f(s_1(\bar{v}), \ldots, s_k(\bar{v}))=t(\bar{v})$ then $\bar{\theta}(\bar{v})$ is

$$\exists y_1, \ldots, y_k, z(\bar{\psi}(\bar{v}, z) \wedge \bigwedge_{i=1}^{k} \bar{\theta}_i(\bar{v}, y_i) \wedge \varphi(y_1, \ldots, y_k, z))$$

for suitable new variables $y_1, \ldots, y_k, z$ not already occurring in $\bar{\psi}$, $\bar{\theta}_i$, or φ.

(b) if $\theta(\bar{v})$ is $t(\bar{v})<f(s_1(\bar{v}), \ldots, s_k(\bar{v}))$ then $\bar{\theta}(\bar{v})$ is

$$\exists y_1, \ldots, y_k, z, w(\bar{\psi}(\bar{v}, z) \wedge \bigwedge_{i=1}^{k} \bar{\theta}_i(\bar{v}, y_i) \wedge \varphi(\bar{y}, w) \wedge z<w)$$

for suitable $\bar{y}$, z, w;

(c) similarly, if $\theta(\bar{v})$ is $f(s_1(\bar{v}), \ldots, s_k(\bar{v}))<t(\bar{v})$ then $\bar{\theta}(\bar{v})$ is

$$\exists y_1, \ldots, y_k, z, w(\bar{\psi}(\bar{v}, z) \wedge \bigwedge_{i=1}^{k} \bar{\theta}_i(\bar{v}, y_i) \wedge \varphi(\bar{y}, w) \wedge w<z).$$

$(3)^f$ The definitions in $(1)^f$ and $(2)^f$ above of $\bar{\theta}(\bar{v})$ for *atomic* $\mathscr{L}''_A$-formulas $\theta(\bar{v})$ are extended to *all* $\mathscr{L}''_A$-formulas $\theta(\bar{v})$ in the obvious way:

$$\begin{array}{lll}
\overline{\neg\theta(\bar{v})} & \text{is} & \neg\bar{\theta}(\bar{v}) \\
\overline{\theta_1 \wedge \theta_2(\bar{v})} & \text{is} & \bar{\theta}_1(\bar{v}) \wedge \bar{\theta}_2(\bar{v}) \\
\overline{\theta_1 \vee \theta_2(\bar{v})} & \text{is} & \bar{\theta}_1(\bar{v}) \vee \bar{\theta}_2(\bar{v}) \\
\overline{\exists y\theta(\bar{v}, y)} & \text{is} & \exists y\bar{\theta}(\bar{v}, y) \\
\overline{\forall y\theta(\bar{v}, y)} & \text{is} & \forall y\bar{\theta}(\bar{v}, y).
\end{array}$$

The following lemma says that this translation is 'correct'.

LEMMA 4.4. Let $\mathscr{L}'_A$, $PA(\mathscr{L}'_A)$, $\varphi(\bar{x}, y)$, f, $\mathscr{L}''_A = \mathscr{L}'_A \cup \{f\}$ be as above. Then for any model $M \vDash PA(\mathscr{L}'_A) + \forall\bar{x}\varphi(\bar{x}, f(\bar{x}))$ for $\mathscr{L}''_A$, we have: for all $\mathscr{L}''_A$-formulas $\theta(\bar{v})$ and all $\bar{a} \in M$

$$M \vDash \theta(\bar{a}) \Leftrightarrow M \vDash \bar{\theta}(\bar{a}).$$

Proof. By induction on the complexity of formulas. We shall prove the induction step corresponding to $(2)^f$ above; all other cases are just as easy.

Suppose $\theta(\bar{v})$ is $t(\bar{v})=f(s_1(\bar{v}), \ldots, (s_k(\bar{v}))$, $\theta_i(\bar{v}, y)$ is $s_i(\bar{v})=y$ $(1 \leq i \leq k)$ and $\psi(\bar{v}, y)$ is $t(\bar{v})=y$, and suppose the lemma is true for the formulas θ_i

and ψ. Then if $\bar{a} \in M$ we have

$$\begin{aligned}
&M \vDash \theta(\bar{a})\\
&\Leftrightarrow M \vDash t(\bar{a}) = f(s_1(\bar{a}), \ldots, s_k(\bar{a}))\\
&\Leftrightarrow M \vDash \varphi(s_1(\bar{a}), \ldots, s_k(\bar{a}), t(\bar{a}))
\end{aligned}$$

(since $M \vDash \forall x \exists ! y \varphi(\bar{x}, y)$ and $M \vDash \forall \bar{x} \varphi(\bar{x}, f(\bar{x}))$

$$\begin{aligned}
&\Leftrightarrow M \vDash \exists y_1, \ldots, y_k, z(y_1 = s_1(\bar{a}) \wedge \ldots \wedge y_k = s_k(\bar{a}) \wedge z = t(\bar{a}) \wedge \varphi(\bar{y}, z))\\
&\Leftrightarrow M \vDash \exists \bar{y}, z(\textstyle\bigwedge_{i=1}^{k} \theta_i(\bar{a}, y_i) \wedge \psi(\bar{a}, z) \wedge \varphi(\bar{y}, z))\\
&\Leftrightarrow M \vDash \exists \bar{y}, z(\textstyle\bigwedge_{i=1}^{k} \bar{\theta}_i(\bar{a}, y_i) \wedge \bar{\psi}(\bar{a}, z) \wedge \varphi(\bar{y}, z))
\end{aligned}$$

(by the induction hypothesis)

$$\Leftrightarrow M \vDash \bar{\theta}(\bar{a}),$$

as required. □

There is also a version for relations: if $\mathscr{L}'_A$ and $PA(\mathscr{L}'_A)$ are as before and $\psi(\bar{x})$ is an $\mathscr{L}'_A$-formula, we may define $\mathscr{L}''_A = \mathscr{L}'_A \cup \{R\}$ where R is a new k-ary relation symbol (where k is the length of $\bar{x}$ again) and we may define a 'translation' of $\mathscr{L}''_A$-formulas to $\mathscr{L}'_A$-formulas by

$(1)^R$ If $\theta(\bar{v})$ is atomic and does not contain any occurrence of R, then $\bar{\theta}(\bar{v})$ is just $\theta(\bar{v})$.

$(2)^R$ If $t_1(\bar{v}), \ldots, t_k(\bar{v})$ are $\mathscr{L}'_A$-terms and $\theta(\bar{v})$ is $R(t_1(\bar{v}), \ldots, t_k(\bar{v}))$ then $\bar{\theta}(\bar{v})$ is $\psi(t_1(\bar{v}), \ldots, t_k(\bar{v}))$.

$(3)^R$

$$\begin{array}{lcl}
\overline{\neg \theta}(\bar{v}) & \text{is} & \neg \bar{\theta}(\bar{v})\\
\overline{\theta_1 \wedge \theta_2}(\bar{v}) & \text{is} & \bar{\theta}_1(\bar{v}) \wedge \bar{\theta}_2(\bar{v})\\
\overline{\theta_1 \vee \theta_2}(\bar{v}) & \text{is} & \bar{\theta}_1(\bar{v}) \vee \bar{\theta}_2(\bar{v})\\
\overline{\exists y \theta(\bar{v}, y)} & \text{is} & \exists y \bar{\theta}(\bar{v}, y)\\
\overline{\forall y \theta(\bar{v}, y)} & \text{is} & \forall y \bar{\theta}(\bar{v}, y).
\end{array}$$

We then have the analogue of Lemma 4.4 for this translation.

LEMMA 4.5. Let $\mathscr{L}'_A$, $PA(\mathscr{L}'_A)$, $\psi(\bar{x})$, R, $\mathscr{L}''_A = \mathscr{L}'_A \cup \{R\}$ be as above. Then for any model $M \vDash PA(\mathscr{L}'_A) + \forall \bar{x}(R(\bar{x}) \leftrightarrow \psi(\bar{x}))$ we have: for any $\mathscr{L}''_A$-formula $\theta(\bar{v})$ and all $\bar{a} \in M$

$$M \vDash \theta(\bar{a}) \Leftrightarrow M \vDash \bar{\theta}(\bar{a}).$$

Proof. By induction on the complexity of formulas in a similar way to the proof of Lemma 4.4. □

From this we can deduce the main theorem of this section.

THEOREM 4.6. Let $\mathscr{L}'_A$ and $PA(\mathscr{L}'_A)$ be as before, let $M \models PA(\mathscr{L}'_A)$, and let $\mathscr{L}''_A$ be an expansion by definition of $\mathscr{L}'_A$ (where *either* $\mathscr{L}''_A = \mathscr{L}'_A \cup \{f\}$, f being defined by the axiom $\forall \bar{x} \varphi(\bar{x}, f(\bar{x}))$, $\varphi(\bar{x}, y)$ an $\mathscr{L}'_A$-formula and $PA(\mathscr{L}'_A) \vdash \forall \bar{x} \exists! y \varphi(\bar{x}, y)$ *or* $\mathscr{L}''_A = \mathscr{L}'_A \cup \{R\}$, where R is the relation with defining axiom $\forall \bar{x}(R(\bar{x}) \leftrightarrow \psi(\bar{x}))$ and $\psi(\bar{x})$ is an $\mathscr{L}'_A$-formula). Then the unique expansion of M satisfying the defining axiom for the new symbol of $\mathscr{L}''_A$ also satisfies all induction axioms $I\theta$ for formulas θ in $\mathscr{L}''_A$.

Proof. Clearly this unique expansion M^+ of M is either (M, f) where $f: M^k \to M$ is given by

$$f(\bar{a}) = \text{the unique } b \in M \text{ s.t. } M \models \varphi(\bar{a}, b)$$

or is (M, R) where $R \subseteq M^k$ is the relation

$$R = \{\bar{a} \in M^k \mid M \models \psi(\bar{a})\}.$$

We must show M^+ satisfies all induction axioms $I_u\theta$ for all $\mathscr{L}''_A$-formulas $\theta(u, \bar{v})$. But if $\bar{a} \in M$ and

$$M^+ \models \theta(0, \bar{a}) \wedge \forall u(\theta(u, \bar{a}) \to \theta(u+1, \bar{a}))$$

then

$$M \models \overline{\theta(0, \bar{a}) \wedge \forall u(\theta(u, \bar{a}) \to \theta(u+1, \bar{a}))}$$

by Lemma 4.4 or Lemma 4.5, so by $(3)^f$ or $(3)^R$,

$$M \models \bar{\theta}(0, \bar{a}) \wedge \forall u(\bar{\theta}(u, \bar{a}) \to \bar{\theta}(u+1, \bar{a})),$$

hence using $PA(\mathscr{L}'_A)$ in M, $M \models \forall u \bar{\theta}(u, \bar{a})$ hence (by Lemmas 4.4 and 4.5 again) $M^+ \models \forall u \theta(u, \bar{a})$, as required. □

COROLLARY 4.7. With $\mathscr{L}'_A$, $PA(\mathscr{L}'_A)$, $\mathscr{L}''_A$ as in Theorem 4.6, let $PA(\mathscr{L}''_A)$ be $PA(\mathscr{L}'_A)$ + all induction axioms for formulas of $\mathscr{L}''_A$ together with the defining axiom for the new symbol of $\mathscr{L}''_A$. Then $PA(\mathscr{L}''_A)$ is a *conservative extension* of $PA(\mathscr{L}'_A)$, i.e., for each $\mathscr{L}'_A$-sentence σ

$$PA(\mathscr{L}''_A) \vdash \sigma \Leftrightarrow PA(\mathscr{L}'_A) \vdash \sigma.$$

Proof. The '$\Leftarrow$' direction is trivial. Suppose $PA(\mathscr{L}''_A) \vdash \sigma$. Then for each model $M \models PA(\mathscr{L}'_A)$ there is a unique expansion $M^+ \models PA(\mathscr{L}''_A)$, by Theorem 4.6. So $M^+ \models \sigma$ and since σ is an $\mathscr{L}'_A$ sentence and M^+ an expansion of M we have $M \models \sigma$. Thus $PA(\mathscr{L}'_A) \vdash \sigma$ by the completeness theorem. □

There is a special case of these expansions of the language $\mathscr{L}_A$ by definitions, namely when the new function or relation is *provably recursive* in PA or $PA(\mathscr{L}'_A)$. To explain this we first extend the idea of Π_1 and Σ_1 formulas to a language $\mathscr{L}'_A$ extending $\mathscr{L}_A$.

Definition. If $\mathscr{L}'_A$ extends $\mathscr{L}_A$, and $\mathscr{L}'_A$-formula $\theta(\bar{x})$ is Δ_0 *in* $\mathscr{L}'_A$ iff all of its quantifiers occur as $\exists v < t(\bar{x})(\ldots)$ or $\forall v < t(\bar{x})(\ldots)$ for some term $t(\bar{x})$ of $\mathscr{L}'_A$. $\psi(\bar{x})$ is Σ_1 *in* $\mathscr{L}'_A$ iff it is $\exists \bar{y}\theta(x, \bar{y})$ for some Δ_0 $\mathscr{L}'_A$-formula θ, and $\psi(\bar{x})$ is Π_1 in $\mathscr{L}'_A$ iff it is $\forall \bar{z}\theta(\bar{x}, \bar{z})$ for some Δ_0 $\mathscr{L}'_A$-formula θ.

Definition. A function f is *provably recursive* in $PA(\mathscr{L}'_A)$ iff it is defined in all models of $PA(\mathscr{L}'_A)$ by $f(\bar{x}) =$ the unique y s.t. $\theta(\bar{x}, y)$, where $\theta(\bar{x}, y)$ is Σ_1 in $\mathscr{L}'_A$ and $PA(\mathscr{L}'_A) \vdash \forall \bar{x} \exists ! y \theta(\bar{x}, y)$. A relation R is *provably recursive* in $PA(\mathscr{L}'_A)$ iff it is defined in all models by $R(\bar{x}) \leftrightarrow \theta(\bar{x})$ where $\theta(\bar{x})$ is a Σ_1 $\mathscr{L}'_A$-formula s.t. there is a formula $\psi(\bar{x})$ that is Π_1 in $\mathscr{L}'_A$ with $PA(\mathscr{L}'_A) \vdash \forall \bar{x}(\theta(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

Defining provably recursive *functions* and *relations* in this way is really a misnomer, since they are simply 'recipes' for functions or relations that apply to an arbitrary model of $PA(\mathscr{L}'_A)$. This will be convenient abuse of language though, as we will be particularly interested in the actual functions and relations on $\mathbb{N}$ that are obtained from these recipes. For example, if $f(\bar{x}) =$ the unique y s.t. $\theta(\bar{x}, y)$ holds, where $\theta(\bar{x}, y)$ is a Σ_1 formula and $PA \vdash \forall \bar{x} \exists ! y \theta(x, y)$, then, by Theorem 3.3, f (as defined on $\mathbb{N}$ using this recipe) is actually recursive, since it is total and has Σ_1 graph. Similarly a provably recursive relation R (considered as a relation on $\mathbb{N}$) is actually recursive, by Corollary 3.5. (Not all recursive functions are provably recursive though: see Exercise 4.8.).

The special feature of expansions of languages by provably recursive functions and relations is that, in an expansion from $\mathscr{L}'_A$ to $\mathscr{L}''_A$, say, Σ_1 and Π_1 formulas of $\mathscr{L}''_A$ can be translated to Σ_1 and Π_1 formulas of $\mathscr{L}'_A$, respectively. For example, suppose $PA(\mathscr{L}'_A) \vdash \forall \bar{x} \exists ! y \varphi(\bar{x}, y)$, φ is Σ_1 in $\mathscr{L}'_A$ and we are considering expanding $\mathscr{L}'_A$ to $\mathscr{L}''_A$ by adding a new function symbol f with defining axiom $\forall \bar{x} \varphi(\bar{x}, f(\bar{x}))$. Then every Δ_0 formula θ of $\mathscr{L}'_A \cup \{f\}$ has both a translation $\bar{\theta}^\Sigma$, that is Σ_1 in $\mathscr{L}'_A$ and a translaton $\bar{\theta}^\Pi$, that is Π_1 in $\mathscr{L}'_A$. The idea is to replace $\theta(\ldots, f(\bar{t}), \ldots)$ with *either* $\exists y(\varphi(\bar{t}, y) \wedge \theta(\ldots, y, \ldots))$ *or* $\forall y(\varphi(\bar{t}, y) \rightarrow \theta(\ldots, y, \ldots))$. We need to know that $\forall \bar{x} \exists ! y \varphi(\bar{x}, y)$ is provable in $PA(\mathscr{L}'_A)$ to show that these two replacements are equivalent, and hence that $\bar{\theta}^\Sigma$, $\bar{\theta}^\Pi$ are equivalent, and we need to know that φ is Σ_1 in $\mathscr{L}'_A$ to show that $\bar{\theta}^\Sigma$ is Σ_1 in $\mathscr{L}'_A$ and $\bar{\theta}^\Pi$ is Π_1 in $\mathscr{L}'_A$. Similarly if $PA(\mathscr{L}'_A) \vdash \forall \bar{x}(\theta(\bar{x}) \leftrightarrow \psi(\bar{x}))$, where θ is Σ_1 in $\mathscr{L}'_A$ and ψ is Π_1

in $\mathscr{L}'_A$, then every Δ_0 formula φ of $\mathscr{L}'_A \cup \{R\}$ has translations $\bar{\theta}^\Sigma$ and $\bar{\theta}^\Pi$ that are Σ_1 and Π_1 in $\mathscr{L}'_A$, by replacing $R(\bar{t})$ *either* by $\theta(\bar{t})$ *or* by $\psi(\bar{t})$, as appropriate. In both cases a Π_1 formula $\forall\bar{y}\theta(\bar{x},\bar{y})$ of the extended language translates to the Π_1 $\mathscr{L}'_A$-formula $\forall\bar{y}\bar{\theta}^\Pi(\bar{x},\bar{y})$, and a Σ_1 formula $\exists\bar{y}\theta(\bar{x},\bar{y})$ translates to the Σ_1 $\mathscr{L}'_A$-formula $\exists\bar{y}\bar{\theta}^\Sigma(\bar{x},\bar{y})$. (But note also that Δ_0 formulas of the extended language may not translate to Δ_0 formulas of $\mathscr{L}'_A$.)

A particular consequence of this is that if $M \subseteq_e N$ are models of $PA(\mathscr{L}'_A)$ and f is a provably recursive function of $PA(\mathscr{L}'_A)$, then M and N agree on their definitions of f (since if f is defined by the Σ_1 $\mathscr{L}'_A$-formula $\varphi(\bar{x},y)$ and $\bar{a},b \in M \models \varphi(\bar{a},b)$, then $N \models \varphi(\bar{a},b)$ since $N \supseteq_e M$, using the analogue of Corollary 2.8(a) and the fact that φ is Σ_1). Thus Δ_0 formulas of the expanded language $\mathscr{L}'_A \cup \{f\}$ are also absolute between N and M. Similar remarks hold for expansions of $\mathscr{L}'_A$ by provably recursive relations too, and Δ_0 formulas in languages that are expansions of $\mathscr{L}'_A$ by provably recursive relations are also absolute for end-extensions.

Exercises for Section 4.2.

4.6* Show that the arguments in Section 4.1 apply to show that, if $\mathscr{L}'_A$ is any expansion of the language $\mathscr{L}_A$, then $PA(\mathscr{L}'_A)$ proves all instances of the least-number principle, total induction, and induction up to z, in the least expanded language $\mathscr{L}'_A$.

4.7 Write down the precise definitions of the Π_1 and Σ_1 translations $\bar{\theta}^\Pi$ and $\bar{\theta}^\Sigma$ of a Δ_0 formula θ in a language $\mathscr{L}''_A$ extending $\mathscr{L}'_A$ by functions and relations that are provable recursive in $PA(\mathscr{L}'_A)$. Verify that your $\bar{\theta}^\Pi$ and $\bar{\theta}^\Sigma$ are equivalent to each other in all models of $PA(\mathscr{L}'_A)$, and are equivalent to θ in the canonical expansion of a model of $PA(\mathscr{L}'_A)$ to the language $\mathscr{L}''_A$.

4.8 Use the Church–Turing thesis to show that the following function is recursive:

$$f(x) = \max_\theta\{(\mu z)\forall y \leqslant x \exists w \leqslant z \theta(y,w)\}$$

where $\max_\theta$ is taken over all Σ_1 formulas $\theta(u,v)$ with Gödel-number $\ulcorner\theta(u,v)\urcorner < x$ such that there is a proof of $\forall u \exists v \theta(u,v)$ in PA, this proof also having Gödel-number less than x.

Show that if $g(\bar{x})$ is provably recursive in PA, then for some $k \in \mathbb{N}$, $g(\bar{n}) \leqslant f(\max(\bar{n}))$ for all $\bar{n} > k$. Deduce that not all recursive functions are provably recursive.

5

Some number theory in PA

It will be clear by now that, if we are to have any chance of making progress, I must produce examples of 'real' mathematical theorems, theorems which every mathematician will admit to be first-rate.

G. H. Hardy
From *A mathematician's apology*

We will now start developing number theory in PA. One of our main goals is to prove a version of the *Chinese remainder theorem*. This will enable us to prove Gödel's lemma (Lemma 3.2) for arbitrary models of PA. As an application we will show how to derive Euclid's theorem on the infinitude of the primes in PA. (This theorem, coincidentally, is the first 'real' mathematical theorem that Hardy proves in his *Apology*!) In fact, equipped with Gödel's lemma, the reader will see that it is routine to transcribe the proofs of many theorems in number theory into PA.

5.1 EUCLIDEAN DIVISION AND BÉZOUT'S THEOREM

Our first theorem concerns *Euclidean division*.

THEOREM 5.1. Let $M \vDash PA$ and $a, b \in M$ with $a \neq 0$. Then there exist unique elements $r, s \in M$ such that $M \vDash b = as + r \wedge r < a$.

Proof. The existence of r, s follows from induction on x in the formula $\exists r, s(x = as + r \wedge r < a)$. Clearly $M \vDash 0 = a \cdot 0 + 0 \wedge 0 < a$, since $a \neq 0$. Suppose $x, r, s \in M \vDash x = as + r \wedge r < a$. Then $M \vDash x + 1 = as + (r + 1)$ and either $r + 1 < a$, or $r + 1 = a$ (hence $M \vDash x + 1 = a(s + 1) + 0$); and in both cases we have $M \vDash \exists r', s'(x + 1 = as' + r' \wedge r' < a)$. By induction, $M \vDash \forall x \exists r, s (x = as + r \wedge r < a)$.

To prove uniqueness, suppose $b, s, r, s', r' \in M$ with $M \vDash b = as + r = as' + r' \wedge r < a \wedge r' < a$. If $s < s'$, then $M \vDash b = as + r < a(s + 1) \leqslant as' + r' = b$, and if $s' < s$ we obtain a similar contradiction. Thus $s = s'$ and $M \vDash as + r = as + r'$, hence $r = r'$, as required. □

We now define several new function and relation symbols: the resulting system consisting of PA^-, the defining axioms, and the induction scheme for all formulas in the expanded language will also be denoted PA. (This abuse of notation is justified by Theorem 4.6 of the last chapter, since we may regard formulas in the new language as abbreviations for formulas in the old language.) We shall usually give informal definitions of the new function and relation symbols, since these are clearer than the corresponding formal ones; the reader will easily be able to work out what the corresponding defining axioms are.

DEFINITIONS. The following definitions are thought of as taking place in PA or some extension of PA.

(a) $\left\lfloor \frac{x}{y} \right\rfloor$ (the *integer part of x divided by y*) is the two-place function

$$\left\lfloor \frac{x}{y} \right\rfloor = \begin{cases} \text{the unique } z \text{ s.t. } zy \leqslant x < (z+1)y, & \text{if } y \neq 0 \\ 0, \quad \text{if } y = 0, \end{cases}$$

so, for example, a suitable defining axiom is

$$\left(y=0 \wedge \left\lfloor \frac{x}{y} \right\rfloor = 0\right) \vee \left(y \neq 0 \wedge y \cdot \left\lfloor \frac{x}{y} \right\rfloor \leqslant x < y \cdot \left(\left\lfloor \frac{x}{y} \right\rfloor + 1\right)\right).$$

(b) $\left(\frac{x}{y}\right)$ (the *remainder on dividing x by y*) is given by

$$\left(\frac{x}{y}\right) = \begin{cases} \text{the unique } z \text{ s.t. } \exists w \leqslant x(yw+z=x \wedge z<y), & \text{if } y \neq 0 \\ 0, \text{ otherwise.} \end{cases}$$

(c) $x|y$ (*x divides y*) is the two-place relation given by

$$x|y \leftrightarrow \exists z \leqslant y(xz=y) \wedge x \neq 0.$$

Notice that $\forall x, y(x|y \leftrightarrow x \neq 0 \wedge (y/x)=0)$. We will usually write $x \nmid y$ in place of $\neg(x|y)$.

(d) $x \equiv y \bmod z$ (*x is congruent to y modulo z*) is the three-place relation

$$x \equiv y \bmod z \leftrightarrow \left(z \neq 0 \wedge \left(\frac{x}{z}\right) = \left(\frac{y}{z}\right)\right)$$

(e) prime(x) (*x is a prime*) is the unary relation

$$\text{prime}(x) \leftrightarrow (x \geqslant 2 \wedge \forall y, z(x|(yz) \rightarrow (x|y \vee x|z)))$$

(f) irred(x) (*x is irreducible*) is the unary relation

$$\text{irred}(x) \leftrightarrow \forall y(y|x \rightarrow (y=1 \vee y=x))$$

(g) $(x, y)=1$ (x, y are *coprime*) is the relation

$$(x, y)=1 \leftrightarrow (x \geqslant 1 \wedge y \geqslant 1 \wedge \forall u(u|x \wedge u|y \rightarrow u=1)).$$

Notice that we have attempted to retain usual mathematical notation, for example writing $(x, y)=1$ in place of coprime(x, y). The only unusual notation is (x/y) rather than rem(x, y), which might be more commonly used for this notion. We will however be using (x/y) a lot, and this notation is rather more suggestive of its intended meaning.

THEOREM 5.2. (Bézout's theorem in PA)

$$PA \vdash \forall x, y((x, y)=1 \rightarrow \exists z<y(zx \equiv 1 \bmod y)).$$

Proof. Let $M \models PA$ with $x, y \in M$ satisfying $M \models (x, y)=1 \wedge x>1 \wedge y>1$. (Notice that if x or y is 1 then the result is trivial.) Now

$$M \models \exists w(w \geqslant 1 \wedge \exists z<y(zx \equiv w \bmod y)) \tag{1}$$

since $w=(x/y)$ certainly satisfies this with $z=1$. By the least-number principle there is a least such w, w_0 say, and clearly

$$M \models 1 \leqslant w_0 \leqslant \left(\frac{x}{y}\right) \leqslant \min(x, y). \tag{2}$$

We claim that $w_0=1$, thus proving the theorem. Our strategy is to show that $w_0|x$ and $w_0|y$, and use $(x, y)=1$ to deduce $w_0=1$.

Now suppose $z_0 \in M$ satisfies

$$M \models z_0<y \wedge (z_0x \equiv w_0 \bmod y).$$

To see that $w_0|x$, let $t=\lfloor z_0 \cdot x/y \rfloor$ and notice that $z_0 \cdot x=t \cdot y+w_0$, since $w_0<y$. Also, let $s=\lfloor x/w_0 \rfloor$ and $r=(x/w_0)$, so $x=s \cdot w_0+r$ and $0 \leqslant r<w_0$.

But then

$$r = x - s \cdot w_0 = x - s(z_0x - ty) = x + sty - sz_0x$$

so

$$r \equiv (1 + uy - sz_0) \cdot x \bmod y$$

where we take u so that $0 \leqslant (1 + uy - sz_0) < y$, i.e., we take

$$u = \left\lfloor \frac{sz_0 + y - 2}{y} \right\rfloor.$$

But $0 \leqslant r < w_0$ and w_0 is the least w satisfying (1), hence $r = 0$, hence $x = s \cdot w_0$ and thus $w_0 | x$.

To see that $w_0 | y$, let $y = cw_0 + d$ where $c = \lfloor y/w_0 \rfloor$ and $d = (y/w_0)$, so $0 \leqslant d < w_0$ and $d = y - cw_0 = y - c(z_0x - ty) = (1 + ct)y - cz_0x$, hence $d \equiv (by - cz_0) \cdot x \bmod y$ for suitable b such that $0 \leqslant (by - cz_0) < y$, that is,

$$b = \left\lfloor \frac{cz_0 + y - 1}{y} \right\rfloor,$$

hence $d = 0$ and $w_0 | y$. This proves the theorem. □

Bézout's theorem says that, if $(x, y) = 1$ then x has a multiplicative inverse modulo y. It is worth noting that there is a converse, namely

$$PA \vdash \forall x, y(x \geqslant 1 \wedge y \geqslant 1 \wedge \exists z < y(zx \equiv 1 \bmod y) \rightarrow (x, y) = 1),$$

for if $x, y \geqslant 1$ and $zx = 1 + wy$ for some z, w, then whenever $u|x \wedge u|y$ we have $u(zx - wy) = 1$, hence $u = 1$.

Bézout's theorem has a number of important consequences and Section 5.2 will be devoted to one of these. We can also use it to deduce that, in models of PA, the notions 'prime' and 'irreducible' are actually equivalent.

Corollary 5.3. $PA \vdash \forall x(\text{prime}(x) \leftrightarrow \text{irred}(x))$.

Proof. Let $M \models PA$ and suppose $x \in M$ is prime, i.e.

$$M \models \forall y, z(x|(yz) \rightarrow (x|y \vee x|z))$$

with $x \geqslant 2$. Then if $x = uv$ we have (trivially) $x|(uv)$, so $x|u$ or $x|v$, hence $u = 1 \wedge v = x$ or $v = 1 \wedge u = x$. This shows that x is irreducible. For the (more difficult) converse, suppose $x \in M$ is irreducible, that is,

$$M \vDash \forall u(u|x \rightarrow (u=1 \vee u=x)) \wedge x \geqslant 2$$

and suppose a, $b \in M$ with $x|(ab)$ and $\neg(x|a) \wedge \neg(x|b)$. Then $(x, a)=1$, since if $u|x \wedge u|a$, then $u=1$ or $u=x$, and this second possibility is ruled out by $\neg(x|a)$. Similarly $(x, b)=1$. Thus by Theorem 5.2 there are $r, s, u, v \in M$ such that

$$ra = ux + 1$$

and

$$sb = vx + 1.$$

If $y \in M \vDash xy = ab$, then

$$rsxy = rsab = uvx^2 + ux + vx + 1,$$

from which it follows that, since $x \geqslant 2$,

$$0 = \left(\frac{rsyx}{x}\right) = \left(\frac{rsab}{x}\right) = 1,$$

a contradiction. □

Exercises for Section 5.1

5.1* By considering the least $v > 1$ s.t. $v|x$, show that

$$PA \vdash \forall x(x > 1 \rightarrow \exists v(\text{irred}(v) \wedge v|x))$$

i.e. 'every $x > 1$ has an irreducible divisor'.

5.2* By considering Corollary 5.3 and Exercise 5.1, show that

$$PA \vdash \forall x, y[(x \geqslant 1 \wedge y \geqslant 1 \wedge \forall z \neg(\text{prime}(z) \wedge z|x \wedge z|y)) \leftrightarrow (x, y) = 1]$$

i.e. $x, y \geqslant 1$ are coprime iff they have no common prime divisors.

5.3 Check that the formula irred(x) is equivalent (in PA^-) to a Δ_0 formula. Conclude that, for any prime $p \in \mathbb{N}$,

$$PA^- \vdash \text{irred}(p)$$

and

$$PA \vdash \text{prime}(p).$$

Exhibit a model of PA^- in which 2 is not prime, and hence conclude that prime(x) is not equivalent to a Δ_0 formula in PA^-.

5.4 Using $PA \vdash \text{prime}(2)$ (from Exercise 5.3) show that $PA \vdash \forall x, y(x^2 = 2y^2 \to x = y = 0)$ ('the irrationality of $\sqrt{2}$') by considering the least x (if such x exists) satisfying $\exists y \leqslant x(x^2 = 2y^2 \wedge x \neq 0)$.

5.5* Show that all the functions and relations introduced in the definitions preceding Theorem 5.2 are provably recursive.

5.2 GÖDEL'S LEMMA

Our goal now is to prove Gödel's lemma (Lemma 3.2). In fact we shall do slightly better: we shall formalize a version of Lemma 3.2 *inside PA*. To help the reader keep track of what is going on in the proof below, we first sketch out Gödel's main idea with reference to the standard model $\mathbb{N}$.

The idea is as follows. Let $x_0, \ldots, x_{n-1}$ be any finite sequence of natural numbers. Let $m = b!$, where $b = \max(n, x_0, \ldots, x_{n-1})$. Then the sequence of numbers

$$m+1, 2m+1, 3m+1, \ldots, nm+1 \tag{3}$$

is pairwise coprime, for if $0 < i < j \leqslant n$ and $u|(im+1) \wedge u|(jm+1)$ then $u|(j-i)m$ and $j-i < n \leqslant b$ so $u|m$, hence $u|im$, hence $u|im+1-im = 1$, so $u = 1$. By the well-known *Chinese remainder theorem* we can find $a \in \mathbb{N}$ such that

$$a \equiv x_i \bmod((i+1)m+1) \tag{4}$$

for each $i < n$. Then we can say that the pair (a, m) *codes* the sequence $x_0, \ldots, x_{n-1}, \ldots$ since given (a, m) we can recover each x_i uniformly as follows:

$$x_i = \left(\frac{a}{m(i+1)+1}\right)$$

because we chose $x_i < m(i+1)+1$ for each i.

The two main problems we must overcome if we are to formalize this argument in PA are: finding a suitable m such that all pairs of numbers in the sequence (3) are coprime; and finding a suitable number a satisfying (4). Both problems are solved by using the induction axioms in PA, and the formalizaton is explained in the next three lemmas. (When reading these lemmas we urge the reader to think of what happens in the standard model $\mathbb{N}$. This will explain where the relevant formulas come from, and why the lemmas are true.)

Informally, the next lemma states that if the numbers

$$x+1, 2x+1, \ldots, yx+1 \tag{5}$$

are all coprime with z then there is a number w, also coprime with z, such that each number in (5) divides w.

LEMMA 5.4.

$$PA \vdash \forall x, y, z((\forall i<y((i+1)x+1, z)=1 \wedge z \neq 0) \\ \rightarrow \exists w((w, z)=1 \wedge \forall i<y(x(i+1)+1)|w)).$$

Proof. In $\mathbb{N}$, we could just take w to be the product of the numbers in (5). Working in an arbitrary model $M \vDash PA$ with

$$x, y, z \in M \vDash \forall i<y((i+1)x+1, z)=1 \wedge z \neq 0,$$

we shall do something similar but using induction to form the y-fold product.

We shall employ induction on k up to y in the formula $\theta(k)$, which is

$$\exists w((w, z)=1 \wedge \forall i<k(x(i+1)+1)|w).$$

For $k=0$ we may take $w=1$ so $M \vDash \theta(0)$. If $k<y$ and

$$w \in M \vDash (w, z)=1 \wedge \forall i<k(x(i+1)+1)|w,$$

we put $w' = w \cdot (\overset{x}{\cancel{w}}(k+1)+1)$. Then by Exercise 5.2 $(w', z)=1$, since if $p|w'$ is prime, then *either* $p|w$ so $p\nmid z$ since $(w, z)=1$, *or* $p|x(k+1)+1$ so $p\nmid z$ since $(z, x(k+1)+1)=1$ as $k<y$. It is clear that for all $i<k+1$ $(x(i+1)+1)|w'$ since if $i<k$, $(x(i+1)+1)|w$. Hence by induction $M \vDash \theta(y)$, i.e.,

$$M \vDash \exists w((w, z)=1 \wedge \forall i<y(x(i+1)+1)|w)$$

as required. □

LEMMA 5.5.

$$PA \vdash \forall x, y \exists z(z>x \wedge \forall i, j<y(i \neq j \rightarrow (z(i+1)+1, z(j+1)+1)=1)).$$

Proof. In $\mathbb{N}$ we could take z to be any multiple of $y!$, chosen so that $z>x$. Working in an arbitrary model of PA we use a similar idea together with the induction scheme.

Suppose $x \in M \models PA$. We show that $M \models \forall y \exists z(z > x \wedge \forall i < y(i+1)|z)$ by induction on y. For $y=0$ we may take $z=x+1$. If $y \geq 0$ and $z \in M \models z > x \wedge \forall i < y\ (i+1)|z$, then put $z' = z \cdot (y+1)$. Clearly $z' \geq z > x$ and if $i < y+1$ then either $i < y$, so $(i+1)|z'$, or $i = y$, so $i+1 = y+1|z'$. Thus $M \models z' > x \wedge \forall i < y+1\ (i+1)|z'$ and so, by induction, for all $y \in M$ there is $z \in M$ s.t.

$$M \models z > x \wedge \forall i < y\ (i+1)|z. \tag{6}$$

Let $z \in M$ be as in (6). We claim that

$$M \models \forall i, j < y(i \neq j \rightarrow (z(i+1)+1, z(j+1)+1) = 1).$$

Using Exercise 5.2, we must check that if $i < j < y$ and $p \in M$ is prime, then

$$M \models \neg(p | (z(i+1)+1) \wedge p | (z(j+1)+1)).$$

But if $p | (z(i+1)+1)$ and $p | (z(j+1)+1)$, then

$$p | ((z(j+1)+1) - (z(i+1)+1)) = z \cdot (j-i)$$

and so $p|z$ or $p|(j-i)$ (since p is prime). But $j-i < y$, so $(j-i)|z$, hence in both cases $p|z$. Thus $p|z(i+1)$, so $p=1$, since p also divides $z(i+1)+1$, contradiction. This proves the lemma. □

The next lemma is the key property of the sequence

$$x_i = \left(\frac{a}{m(i+1)+1}\right)$$

coded by a, m. It says that any such (finite) sequence can be extended.

LEMMA 5.6. $PA \vdash \forall a, m, x, y \exists a', m'$

$$\left(\forall i < y \left(\frac{a'}{m'(i+1)+1}\right) = \left(\frac{a}{m(i+1)+1}\right) \wedge \left(\frac{a'}{m'(y+1)+1}\right) = x\right).$$

Proof. Let $M \models PA$ with $a, m, x, y \in M$. Using Lemma 5.5 let $m' > \max(x, m)$ satisfy

$$M \models \forall i, j \leq y(i \neq j \rightarrow (m'(i+1)+1, m'(j+1)+1) = 1).$$

We first show by induction on k up to y that

$$M \vDash \forall k \leq y \exists b \forall i < k\left(\left(\frac{b}{m'(i+1)+1}\right) = \left(\frac{a}{m(i+1)+1}\right)\right). \tag{7}$$

This is trivial for $k=0$. Suppose $k<y$ and

$$b \in M \vDash \forall i < k\left(\left(\frac{b}{m'(i+1)+1}\right) = \left(\frac{a}{m(i+1)+1}\right)\right).$$

Then using Lemma 5.4 there exists $w \in M$ such that

$$M \vDash (w, m'(k+1)+1) = 1$$

and

$$M \vDash \forall i < k((m'(i+1)+1) | w).$$

By Bézout's theorem there is $u < m'(k+1)+1$ in M such that

$$M \vDash uw \equiv 1 \bmod m'(k+1)+1,$$

and let $v = bm'(k+1)+z$, where

$$z = \left(\frac{a}{m(k+1)+1}\right).$$

Finally, let $b' = b + uvw$. Then if $i<k$ we have

$$\begin{aligned} \left(\frac{b'}{m'(i+1)+1}\right) &= \left(\frac{b}{m'(i+1)+1}\right), && \text{since } (m'(i+1)+1) | w \\ &= \left(\frac{a}{m(i+1)+1}\right), && \text{by the induction hypothesis;} \end{aligned}$$

and also

$$\left(\frac{b'}{m'(k+1)+1}\right) = z = \left(\frac{a}{m(k+1)+1}\right)$$

since $m' > m$, so $z < m'(k+1)+1$, and

$$\begin{aligned} b' = b + uvw &\equiv b + v \bmod m'(k+1)+1 \\ &\equiv b + bm'(k+1) + z \bmod m'(k+1)+1 \\ &\equiv b(m'(k+1)+1) + z \bmod m'(k+1)+1 \\ &\equiv z \bmod m'(k+1)+1. \end{aligned}$$

Hence by induction (7) is true.

Now to prove the lemma we construct a' from b in exactly the same way, using Bézout's theorem, but this time using x in place of z, i.e., if b satisfies

$$M \models \forall i < y \left(\frac{b}{m'(i+1)+1} \right) = \left(\frac{a}{m(i+1)+1} \right)$$

then we put $a' = b + uvw$, where $uw \equiv 1 \bmod m'(y+1)+1$ and $v = bm'(y+1)+x$, so that

$$M \models \forall i < y \left(\frac{a'}{m'(i+1)+1} \right) = \left(\frac{a}{m(i+1)+1} \right)$$

and

$$M \models \left(\frac{a'}{m'(y+1)+1} \right) = x. \qquad \square$$

We now have almost everything we need to prove Gödel's lemma: we can now code arbitrary finite sequences $x_0, x_1, \ldots, x_{n-1}$ by a pair of numbers (a, m). To reduce this pair to a single number we need the pairing function of Chapter 3. We define

$$\langle x, y \rangle = \frac{(x+y+1)(x+y)}{2} + y$$

LEMMA 5.7. *PA* proves the following:
(a) $\forall z (2|z \vee 2|(z+1))$;
(b) $\forall z \exists x, y (\langle x, y \rangle = z)$;
(c) $\forall x, y, u, v (\langle x, y \rangle = \langle u, v \rangle \rightarrow x = u \wedge y = v)$.

Proof. (a) is by induction on z in the formula $2|z \vee 2|(z+1)$ and is easy. This shows that $\langle x, y \rangle$ is a definable function in *PA*, so in particular the statements in (b) and (c) above make sense.

(b) is proved by induction on z. Notice that $0 = \langle 0, 0 \rangle$ and if $z = \langle x, y \rangle$ then *either* $x > 0$ so

$$z+1 = \frac{((x-1)+(y+1)+1)((x-1)+(y+1))}{2} + (y+1)$$

i.e. $z+1=\langle x-1, y+1\rangle$, *or* $x=0$, so

$$z+1=\frac{(y+1)y}{2}+y+1=\frac{(y+1)(y+2)}{2}=\langle y+1, 0\rangle.$$

For (c), suppose $\langle x, y\rangle=\langle u, v\rangle$. We first show $x+y=u+v$. So suppose $x+y<u+v$; then $x+y+1\leqslant u+v$ and

$$\begin{aligned}\frac{(x+y+1)(x+y)}{2}+y&<\frac{(x+y+1)(x+y)}{2}+x+y+1\\ &=\frac{(x+y+1)(x+y+2)}{2}\leqslant\frac{(u+v)(u+v+1)}{2}+v\\ &=\frac{(x+y)(x+y+1)}{2}+y,\end{aligned}$$

a contradiction, hence $u+v\leqslant x+y$. Similarly $x+y\leqslant u+v$ and hence $x+y=u+v$. Therefore

$$\begin{aligned}v&=\langle u, v\rangle-\frac{(u+v)(u+v+1)}{2}\\ &=\langle x, y\rangle-\frac{(x+y)(x+y+1)}{2}=y.\end{aligned}$$

So $u=x$ and $v=y$, as required. □

DEFINITIONS. The pairing function is extended to n-tuples (for each $n\in\mathbb{N}$) by setting $\langle x_1, x_2, \ldots, x_n\rangle=\langle x_1, \langle x_2, \ldots, x_n\rangle\rangle$. We define $(x)_i$ to be

$$(x)_i=\left(\frac{a}{m(i+1)+1}\right)$$

where a, m are the unique numbers satisfying $x=\langle a, m\rangle$. (Notice that these functions are all definable in PA.)

With this new notation Lemma 5.6 becomes

LEMMA 5.8. PA proves
(a) $\forall x\exists y\,(y)_0=x$
(b) $\forall x, y, z\exists w(\forall i<z((w)_i=(y)_i)\wedge(w)_z=x)$
(c) $\forall x, y\,(x)_y\leqslant x$. □

Notice also that the formula $(x)_y = z$ is equivalent to a Δ_0 formula, namely

$$\exists u, v \leqslant x \begin{pmatrix} (u+v)(u+v+1)+2u=2x \wedge \\ z<yv+1 \wedge \\ \exists w \leqslant u(u=w\cdot(yv+1)+z) \end{pmatrix}.$$

Thus we immediately have the following version of Gödel's Lemma 3.2.

LEMMA 5.9 (Gödel's lemma). Let $M \vDash PA$ and let $n \in \mathbb{N}$ with $x_0, x_1, \ldots, x_{n-1} \in M$. Then there exists $u \in M$ s.t.

$$M \vDash (u)_i = x_i$$

for all $i<n$; moreover the formula $(u)_y = x$ is Δ_0.

Proof. Induction on n, using Lemma 5.8. □

(*Note:* The function $(x)_y$ is often denoted $\beta(x, y)$, and referred to as 'Gödel's β-function'. We shall use the $(x)_y$ notation as it is more suggestive.)

Lemma 5.9 is, as we have seen, very useful. The real strength of Lemma 5.8, however, is due to the fact that it shows how to code sequences of 'length' corresponding to elements in whatever nonstandard model $M \vDash PA$ that we are considering. These sequences behave like finite sequences as far as M is concerned, but they may be infinite sequences when viewed from the outside world. (Such sequences will be referred to as *M-finite*.) We will see in the next section how these sequences allow us to define many familiar functions such as exponentiation in arbitrary models of PA.

Exercises for Section 5.2

5.6* Let $M \vDash PA$, let $\theta(x, y, \bar{z})$ be an $\mathscr{L}_A$-formula and let $\bar{a} \in M$. Suppose further that

$$M \vDash \forall x \exists! y \theta(x, y, \bar{a}).$$

Using the properties of $(u)_v$ described in Lemma 5.8, show that

by ind on x

$$M \vDash \forall x \exists w \forall i < x \theta(i, (w)_i, \bar{a}).$$

Now suppose $f(x, y)$ is a function defined by a first-order formula in M (i.e.

$$M \vDash \forall x, y \exists! z \psi(x, y, z, \bar{b})$$

for some $\mathcal{L}_A$-formula ψ and some $\bar{b} \in M$ and $f(x, y)=$ the unique $z \in M$ s.t. $M \models \psi(x, y, z, \bar{b})$) that also satisfies Lemma 5.6, i.e.

$$M \models \forall x, y, z \exists w(\forall i < z(f(w, i) = f(y, i)) \wedge f(w, z) = x).$$

Show that the M-finite sequences coded in the sense of f are exactly the M-finite sequences coded in the sense of $(u)_v$, i.e.

$$M \models \forall x, y \exists w \forall i < x((y)_i = f(w, i))$$

and

$$M \models \forall x, w \exists y \forall i < x(f(w, i) = (y)_i).$$

5.7 Let $M \models PA$ be countable and nonstandard and let $a > \mathbb{N}$ be in M. Show that there are $2^{\aleph_0}$ functions

$$f: S = \{x \in M \mid M \models x < a\} \to \{0, 1\}$$

and hence not all such functions are coded in M in the sense of Lemma 5.8.
(We will be increasingly interested later on in *which* functions f of the above form are coded in M.)
5.8 Prove the following two forms of the Chinese remainder theorem for an arbitrary model $M \models PA$:
(a) for all $n \in \mathbb{N}$, for all $x_0, \ldots, x_{n-1}, m_0, \ldots, m_{n-1} \in M$, if $M \models (m_i, m_j) = 1$ for each $i < j < n$, then there exists $a \in M$ such that

$$M \models a \equiv x_i \bmod m_i$$

for each $i < n$;

(b)
$$M \models \forall x, m, y[\forall i, j < y(i \neq j \to ((m)_i, (m)_j) = 1) \to \exists a \forall i < y(a \equiv (x)_i \bmod (m)_i)].$$

Compare these two statements.

5.3 THE INFINITUDE OF PRIMES

We have now developed PA sufficiently to be able to routinely transcribe many 'standard' proofs in number theory into proofs that formalize in PA. In this section we will illustrate how this is done by means of an example, and (just as for Hardy in his *Mathematician's apology*) to find a suitable example we can hardly do better than go back to the Greeks. We shall show that PA proves there are unboundedly many primes.

This first step is to define the factorial function $x!$ in PA. This is done by recalling the recursive definition of factorial,

$$0! = 1$$
$$(n+1)! = (n+1) \cdot n!$$

and using Lemma 5.8. We let $y = x!$ denote the formula

$$\exists w((w)_0 = 1 \wedge \forall i < x((w)_{i+1} = (i+1) \cdot (w)_i) \wedge (w)_x = y).$$

We claim $PA \vdash \forall x \exists ! y(y = x!)$. To see this let $M \vDash PA$. We prove $M \vDash \forall x \exists y(y = x!)$ by induction on x in the formula $\exists y(y = x!)$ using Lemma 5.8. Clearly $M \vDash \exists w(w)_0 = 1$, so $M \vDash \exists y(y = 0!)$; suppose $x, y, w \in M$ and

$$M \vDash (w)_0 = 1 \wedge \forall i < x((w)_{i+1} = (i+1) \cdot (w)_i) \wedge (w)_x = y.$$

Then by Lemma 5.8

$$M \vDash \exists w'((w')_{x+1} = (x+1) \cdot (w)_x \wedge \forall i < x((w')_i = (w)_i))$$

so $M \vDash (x+1) \cdot y = (x+1)!$, thus $M \vDash \forall x \exists y \; y = x!$ by induction on x. To prove uniqueness suppose

$$x, w, w' \in M \vDash (w)_0 = (w')_0 = 1 \wedge \forall i < x((w)_{i+1} = (i+1) \cdot (w)_i)$$
$$\wedge \forall i < x((w')_{i+1} = (i+1)(w')_i),$$

then by induction on i up to x we can see that $M \vDash \forall i \leqslant x((w)_i = (w')_i)$ and hence $(w)_x = (w')_x$, so

$$M \vDash \forall y, y', x(y = x! \wedge y' = x! \rightarrow y = y').$$

This shows that the factorial function is *well-defined* in PA, and in such a way that *PA proves its recursive definition*, i.e.

$$PA \vdash 0! = 1 \wedge \forall x((x+1)! = (x+1) \cdot x!). \tag{8}$$

(In a similar way, PA defines every primitive recursive function; see Exercise 5.10.) In fact, it is easy to check from the above definition that the factorial function is *provably recursive* in PA (as are all of the primitive recursive functions, see Exercise 5.10).

Armed with this definition and the fact (8), we can now start proving various properties f the factorial function. For example $PA \vdash \forall x(x! \neq 0)$ is proved easily by induction on x, and

$$PA \vdash \forall x, y(0 < y \leqslant x \rightarrow y \mid x!) \tag{9}$$

is proved by induction on z in the formula $(y>0\to y|(y+z)!)$, for if $y>0$ then $y!=y\cdot(y-1)!$ and $(y-1)!\neq 0$ so $y|y!$, and if $y|(y+z)!$ then $(y+(z+1))!=(y+z)!\cdot(y+z+1)$ so $y|(y+z+1)!$ (Notice how the induction always follows the form of the recursive definition: this is a very useful rule of thumb when trying to prove anything in PA.)

Fact (9) and various facts already proved about primes now suffice to allow us to prove Euclid's theorem on the infinitude of primes in PA. Let $M\models PA$ be arbitrary and let $a\in M$. Then $a!+1\geqslant 2$ and so, by Exercise 5.1, there is an irreducible factor p of $a!+1$ with $p>1$, which by Corollary 5.3 is also prime. If $p\leqslant a$ then (9) would tell us that $p|a!$ and since $p|(a!+1)$ this would mean $p|1$, a contradiction. Thus $p>a$ and we have proved

THEOREM 5.10. $PA\vdash\forall x\exists y(y>x\wedge \mathrm{prime}(y))$. □

Exercises for Section 5.3

5.9* Define *exponentiation*, $z=x^y$, in PA by considering the formula

$$\chi(x,y,z)=\exists w[(w)_0=1\wedge\forall i<y((w)_{i+1}=x\cdot(w)_i)\wedge z=(w)_y]$$

(you will need to show both existence and uniqueness of z s.t. $\chi(x,y,z)$ holds). Verify for your definition that

$$PA\vdash\forall x(x^0=1)\wedge\forall x,y(x^{(y+1)}=x\cdot(x^y)).$$

Use induction to prove the following properties of exponentiation in PA:
(a) $\forall x,y,z\ (x^y)\cdot(x^z)=x^{(y+z)}$
(b) $\forall x,y,z\ (x^y)^z=x^{(y\cdot z)}$
(c) $\forall x,y,z\ (x\geqslant y\to x^z\geqslant y^z)$
(d) $\forall x,y,z\ (x\geqslant y\wedge z>0\to z^x\geqslant z^y)$
(e) $\forall x\ (x^n=\overbrace{x\cdot x\cdots x}^{n})$ for each $n\geqslant 1$ in $\mathbb{N}$.

5.10 Show that the collection of functions that are definable in PA is closed under primitive recursion, i.e. show that if $\varphi(\bar{x},y)$ and $\psi(\bar{x},y,z,w)$ are $\mathscr{L}_A$-formulas such that $PA\vdash\forall\bar{x}\exists!y\varphi(\bar{x},y)$ and $PA\vdash\forall\bar{x},y,z\exists!w\psi(\bar{x},y,z,w)$, φ, ψ defining functions $f(\bar{x})$ and $g(\bar{x},y,z)$ respectively, then there is another $\mathscr{L}_A$-formula $\theta(\bar{x},y,z)$ such that $PA\vdash\forall\bar{x},y\exists!z\theta(\bar{x},y,z)$ and

$$PA\vdash\forall\bar{x}\theta(\bar{x},0,f(\bar{x}))$$

$$PA\vdash\forall\bar{x},y,z(\theta(\bar{x},y,z)\to\theta(\bar{x},y+1,g(\bar{x},y,z))).$$

Prove also that, if both ψ, φ are Σ_1, then θ can be taken to be Σ_1, and deduce that each function $f\in\mathscr{PR}$ is provably recursive in PA.

5.11⁺ The *Ackermann function* $A(x, y)$, is defined by

$$A(0, y) = y + 1$$
$$A(x+1, 0) = A(x, 1)$$
$$A(x+1, y+1) = A(x, A(x+1, y)).$$

(a) Show that $A(x, y)$ is not in $\mathscr{PR}$.
(*Hint:* Using induction, show that for each $f(\bar{x}) \in \mathscr{PR}$ there is a fixed $c \in \mathbb{N}$ s.t. $f(x_1, \ldots, x_k) < A(c, x_1 + \cdots + x_k)$ for all $\bar{x} \in \mathbb{N}$.)
(b) Consider the $\mathscr{L}_A$-formula $\psi(x, y) =$

$$\begin{aligned}\exists w[&\forall v, z \leq w \forall i < (w)_0((w)_{i+1} = \langle 0, v, z\rangle \to z = v + 1) \wedge \\ &\forall u, z \leq w \forall i < (w)_0((w)_{i+1} = \langle u+1, 0, z\rangle \to \exists j < i (w)_j = \langle u, 1, z\rangle) \wedge \\ &\forall u, v, z \leq w \forall i < (w)_0((w)_{i+1} = \langle u+1, v+1, z\rangle \to \\ &\quad \exists j, k < i \exists r \leq w((w)_{j+1} = \langle u+1, v, r\rangle \wedge (w)_k = \langle u, r, z\rangle)) \wedge \\ &\exists z \leq w \exists i < (w)_0((w)_i = \langle x, y, z\rangle)].\end{aligned}$$

Show by induction on y that

$$PA \vdash \forall x, y(\forall z \chi(x, z) \to \chi(x+1, y));$$

hence show

$$PA \vdash \forall x, y \chi(x, y).$$

Deduce that $A(x, y)$ is provably recursive in PA.
5.12* The *pigeonhole principle* is the scheme

$$\begin{aligned}&\forall \bar{a}, s(\forall x < s + 1 \exists y < s \psi(x, y, \bar{a}) \\ &\qquad \to \exists x_1, x_2 < s + 1 \exists y < s(x_1 \neq x_2 \wedge \psi(x_1, y, \bar{a}) \wedge \psi(x_2, y, \bar{a})))\end{aligned}$$

for all $\mathscr{L}_A$-formulas ψ. (This says that $s+1$ letters cannot be put into s pigeonholes without putting two letters into the same pigeonhole.) Show that all instances of this principle are provable in PA.
(*Hint:* If $\forall x < s + 1 \exists y < s \psi(x, y, \bar{a})$, use induction to construct w s.t.

$$\forall x < s + 1(\psi(x, (w)_x) \wedge (w)_x < s),$$

and if s exists such that

$$\exists w(\forall x < s + 1((w)_x < s) \wedge \forall y, z < s + 1(y \neq z \to (w)_y \neq (w)_z))$$

consider the least such s with this property.)

6

Models of PA

I must Create a System or be enslav'd by another Man's;
I will not Reason and Compare: my business is to Create.

William Blake
Jerusalem, plate 10, line 20

Although it is very instructive (and useful!) to see how number theory can be formalized *within* PA, the study of PA starts to become interesting when we consider the formal system from *without*. In this chapter we take our first look at nonstandard models of PA, and our investigation is focused on the order-type of these models, and in particular their initial segments.

Our first observation is that proper initial segments always exist. (There are in fact at least $2^{\aleph_0}$ of them, as we shall soon see.) Let $M \models PA$ be nonstandard, and let $a \in M \setminus \mathbb{N}$. The initial segment $a^{\mathbb{N}}$ is defined to be

$$\{b \in M \mid M \models b < a^n \text{ for some } n \in \mathbb{N}\}$$

and is considered as an $\mathscr{L}_A$-structure in the obvious way by considering the restrictions of $+, \cdot, <$ in M to $a^{\mathbb{N}}$. (It is easy to see that $a^{\mathbb{N}}$ is closed under $+, \cdot$; indeed if $x, y < a^n$, then $x+y$, $x \cdot y < a^n \cdot a^n = a^{2n}$. In fact $a^{\mathbb{N}}$ is the *smallest* initial segment of M that contains a and is closed under $+, \cdot$.) Clearly $a^{\mathbb{N}} \neq M$, since by Exercise 5.9 M contains $b = a^a$ and (as a is non-standard) $a^a > a^n$ for each $n \in \mathbb{N}$, so $b > a^{\mathbb{N}}$.

In fact, this argument shows that $a^{\mathbb{N}}$ cannot satisfy PA, for if $a^{\mathbb{N}} \models PA$ then, by Exercise 5.9 again, $a^{\mathbb{N}}$ would contain an element c such that

$$a^{\mathbb{N}} \models (c)_0 = 1 \wedge \forall i < a((c)_{i+1} = a \cdot (c)_i).$$

This formula is Δ_0, so by Corollary 2.8

$$M \models (c)_0 = 1 \wedge \forall i < a((c)_{i+1} = a \cdot (c)_i)$$

so $M \models (c)_{a-1} > a^n$ for each $n \in \mathbb{N}$, so $c > a^{\mathbb{N}}$, contradiction. $a^{\mathbb{N}}$ does, however, satisfy all Π_1 sentences that are true in M, by Corollary 2.8 again. In particular this means that $a^{\mathbb{N}}$ satisfies $I\Delta_0$, that is, the theory PA^- together with all instances of the axiom of induction up to b,

$$\forall \bar{a}, b(\theta(\bar{a}, 0) \wedge \forall x < b(\theta(\bar{a}, x) \rightarrow \theta(\bar{a}, x+1)) \rightarrow \forall x \leqslant b\theta(\bar{a}, x))$$

restricted to Δ_0 formulas $\theta(\bar{a}, x)$.* (It follows from this that $I\Delta_0$ cannot prove the existence of functions of exponential growth—see Exercise 6.5 below. There is a great deal of interest in finding out which number-theoretic statements are provable in $I\Delta_0$ alone. For example, Wilkie has asked whether $I\Delta_0$ proves the infinitude of primes. In spite of a great deal of effort, this question is still unanswered.)

It is clear that $a^{\mathbb{N}}$ failed to satisfy PA because it was 'too small' (relative to a). Our second attempt might be to consider

$$I = \exp_{\mathbb{N}}(a) = \{b \in M \mid M \models b < \exp_n(a) \text{ for some } n \in \mathbb{N}\}$$

where $\exp_0(a) = a$ and $\exp_{n+1}(a) = a^{\exp_n(a)}$. This initial segment is closed under exponentiation as well as $+, \cdot$, but it is still too small to satisfy PA since it does not contain the number $\exp_a(a)$. (Exercise: Make this argument rigorous by defining the primitive-recursive function $\exp_b(a)$ in PA, using Exercise 5.10.)

The moral of all this is, whilst this approach allows us to construct many interesting initial segments of M, we must be rather more clever if we want to construct nonstandard initial segments of M satisfying PA itself. This and the next few chapters are devoted to developing the machinery for doing this, and of the utmost importance is the main result of the next section: *overspill*.

6.1 OVERSPILL

Recall from p. 22 that a non-empty subset I of a model $M \models PA$ is a *cut* of M M iff $x < y \in I \Rightarrow x \in I$ and I is closed under the successor function.

Suppose that I is a *proper* cut of M (i.e. I is not equal to the domain of M); then I is not a *definable* subset of M; for if $\bar{a} \in M$ and $\varphi(x, \bar{a})$ is an $\mathscr{L}_A$-formula with

$$I = \{b \in M \mid M \models \varphi(b, \bar{a})\},$$

then $M \models \varphi(0, \bar{a}) \wedge \forall x(\varphi(x, \bar{a}) \rightarrow \varphi(x+1, \bar{a}))$, since I is non-empty and is closed under the successor function, and so by induction in M, $M \models \forall x \varphi(x, \bar{a})$, contradicting the assumption that I is a proper cut. The *overspill lemma*, due to Abraham Robinson, essentially says just this.

LEMMA 6.1 (Overspill). Let $M \models PA$ be non-standard and let I be a proper

* By an argument similar to that in Proposition 4.1, this theory turns out to be equivalent to the theory $PA^- + \{I\theta \mid \theta \in \Delta_0\}$, so we can, without ambiguity use the notation '$I\Delta_0$' for it.

cut of M. Suppose $\bar{a} \in M$ and $\varphi(x, \bar{a})$ is an $\mathscr{L}_A$-formula such that

$$M \vDash \varphi(b, \bar{a}) \text{ for all } b \in I.$$

Then there is $c > I$ in M such that

$$M \vDash \forall x \leqslant c \varphi(x, \bar{a}).$$

Proof. If the assumptions of the theorem were true, but not the conclusion, then I would be defined by the formula

$$\forall y < x \varphi(y, \bar{a}). \qquad \square$$

The overspill lemma is used a great deal, usually in the case when $I = \mathbb{N}$. There are also two useful variants.

LEMMA 6.2. Let $M \vDash PA$ be nonstandard and I a proper cut of M. Suppose $\varphi(x, \bar{a})$ is an $\mathscr{L}_A$-formula with $\bar{a} \in M$, and that for all $x \in I$ there exists $y \in I$ such that

$$M \vDash y \geqslant x \wedge \varphi(y, \bar{a})$$

(i.e. there are 'unboundedly many $y \in I$ satisfying $\varphi(y, \bar{a})$'). Then for each $c > I$ in M there exists $b \in M$ with $I < b < c$ and

$$M \vDash \varphi(b, \bar{a})$$

(i.e. 'there are arbitrarily small $b > I$ satisfying $\varphi(b, \bar{a})$').

Proof. Apply Lemma 6.1 to the formula

$$\exists y(x \leqslant y < c \wedge \varphi(y, \bar{a})),$$

where $c \in M$ is an arbitrary element satisfying $c > I$. $\square$

LEMMA 6.3 (Underspill). Let $\bar{a} \in M \vDash PA$, $I \subseteq M$ a proper cut and $\varphi(x, \bar{y})$ an $\mathscr{L}_A$-formula. Then:
(a) if $M \vDash \varphi(c, \bar{a})$ for all $c > I$ in M, then $M \vDash \forall x \geqslant b \varphi(x, \bar{a})$ for some $b \in I$;
(b) if for all $c > I$ in M there exists $x > I$ with $M \vDash \varphi(x, \bar{a}) \wedge x < c$, then for all $b \in I$ there exists $y \in I$ with $M \vDash y \geqslant b \wedge \varphi(y, \bar{a})$.

Proof. (a) Suppose there are unboundedly many $x \in I$ satisfying $M \vDash \neg\varphi(x, \bar{a})$. Then $M \vDash \neg\varphi(x, \bar{a})$ for some $x > I$ by Lemma 6.2.
(b) Suppose $b \in I$ and $M \vDash \neg\varphi(y, \bar{a})$ for all $y \geqslant b$ in I. Then $M \vDash y < b \vee \neg\varphi(y, \bar{a})$ for all $y \in I$, so by Lemma 6.1 there is $c > I$ with $M \vDash \forall y \leqslant c (y < b \vee \neg\varphi(y, \bar{a}))$, so the assumption to (b) fails. $\square$

As a simple application of these lemmas, let $M \vDash PA$ be nonstandard and let $a \in M \setminus \mathbb{N}$. Consider the cut $a^{1/\mathbb{N}}$ defined by

$$a^{1/\mathbb{N}} = \{b \in M \mid M \vDash b^n < a \text{ for all } n \in \mathbb{N}\}.$$

It is a simple matter to see that $a^{1/\mathbb{N}}$ is closed under $+$ and $\cdot$, that $a \notin a^{1/\mathbb{N}}$, and that $\mathbb{N} \subseteq_e a^{1/\mathbb{N}} \subseteq_e M$. (In fact $a^{1/\mathbb{N}}$ is the largest segment of M not containing a.)

We can prove $\mathbb{N} \neq a^{1/\mathbb{N}}$ using overspill by considering the formula $\psi(y, a) = \exists x(x \geqslant y \wedge x^y < a)$. Clearly $M \vDash \psi(n, a)$ for each $n \in \mathbb{N}$, since $n^n \in \mathbb{N} < a$. By overspill $M \vDash \psi(b, a)$ for some $b > \mathbb{N}$, i.e. for some $c \in M$ $M \vDash c \geqslant b \wedge c^b < a$, so as $b > \mathbb{N}$ $c^b > c^n$ for all $n \in \mathbb{N}$, hence $c \in a^{1/\mathbb{N}} \setminus \mathbb{N}$. Exercise 6.4 takes this much further, using overspill to show that $a^{1/\mathbb{N}} \nvDash PA$ and also $a^{1/\mathbb{N}} \neq b^{\mathbb{N}}$ for all $b \in M$.

Exercises for Section 6.1

6.1 Show that overspill for arbitrary cuts I is equivalent to induction by showing that, if $M \vDash PA^-$ is nonstandard and satisfies the conclusion of Lemma 6.1, then $M \vDash PA$.
(*Hint:* If $\bar{a} \in M$ and φ is an $\mathscr{L}_A$-formula so that

$$M \vDash \varphi(0, \bar{a}) \wedge \forall x(\varphi(x, \bar{a}) \rightarrow \varphi(x+1, \bar{a}))$$

consider $I = \{x \in M \mid M \vDash \forall y < x\varphi(y, \bar{a})\}$.)
6.2 In contrast to Exercise 6.1, overspill for the standard cut $\mathbb{N}$ is not first-order. Show that any model $M \vDash PA^-$ has an elementary extension $\bar{M} > M$ such that, for every $\bar{a} \in \bar{M}$ and every $\varphi(x, \bar{a})$

$$\bar{M} \vDash \varphi(n, \bar{a}) \text{ for all } n \in \mathbb{N} \Rightarrow \bar{M} \vDash \forall x < b\varphi(x, \bar{a}) \text{ for some } b > \mathbb{N} \text{ in } \bar{M}.$$

(*Hint*: Show first that if $M \vDash PA^-$, $\bar{a} \in M$ and $\varphi(x, \bar{y})$ is an $\mathscr{L}_A$-formula, then there is an $M' > M$ such that

$$M \vDash \varphi(n, \bar{a}) \text{ for all } n \in \mathbb{N} \Rightarrow M' \vDash \forall x < b\varphi(x, \bar{a}) \text{ for some } b > \mathbb{N} \text{ in } M'.$$

Now iterate this construction over and over again for different $\bar{a}$, φ (using the lemma on elementary chains, Lemma 0.6, where necessary) to obtain the required model $\bar{M}$.)
6.3* Let $M \vDash PA$ be nonstandard and suppose $\theta(x, \bar{y})$ is a Δ_0 formula, $\bar{k} \in \mathbb{N}$ and $\mathbb{N} \vDash \theta(n, \bar{k})$ for all $n \in \mathbb{N}$. Deduce that there is $b > \mathbb{N}$ in M s.t.

$$M \vDash \forall x \leqslant b\theta(x, \bar{k}).$$

Where in your argument do you need $\theta(x, \bar{y})$ to be Δ_0?
6.4 Let $M \vDash PA$ be nonstandard with $a > \mathbb{N}$.
(a) For each $n > 0$ in $\mathbb{N}$ show that M contains an element b s.t. $M \vDash b^n \leqslant a < (b+1)^n$, and for each such b, $M \vDash b^b > a$. Hence show that $a^{1/\mathbb{N}} \nvDash PA$ by showing that there is $c \in a^{1/\mathbb{N}}$ such that $M \vDash c^c > a$.

(b) Show that for no $b \in M$ can it be the case that $b^{\mathbb{N}} = a^{1/\mathbb{N}}$. (*Hint:* Consider the formula $\exists y(b^x < y \wedge y^x < a)$.)

6.5(a) Let T be a Π_1 axiomatized $\mathscr{L}_A$-theory containing PA^-. Suppose $T + \exists \bar{x} \forall \bar{y} \leqslant t(x) \neg \theta(\bar{x}, \bar{y})$ is consistent for all $\mathscr{L}_A$-terms $t(\bar{x})$, where θ is a fixed Δ_0 formula. Deduce from the compactness theorem that there is a model $M \models T$ with a fixed tuple $\bar{a} \in M$ such that $M \models \forall \bar{y} \leqslant t(\bar{a}) \neg \theta(\bar{a}, \bar{y})$ for all terms $t(\bar{a})$.

Now consider $b^{\mathbb{N}} \subseteq_e M$, where $b = \max(\bar{a}, 2)$. Show that

$$b^{\mathbb{N}} \models T + \exists \bar{x} \forall \bar{y} \neg \theta(\bar{x}, \bar{y}).$$

Deduce *Parikh's theorem*, that if $\theta(\bar{x}, \bar{y})$ is Δ_0 and $T \vdash \forall \bar{x} \exists \bar{y} \theta(\bar{x}, \bar{y})$, then $T \vdash \forall \bar{x} \exists \bar{y} \leqslant t(\bar{x}) \theta(\bar{x}, \bar{y})$ for some $\mathscr{L}_A$-term $t(\bar{x})$.

(b) Show that $I\Delta_0$ is a Π_1 axiomatized theory. Conclude from Parikh's theorem that $I\Delta_0 \nvdash \forall x \exists y \eta(x, y)$, where η is any Σ_1 formula representing '$y = 2^x$' in $\mathbb{N}$.

(c) By considering the formula

$$\theta(x, y) = (\forall z \neg \eta(x, z)) \vee \eta(x, y),$$

where $\eta(x, y)$ again represents '$y = 2^x$', show that the conclusion to (a) may fail if the assumption that $\theta \in \Delta_0$ is dropped.

6.2 THE ORDER-TYPE OF A MODEL OF *PA*

If M is an $\mathscr{L}_A$-structure then $M \upharpoonright <$ denotes the reduct of M to the language $\{<\}$, i.e. the structure with the same domain as M but with only one relation, namely $<$. It is evident that if $M \models PA^-$ the $M \upharpoonright <$ satisfies the theory T_{DIS} of a *discrete linear order with first element but no last element*, axiomatized by

$T_{\mathrm{DIS}}1$: $\quad \forall x \neg x < x$

$T_{\mathrm{DIS}}2$: $\quad \forall x, y, z(x < y \wedge y < z \rightarrow x < z)$

$T_{\mathrm{DIS}}3$: $\quad \forall x, y(x < y \vee y < x \vee x = y)$

$T_{\mathrm{DIS}}4$: $\quad \forall x(\exists y(y < x) \rightarrow \exists y(y < x \wedge \forall w(w < x \rightarrow (w < y \vee w = y))))$

$T_{\mathrm{DIS}}5$: $\quad \forall x \exists z(x < z \wedge \forall w(x < w \rightarrow z < w \vee z = w))$

$T_{\mathrm{DIS}}6$: $\quad \exists z \forall x(z < x \vee x = z)$.

In fact T_{DIS} is a *complete* theory, i.e., for any sentence σ of the language $\{<\}$ either $T_{\mathrm{DIS}} \vdash \sigma$ or $T_{\mathrm{DIS}} \vdash \neg \sigma$. (For a proof of this see Exercise 6.6 or (Bridge, 1977, Section 4.3, p. 129–33).) T_{DIS} has $2^{\aleph_0}$ non-isomorphic countable models, for if $(A, <_A)$ is any (possibly empty) linearly ordered

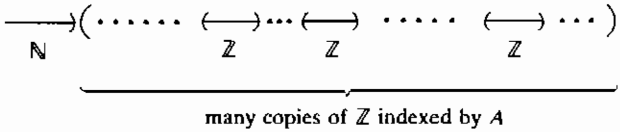

Fig. 3 The order-type $\mathbb{N}+\mathbb{Z}\cdot A$

set then $\mathbb{N}+\mathbb{Z}\cdot A \models T_{\mathrm{DIS}}$, where the model $\mathbb{N}+\mathbb{Z}\cdot A$ is that with domain $\mathbb{N}\cup(\mathbb{Z}\times A)$ (this union will be assumed to be disjoint) and order $<$ defined by:

(a) $n<(z, a)$ for all $n\in\mathbb{N}$ and $(z, a)\in\mathbb{Z}\times A$;
(b) $<$ restricted to $\mathbb{N}$ is the natural order on $\mathbb{N}$;
(c) $(z_1, a_1)<(z_2, a_2)$ iff $(a_1<_A a_2$ or $(a_1=a_2$ and $z_1<z_2))$ for all (z_1, a_1), $(z_2, a_2)\in\mathbb{Z}\times A$, and where $z_1<z_2$ is the usual order on $\mathbb{Z}$.

The order $\mathbb{N}+\mathbb{Z}\cdot A$ is the linear order shown in Fig. 3.

We shall now sketch out a proof that if $A\not\cong B$ then $\mathbb{N}+\mathbb{Z}\cdot A\not\cong\mathbb{N}+\mathbb{Z}\cdot B$, and that all models of T_{DIS} are isomorphic to some $\mathbb{N}+\mathbb{Z}\cdot A$ for some linearly ordered set A. Let $(M, <_M)\models T_{\mathrm{DIS}}$, and let 0_M be the $<_M$-least element of M (given by the axiom $T_{\mathrm{DIS}}6$) and define functions S and P on M by

$$\begin{aligned} S(x) &= \text{the unique } y\in M \text{ s.t.}\\ &\quad M\models x<y\wedge\forall w(x<w\rightarrow(y<w\vee y=w))\\ P(x) &= \text{the unique } y\in M \text{ s.t.}\\ &\quad M\models y<x\wedge\forall w(w<x\rightarrow(w<y\vee w=y)),\ \text{if } x\neq 0_M \end{aligned}$$

and $P(0_M)=0_M$.

Next we define an equivalence relation $\sim$ on M by $a\sim b$ iff $b=P^{(n)}(a)$ or $b=S^{(n)}(a)$ for some $n\in\mathbb{N}$. We leave it to the reader to check the following facts:

(a) $\sim$ *is* an equivalence relation (we denote the equivalence class of $a\in M$ by $[a]$);

(b) $<_M$ induces a linear order $<$ on $M/{\sim}$ given by $[a]<[b]\Leftrightarrow a<_M b$ and $[a]\neq[b]$;

(c) if $A=(M\setminus[0_M])/{\sim}$ and $<_A$ is the linear order given in (b), then $(M, <_M)\cong\mathbb{N}+\mathbb{Z}\cdot A$.

It follows that all models of T_{DIS} are of the form $\mathbb{N}+\mathbb{Z}\cdot A$, and by examining the construction above it is easy to see that the ordered set $(A, <_A)$ obtained from $(M, <_M)$ is unique (up to isomorphism). Thus $\mathbb{N}+\mathbb{Z}\cdot A\cong\mathbb{N}+\mathbb{Z}\cdot B$ iff $A\cong B$. From this we can deduce that, since there are up to isomorphism $2^{\aleph_0}$ countable linearly ordered sets A, there are $2^{\aleph_0}$ countable models of T_{DIS}.

Returning to models of arithmetic now, we would like to know: given a nonstandard model $M \vDash PA$, which of the possible models of T_{DIS} can $M \restriction <$ be? The rather surprising answer is that, if M is countable, there is only one possibility for the order-type of $M \restriction <$.

THEOREM 6.4. Let $M \vDash PA$ be nonstandard. Then $M \restriction < \cong \mathbb{N} + \mathbb{Z} \cdot A$ for some linearly ordered set $(A, <_A)$ satisfying the theory DLO axiomatized by

$$\begin{array}{ll} DLO1: & \forall x \neg x < x \\ DLO2: & \forall x, y, z(x<y \wedge y<z \rightarrow x<z) \\ DLO3: & \forall x, y(x<y \vee x=y \vee y<x) \\ DLO4: & \forall x, y(x<y \rightarrow \exists z(x<z \wedge z<y)) \\ DLO5: & \forall x \exists y, z(y<x \wedge x<z). \end{array}$$

In particular, if M is countable, then $M \cong \mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$, where $\mathbb{Q}$ is the set of rationals with its natural order.

Proof. It is a well-known theorem of Cantor's (and a very popular exercise to give to undergraduate mathematicians!) that any countable structure $(A, <_A)$ satisfying DLO is isomorphic to $(\mathbb{Q}, <)$. A proof appears in Bridge (1977, pp. 107–9), for example. (DLO, by the way, stands for dense linear order.) This shows it is sufficient to prove that $M \restriction < \cong \mathbb{N} + \mathbb{Z} \cdot A$ for some $(A, <_A) \vDash DLO$.

Since the functions $S(x)$ and $P(x)$ defined above are obviously $x+1$ and $x \dot{-} 1$ in M we can re-interpret the equivalence relation $\sim$ by: $a \sim b$ iff $M \vDash a+n=b \vee a=b+n$ for some $n \in \mathbb{N}$. So, as before, let

$$A = (M \setminus [0])/\sim$$

with its order $<_A$ induced from $M \restriction <$ by

$$[a] <_A [b] \Leftrightarrow [a] \neq [b] \ \&\ M \vDash a<b.$$

We must check $(A, <_A) \vDash DLO$.

Notice that $[a] \in A$ iff $a > \mathbb{N}$. Suppose $[a] <_A [b]$ where $a, b \in M$ are nonstandard. Then $M \vDash a<b$ and

$$M \vDash \exists x(a+n<x \wedge x+n<b)$$

for each $n \in \mathbb{N}$. Hence, by overspill, there is $c \in M$ with $c > \mathbb{N}$ and $d \in M$ s.t.

$$M \vDash a+c<d \wedge d+c<b$$

so $[a] <_A [d] <_A [b]$, thus $A \vDash DLO4$. To check $DLO5$ holds, suppose $a > \mathbb{N}$. Then

$$M \vDash \left\lfloor \frac{a}{2} \right\rfloor < \left\lfloor \frac{a}{2} \right\rfloor + n < a < a + n < 2a$$

for each $n \in \mathbb{N}$, so

$$\left[\left\lfloor \frac{a}{2} \right\rfloor \right] <_A [a] <_A [2a].$$

$DLO1$, $DLO2$, $DLO3$ are just the axioms for a linear order, so they are automatically satisfied. This proves the theorem. □

Theorem 6.4 is also very useful: we can deduce from it that countable models of PA have the maximum possible number of cuts.

COROLLARY 6.5. Let $M \vDash PA$ be countable and nonstandard. Then M has $2^{\aleph_0}$ proper cuts I.

Proof. By the theorem, there is an order-isomorphism $f: M \upharpoonright < \to \mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$. If $S \subseteq \mathbb{Q}$ satisfies $\forall x \in S \forall y \in \mathbb{Q}(y < x \to y \in S)$ then the inverse image under f of $\mathbb{N} + \mathbb{Z} \cdot S$ is a cut I of M, and clearly distinct such $S_1, S_2 \subseteq \mathbb{Q}$ give rise in this way to distinct cuts I_1, I_2 of M. The corollary follows, since the reals $\mathbb{R}$ are in 1–1 correspondence with the cuts they make in $\mathbb{Q}$, via the map

$$r \mapsto \{q \in \mathbb{Q} \mid q < r\}$$

and the cardinality of $\mathbb{R}$ is $2^{\aleph_0}$. □

Compare Corollary 6.5 with Exercise 2.13 which showed that non-standard models of PA^- need only have $\aleph_0$ proper cuts!

The argument just given can be easly modified to give

THEOREM 6.6. Let $M \vDash PA$ be countable and nonstandard. Then M has $2^{\aleph_0}$ proper cuts $I \subseteq_e M$ that are closed under $+, \cdot$.

Proof. Redefine $\sim$ on M by $\wedge$?

$$a \sim b \text{ iff } M \vDash b < (a+2)^n \vee a < (b+2)^n \text{ for some } n \in \mathbb{N}.$$

Then $\sim$ is again an equivalence relation, $A = (M \setminus [0])/\sim$ is linearly ordered by $[a] <_A [b]$ iff $[a] \neq [b]$ and $M \vDash a < b$. In fact $(A, <_A) \vDash DLO$

again, since if $[a] <_A [b]$, then $\mathbb{N} < a < b$ and

$$M \models \exists x((a+2)^n < x < (x+2)^n < b)$$

for each $n \in \mathbb{N}$. So by overspill there are c, $d > \mathbb{N}$ in M, so that

$$M \models (a+2)^c < d < (d+2)^c < b$$

(here we are using Exercise 5.8 and the fact that exponentiation can be defined in PA by a single formula). It follows that $[a] <_A [d] <_A [b]$, and so $(A, <_A) \models DLO4$. $DLO5$ is proved in a similar way.

Thus there is an order-isomorphism $f: (A, <_A) \rightarrow (\mathbb{Q}, <)$. Inverse images of cuts $S_r = \{q \in \mathbb{Q} \mid q < r\}$ (where $r \in \mathbb{R}$) yield initial segments of M that are closed under $+$ and $\cdot$, and distinct cuts S_r, S_s yield distinct initial segments, as required. □

Exercises for Section 6.2

6.6 Prove that any countable model $(A, <_A)$ of DLO is isomorphic to $(\mathbb{Q}, <)$. Deduce that DLO is complete.

6.7 Show that T_{DIS} is complete as follows:

(a) Let $M \models T_{\mathrm{DIS}}$ be countable, and define 0, $P(x)$, and $S(x)$ on M using $T_{\mathrm{DIS}}6$, $T_{\mathrm{DIS}}4$ and $T_{\mathrm{DIS}}5$ respectively, by *first-order* formulas in the language of T_{DIS}.

(b) Now define $\sim$ on arbitrary models of T_{DIS} by

$$a \sim b \text{ iff } \bigvee\!\!\!\bigvee_{n \in \mathbb{N}} (P^{(n)}(a) = b \vee S^{(n)}(a) = b).$$

Show that if $M \models T_{\mathrm{DIS}}$, $a, b \in M \models a < b$ and $[a] \neq [b]$, then there is an elementary extension $M' > M$ with $c \in M'$ satisfying $M' \models a < c < b$ and $[a] \neq [c] \neq [b]$. Similarly, if $a \in M$ and $[a] \neq [0]$ show that there is an elementary extension $M' > M$ with b, $c \in M' \models b < a < c$ and $[0] \neq [b] \neq [a] \neq [c]$.

(c) Using (b) and a union of chains argument, show that any countable model T_{DIS} is an elementary substructure of a model $\bar{M} \cong \mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$. Deduce that any two countable models M_1, $M_2 \models T_{\mathrm{DIS}}$ can be elementarily embedded in a common extension $\bar{M}$, and hence T_{DIS} is complete.

6.8 Modify the proof of Theorem 6.4 to show that any nonstandard $M \models PA^-$ satisfying overspill for the standard cut $\mathbb{N}$ has order-type $\mathbb{N} + \mathbb{Z} \cdot A$ where $A \models DLO$.

7

Collection

'Are five nights warmer than one night, then?' Alice ventured to ask.
'Five times as warm of course.'
'But they should be five times as *cold*, by the same rule—'
'Just so!' cried the Red Queen. 'Five times as warm, *and* five times as cold—just as I'm five times as rich as you, *and* five times as clever!'

Lewis Carroll
From: *Through the looking glass and what Alice found there*

The *collection axioms* are the arithmetic analogue of Fraenkel's *collection principle* in set theory. Fraenkel's principle says that if f is a function (defined by a first-order formula) whose domain is a set x, then the range of f can be 'collected' together as a subset of some set y. This principle (which was a very powerful addition to Zermelo's original set theory), when put into the context of arithmetic, becomes:

$$\text{an } \mathscr{L}_{\mathrm{A}}\text{-definable function } f \text{ with finite domain has finite range.} \tag{*}$$

it is easy to see that (*) is true in $\mathbb{N}$, and in fact these arithmetic collection axioms are provable from PA. Surprisingly, they are very powerful, and are actually equivalent to the induction scheme over the weak theory $I\Delta_0$. Collection plays an important role in the model theory of PA, and we will explore aspects of this in both this chapter and the next.

Another application of the collection axioms is to show that the formula classes Σ_1 and Π_1 are closed under bounded quantification. More generally, Σ_1 and Π_1 form the bottom level of a hierarchy of formula classes Σ_2, Π_2, Σ_3, Π_3, . . . , and collection tells us that each of these classes is closed under bounded quantification. We shall start by describing this hierarchy and the role of collection there.

7.1 THE ARITHMETIC HIERARCHY

Recall that Δ_0 is the class of all *bounded* formulas, i.e. formulas whose quantifiers all occur as $\forall x < t \ldots$ or $\exists x < t \ldots$. For convenience (and to make the following definition more elegant) we also denote Δ_0 by Σ_0 and

Π_0. The following definition defines classes Σ_n and Π_n for all $n \in \mathbb{N}$ by induction on n.

DEFINITION. A formula is Σ_{n+1} iff it is of the form $\exists \bar{x} \varphi(\bar{x}, \bar{y})$ with $\varphi \in \Pi_n$; a formula is Π_{n+1} iff it is of the form $\forall \bar{x} \varphi(\bar{x}, \bar{y})$ with $\varphi \in \Sigma_n$.

Thus a formula is Σ_n iff it looks like

$$\exists \bar{x}_1 \forall \bar{x}_2 \exists \bar{x}_3 \ldots Q \bar{x}_n \varphi(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n, \bar{y})$$

where all quantifiers in φ are bounded (and Q is $\exists$ or $\forall$ according to whether n is odd or even). Similarly a formula is Π_n iff it looks like

$$\forall \bar{x}_1 \exists \bar{x}_2 \forall \bar{x}_3 \cdots Q \bar{x}_n \varphi(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n, \bar{y}).$$

We allow blocks of quantifiers to be empty, so that for example any Π_n formula is automatically both Π_{n+1} and Σ_{n+1}. We shall also say that a formula $\theta(\bar{x})$ is Σ_n or Π_n if it is *equivalent* to a Σ_n or a Π_n formula $\varphi(\bar{x})$ in the theory T or the model M we are considering, i.e. if

$$T \vdash \forall \bar{x}(\theta(\bar{x}) \leftrightarrow \varphi(\bar{x}))$$

or

$$M \vDash \forall \bar{x}(\theta(\bar{x}) \leftrightarrow \varphi(\bar{x})).$$

When it is important to indicate the theory or model in which this equivalence takes place we shall do this thus: $\Sigma_n(T)$, $\Pi_n(T)$, $\Sigma_n(M)$ or $\Pi_n(M)$. Usually, however, T or M will be clear from the context.

A formula is Δ_n iff it is equivalent to both a Σ_n formula and a Π_n formula. As before, these equivalences take place in some fixed theory T or model M, and we indicate this (where necessary) by $\Delta_n(T)$ or $\Delta_n(M)$. Thus, for example, by Corollary 3.5 all recursive sets are represented by $\Delta_1(\mathbb{N})$ formulas.

Notice that every formula is equivalent (in the predicate calculus) to some Σ_n or Π_n formula. Indeed, by prenex normal form (see, for example, Bridge 1977, pp. 54–6) any formula $\theta(\bar{x})$ is equivalent to a formula of the form

$$Q_1 y_1 Q_2 y_2 \cdots Q_n y_n \varphi(\bar{x}, \bar{y})$$

for some $n \in \mathbb{N}$, where φ is quantifier-free and Q_i is either $\forall$ or $\exists$ for each i. Thus θ is Π_n or Σ_n for some $n \in \mathbb{N}$. Also, since we allow quantifier blocks $\exists \bar{x}$ or $\forall \bar{x}$ to be empty in the above definition, we have the obvious inclusions $\Sigma_n \subseteq \Delta_{n+1} \subseteq \Sigma_{n+1}$ and $\Pi_n \subseteq \Delta_{n+1} \subseteq \Pi_{n+1}$. Figure 4 shows these inclusions in diagramatic form. The hierarchy of formula classes Σ_n, Π_n, Δ_n, . . . is called the *arithmetic hierarchy*.

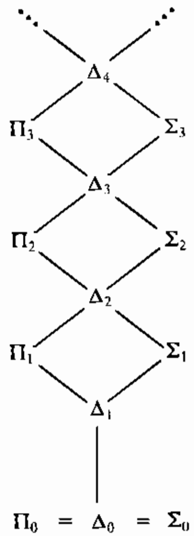

Fig. 4. The arithmetic hierarchy

The classes Σ_n and Π_n are closed under (finite) conjunctions and disjunctions. For example, whenever $\theta_1(\bar{x})$ and $\theta_2(\bar{x})$ are Σ_n there are Σ_n formulas equivalent (in the predicate calculus) to $\theta_1(\bar{x}) \wedge \theta_2(\bar{x})$, and similarly for Π_n. To see this suppose $\theta_1(\bar{x})$ is $\exists \bar{y}_1 \forall \bar{y}_2 \cdots \mathsf{Q}\bar{y}_n \varphi_1(\bar{x}, \bar{y})$ and $\theta_2(\bar{x})$ is $\exists \bar{z}_1 \forall \bar{z}_2 \cdots \mathsf{Q}\bar{z}_n \varphi_2(\bar{x}, \bar{z})$ with $\varphi_1, \varphi_2 \in \Delta_0$ and $\mathsf{Q} = \exists$ or $\forall$ according to whether n is odd or even, then by renaming variables if necessary so that there are no variables common to $\bar{y}$ and $\bar{z}$ we see that $\theta_1(\bar{x}) \wedge \theta_2(\bar{x})$ is equivalent to

$$\exists \bar{y}_1, \bar{z}_1 \forall \bar{y}_2, \bar{z}_2 \cdots \mathsf{Q}\bar{y}_n, \bar{z}_n (\varphi_1(\bar{x}, \bar{y}) \wedge \varphi_2(\bar{x}, \bar{z}));$$

a similar argument applies to all the other cases.

If $\theta(\bar{x})$ is Σ_n, then $\neg\theta(\bar{x})$ is equivalent (in the predicate calculus again) to a Π_n formula. For example if $\theta(\bar{x})$ is $\exists \bar{y}_1 \forall \bar{y}_2 \cdots \mathsf{Q}\bar{y}_n \varphi(\bar{x}, \bar{y})$ with $\varphi \in \Delta_0$ then $\neg\theta$ is equivalent to $\forall \bar{y}_1 \exists \bar{y}_2 \cdots \mathsf{Q}'\bar{y}_n \neg\varphi(\bar{x}, \bar{y})$ where Q' is the 'dual' of Q, i.e. Q' is $\forall$ if Q is $\exists$ and Q' is $\exists$ otherwise. Similarly if $\theta(\bar{x})$ is Π_n then $\neg\theta(\bar{x})$ is Σ_n. This shows that Δ_n is closed under negations, and by the last paragraph Δ_n is also closed under $\wedge$ and $\vee$. (However, for $n \geqslant 1$ neither Σ_n nor Π_n are closed under $\neg$. For example if $\Sigma_1(\mathbb{N})$ was closed under $\neg$, every r.e. set $A \subseteq \mathbb{N}$ would have an r.e. complement, and so would be recursive!)

Using the *pairing function* $\langle x_1, x_2 \rangle = (x_1 + x_2)(x_1 + x_2 + 1)/2 + x_2$ (see p. 62) and the derived functions $\langle x_1, \ldots, x_n \rangle$ coding tuples in PA we see that a Σ_n formula $\exists \bar{y}_1 \forall \bar{y}_2 \cdots \mathsf{Q}\bar{y}_n \varphi(\bar{x}, \bar{y})$ is equivalent (in PA) to a formula in Σ_n

form but with *only one quantifier in each block*. For example the formula above is equivalent to

$$\exists z_1 \forall z_2 \cdots Qz_n[\exists \bar{y}_1 \leqslant z_1 \exists \bar{y}_2 \leqslant z_2 \cdots \exists \bar{y}_n \leqslant z_n \bigwedge_{i \leqslant n}(z_i = \langle \bar{y}_i \rangle) \wedge \varphi(\bar{x}, \bar{y})]$$

where the formula in square brackets is clearly Δ_0. Similarly every Π_n formula is equivalent (in PA) to a formula in Π_n form with only one quantifier in each block.

In fact the equivalences in the last paragraph are provable in a rather weak fragment of PA. All that is needed is

$$\forall x_1, x_2 \exists ! y \langle x_1, x_2 \rangle = y$$

$$\forall y \exists ! x_1, x_2 \langle x_1, x_2 \rangle = y$$

$$\forall x_1, x_2, y(\langle x_1, x_2 \rangle = y \rightarrow x_1, x_2 \leqslant y)$$

to be provable, and it is a simple exercise to check that these sentences are indeed provable in $I\Delta_0$ only.

The collection axioms

The collection axioms allow us to show that the formula classes Σ_n, Π_n and Δ_n are closed under bounded quantification.

DEFINITION. Let $\varphi(x, \bar{y}, \bar{z})$ be a formula of $\mathscr{L}_A$. The *collection axiom for* φ is the sentence $B\varphi$, which is

$$\forall \bar{z}, t(\forall x < t \exists \bar{y} \varphi(x, \bar{y}, \bar{z}) \rightarrow \exists s \forall x < t \exists \bar{y} < s \varphi(x, \bar{y}, \bar{z})).$$

(Strictly, we should indicate $x, \bar{y}$ in the notation by '$B_{x,\bar{y}}\varphi$'; however, no confusion will arise from just using '$B\varphi$'.)

Notice that the 'converse' of $B\varphi$,

$$\forall \bar{z}, t(\exists s \forall x < t \exists \bar{y} < s \varphi(x, \bar{y}, \bar{z}) \rightarrow \forall x < t \exists \bar{y} \varphi(x, \bar{y}, \bar{z}))$$

is valid in all $\mathscr{L}_A$-structures. Also notice that $B(\neg\varphi)$ implies

$$\forall \bar{z}, t(\exists x < t \forall \bar{y} \varphi(x, \bar{y}, \bar{z}) \leftrightarrow \forall s \exists x < t \forall \bar{y} < s \varphi(x, \bar{y}, \bar{z})).$$

Thus the collection axioms say that a bounded quantifier can be 'pushed inside' an unbounded quantifier.

We shall denote the induction-free theory

$$PA^- + \{B\varphi \mid \varphi \text{ an } \mathscr{L}_A\text{-formula}\}$$

by Coll. The subtheory

$$PA^- + \{B\varphi \mid \varphi \text{ is a } \Sigma_n \ \mathscr{L}_A\text{-formula}\}$$

of Coll will be denoted Coll_n. (Here 'φ is a Σ_n formula' is understood in its *strict* sense, i.e. meaning 'φ is in Σ_n form', and *not* meaning 'φ is equivalent to a Σ_n formula'. However, if $M \vDash \text{Coll}_n$ and $\theta(x, \bar{y}, \bar{z})$ is $\Sigma_n(M)$, then

$$M \vDash \forall x, \bar{y}, \bar{z}(\theta(x, \bar{y}, \bar{z}) \leftrightarrow \varphi(x, y, z))$$

for some $\varphi(x, \bar{y}, \bar{z})$ in Σ_n in the strict sense, so since $M \vDash B\varphi$ we have $M \vDash B\theta$ also; thus this distinction makes little difference. But see also Exercise 7.1.) The theory $B\Sigma_n$ is $I\Delta_0 + \text{Coll}_n$, i.e. the theory with axioms: PA^-; induction on Δ_0 formulas; and collection for Σ_n formulas.

PROPOSITION 7.1. Let $n \in \mathbb{N}$, $\theta(x, \bar{y})$ be a Σ_n formula, $\psi(x, \bar{y})$ be a Π_n formula, and $t(\bar{y})$ be an $\mathscr{L}_A$-term. Then the formulas $\forall x < t(\bar{y})\theta(x, \bar{y})$ and $\exists x < t(\bar{y})\psi(x, \bar{y})$ are $\Sigma_n(\text{Coll}_n)$ and $\Pi_n(\text{Coll}_n)$ respectively. Thus $\Sigma_n(\text{Coll}_n)$, $\Pi_n(\text{Coll}_n)$ and $\Delta_n(\text{Coll}_n)$ are closed under bounded quantification, for all $n \in \mathbb{N}$.

Proof. By induction on n.

Clearly $\Sigma_0 = \Pi_0 = \Delta_0$ is closed under bounded quantification, so the proposition is true for $n = 0$. Now suppose $\theta(x, \bar{y})$ is $\exists \bar{z}\varphi(x, \bar{y}, \bar{z})$ where φ is Π_{n-1} and $n \geq 1$. Then

$$\text{Coll}_n \vdash \forall \bar{y}(\forall x < t\theta(x, \bar{y}) \leftrightarrow \exists s \forall x < t \exists \bar{z} < s\varphi(x, \bar{y}, \bar{z}))$$

and by the induction hypothesis $\exists \bar{z} < s\varphi(x, \bar{y}, \bar{z})$ *is* $\Pi_{n-1}(\text{Coll}_{n-1})$, i.e. there is a Π_{n-1} formula $\chi(x, \bar{y}, s)$ such that

$$\text{Coll}_{n-1} \vdash \forall x, \bar{y}, s(\chi(x, \bar{y}, s) \leftrightarrow \exists \bar{z} < s\varphi(x, \bar{y}, \bar{z})).$$

Thus

$$\text{Coll}_n \vdash \forall y(\forall x < t\theta(x, \bar{y}) \leftrightarrow \exists s \forall x < t\chi(x, \bar{y}, s))$$

and $\exists s \forall x(x < t \rightarrow \chi(x, \bar{y}, s))$ is clearly Σ_n, thus $\forall x < t\theta(x, \bar{y})$ is $\Sigma_n(\text{Coll}_n)$. The induction step showing that $\Pi_n(\text{Coll}_n)$ is closed under bounded quantification is proved in exactly the same way. □

Having shown the importance of the collection principle we shall show that it is provable in PA. Recall that for a class of formulas Γ, $I\Gamma$ is the theory axiomatized by PA^- together with all induction axioms $I\theta$ for all

$\theta \in \Gamma$. In particular $I\Sigma_n$ is PA^- + induction for all Σ_n formulas, and $I\Pi_n$ is PA^- + induction for all Π_n formulas. Then, since every formula is equivalent (in the predicate calculus) to a Σ_n formula for some $n \in \mathbb{N}$, PA is equivalent to the theory $I\Sigma_1 + I\Sigma_2 + I\Sigma_3 + \cdots$, which is also equivalent to $I\Pi_1 + I\Pi_2 + I\Pi_3 + \cdots$. We prove

PROPOSITION 7.2. $I\Sigma_n \vdash \mathrm{Coll}_n$ for all $n \geq 1$. Hence $PA \vdash \mathrm{Coll}$.

Proof. Let $M \models I\Sigma_n$, let $\varphi(x, \bar{y}, \bar{z})$ be a Σ_n formula, and suppose $\bar{a}, t \in M$ satisfy

$$M \models \forall x < t \exists \bar{y} \varphi(x, \bar{y}, \bar{a}).$$

In the case $n \geq 2$ we shall also assume inductively that $I\Sigma_{n-1} \vdash \mathrm{Coll}_{n-1}$. We must show that $M \models \exists s \forall x < t \exists \bar{y} < s \varphi(x, \bar{y}, \bar{a})$. Clearly we may assume that φ is Π_{n-1} since if φ is $\exists \bar{w} \theta(x, \bar{y}, \bar{w}, \bar{a})$ with $\theta \in \Pi_{n-1}$ then we may consider θ in place of φ and the tuple $\bar{y}, \bar{w}$ in place of $\bar{y}$.

We consider the formula $\psi(w, \bar{a})$, which is

$$(\exists s \forall x < w \exists \bar{y} < s \varphi(x, \bar{y}, \bar{a})) \vee w > t.$$

Now if $n = 1$ $\psi(w, \bar{a})$ is Σ_1 since φ is $\Pi_{n-1} = \Delta_0$, and if $n > 1$ then $\forall x < w \exists \bar{y} < s$ $\varphi(x, \bar{y}, \bar{a})$ is the result of applying bounded quantifiers to a Π_{n-1} formula, so is equivalent to a Π_{n-1} formula in Coll_{n-1} and hence in $I\Sigma_n$ by our induction hypothesis. (Note that as $\Sigma_n \supseteq \Sigma_{n-1}$, $I\Sigma_n \vdash I\Sigma_{n-1}$.) Thus ψ is $\Sigma_n(I\Sigma_n)$ and we may use induction on ψ.

Clearly $M \models \psi(0, \bar{a})$, since any value at all for s will do. If $M \models \psi(w, \bar{a})$ with $w < t$ then there are $s_1, s_2 \in M$ such that $M \models \forall x < w \exists \bar{y} < s_1 \varphi(x, y, \bar{a})$ and $M \models \exists \bar{y} < s_2 \varphi(w, \bar{y}, \bar{a})$. ($s_2$ comes from considering $\forall x < t \exists \bar{y} \varphi(x, \bar{y}, \bar{a})$ for $x = w$ and putting $s_2 = \max(\bar{y}) + 1$ for a suitable $\bar{y}$.) Then $s = \max(s_1, s_2)$ clearly satisfies

$$M \models \forall x < w + 1 \exists \bar{y} < s \varphi(x, \bar{y}, \bar{a})$$

proving the induction step, $M \models \forall w < t(\psi(w) \to \psi(w + 1))$. Thus $M \models \forall w \psi(w)$ by $I\Sigma_n$, and in particular $M \models \exists s \forall x < t \exists \bar{y} < s \varphi(x, \bar{y}, \bar{a})$, as required. □

The remarkable fact about collection is that it is actually *equivalent* to PA over the weak theory $I\Delta_0$.

THEOREM 7.3. PA is equivalent to $I\Delta_0 + \mathrm{Coll}$.

Proof. We have already seen that $I\Sigma_n \vdash I\Delta_0 + \mathrm{Coll}_n$ for each $n \in \mathbb{N}$. For the other direction, we prove $I\Delta_0 + \mathrm{Coll}_{n+1} \vdash I\Sigma_n$. This follows from the next two lemmas.

LEMMA 7.4. For all $n \geqslant 0$ $I\Pi_n + \mathrm{Coll}_{n+2} \vdash I\Sigma_{n+1}$.

Proof. Let $\bar{a} \in M \vDash I\Pi_n + \mathrm{Coll}_{n+\cancel{1}^{2}}$, and let $\theta(x, \bar{a}) \in \Sigma_{n+1}$, where $\theta(x, \bar{a})$ is $\exists \bar{y} \psi(x, \bar{y}, \bar{a})$ and $\psi \in \Pi_n$. Suppose also that

$$M \vDash \theta(0, \bar{a}) \wedge \forall x(\theta(x, \bar{a}) \rightarrow \theta(x+1, \bar{a})). \tag{*}$$

We must show that $M \vDash \theta(b, \bar{a})$ for all $b \in M$.

So let $b \in M$ be arbitrary. Clearly we have

$$M \vDash \forall x < b+1 \exists \bar{y}[\psi(x, \bar{y}, \bar{a}) \vee \forall \bar{z} \neg \psi(x, \bar{z}, \bar{a})],$$

and the formula in square brackets here in Π_{n+1}, so by the collection axioms in M there exists $c \in M$ such that

$$M \vDash \forall x < b+1 \exists \bar{y} < c(\psi(x, \bar{y}, \bar{a}) \vee \forall \bar{z} \neg \psi(x, \bar{z}, \bar{a})).$$

This shows that

$$M \vDash \forall x \leqslant b(\exists \bar{y} \psi(x, \bar{y}, \bar{a}) \leftrightarrow \exists \bar{y} < c \psi(x, \bar{y}, \bar{a})).$$

But $\exists \bar{y} < c \psi(x, \bar{y}, \bar{a})$ is equivalent (in M) to a Π_n formula $\varphi(x, c, \bar{a})$ (by Proposition 7.1), and hence from (*) we have

$$M \vDash \varphi(0, c, \bar{a}) \wedge \forall x < b(\varphi(x, c, \bar{a}) \rightarrow \varphi(x+1, c, \bar{a})),$$

so by Π_n-induction on the formula

$$x > b \vee (x \leqslant b \wedge \varphi(x, c, \bar{a}))$$

we have $M \vDash \varphi(b, c, \bar{a})$, hence $M \vDash \exists \bar{y} \psi(b, \bar{y}, \bar{a})$, as required. □

LEMMA 7.5. For all $n \geqslant 0$,

$$I\Sigma_n \vdash I\Pi_n \quad \text{and} \quad I\Pi_n \vdash I\Sigma_n.$$

Proof. To prove $I\Sigma_n \vdash I\Pi_n$, let $M \vDash I\Sigma_n$, $\varphi(x, \bar{y})$ a Π_n formula, $\bar{a} \in M$ and suppose $M \vDash \varphi(0, \bar{a}) \wedge \forall x(\varphi(x, \bar{a}) \rightarrow \varphi(x+1, \bar{a}))$. We must show that, for all $b \in M$, $M \vDash \varphi(b, \bar{a})$. So suppose this is not the case, i.e., that $b \in M \vDash \neg \varphi(b, \bar{a})$, and consider the formula $\psi(x, b, \bar{a})$, which is

$$(x > b) \vee (x \leqslant b \wedge \exists y(y + x = b \wedge \neg \varphi(y, \bar{a}))).$$

Since φ is Π_n, $\neg\varphi$ is a Σ_n formula, and as Σ_n is closed under $\wedge$, $\vee$ and existential quantification, ψ is equivalent to a Σ_n formula. Moreover, since $M \models \neg\varphi(b, \bar{a})$ we have $M \models \varphi(0, b, \bar{a})$, and since

$$M \models \forall x(\neg\varphi(x+1, \bar{a}) \to \neg\varphi(x, \bar{a}))$$

we have

$$M \models \forall x(\psi(x, b, \bar{a}) \to \psi(x+1, b, \bar{a})).$$

Thus $M \models \forall x \psi(x, b, \bar{a})$ and in particular $M \models \psi(b, b, \bar{a})$, hence $M \models \neg\varphi(0, \bar{a})$, a contradiction.

The converse direction (which is not needed for the proof of Theorem 7.3 anyway) is proved in a similar fashion by considering

$$x > b \vee (x \leqslant b \wedge \forall y(y + x = b \to \neg\varphi(y, \bar{a})))$$

for $\varphi \in \Sigma_n$. □

Exercises for Section 7.1

7.1. (a) For some suitable Gödel numbering $\ulcorner \cdot \urcorner$ on formulas of $\mathscr{L}_{\mathrm{A}}$ as in Chapter 3, show that for each $n \in \mathbb{N}$

$$\{\ulcorner\varphi\urcorner \mid \varphi \text{ is a (strict) } \Sigma_n \text{ formula}\}$$

is recursive, and hence the theories $I\Sigma_n$, $B\Sigma_n$, $I\Pi_n$, Coll_n, etc. are recursively axiomatized.

(b) Assume that there exist sets $A \subseteq \mathbb{N}$ that are r.e. but not recursive. (This will be proved in Chapter 9.) Using Corollaries 3.4 and 3.5 show that there is no Σ_1 formula $\psi(x)$, such that, for all $\mathscr{L}_{\mathrm{A}}$-formulas $\varphi(x)$, $\mathbb{N} \models \psi(\ulcorner\varphi(x)\urcorner)$ iff $\varphi(x)$ is $\Sigma_1(\mathbb{N})$. (*Hint:* Let $A = \{n \in \mathbb{N} \mid \mathbb{N} \models \theta(n)\}$, where $\theta(x)$ is Σ_1 and consider $\neg\psi(\ulcorner\neg\theta\underline{n}) \vee \neg\theta(x)\urcorner)$ as n varies over $\mathbb{N}$.)

7.2 Define formula classes Σ_n^+, Π_n^+ as follows: $\Sigma_0^+ = \Pi_0^+ = \Delta_0$; for all $n \in \mathbb{N}$ Σ_{n+1}^+ is the least class of $\mathscr{L}_{\mathrm{A}}$-formulas containing Π_n^+ and closed under conjunctions, disjunctions, bounded quantification, and (unbounded) existential quantification; Π_{n+1}^+ is the least class containing Σ_n^+ and closed conjunctions, disjunctions, bounded quantification and (unbounded) universal quantification.

Show that for all $n \in \mathbb{N}$ and for all formulas $\theta(\bar{x}) \in \Sigma_n^+$ (respectively Π_n^+) there is a formula $\tilde{\theta}(\bar{x}) \in \Sigma_n$ (respectively Π_n) such that

$$PA^- + \mathrm{Coll} \vdash \forall \bar{x}(\theta(\bar{x}) \leftrightarrow \tilde{\theta}(\bar{x})).$$

Show that the function f: $\ulcorner\theta(\bar{x})\urcorner \mapsto \ulcorner\tilde{\theta}(\bar{x})\urcorner$ may be taken to be recursive.

7.3* (a) Given that $B\Pi_n$ is $I\Delta_0 + \{B\theta \mid \theta \in \Pi_n\}$, show that $B\Pi_n$ is equivalent to $B\Sigma_{n+1}$ for all $n \geqslant 0$.

(b) If $L\Gamma$ denotes the theory consisting of PA^- together with all instances $L\theta$ of the least number principle for $\theta \in \Gamma$, show that $L\Sigma_n \vdash I\Pi_n$ and $L\Pi_n \vdash L\Sigma_n$ for all $n \geq 0$.
(c) Modify the proof of Proposition 4.2 to show that $I\Pi_n \vdash L\Sigma_n$ and (using Propositions 7.1 and 7.2) that $I\Sigma_n \vdash L\Pi_n$.
(d) Deduce that the following implications between the theories $I\Sigma_n$, $L\Sigma_n$, $B\Sigma_n$, etc. hold for all $n \geq 0$:

$$\begin{array}{l} I\Sigma_{n+1} \\ \Downarrow \\ B\Sigma_{n+1} \Leftrightarrow B\Pi_n \\ \Downarrow \\ I\Sigma_n \Leftrightarrow I\Pi_n \Leftrightarrow L\Sigma_n \Leftrightarrow L\Pi_n. \end{array}$$

7.4* Verify the following version of the Tarski–Vaught test for Σ_n-elementary extensions:

$M \prec_{\Sigma_n} N$ iff $M \prec_{\Delta_0} N$ and, for all $\bar{a} \in M$ and for all $\mathscr{L}_A$-formulas $\psi(\bar{x}, \bar{a})$, if $\exists \bar{x}\psi(\bar{x}; \bar{a})$ is Σ_n and $N \models \exists \bar{x}\psi(\bar{x}, \bar{a})$, then $N \models \psi(\bar{b}, \bar{a})$ for some $\bar{b} \in M$.

7.2 COFINAL EXTENSIONS

Collection is a powerful tool in the model theory of PA. We will illustrate this here by studying *cofinal extensions.*

If $M \subseteq N$ are models of PA^- we say that M is *cofinal* in N, N is a *cofinal* extension of M (in symbols: $M \subseteq_{cf} N$) iff $\forall a \in N \exists b \in M \; (N \models b \geq a)$. Thus $M \subseteq_{cf} N)$ iff there are elements of M arbitrarily high up N.

Example. If $M \models PA^-$ is nonstandard, let $\mathscr{L}_{M,c}$ be the language obtained by adding constants for each $a \in M$ and a further new constant c. Let $b \in M$ be nonstandard. Then by compactness the theory

$$\{\varphi(\bar{a}) \mid \bar{a} \in M \models \varphi(\bar{a}), \varphi \text{ an } \mathscr{L}_A \text{ formula}\} \cup \{c \neq a \mid a \in M\} \cup \{c < b\}$$

has a model K which (by considering the map sending each $a \in M$ to the element realizing the constant corresponding to a) we may assume to be an elementary extension of M. Let $N \subseteq K$ be the initial segment with domain $\{d \in K \mid K \models d < a\}$ for some $a \in M\}$. It is easy to check that $M \subseteq_{cf} N$ and N is an $\mathscr{L}_A$-structure. By Exercise 2.11, $N \models PA^-$, and since $c \in N \setminus M$, N is a proper cofinal extension of M.

On the other hand Theorem 2.2 shows that the standard model $\mathbb{N}$ has no proper cofinal extensions satisfying PA^-. The question that we will consider now is whether every nonstandard model of PA has a proper cofinal extension satisfying PA. The collection axioms enter the discussion via the following proposition.

Proposition 7.6. Suppose $M \subseteq_{cf} N$ are models of PA^- and $M \prec_{\Delta_0} N$. Suppose further that both M and N satisfy the collection axioms. Then $M \prec N$.

Proof. We show that $M \prec N$ by induction on the complexity of formulas as measured by their position in the arithmetic hierarchy. Suppose $M \prec_{\Sigma_n} N$ (which is true for $n=0$ by assumption). We shall show that whenever $\theta(\bar{x}, \bar{y})$ is Π_n, $\bar{a} \in M$ and $N \models \exists \bar{x} \theta(\bar{x}, \bar{a})$, then $M \models \exists \bar{x} \theta(\bar{x}, \bar{a})$, which clearly suffices (by Exercise 7.4) to show that $M \prec_{\Sigma_{n+1}} N$.

If $\theta \in \Pi_n$, $N \models \exists \bar{x} \theta(\bar{x}, \bar{a})$, and $\bar{a} \in M$, then since $N \supseteq_{\mathrm{cf}} M$ we have $N \models \exists \bar{x} < b\, \theta(x, \bar{a})$ for some $b \in M$. Now the formula $\exists \bar{x} < b \theta(\bar{x}, \bar{a})$ is $\Pi_n(\mathrm{Coll})$ by Proposition 7.1, that is

$$\mathrm{Coll} \vdash \forall \bar{y}, z(\exists \bar{x} < z \theta(\bar{x}, \bar{y}) \leftrightarrow \chi(\bar{y}, z))$$

for some Π_n formula $\chi(\bar{y}, z)$. Thus

$$N \models \chi(\bar{a}, b)$$

so $M \models \chi(\bar{a}, b)$ (since if $M \models \neg\chi(\bar{a}, b)$ then $N \models \neg\chi(\bar{a}, b)$ as $M \prec_{\Sigma_n} N$ and $\neg\chi$ is Σ_n), hence $M \models \exists \bar{x} < b \theta(\bar{x}, \bar{a})$, since $M \models \mathrm{Coll}$. $\square$

Proposition 7.6 works because both M and N satisfy collection. If M satisfies PA the same result holds but in this case the assumption that $N \models \mathrm{Coll}$ is actually unnecessary, as observed by Gaifman (1972).

THEOREM 7.7. Suppose $M \subseteq_{\mathrm{cf}} N$ are models of PA^-. Suppose further that $M \prec_{\Delta_0} N$ and that $M \models PA$. Then $M \prec N$, and hence $N \models PA$ also.

Proof. We first show that $M \prec_{\Sigma_2} N$. It is sufficient to show that if $\bar{a} \in M$ and $\theta(\bar{x}) \in \Sigma_2$, then $N \models \theta(\bar{a}) \Rightarrow M \models \theta(\bar{a})$, for clearly, if $\bar{a} \in M \models \psi(\bar{a})$ with $\psi(\bar{a}) \in \Sigma_2$ of the form $\exists \bar{x} \varphi(\bar{a}, \bar{x})$ and $\varphi(\bar{a}, \bar{x}) \in \Pi_1$, then $M \models \varphi(\bar{a}, \bar{b})$ for some $\bar{b} \in M$, $\neg\varphi(\bar{a}, \bar{b})$ is Σ_2, so $N \not\models \neg\varphi(\bar{a}, \bar{b})$ i.e. $N \models \varphi(\bar{a}, \bar{b})$.

So suppose $\theta(\bar{x})$ is $\exists \bar{y} \forall \bar{z} \psi(\bar{x}, \bar{y}, \bar{z})$ with $\psi \in \Delta_0$ and $\bar{a} \in M$, $N \models \theta(\bar{a})$. Then for some $b \in M$ $N \models \exists \bar{y} < b \forall \bar{z} \psi(\bar{a}, \bar{y}, \bar{z})$ since $M \subseteq_{\mathrm{cf}} N$, hence for all $c \in M$ $N \models \exists \bar{y} < b \forall \bar{z} < c \psi(\bar{a}, \bar{y}, \bar{z})$. As ψ is Δ_0 and $N \succ_{\Delta_0} M$, this means that

$$M \models \forall w \exists \bar{y} < b \forall \bar{z} < w \psi(\bar{a}, \bar{y}, \bar{z})$$

and hence by collection in M

$$M \models \exists \bar{y} < b \forall \bar{z} \psi(\bar{a}, \bar{y}, \bar{z})$$

as required.

We will now show that, for all $n \geq 2$, $M \prec_{\Sigma_n} N \Rightarrow M \prec_{\Sigma_{n+1}} N$, completing the proof. (This is where we need the extra assumption that M satisfies induction, and not just collection.) Rather than showing that, for all $\theta(\bar{a}) \in \Sigma_{n+1}$ with $\bar{a} \in M$, $N \models \theta(\bar{a}) \Rightarrow M \models \theta(\bar{a})$, we shall prove the equivalent assertion that, for all $\bar{a} \in M$ and $\psi(\bar{a}) \in \Pi_{n+1}$, $M \models \psi(\bar{a}) \Rightarrow N \models \psi(\bar{a})$.

So suppose $\bar{a} \in M$, $\psi(\bar{a})$ is $\forall y \exists z \varphi(\bar{a}, y, z)$ with $\varphi \in \Pi_{n-1}$. (We may assume y, z are both single variables, using the pairing function $\langle x_1, x_2 \rangle = y$ which has the required properties stated on p. 81 in both M and N since $M \prec_{\Sigma_2} N$.) Let $b \in M$ be arbitrary. We shall show that

$$M \vDash \forall y < b \exists z \varphi(\bar{a}, y, z) \Rightarrow N \vDash \forall y < b \exists z \varphi(\bar{a}, y, z)$$

which suffices since $M \subseteq_{\mathrm{cf}} N$. But if $M \vDash \forall y < b \exists z \varphi(\bar{a}, y, z)$ then by induction in M we have

$$M \vDash \exists w \forall y < b \forall z (z = (w)_y \rightarrow \varphi(\bar{a}, y, z))$$

hence (since this formula is Σ_n and $M \prec_{\Sigma_n} N$)

$$N \vDash \exists w \forall y < b \forall z (z = (w)_y \rightarrow \varphi(\bar{a}, y, z)).$$

Also, the sentence $\forall y, w \exists z (z = (w)_y)$ is Π_2 and true in M, hence it is true in N since $M \prec_{\Sigma_2} N$, so it follows that

$$N \vDash \forall y < b \exists z \varphi(\bar{a}, y, z)$$

as required. $\square$

This result can be strengthened even further: we say an $\mathscr{L}_A$-formula is $\exists_1$ or *existential* iff it is of the form $\exists \bar{y} \theta(\bar{x}, \bar{y})$ with θ quantifier-free. A very beautiful and deep result due to Matijasevič, J. Robinson, Davis, and Putnam in answer to Hilbert's 10th problem shows that every Σ_1 formula $\varphi(\bar{x})$ is equivalent (in $\mathbb{N}$) to a $\exists_1$ formula $\psi(\bar{x})$. This can be sharpened slightly to give

RESULT 7.8. (The MRDP theorem.) For every Σ_1 formula $\varphi(\bar{x})$ of $\mathscr{L}_A$ there is an $\exists_1$ formula $\psi(\bar{x})$ of $\mathscr{L}_A$ such that

$$PA \vdash \forall \bar{x} (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})). \qquad \square$$

A proof of the MRDP theorem is beyond the scope of this book. (The reader is encouraged to read one of the accounts in Manin 1977, Bell and Machover 1977 or Davis 1973.) The consequence of Result 7.8 that interests us here is that, if $M \subseteq N$ are *any* models of PA, then this extension is automatically Δ_0-elementary, for if $\bar{a} \in M$ and $\theta(\bar{x})$ is Δ_0 then both $\theta(\bar{x})$ and $\neg\theta(\bar{x})$ are (trivially) Σ_1, so by the MRDP theorem there are quantifier-free fromulas $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{z})$ such that

$$PA \vdash \forall \bar{x} (\theta(\bar{x}) \leftrightarrow \exists \bar{y} \varphi(\bar{x}, \bar{y}))$$

and

$$PA \vdash \forall \bar{x}(\neg\theta(\bar{x}) \leftrightarrow \exists \bar{z}\psi(\bar{x}, \bar{y})).$$

Thus if $M \models \theta(\bar{a})$, then $M \models \varphi(\bar{a}, \bar{b})$ for some $\bar{b} \in M$, hence $N \models \varphi(\bar{a}, \bar{b})$, hence $N \models \theta(\bar{a})$, since $N \models PA$ also. Similarly $M \models \neg\theta(\bar{a}) \Rightarrow \exists \bar{c} \in M \models \psi(\bar{a}, \bar{c}) \Rightarrow \exists \bar{c} \in N \models \psi(\bar{a}, \bar{c}) \Rightarrow N \models \neg\theta(\bar{a})$. So $M \prec_{\Delta_0} N$. This gives us

THEOREM 7.9 (Gaifman 1972). Let $M \models PA$ and $M \subseteq_{cf} N \models PA^-$. Then the following are equivalent:
(a) $M \prec_{\Delta_0} N$;
(b) $N \models PA$;
(c) $M \prec N$. □

COROLLARY 7.10 (Gaifman's splitting theorem). If $M \subseteq N$ models of PA then there is a unique $K \models PA$ such that $M \subseteq_{cf} K \subseteq_e N$; and moreover $M \prec K$.

Proof. The choice of the K is forced. It is

$$K = \{a \in N \mid N \models a < b \quad \text{for some } b \in M\},$$

and clearly $M \subseteq_{cf} K$. To show that $K \models PA$ it is sufficient to prove $M \prec_{\Delta_0} K$; but $M \prec_{\Delta_0} N$ by the MRDP theorem, and $K \prec_{\Delta_0} N$ since $K \subseteq_e N$, hence if $\theta(\bar{x}) \in \Delta_0$ and $\bar{a} \in M$, then

$$M \models \theta(\bar{a}) \text{ iff } N \models \theta(\bar{a}) \text{ iff } K \models \theta(\bar{a})$$

so $M \prec_{\Delta_0} K$ as required. □

We can now give the solution to our question about the existence of proper cofinal extensions at the beginning of this section.

COROLLARY 7.11. Any nonstandard model $M \models PA$ has a proper elementary cofinal extension N.

Proof. Following the argument in the example on p. 86, let $K \succ M$ with some element $c \in K \setminus M$ satisfying $K \models c < b$ for some nonstandard $b \in M$. Then set

$$N = \{d \in K \mid K \models d < a \text{ for some } a \in M\};$$

by the proof of Corollary 7.10 $N \succ_{\Delta_0} M$ (although the use of the MRDP theorem can be avoided here since we chose $K \succ M$, see Exercise 7.5) and so by Theorem 7.7, $M \prec N \models PA$. □

Exercises for Section 7.2

7.5 Let M, K, N be models of PA^- such that $M \prec K \supseteq_e N \supseteq M$. Show that $M \prec_{\Delta_0} N$, and hence the use of the MRDP theorem in Corollary 7.11 can be eliminated.

7.6 Show that Corollary 7.10 is actually *equivalent* to Result 7.8 in the following way. Suppose Result 7.8 were false. Then:

(a) there would be a Δ_0 formula $\theta(\bar{x})$ that is not equivalent in PA to any $\forall_1$ formula (i.e. one of the form $\forall \bar{y} \psi(\bar{x}, \bar{y})$ with $\psi(\bar{x}, y)$ quantifier-free);

(b) hence the theory

$$T = PA + \{\varphi(\bar{c}) \mid \varphi \text{ is } \forall_1 \text{ and } PA + \theta(\bar{c}) \vdash \varphi(\bar{c})\} + \neg\theta(\bar{c})$$

(in the language $\mathscr{L}_A$ expanded by adding new constant symbols $\bar{c}$) is consistent;

(c) for any $M \vDash T$, by considering the theory

$$S = PA + \theta(\bar{c}) + \{\chi(\bar{a}, \bar{c}) \mid \chi \text{ quantifier-free}, \bar{a} \in M \vDash \psi(\bar{a}, \bar{c})\}$$

in $\mathscr{L}_A \cup M \cup \{\bar{c}\}$, there exists $N \supseteq M$ with $N \vDash PA + \theta(\bar{c})$, and hence for no K satisfying $M \subseteq_{cf} K \subseteq_e N$ is $M \prec K$.

7.7 (a) Let $M \vDash PA^-$ and let R be the corresponding discretely ordered ring. By considering the non-negative elements of $R[X]$ for some suitable order, show that M has a proper end-extension $K \vDash PA^-$.

(b) Let $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ be models of PA^-. Show that $M = \bigcup_i M_i$ is also a model of PA^-.

(c) A model $M \vDash PA^-$ is said to be *ω_1-like* iff $\text{card}(M) = \omega_1$ and for all $a \in M$ $\text{card}\{b \in M \mid M \vDash b < a\} \leqslant \aleph_0$. By constructing a suitable chain $(M_\alpha)_{\alpha < \omega_1}$ of models of PA^- (where ω_1 is the first uncountable ordinal and $\alpha < \beta < \omega_1 \Rightarrow M_\alpha \subseteq_e M_\beta$) using (a) and (b) above, show that any countable model $M \vDash PA^-$ has an ω_1-like end-extension $K \vDash PA^-$.

(d) Show that every ω_1-like $K \vDash PA^-$ also satisfies Coll.

(e) By considering Σ_1 sentences true in the model $\mathbb{Z}[X]^+ \vDash PA^-$ show that $PA^- + \text{Coll} \nvdash PA$.

7.8 Show that the collection axioms are equivalent (in PA^-) to the scheme

$$\forall \bar{a}, b[\forall w \exists x > w \exists y < b\theta(x, y, \bar{a}) \to \exists y < b \forall w \exists x > w\theta(x, y, \bar{a}))$$

over all $\mathscr{L}_A$-formulas $\theta(x, y, \bar{a})$.

7.9 Let $\mathscr{L}$ extend the language $\mathscr{L}_A$, and let $M \prec N$ be $\mathscr{L}$-structures satisfying PA^-, such that $M \subseteq_e N$. If $M \neq N$, show that M and N satisfy all collection axioms $B\varphi$ for all $\mathscr{L}$-formulas φ.

8

Prime models

boss the other day
i heard an
ant conversing
with a flea
small talk i said
disgustedly
and went away
from there

Don Marquis
From 'certain maxims of archy', *Archy and Mehitabel*, 1934.

If T is a complete consistent extension of PA there is a 'smallest' model of T having some very pleasant features. This chapter is about these models and the various ways in which they can be regarded as 'small'. Section 8.2 contains an application (the famous MacDowell–Specker theorem) showing that any countable model of PA has an elementary end-extension.

8.1 DEFINABLE ELEMENTS

Let $M \models PA$ and let $A \subseteq M$. An element $b \in M$ is *definable in M over A* iff there is an $\mathcal{L}_A$-formula $\theta(x, \bar{y})$ and a tuple $\bar{a} \in A$ such that $M \models \exists! x \theta(x, \bar{a})$ and b is this unique element of M, i.e. $M \models \theta(b, \bar{a})$. $K(M; A)$ denotes the set of all elements of M that are definable over A. If A is empty we denote $K(M; A)$ by just $K(M)$, and if $A = \{a_1, \ldots, a_n\}$ is finite we denote $K(M; A)$ by $K(M; a_1, \ldots, a_n)$.

THEOREM 8.1. If $M \models PA$ and $A \subseteq M$ then $A \subseteq K(M; A) \prec M$, and so $K(M; A)$ satisfies PA too.

Proof. Clearly each $a \in A$ is definable over A by the formula $x = a$. If $x, y \in K(M; A)$ are defined by the formulas $\eta(x, \bar{a})$ and $\xi(y, \bar{b})$ respectively (where $\bar{a}, \bar{b} \in A$), then $x + y$ and $x \cdot y$ are defined by

$$\exists u, v(\eta(u, \bar{a}) \wedge \xi(v, \bar{b}) \wedge z = u + v)$$

and

$$\exists u, v(\eta(u, \bar{a}) \wedge \xi(v, \bar{b}) \wedge z = u \cdot v)$$

respectively, so $K(M; A)$ can be regarded as a substructure of M.

To show $K(M;A) \prec M$ we use the Tarski–Vaught test, showing that whenever $M \vDash \exists x \varphi(x, \bar{c})$ with $\bar{c} \in K(M;A)$ then $M \vDash \varphi(d, \bar{c})$ for some $d \in K(M;A)$. Suppose $\bar{c} = c_1, \ldots, c_n$ and each c_i is defined by the formula $\eta_i(x, \bar{a})$, where the parameters $\bar{a}$ are in A. Then

$$M \vDash \exists x, \bar{y}(\bigwedge_{i=1}^{n} \eta_i(y_i, \bar{a}) \wedge \varphi(x, \bar{y}))$$

so by the least-number principle in M

$$M \vDash \exists x\, [(\exists \bar{y} \bigwedge_{i=1}^{n} \eta_i(y_i, \bar{a}) \wedge \varphi(x, \bar{y})) \wedge \\ \forall z < x \forall \bar{w}(\bigwedge_{i=1}^{n} \eta_i(w_i, \bar{a}) \rightarrow \neg \varphi(z, \bar{w}))]$$

and the formula in square brackets here clearly defines an element $d \in K(M;A)$ satisfying $M \vDash \varphi(d, \bar{c})$, as required. □

For each complete consistent theory T extending PA, let $M \vDash T$ be an arbitrary model and let $K_T = K(M)$. K_T is called the *prime model* for T. The definition of K_T only depends on T (at least, up to isomorphism), and not on our choice of the model M, as the next theorem shows.

THEOREM 8.2. Let T be a complete consistent extension of PA and let N be a model of T. Then there is a unique elementary embedding $K_T \hookrightarrow N$, and moreover the image of this embedding is $K(N)$.

Proof. Suppose $K_T = K(M)$ where $M \vDash T$, and suppose $N \vDash T$. Then, since T is complete and $K_T \vDash T$ (by Theorem 8.1), we have

$$N \vDash \exists ! x \theta(x) \text{ iff } T \vdash \exists ! x \theta(x) \text{ iff } K_T \vDash \exists ! x \theta(x)$$

for any $\mathscr{L}_A$-formula $\theta(x)$. Now for each $a \in K_T$ let $\theta_a(x)$ be a formula defining a in M (and hence also in K_T), and define $h\colon K_T \rightarrow N$ by

$$h(a) = \text{the unique element of } N \text{ satisfying } \theta_a(x).$$

h is an embedding (i.e. it is one-to-one and preserves $0, 1, +, \cdot$, and $<$) since if $a_1 \neq a_2$ in K_T then T proves

$$\forall x, y(\theta_{a_1}(x) \wedge \theta_{a_2}(y) \rightarrow x \neq y),$$

so $h(a_1) \neq h(a_2)$, and if $a_1 + a_2 = a_3$ in K_T then T proves

$$\forall x, y, z(\theta_{a_1}(x) \wedge \theta_{a_2}(y) \wedge \theta_{a_3}(z) \rightarrow x + y = z),$$

so $h(a_1)+h(a_2)=h(a_3)$ in N, and similarly for $\cdot$ and $<$. Clearly the image of h is contained in $K(N)$, but in fact the image of h is *exactly* $K(N)$, since if $b \in K(N)$ is defined in N by $\varphi(x)$ then $N \vDash \exists! x\varphi(x)$ so $T \vdash \exists! x\varphi(x)$ and $K_T \vDash \exists! x\varphi(x)$, and hence for some $a \in K_T$, $K_T \vDash \varphi(a)$ and $T \vdash \forall x, y(\varphi(x) \wedge \theta_a(y) \to x=y)$, thus $b=h(a)$. Thus h is an embedding $h: K_T \to N$ with image $K(N) < N$, so it is an elementary embedding by Theorem 8.1.

To prove that h above is unique, let $k: K_T \to N$ be an arbitrary elementary embedding. Then for each $a \in K_T$, $K_T \vDash \theta_a(a)$ and so $N \vDash \theta_a(k(a))$ since k is an elementary embedding. Moreover $T \vdash \exists! x\theta_a(x)$ and $K_T, N \vDash T$ so this tells us that $k(a)=h(a)$ for all a. □

In particular Theorem 8.2 tells us that the definition of K_T did not depend on M, for it shows that there is a canonical isomorphism $K(N) \cong K(M)$ for all models $N, M \vDash T$, where T is a complete consistent extension of PA.

It is obvious that K_T should be regarded as being 'small'. The next three corollaries are devoted to saying this in more precise ways.

COROLLARY 8.3. Let T be a complete consistent extension of PA. Then K_T is *minimal*, i.e. it has no proper elementary substructures.

Proof. Let T be as above, and for each $a \in K_T$ let $\theta_a(x)$ be an $\mathscr{L}_A$-formula that defines it in K_T. If $N < K_T$ then Theorem 8.2 shows that there is a canonical embedding $h: K_T \to N$ given by $h(a)=$ the unique element of N satisfying $\theta_a(x)$. But if $b \in N \vDash \theta_a(b)$, then $K_T \vDash \theta_a(b)$, so $b=a$ and h is the identity function, so $N=K_T$. □

COROLLARY 8.4. Let T be a complete consistent extension of PA. Then K_T is *rigid*, i.e., it has no non-trivial automorphisms.

Proof. If h is an automorphism of K_T (i.e., h is an embedding $K_T \to K_T$ that is also onto), then h must be the unique elementary embedding $K_T \to K_T$ of Theorem 8.2, i.e., h must be the identity function. □

We should also note that if $T=\mathrm{Th}(\mathbb{N})$, then K_T is the standard model $\mathbb{N}$, since $\mathbb{N} \vDash T$ and has no proper substructures.

The last corollary to Theorems 8.1 and 8.2 is a version of the well-known *omitting types theorem*, and is perhaps the most useful measure of how 'small' K_T and the associated $K(M; A)$s are. To state the corollary we must first introduce some very important definitions.

If T is a theory, a *type* over T is a set $p(\bar{x})$ of formulas $\varphi(\bar{x})$ containing finitely many free-variables $\bar{x}$, such that $T+\{\varphi(\bar{c}) \mid \varphi(\bar{x}) \in p(\bar{x})\}$ is consistent, where $\bar{c}$ is a tuple of new constants. (We shall write this more simply by saying '$T+p(\bar{x})$ is consistent', thinking of the $\bar{x}$ as constants rather than variables when convenient.) $p(\bar{x})$ is a *complete* type over T iff $T+p(\bar{x})$ is a complete theory, that is if for all $\theta(\bar{x})$, either $T+p(\bar{x})\vdash\theta(\bar{x})$ or $T+p(\bar{x})\vdash\neg\theta(\bar{x})$. The type $p(\bar{x})$ is *principal* iff there is a single formula $\psi(\bar{x})$ such that $T+p(\bar{x})\vdash\psi(\bar{x})$ and $T\vdash\forall\bar{x}(\psi(\bar{x})\rightarrow\varphi(\bar{x}))$ for all $\varphi(\bar{x})\in p(\bar{x})$; thus $p(\bar{x})$ is principal iff it is 'generated' by a single formula $\psi(\bar{x})$. Finally, if $M\vDash T$ we say that a type $p(\bar{x})$ over T is *realized* in M iff there is $\bar{a}\in M$ s.t. $M\vDash\varphi(\bar{a})$ for all $\varphi(\bar{x})\in p(\bar{x})$. Otherwise we say that the model M *omits* $p(\bar{x})$.

It is easy to see from the completeness theorem that, if $p(\bar{x})$ is a type over the theory T, then T has a model realizing $p(\bar{x})$. Similarly if $p_1(\bar{x}), p_2(\bar{x}), \ldots$ are all types over the theory $\mathrm{Th}(M)$ of a model M, then there is an elementary extension $N>M$ realizing all the types $p_i(\bar{x})$. In general it is much harder to show that a type can be omitted. We have, however:

Corollary 8.5 (Omitting types). Let $\mathscr{L}_C$ be an expansion of the language $\mathscr{L}_A$ of arithmetic obtained by adding a new (not necessarily countable) set of constants C. Let T be a complete consistent $\mathscr{L}_C$-theory containing PA. Then there is a model K of T such that, for each complete type $p(\bar{x})$ over T, K realizes $p(\bar{x})$ iff $p(\bar{x})$ is principal.

Proof. Let $M\vDash T$ and let $A\subseteq M$ be the set of elements of M realizing the constants C. Let $K=K(M;A)$. Clearly, since $A\subseteq K\prec M$, $K\vDash T$. Let $p(\bar{x})$ be a complete type over T. Suppose first that $p(\bar{x})$ is realized by $\bar{a}\in K$. Let $\bar{a}=a_1,\ldots,a_n$, where each a_i is defined in M (and hence also in K) by the formula $\eta_i(x,\bar{c})$, where $\bar{c}\in A$. Then

$$K\vDash\exists\bar{x}\textstyle\bigwedge_{i=1}^{n}\eta_i(x_i,\bar{c})$$

and

$$K\vDash\forall\bar{x}(\textstyle\bigwedge_{i=1}^{n}\eta_i(x_i,\bar{c})\rightarrow\varphi(\bar{x}))$$

for all $\varphi(\bar{x})\in p(\bar{x})$, and moreover $T+p(\bar{x})\vdash\bigwedge_{i=1}^{n}\eta_i(x_i,\bar{c})$ since $p(\bar{x})$ is complete. Thus $p(\bar{x})$ is principal, since $K\vDash T$ and T is complete.

Conversely, suppose $p(\bar{x})$ is principal and $\theta(\bar{x})$ satisfies $T+p(\bar{x})\vdash\theta(\bar{x})$ and $T\vdash\forall\bar{x}(\theta(\bar{x})\rightarrow\varphi(\bar{x}))$ for all $\varphi(\bar{x})\in p(\bar{x})$. Then $T\vdash\exists\bar{x}\theta(\bar{x})$, since T is complete and $T+p(\bar{x})$ is consistent, and $K\vDash\exists\bar{x}\theta(\bar{x})$, hence $p(\bar{x})$ is realized in K. $\square$

Notice that, if $p(\bar{x})$ is a principal type over a complete theory T, then $p(\bar{x})$ is realized in all models $M\vDash T$. The moral to Corollary 8.5 is that the models $K(M;A)$ of PA realize the smallest possible set of types.

The omitting types theorem is better known in its more general version: if T is any complete theory in a countable language $\mathscr{L}$ and $p_1(\bar{x}), p_2(\bar{x}), \ldots$ are *countably many* non-principal types over T, then T has a model omitting each of $p_1(\bar{x})$, $p_2(x)$, . . . (See. eg. Chang and Keisler 1973 for a proof.) It is instructive to contrast this with our Corollary 8.5. The main difference is that, in 8.5, we are able to omit *uncountably many* types, whereas the general omitting types theorem only allows us to omit at most *countably* many types at one go. The extra power of Corollary 8.5 is derived from PA and its very special feature of having these 'pointwise definable models' $K(M; A)$.

Exercises for Section 8.1

8.1 'Certain natural numbers are interesting. For example: zero is the smallest natural number; one is the only non-zero natural number that is equal to its square; two is the only non-zero natural number whose square is twice itself; three is the smallest odd prime; and so on. If uninteresting numbers exist, consider

$$n = \text{the least uninteresting number}$$

and what could be more interesting than n?'

Interpret 'interesting' by 'definable in $\mathscr{L}_A$' and show that, for any model $M \vDash PA$, there is an $\mathscr{L}_A$-formula defining exactly the set of interesting numbers in M iff $M = K(M)$.

8.2 Let $M \vDash PA$ and let $A \subseteq M$. Define $I(M; A)$ to be the substructure with domain

$$\{x \in M \mid \exists a \in K(M; A) M \vDash x \leqslant a\}.$$

Write $I(M; a_1, \ldots, a_n)$ for $I(M; \{a_1, \ldots, a_n\})$ and $I(M)$ for $I(M; \varnothing)$.

(a) Use Theorem 7.7 to show that $K(M; A) \prec I(M; A)$, and hence deduce $I(M; A) \vDash PA$.

(b) Prove that $I(M; A) \prec M$.

(*Hint:* If $M \vDash \exists x \varphi(x, a)$ with $a \in I(M; A)$, suppose $a < b \in K(M; A)$ and b is defined by $\eta(y)$. Consider the formula

$$\exists z \forall u < b(\exists x \varphi(x, u) \rightarrow \varphi((z)_u, u)).)$$

(c) Use the compactness theorem to find $M \vDash PA$ for which $K(M) \neq I(M) \neq M$.

8.3 *Parameter-free PA* is the theory PA^- together with the scheme of induction

$$(\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1))) \rightarrow \forall x \varphi(x)$$

for all $\mathscr{L}_A$-formulas $\varphi(x)$ with only one free-variable. Parameter-free least-number principle (*LNP*) is PA^- together with

$$\exists x \varphi(x) \rightarrow \exists x(\varphi(x) \wedge \forall y < x \neg \varphi(y))$$

for all $\varphi(x)$ with only one free-variable.

(a) Show that parameter-free *PA* proves all axioms of parameter-free *LNP*.

(b) Let $M \models$ parameter-free PA. Using LNP show that $K(M) \prec M$ and that $K(M) \models PA$. Deduce that parameter-free PA and PA are equivalent.

8.4 A theory T in a countable language $\mathscr{L}$ extending the language $\mathscr{L}_A$ is said to be *Peano-like* if T extends PA and proves all instances of the induction axiom

$$\forall \bar{y}((\varphi(0,\bar{y}) \wedge \forall x(\varphi(x,\bar{y}) \rightarrow \varphi(x+1,\bar{y}))) \rightarrow \forall x \varphi(x,\bar{y}))$$

for all formulas $\varphi(x,\bar{y})$ of $\mathscr{L}$.

Using a modified version of the $K(M;A)$ construction, verify that Theorems 8.1, 8.2 and Corollaries 8.3, 8.4, and 8.5 hold for any Peano-like theory in place of PA.

8.5 If $M \models PA$ and $A \subseteq M$, $K^n(M; A)$ is the set of all Σ_n-definable elements of M with parameters from A (i.e., the defining formula must be a Σ_n formula). If $n \geqslant 1$ show that:

(a) $A \subseteq K^n(M;A) \prec_{\Sigma_n} M$;

(b) $K^n(M;\varnothing)$ is minimal and rigid;

(c) if $N \models \mathrm{Th}(K^n(M;\varnothing))$ then there is a unique elementary embedding

$$h\colon K^n(M;\ \varnothing) \rightarrow \mathrm{N}.$$

8.2 END-EXTENSIONS

In this section we will prove

THEOREM 8.6 (MacDowell and Specker 1961). Every model $M \models PA$ has a proper elementary end-extension.

The argument we use here to prove this theorem is due to Gaifman (1976) and gives extra information.

To motivate the ideas in the proof, notice that if $M \prec K$ with K an end-extension of M and $c \in K \setminus M$ then K would satisfy the theory

$$\{\varphi(\bar{a}) \mid \bar{a} \in M \models \varphi(\bar{a}),\ \varphi \text{ an } \mathscr{L}_A\text{-formula}\} \cup \{c > a \mid a \in M\}$$

in the language obtained by adding a constant c and, for each $a \in M$, a constant naming the element a. Moreover, since K is to be an end-extension, K has to omit each of the types

$$p_a(x) = \{x < a\} \cup \{x \neq b \mid b \in M \models b \leqslant a\}$$

for each $a \in M$. (Notice that there may be uncountably many such types $p_a(x)$ that we must omit.) In any case it is clear that K will have to be a 'small' model. We will in fact define $K = K(N;M \cup \{c\})$ for some suitable $c \in N > M$. Most of the work will be involved in constructing N and c.

It is easy to check that, were we to succeed in finding $K > M$ with $K \supseteq_e M$ and $c \in K \setminus M$, then for every formula $\varphi(x,\bar{y})$ and every $\bar{b} \in M$, $K \models \varphi(c,\bar{b})$ would imply $M \models \forall z \exists x(x > z \wedge \varphi(x,\bar{b}))$, i.e. 'there are unboundedly many x in M satisfying $\varphi(x,\bar{b})$'. This notion will be important, and we will abbreviate $\forall z\exists x(x>z\wedge\varphi(x,\bar{b}))$ to $\mathrm{Q}x\varphi(x,\bar{b})$. We also denote the language $\mathscr{L}_A$ expanded by adding constant symbols for each element of M by $\mathscr{L}_A(M)$ and we consider M as an $\mathscr{L}_A(M)$-structure in the obvious way.

LEMMA 8.7. With $\mathrm{Q}x\varphi(x)$ having the above meaning, suppose $\varphi(x)$ is an $\mathscr{L}_A(M)$-formula such that $M\vDash\mathrm{Q}x\varphi(x)$, and let $\theta(x,\bar{y})$ be an arbitrary $\mathscr{L}_A(M)$-formula. Then there is a third $\mathscr{L}_A(M)$-formula $\psi(x)$ such that:

(a) $M\vDash\mathrm{Q}x\psi(x)$;
(b) $M\vDash\forall x(\psi(x)\rightarrow\varphi(x))$;
(c) for all $\bar{a}\in M$,
either: $M\vDash\exists y\forall x(x>y\wedge\psi(x)\rightarrow\theta(x,\bar{a}))$
or: $M\vDash\exists y\forall x(x>y\wedge\psi(x)\rightarrow\neg\theta(x,\bar{a}))$.

Proof. To simplify things we may assume that the tuple $\bar{y}$ in $\theta(x,\bar{y})$ is actually a single variable. We can do this by using the function $\langle\bar{y}\rangle$ coding tuples, replacing $\theta(x,\bar{y})$ with $\forall\bar{w}(\langle\bar{w}\rangle=y\rightarrow\theta(x,\bar{w}))$.

The idea is to 'define' formulas $\psi_y(x)$ where $\psi_0(x)$ is $\varphi(x)$, and for each $y\in M$ (supposing that ψ_y has already been 'defined' such that $M\vDash\mathrm{Q}x\psi_y(x)$), then either $M\vDash\mathrm{Q}x[\psi_y(x)\wedge\theta(x,y)]$ or $M\vDash\mathrm{Q}x[\psi_y(x)\wedge\neg\theta(x,y)]$, and we 'define' $\psi_{y+1}(x)$ to be whichever of the two formulas in square brackets is true for unboundedly many x. Finally we 'define' $\psi(x)$ to be the formula that says

$$\exists y \text{ `}x\text{ is the }(y+1)\text{st element satisfying }\psi_y\text{'}.$$

It is necessary to show that $\psi(x)$ turns out to be a genuine $\mathscr{L}_A(M)$-formula with the three required properties; however, the intuitive reason why $\psi(x)$ works is this. $M\vDash\forall x(\psi_{y+1}(x)\rightarrow\psi_y(x))$ and $M\vDash\mathrm{Q}x\psi_y(x)$ for all y, so the $(y+2)$nd element satisfying ψ_{y+1} is greater than the $(y+1)$st element satisfying ψ_y, hence $M\vDash\mathrm{Q}x\psi(x)$. Also, for all $a\in M$, since either $M\vDash\forall x(\psi_y(x)\rightarrow\theta(x,a))$ for all $y>a$ or $M\vDash\forall x(\psi_y(x)\rightarrow\neg\theta(x,a))$ for all $y>a$, it is the case that whenever x is greater than the $(a+2)$nd element of M satisfying ψ_{a+1} and $M\vDash\psi(x)$ then the truth value of '$\theta(x,a)$' in M does not depend on x.

We must now make this argument precise. By results from Chapter 5, any element $s\in M$ may be considered as coding a sequence $(s)_0,(s)_1,\ldots$. We define $\chi(x,y,s)$ to be the $\mathscr{L}_A(M)$-formula

$$\varphi(x)\wedge\forall u<y[((s)_u=0\rightarrow\theta(x,u))\wedge$$
$$((s)_u\neq0\rightarrow\neg\theta(x,u))].$$

Intuitively $\chi(x,y,s)$ expresses the truth of $\varphi(x)\wedge\bigwedge_{u<y}(\neg)^{(s)_u}\theta(x,u)$ where

$(\neg)^{(s)_u}$ is empty if $(s)_u=0$ and $\neg$ otherwise. Next we define $\delta(x,y)$ to be

$$\exists s[\forall u<y(\mathsf{Q}z(\chi(z,u,s)\wedge\theta(z,u))\leftrightarrow(s)_u=0)\wedge\chi(x,y,s)]$$

and the idea is that $\delta(x,y)$ expresses '$\psi_y(x)$ is true'. Before we define $\psi(x)$ we state some properties of χ and δ.

Notice first that the truth value of $\chi(x,y,s)$ in M only depends on x,y and $(s)_0,\ldots,(s)_{y-1}$. Secondly, for all $x,y\in M$ there is $s\in M$ s.t.

$$M\vDash\forall u<y(\mathsf{Q}z(\chi(z,u,s)\wedge\theta(z,u))\leftrightarrow(s)_u=0).$$

This is proved by induction on u up to $y-1$. *If* s,s' both satisfy this formula then we can prove

$$M\vDash\forall u<y((s)_u=0\leftrightarrow(s')_u=0)$$

by an equally straightforward induction on u, and hence $M\vDash\forall u\leqslant y(\chi(x,u,s)\leftrightarrow\chi(x,u,s'))$.

We can now show that, for all $y\in M$, $M\vDash\mathsf{Q}x\delta(x,y)$. The proof is by induction on y. It is easy to check that $M\vDash\forall x(\delta(x,0)\leftrightarrow\varphi(x))$ so $M\vDash\mathsf{Q}x\delta(x,0)$. Now suppose $M\vDash\mathsf{Q}x\delta(x,y)$. Then either $M\vDash\mathsf{Q}x(\delta(x,y)\wedge\theta(x,y))$ or $M\vDash\mathsf{Q}x(\delta(x,y)\wedge\neg\theta(x,y))$ (for if neither is true then

$$M\vDash\exists w\forall x(\delta(x,y)\wedge\theta(x,y)\rightarrow x<w)$$

and

$$M\vDash\exists w'\forall x(\delta(x,y)\wedge\neg\theta(x,y)\rightarrow x<w')$$

so $M\vDash\neg\mathsf{Q}x\delta(x,y)$.) If $M\vDash\mathsf{Q}x(\delta(x,y)\wedge\theta(x,y))$ let $s'\in M$ satisfy

$$M\vDash\forall u<y((s)_u=(s')_u)\wedge(s')_y=0;$$

otherwise let $s'\in M$ satisfy

$$M\vDash\forall u<y((s)_u=(s')_u)\wedge(s')_y=1.$$

(Such s' exists by Lemma 5.8.) Now, since $(s)_u=(s')_u$ for all $u<y$, $M\vDash\forall x(\chi(x,y,s)\leftrightarrow\chi(x,y,s'))$ and so by our choice of $(s')_y$

$$M\vDash\forall x(\delta(x,y+1)\leftrightarrow\chi(x,y,s)\wedge(\neg)^{(s')_y}\theta(x,y))$$

hence $M\vDash\mathsf{Q}x\delta(x,y+1)$, as required.

We define $\gamma(x,y)$ to be

$$\begin{aligned}&\exists w[\delta((w)_0,y)\wedge\forall v<(w)_0\neg\delta(v,y)\wedge\\&\quad\forall u<y(\delta((w)_{u+1},y)\wedge\forall v<(w)_{u+1}(\delta(v,y)\rightarrow v\leqslant(w)_u))\wedge x=(w)_y]\end{aligned}$$

so $\gamma(x,y)$ expresses 'x is the $(y+1)$st element satisfying $\delta(x,y)$'. Since $\mathrm{Q}x\delta(x,y)$ holds in M for each y, we have

$$M \models \forall y \exists! x \gamma(x,y).$$

We set $\psi(x)$ to be the formula $\exists y \gamma(x,y)$.

From the definition of $\delta(x,y)$ it is clear that $M \models \forall x, y(\delta(x, y+1) \rightarrow \delta(x,y))$. It follows that if $M \models \gamma(x,y)$ there are at most $y+1$ elements $w \leqslant x$ s.t. $M \models \delta(w, y+1)$. Hence

$$M \models \forall x, y, z(\gamma(x,y) \wedge \gamma(z, y+1) \rightarrow z > x) \qquad (+)$$

and in particular $M \models \mathrm{Q}x \exists y \gamma(x,y)$. Thus (a) holds.

The second property that we require of ψ, that $M \models \forall x(\psi(x) \rightarrow \varphi(x))$, is easily seen to be true using $M \models \forall x(\delta(x,0) \leftrightarrow \varphi(0))$, $M \models \forall x, y(\delta(x, y+1) \rightarrow \delta(x,y))$ and induction.

Finally, to show (c), let $a \in M$ be arbitrary and let $y \in M$ satisfy $M \models \gamma(y, a)$. We must show that the truth value of $x > y \wedge \psi(x) \rightarrow \theta(x, a)$ is constant (i.e., independent of the choice of $x \in M$). But if $x > y$ and $M \models \psi(x)$ then $M \models \gamma(x, w)$ for some $w > a$, by $(+)$ and an induction, and hence $M \models \delta(x, w)$. But as we have already observed, if $s \in M$ satisfies

$$M \models \forall u \leqslant \bar{a}(\mathrm{Q}z(\chi(z,u,s) \wedge \theta(z,u)) \leftrightarrow (s)_u = 0)$$

then $(s)_0, (s)_1, \ldots, (s)_a$ is essentially unique, and for our $x \in M$ satisfying $M \models \gamma(x, w) \wedge w > a$ the truth value of $\theta(x, a)$ depends only on $(s)_a$, and hence is constant, as required. $\square$

Having obtained this lemma we return to the proof of Theorem 8.6. We use Lemma 8.7 to find all the first-order properties we require our $c \in K \setminus M$ to satisfy. To do this, enumerate all formulas in the language $\mathscr{L}_c = \mathscr{L}_\mathrm{A} \cup \{c\}$ (where c is the new constant symbol) by $\theta_0(c, \bar{y})$, $\theta_1(c, \bar{y}), \ldots, \theta_i(c, \bar{y}), \ldots$ $(i \in \mathbb{N})$. (The number of free-variables $\bar{y}$ in $\theta_i(c, \bar{y})$ will of course vary with i.) We construct a second sequence of $\mathscr{L}_\mathrm{A}$-formulas $\varphi_0(x), \varphi_1(x), \ldots$ (with only one free-variable x) such that $M \models \mathrm{Q}x\varphi_i(x)$ for all $i \in \mathbb{N}$.

The $\varphi_i(x)$s are constucted by first setting $\varphi_0(x)$ to be '$x = x$' (so $M \models \mathrm{Q}x\varphi_0(x)$ is automatically true). If $\varphi_i(x)$ is given satisfying $M \models \mathrm{Q}x\varphi_i(x)$ we may apply the lemma to obtain $\varphi_{i+1}(x)$ such that $M \models \mathrm{Q}x\varphi_{i+1}(x)$, $M \models \forall x(\varphi_{i+1}(x) \rightarrow \varphi_i(x))$ and for all $\bar{a} \in M$

either $\quad M \models \exists y \forall x(x > y \wedge \varphi_{i+1}(x) \rightarrow \theta_i(x, \bar{a}))$

or $\quad M \models \exists y \forall x(x > y \wedge \varphi_{i+1}(x) \rightarrow \neg\theta_i(x, \bar{a}))$.

This process is repeated until $\varphi_i(x)$ is defined for all $i \in \mathbb{N}$.

The idea now is to take c satisfying all the $\varphi_i(x)$, i.e., to consider the theory $T=$

$$\{\theta(\bar{a}) \mid M \vDash \theta(\bar{a}) \text{ and } \theta(\bar{a}) \text{ is an } \mathscr{L}_{\mathrm{A}}(M)\text{-sentence}\} \\ \cup \{c > a \mid a \in M\} \cup \{\varphi_i(c) \mid i \in \mathbb{N}\}$$

in the language $\mathscr{L}_{\mathrm{A}}(M) \cup \{c\}$. This theory is consistent since any finite fragment of it is satisfied by M, where c is interpreted as a sufficiently large element of M satisfying $\varphi_i(x)$ for large enough i. We now take any model $N \vDash T$ which, by the diagram lemma, we can regard as an elementary extension of M by identifying an element $a \in M$ with the element of N realizing the constant corresponding to a. The model K we are interested in is $K = K(N; M \cup \{c\}) \leq N$, and clearly $M \cup \{c\} \subseteq K$ and $M < K$, since $M \vDash \theta(\bar{a})$ iff $N \vDash \theta(\bar{a})$ iff $K \vDash \theta(\bar{a})$ for all $\mathscr{L}_{\mathrm{A}}$-formulas $\theta(\bar{x})$ and all $\bar{a} \in M$. To complete the proof of Theorem 8.6 we must show that K is an end-extension of M.

In fact, K is a *conservative extension* of M, meaning that for any $\bar{b} \in K$ and any $\mathscr{L}_{\mathrm{A}}$-formula $\theta(u, \bar{v})$ there is $\bar{a} \in M$ and $\psi(u, \bar{w})$ such that

$$\{u \in K \mid K \vDash \theta(u, \bar{b})\} \cap M = \{u \in M \mid M \vDash \psi(u, \bar{a})\},$$

i.e., every subset of M that is definable in K is already definable in M. To see this suppose $b_1, \ldots, b_n \in K$ where b_j is defined by the formula $\eta_j(v, a_1, \ldots, a_k, c)$ for each $j = 1, 2, \ldots, n$, $a_1, \ldots, a_k \in M$ and $\theta(x, v_1, v_2, \ldots, v_n)$ is a given $\mathscr{L}_{\mathrm{A}}$-formula. Then the formula

$$\exists v_1, \ldots, v_n \left(\bigwedge_{j=1}^{n} \eta_j(v_j, y_1, \ldots, y_k, x) \wedge \theta(y_0, \bar{v}) \right)$$

is $\theta_i(x, y_0, y_1, \ldots, y_k)$ for some $i \in \mathbb{N}$. We claim that, for all $d \in M$, $K \vDash \theta(d, b_1, \ldots, b_n)$ iff

$$M \vDash \exists w \forall x \left(x > w \wedge \varphi_{i+1}(x) \rightarrow \exists \bar{v} \left(\bigwedge_{j=1}^{n} \eta_j(v_j, \bar{a}, x) \wedge \theta(d, \bar{v}) \right) \right).$$

One direction is easy, for if $w \in M$ and

$$M \vDash \forall x \left(x > w \wedge \varphi_{i+1}(x) \rightarrow \exists \bar{v} \left(\bigwedge_j \eta_j(v_j, \bar{a}, x) \wedge \theta(d, \bar{v}) \right) \right),$$

then this holds in any elementary extension of M, and in particular it holds in K. Putting $x = c$ we get $K \vDash \theta(d, \bar{b})$. Conversely, by the construction of φ_{i+1},

either $\quad M \vDash \exists w \forall x (x > w \wedge \varphi_{i+1}(x) \rightarrow \theta_i(x, d, \bar{a}))$

or $\quad M \vDash \exists w \forall x (x > w \wedge \varphi_{i+1}(x) \rightarrow \neg \theta_i(x, d, \bar{a}))$.

and the second possibility cannot happen if $K \models \theta(d, \bar{b})$, since $M < K \models c > w \wedge \varphi_{i+1}(c) \wedge \theta_i(c, d, \bar{a})$ for all $w \in M$.

Finally we note that any conservative extension $K > M \models PA$ is an end-extension, since if $a \in M$, $b \in K$ with $b \leq a$, then the set $S = \{u \in K \mid K \models u \leq b\} \cap M$ would then be definable in M, $S = \{u \in M \mid M \models \psi(u, \bar{a})\}$ say, for some ψ and some $\bar{a} \in M$. But then S is a bounded definable non-empty subset of M, so has a greatest element, e (which is found by considering the least number principle for $\neg\psi(u, \bar{a})$). This greatest element must be equal to b (since if $e < b$ then $e+1 \in M$ and $e+1 \leq b$, so $e+1 \in S$) hence $b \in S \subseteq M$, as required.

Thus we have proved the following improvement of Theorem 8.6.

THEOREM 8.8 (Gaifman 1976). Every model of PA has a conservative extension. □

Exercise 8.7 examines the proof of this theorem in a little more detail. We conclude this chapter with a corollary on the existence of κ-like models of PA. If κ is an infinite cardinal, a model $M \models PA^-$ is said to be κ-like iff $\mathrm{card}(M) = \kappa$ and for all $a \in M$ $\mathrm{card}\{b \in M \mid M \models b < a\} < \kappa$. (This generalizes the notion of ω_1-like models in Exercise 7.7.)

COROLLARY 8.9. If κ is a cardinal, $\kappa > \aleph_0$, then every $M \models PA$ of cardinality $\mathrm{card}(M) < \kappa$ has a κ-like elementary end-extension.

Proof. Notice the model K constructed in the proof of Theorem 8.6 has the same cardinality as M. Identity κ with the least ordinal in 1–1 correspondence with a set of cardinality κ, and define a chain of models M indexed by ordinals $\alpha \leq \kappa$ as follows:

(a) $M_0 = M$;
(b) at successor stages, $M_{\alpha+1}$ is a proper elementary end-extension of M_α with $\mathrm{card}(M_{\alpha+1}) = \mathrm{card}(M_\alpha)$ as given in Theorem 8.6;
(c) at limit stages $\lambda \leq \kappa$, $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$.

By the lemma on the union of an elementary chain of models, $M_\alpha < M_\beta$ for all $\alpha < \beta \leq \kappa$, and obviously $M_\alpha \supseteq_e M_0$ for all $\alpha \leq \kappa$. Also $\mathrm{card}(M_\alpha) = \max(\mathrm{card}(M_0), \mathrm{card}(\alpha))$, so in particular if $\alpha < \kappa$ then $\mathrm{card}(M_\alpha) < \kappa$. It follows that M_κ is a κ-like elementary end-extension of M, since for each $a \in M$,

$$\mathrm{card}\{b \in M_\kappa \mid M_\kappa \models b < a\} \leq \mathrm{card}(M_\alpha) < \kappa$$

where $\alpha \neq \kappa$ is any ordinal large enough so that $a \in M_\alpha$. □

Exercises for Section 8.2

8.6 Let $M < N$ be models of PA^- where N contains an element a satisfying $N \models a > b$ for all $b \in M$ (but N is not necessarily an end-extension of M). Show that for all $\theta(x) \in \mathscr{L}_A(M)$

$$\exists c \in N(N \models c > b \text{ for all } b \in M \text{ and } N \models \theta(c))$$

holds iff $M \models Qx\theta(x)$.
(*Hint:* For right-to-left consider

$$I = \{x \in N \mid N \models x < b \text{ for some } b \in M\}$$

and use overspill.)
8.7 Let $K > M$ be the model constructed in the proof of Theorem 8.6. Show that the type of c is *definable over* M, i.e.,

for each $\mathscr{L}_A$-formula $\theta(x, \bar{y})$ there is an $\mathscr{L}_A$-formula $\psi(\bar{y})$ such that, for all $\bar{a} \in M$,

$$K \models \theta(c, \bar{a}) \Leftrightarrow M \models \psi(\bar{a}).$$

Use this to show that the type of every $d \in K \setminus M$ is definable over M.
8.8 Construct a non-conservative elementary end-extension of the standard model $\mathbb{N}$.
8.9$^+$ Let $\kappa > \aleph_0$ be a cardinal, and let M be a κ-like model of PA. Show that M satisfies the *second-order collecton axiom*

$$\forall X \forall t(\forall x < t \exists y \langle x, y\rangle \in X \rightarrow \exists z \forall x < t \exists y < z \langle x, y\rangle \in X) \qquad (*)$$

where X ranges over subsets of M. Hence deduce that there exists models of $PA + (*)$ of all possible cardinalities.

Show that, up to isomorphism, there is exactly one *countable* model of $PA + (*)$. (*Hint:* Let $S \subseteq \mathbb{N}$ be any set. Suppose $M \models PA + (*)$ is countable and let $a_0, a_1, \ldots \in M$ satisfy $M \models a_i < a_{i+1}$ for each i, and $\{a_i \mid i \in \mathbb{N}\} \subseteq_{cf} M$. Consider

$$X = \{\langle x, y\rangle \in M \mid M \models a_i < y \wedge \neg(p_i \mid x \leftrightarrow i \in S) \text{ some } i \in \mathbb{N}\}$$

where p_i is the ith prime, and deduce that if M is nonstandard then S is coded in M and hence $\operatorname{card}(M) \geqslant 2^{\aleph_0}$.]
8.10$^+$ If we only want to construct an end-extension of a *countable* model $M \models PA$ the tricky Lemma 8.7 can be avoided. Reconstruct the argument as follows.

Enumerate all $\mathscr{L}_A(M)$-formulas in one free-variable x as $\theta_0(x), \theta_1(x), \ldots$ (using the countability of M). Construct a sequence $\varphi_0(x), \varphi_1(x), \ldots$ of $\mathscr{L}_A(M)$-formulas by setting $\varphi_0(x)$ to be $x = x$, and for each $i \geqslant 0$, if $\theta_i(x)$ is $\exists y < b\psi(x, y)$ for some ψ and some $b \in M$ then either $\varphi_{i+1}(x)$ is $\varphi_i(x) \wedge \psi(x, c) \wedge c < b$ for some $c \in M$, or $\varphi_{i+1}(x)$ is $\varphi_i(x) \wedge \neg\theta_i(x)$. If $\theta_i(x)$ is any other $\mathscr{L}_A(M)$-formula $\varphi_{i+1}(x)$ is either $\varphi_i(x) \wedge \theta_i(x)$ or $\varphi_i(x) \wedge \neg\theta_i(x)$.

(a) Show that this construction can be carried out so that $M \vDash \mathrm{Q}x\varphi_i(x)$ for all i. (*Hint:* Consider Exercise 7.8.)
(b) Show that

$$\{\theta(\bar{a}) \mid \bar{a} \in M \vDash \theta(\bar{a}),\ \theta(\bar{x}) \text{ an } \mathscr{L}_{\mathrm{A}}\text{-formula}\} \cup \{c > a \mid a \in M\} \cup \{\varphi_i(c) \mid i \in \mathbb{N}\}$$

is a consistent theory in $\mathscr{L}_{\mathrm{A}}(M) \cup \{c\}$.
(c) If $N > M$ satisfies the theory in (b) above, then $K = K(N; M \cup \{c\})$ is an elementary end-extension of M.
(d) Deduce a partial converse to Exercise 7.9, that any *countable* model M for a countable language $\mathscr{L} \supseteq \mathscr{L}_{\mathrm{A}}$ satisfying PA^- together with all collection axioms $B\varphi$ for $\mathscr{L}$-formulas φ has a proper elementary end-extension.

(This exercise shows that, at least for countable models, the collection axioms play a very important role in end-extensions of models of arithmetic, as well as in cofinal extensions. For similar constructions, and to put these techniques in their proper model-theoretic setting, see also (Keisler 1970) and (Hodges 1985).)

9

Satisfaction

The truth is rarely pure, and never simple.

Oscar Wilde
The Importance of Being Earnest

This is the chapter that no one wanted to have to write: the material here is technical and difficult to describe clearly, yet it is very necessary for later work. I have put it off as long as possible, but cannot do so any longer.

Roughly, what is going on here is this: by the Church–Turing thesis the set $S \subseteq \mathbb{N}^2$ defined by

$$S = \{(\ulcorner\theta(\bar{v})\urcorner, a) \mid \theta(v_0, \ldots, v_{n-1}) \text{ is } \Sigma_1 \text{ and } \mathbb{N} \vDash \theta((a)_1, \ldots, (a)_n)\}$$

is r.e. (exercise: verify this) and so by Corollary 3.4 there is a Σ_1 formula $\mathrm{Sat}_{\Sigma_1}(x, y)$ representing S in $\mathbb{N}$. $\mathrm{Sat}_{\Sigma_1}(x, y)$ can be considered as representing *Σ_1-truth*, or, as we shall say, of *Σ_1-satisfaction*. It is a 'Σ_1-complete' formula in the sense that any r.e. decision problem can be reduced to deciding for what $y \in \mathbb{N}$ is $\mathrm{Sat}_{\Sigma_1}(n, y)$ true (for some fixed $n \in \mathbb{N}$ depending only on the decision problem being considered). What we have to do here is to define the formula $\mathrm{Sat}_{\Sigma_1}(x, y)$ precisely, *and show it has all the expected properties in arbitrary models of PA*. The important property that we must check holds is Tarski's inductive definition of truth (restricted to Σ_1 formulas in the natural way). It will follow from this that

$$PA \vdash \forall a(\varphi((a)_1, (a)_2, \ldots, (a)_n) \leftrightarrow \mathrm{Sat}_{\Sigma_1}(\ulcorner\varphi(v_0, \ldots, v_{n-1})\urcorner, a))$$

for each Σ_1 formula $\varphi(v_0, \ldots v_{n-1})$.

In a similar way, we will also define Σ_n-complete and Π_n-complete formulas $\mathrm{Sat}_{\Sigma_n}(x, y)$ and $\mathrm{Sat}_{\Pi_n}(x, y)$ for each $n \geqslant 1$. The bulk of the work, however, is in describing the $\Delta_1(PA)$ formula $\mathrm{Sat}_{\Delta_0}(x, y)$. Notice that, by Tarski's theorem on the undefinability of truth (Exercise 3.7) there is no $\mathscr{L}_A$-formula $\mathrm{Sat}(x, y)$ defining satisfaction for *all* $\mathscr{L}_A$-formulas (thus: we can't get any satisfaction!) This problem will be taken up in Chapter 15, where we consider the consequences of adding a new binary relation $\mathrm{Sat}(x, y)$ to the language.

Historically, the motivation for this work on Σ_n- and Π_n-complete formulas was in showing that the formula classes Σ_n and Π_n do indeed form a hierarchy—that is, for each $n \geqslant 1$ there are formulas $\theta(x) \in \Sigma_n$ not equivalent to any Π_n formula, and $\psi(x) \in \Pi_n$ not equivalent to any Σ_n formula. However, the consequences for the model theory of PA go very much deeper, as we shall see in the rest of this book.

9.1 SEQUENCES

We now get down to the nitty-gritty of defining $\mathrm{Sat}_{\Sigma_n}(x, y)$ in PA, and showing that it has all the required properties there. *Throughout Sections 9.1, 9.2, 9.3 we will work in PA* (or some fixed model of PA).

This section is devoted to expanding our repertoire of functions used to handle codes for sequences. We assume we are given a Δ_0 formula $(x)_y = z$ with the following sentences

$$\forall x, y \exists ! \overset{z}{\cancel{y}} \quad (x)_y = z$$

$$\forall x, y \quad (x)_y \leqslant x$$

$$\forall x \exists y \quad (y)_0 = x$$

$$\forall x, y, z \exists w (\forall i < z ((w)_i = (y)_i) \wedge (w)_z = x)$$

all provable in PA. (Obviously we could take $(x)_y = z$ to be the formula of Chapter 5, but the exact choice of formula is unimportant.) We shall use this to develop the means to handle codes for sequences where we simultaneously keep track of the *length* of the sequence. For reasons explained in Chapter 4, p. 51, it will be important that all the new functions we define are provably recursive.

DEFINITION

$$\mathrm{len}(x) = y \leftrightarrow (x)_0 = y$$

$$[x]_y = z \leftrightarrow (y \geqslant \mathrm{len}(x) \wedge z = 0) \vee (y < \mathrm{len}(x) \wedge (x)_{y+1} = z)$$

Thus any number codes a sequence $[x]_0, [x]_1, \ldots, [x]_{l-1}$ where $l = \mathrm{len}(x) = (x)_0$ and $(x)_{i+1} = [x]_i$ for each $i < l$. Notice that $\forall x \exists ! y\ (\mathrm{len}(x) = y)$ and $\forall x, y \exists ! z [x]_y = z$ are both provable in PA, so $\mathrm{len}(x)$ and $[x]_y$ are both provably recursive functions, since they both have Δ_0 graph.

It is clear from this definition and the properties of $(x)_y$ listed above that for each $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_{n-1}$ there is a least z coding the sequence $x_0, x_1, \ldots, x_{n-1}$ in this way, with $\mathrm{len}(z) = n$. Thus we can make the following definition.

DEFINITION. For all $n \in \mathbb{N}$ (including $n=0$)

$$[x_0, x_1, \ldots, x_{n-1}] = z \leftrightarrow \mathrm{len}(z) = n \wedge \bigwedge_{i<n}([z]_i = x_i) \wedge \\ \forall w < z(\mathrm{len}(w) \neq n \vee \bigvee_{i<n}[w]_i \neq x_i).$$

Thus $[x_0, x_1, \ldots, x_{n-1}] = z$ defines a family of functions, each of which is provably recursive in PA. In the case $n=0$, the definition above provides us with the notation $[\,]$ for the least code for the sequence of length 0.

In a similar way we define the *concantenation operation* $\frown$ by

DEFINITION. $x \frown y = z \leftrightarrow z$ is least s.t.

$$\mathrm{len}(z) = \mathrm{len}(x) + \mathrm{len}(y) \wedge \forall i < \mathrm{len}(x)([z]_i = [x]_i) \wedge \\ \forall j < \mathrm{len}(y)([z]_{\mathrm{len}(x)+j} = [y]_j).$$

The idea is, of course, that $z = x \frown y$ codes the sequence

$$[x]_0, [x]_1, \ldots, [x]_{\mathrm{len}(x)-1}, [y]_0, [y]_1, \ldots, [y]_{\mathrm{len}(y)-1}$$

of length $\mathrm{len}(z) = \mathrm{len}(x) + \mathrm{len}(y)$, and again it is easy to check that $x \frown y$ is a provably recursive function in PA.

We will also require the *restriction* of a sequence x to length y, $x \upharpoonright y$, defined by

DEFINITION: $w = x \upharpoonright y \leftrightarrow w$ is least such that

$$\mathrm{len}(w) = y \wedge \forall i < \mathrm{len}(w)([w]_i = [x]_i),$$

so that, if x codes $[x]_0, [x]_1, \ldots, [x]_{\mathrm{len}(x)-1}$ and $y \leqslant \mathrm{len}(x)$, then $x \upharpoonright y$ is the least code for the sequence $[x]_0, [x]_1, \ldots, [x]_{y-1}$ of length y. Notice however that if $y > \mathrm{len}(x)$, then (since we defined $[x]_i = 0$ for $i \geqslant \mathrm{len}(x)$) $x \upharpoonright y$ is the least code for the sequence $[x]_0, [x]_1, \ldots, [x]_{\mathrm{len}(x)-1}, 0, \ldots, 0$ of length y, 'padded out' with zeros, as necessary. Once again it is easy to check that $x \upharpoonright y$ is provably recursive in PA.

The next definition is that of $x[y/z]$ which is the least code for the sequence

$$[x]_0, [x]_1, \ldots, [x]_{z-1}, y, [x]_{z+1}, \ldots, [x]_{\mathrm{len}(x)-1}$$

if $z < \mathrm{len}(x)$, or the least code for

$$[x]_0, [x]_1, \ldots, [x]_{\mathrm{len}(x)-1}, 0, \ldots, 0, y$$

if $z \geqslant \mathrm{len}(x)$ (where this last sequence has length z). Thus $x[y/z]$ is the result of replacing the zth entry of the sequence coded by x, $[x]_z$, with y—padding the sequence out with zeros if necessary. More formally:

DEFINITION. $w = x[y/z] \leftrightarrow w$ is least such that

$$\mathrm{len}(w) = \max(\mathrm{len}(x), z+1) \wedge$$
$$\forall i < \mathrm{len}(w) \begin{Bmatrix} (i = z \rightarrow [w]_i = y) \wedge \\ (i \neq z \rightarrow [w]_i = [x]_i) \end{Bmatrix}.$$

Once again $x[y/z]$ is a provably recursive function in PA.

The last definition of this section is that of the 'part-of' relation, $x \subseteq_p y$. $x \subseteq_p y$ is to hold if x codes a sequence that forms a 'contiguous block' of the sequence coded by y. Thus: $[1, 3, 2, 7] \subseteq_p [9, 8, 1, 3, 2, 7, 11]$. However, since a given sequence may have more than one code, there is an annoying technical consideration that must be made, and the full definition is

DEFINITION. $x \subseteq_p y \leftrightarrow$

$$\mathrm{len}(y) \geqslant \mathrm{len}(x) \wedge \exists i \leqslant \mathrm{len}(y) - \mathrm{len}(x) [\forall j < \mathrm{len}(x)([x]_j = [y]_{i+j})] \wedge$$
$$\forall z < x(\mathrm{len}(z) \neq \mathrm{len}(x) \vee \exists i < \mathrm{len}(x)([x]_i \neq [z]_i))$$

The technical restriction needed is in the last line of this definition: we insist that $x \subseteq_p y$ can only be true if x is the least code for the sequence it codes. The reason for this restriction is to make the next lemma hold.

LEMMA 9.1: For each $n \geqslant 1$, $\Sigma_n(PA)$ and $\Pi_n(PA)$ are closed under 'part-of' quantification

$$\forall x(x \subseteq_p y \rightarrow \ldots) \qquad \text{and} \qquad \exists x(x \subseteq_p y \wedge \ldots).$$

Proof. If $\theta(x, y, z)$ is Σ_n and $n \geqslant 1$ then $\forall x(x \subseteq_p y \rightarrow \theta(x, y, z))$ is equivalent in PA to the formula

$$\exists l[l = \mathrm{len}(y) \wedge \forall i, j < l[i + j < l \rightarrow \exists x\{\mathrm{len}(x) = j \wedge \forall k < j([x]_k = [y]_{i+k})$$
$$\wedge \forall u < x(\mathrm{len}(u) \neq j \vee \exists k < j([u]_k \neq [y]_{i+k})) \wedge \theta(x, y, \bar{z})\}]]$$

which in turn is equivalent in PA to a Σ_n formula by Proposition 7.1.

The case for Π_n formulas is similar. □

Exercises for Section 9.1

9.1 Define a provably recursive function $F(x, y)$ such that the following are provable in PA:

(a) $$\forall u, x, y[\text{len}(u) \leqslant y \wedge \forall i < \text{len}(u)([u]_i \leqslant x) \rightarrow$$
$$\exists v \leqslant F(x, y)(\text{len}(v) = \text{len}(u) \wedge \forall i < \text{len}(u)([u]_i = [v]_i))]$$

(b) $$\forall x, y, z[z = F(x, y) \rightarrow (\text{len}(z) \leqslant y \wedge \forall i < \text{len}(z)([z]_i \leqslant x)$$
$$\wedge \forall w < z(\text{len}(w) \neq \text{len}(z) \vee \exists i < \text{len}(z)([z]_i \neq [w]_i)))].$$

9.2 Define a provably recursive function $G(x)$ such that the following is provable in PA:

$$\forall x(G(x) \subseteq_p x \wedge \forall y \subseteq_p x(y \leqslant G(x)))$$

9.3 (a) Define $\{x\}_y$ = the yth digit of the base v representation of u, where $x = \langle u, v\rangle = (u+v)(u+v+1)/2 + v$. Show that this has the properties of $(x)_y$ described on p. 105. (*Hint*: use $(x)_y$ to define $\{x\}_y$ in PA.)

(b) Define $[x]_y$, $\text{len}(x)$, $\text{len}(x)$, $x \subseteq_p y$, etc. as in Section 9.1 but using $\{x\}_y$ in place of $(x)_y$, and show that the following useful property, $\forall x, y(x \subseteq_p y \rightarrow x \leqslant y)$ is now provable in PA.

9.2 SYNTAX

We are now ready to define our Gödel-numbering, and to make various syntactic definitions (such as: what a formula is; what a term is; what a Δ_0 formula is, etc.) inside PA. The first step is to assign to each symbol s of the first-order language $\mathscr{L}_A$ a unique natural number $\nu(s)$. This can be done, for example, in the way indicated in Table 1, where $\langle x, y\rangle = (x+y)(x+y+1)/2 + y$ is our usual pairing function. (Note that $\langle 13, i\rangle \geqslant 13$ for each i, so that each symbol of $\mathscr{L}_A$ does indeed get a unique number!)

The idea now is to define the Gödel-number $\ulcorner\sigma\urcorner$ of a string of symbols $\sigma = s_0 s_1 s_2 \cdots s_{n-1}$ to be the least natural number x coding the sequence $\nu(s_0), \nu(s_1), \ldots \nu(s_{n-1})$. More precisely

DEFINITION. $GN(x)$ ('x is a Gödel-number') is the $\mathscr{L}_A$-formula

$$\forall i < \text{len}(x)([x]_i \leqslant 12 \vee \exists j \leqslant x[x]_i = \langle 13, j\rangle) \wedge$$
$$\forall w < x(\text{len}(w) \neq \text{len}(x) \vee \exists i < \text{len}(x)[x]_i \neq [w]_i).$$

Since $\text{len}(x) \leqslant x$ and $[x]_i \leqslant x$ for all x, i, it is easy to check that $GN(x)$ is equivalent to a Δ_0 formula in PA. Moreover any string of $\mathscr{L}_A$-symbols, σ,

Table 1 Assignment of natural numbers to symbols of $\mathscr{L}_A$

$\mathscr{L}_A$-symbol, s	natural number, $\nu(s)$
0	0
1	1
$+$	2
$\cdot$	3
$<$	4
$=$	5
$\wedge$	6
$\vee$	7
$\neg$	8
$\exists$	9
$\forall$	10
(	11
)	12
v_i	$\langle 13, i\rangle$

has a unique Gödel-number $\ulcorner\sigma\urcorner \in \mathbb{N}$. For example, $\ulcorner + \urcorner$ is $[2]$, $\ulcorner 0 = 1\urcorner$ is $[0, 5, 1]$; $\ulcorner \mathsf{v}_j\urcorner$ is $[\langle 13, j\rangle]$; $\ulcorner \exists(\urcorner$ is $[9, 11]$, and if x, y are Gödel-numbers then so is their concatenation $x^\frown y$. (Here $[x_0, \ldots, x_{n-1}]$, $^\frown$, etc. are the recursive functions defined in the previous section.) In particular if $GN(x) \wedge GN(y)$, we can form the Gödel-number $z = \ulcorner(\urcorner{}^\frown x^\frown\ulcorner+\urcorner{}^\frown y^\frown\ulcorner)\urcorner$. We shall simplify the notation in this last example considerably by writing $\ulcorner(x+y)\urcorner$ for $\ulcorner(\urcorner{}^\frown x^\frown\ulcorner+\urcorner{}^\frown y^\frown\ulcorner)\urcorner$, with similar abbreviations for other commonly used constructions, so $\ulcorner(x+y)\urcorner$ defines a (provably recursive) function with two arguments x, y. (Thus, the symbols $\ulcorner\,\urcorner$ are used in two different ways: firstly as a function sending a string $s_0, \ldots, s_{n-1}$ of $\mathscr{L}_A$-symbols to their unique Gödel-number $\ulcorner s_0 \cdots s_{n-1}\urcorner \in \mathbb{N}$; and secondly as a device for introducing new provably recursive functions into PA, such as $\ulcorner(x+y)\urcorner$.)

The notation in the last paragraph could lead to confusion in the case of expressions such as $\ulcorner(x = \mathsf{v}_i)\urcorner$, so we make the convention that *sans-serif* v are *fixed symbols of* $\mathscr{L}_A$ (i.e. not arguments to functions such as $\ulcorner(x = v_i)\urcorner$), whereas *all subscripts to* v (such as i in the case above) *and all other variables* ($x, y, u, v, w, i, j, k, \ldots$) *are arguments to functions introduced by the* $\ulcorner\cdots\urcorner$ *notation*. So with this convention, $\ulcorner(x = \mathsf{v}_i)\urcorner$ becomes a convenient notation for the function

$$x, i \mapsto \ulcorner(\urcorner{}^\frown x^\frown\ulcorner=\urcorner{}^\frown\ulcorner\langle 13, i\rangle\urcorner{}^\frown\ulcorner)\urcorner$$

and *not* the function

$$x, \mathsf{v}_i \mapsto \ulcorner(\urcorner{}^\frown x^\frown \ulcorner = \urcorner{}^\frown \mathsf{v}_i{}^\frown \ulcorner)\urcorner.$$

Also: inside the $\ulcorner \cdots \urcorner$ signs, *we interpret* 0, 1, +, ·, (,) *where possible as fixed $\mathscr{L}_A$-symbols*, so $\ulcorner(x+y)\urcorner$ denotes the function $x, y \mapsto \ulcorner(\urcorner{}^\frown x^\frown \ulcorner + \urcorner{}^\frown y^\frown \ulcorner)\urcorner$ applied to x, y, and *not* the function $z \mapsto \ulcorner(\urcorner{}^\frown z^\frown \ulcorner)\urcorner$ applied to $z = x+y$.

Note also that, if $\sigma = s_0 s_1 \cdots s_{n-1}$ is a string of fixed $\mathscr{L}_A$-symbols $\ulcorner\sigma\urcorner$ denotes both the Gödel-number of σ and the constant function (with no arguments) whose value is the number $\ulcorner\sigma\urcorner \in \mathbb{N}$. Thus the use of $\ulcorner \cdots \urcorner$ to denote functions can be regarded as an extension of the Gödel-numbering function if we identify 0-ary functions and constant terms in the obvious way.

With these preliminaries out of the way we can now show how to define an $\mathscr{L}_A$-formula term(x) denoting 'x is the Gödel-number of an $\mathscr{L}_A$-term'.

DEFINITION: termseq(s) denotes the $\mathscr{L}_A$-formula

$$\forall i < \mathrm{len}(s) \left\{ \begin{array}{l} [s]_i = \ulcorner 0 \urcorner \vee \\ [s]_i = \ulcorner 1 \urcorner \vee \\ \exists j \leqslant s([s]_i = \ulcorner \mathsf{v}_j \urcorner) \vee \\ \exists j, k < i([s]_i = \ulcorner([s]_j + [s]_k)\urcorner) \vee \\ \exists j, k < i([s]_i = \ulcorner([s]_j \cdot [s]_k)\urcorner) \end{array} \right\}$$

term(x) denotes the $\mathscr{L}_A$-formula $\exists s$ termseq($s^\frown [x]$).

Thus termseq(s) is a $\Delta_1(PA)$ formula, and term(x) is $\Sigma_1(PA)$. We shall show that term(x) is actually $\Delta_1(PA)$, and has various expected properties in all models of PA.

One way by which we could show that term(x) is $\Delta_1(PA)$ is to find a provably recursive function $f(x)$ such that

$$PA \vdash \forall x(\mathrm{term}(x) \to \exists s \leqslant f(x)\ \mathrm{termseq}(s^\frown \ulcorner x \urcorner)),$$

for then term(x) would be equivalent to the Π_1 formula

$$\forall t(t = f(x) \to \exists s \leqslant t\ \mathrm{termseq}(s^\frown \ulcorner x \urcorner)).$$

(Exercise 9.4 discusses this approach.) An alternative is to define s *determines* term(x) to be the $\mathscr{L}_A$-formula

$$\begin{array}{l} \forall i < \mathrm{len}(s)([s]_i \subseteq_{\mathrm{p}} x) \wedge \\ \mathrm{termseq}(s) \wedge \\ \forall y(y \subseteq_{\mathrm{p}} x \wedge \mathrm{termseq}(s^\frown \ulcorner y \urcorner) \to \exists i < \mathrm{len}(s) y = [s]_i). \end{array}$$

Thus, if 's determines term(x)' is true, then s codes a maximal collection of subsequences of x, each of which is a term. (Note that x itself need not be a term.) The idea is that it should be possible to prove in PA that $\forall x \exists s$ (s determines term(x)), and that the property 'term(x)' can be read off any s determining term(x), by examining whether or not $x=[s]_i$ for some $i<\text{len}(s)$. Notice too that 's determines term(x)' is $\Delta_1(PA)$ by Lemma 9.1 since the quantifier $\forall y$ is bounded by $y \subseteq_p x$. We aim to prove that term(x) is equivalent to $\forall s(s$ determines term$(x) \to \exists i<\text{len}(s)\ x=[s]_i)$, which is $\Pi_1(PA)$. To do this we need a technical lemma.

LEMMA 9.2 (Unique readability of terms). *PA* proves the following sentences:

(a) $\forall x, y(\text{term}(x) \wedge \text{term}(x^\frown y) \to \text{len}(y)=0)$

(b) $\forall x, y(\text{term}(x) \wedge \text{term}(y^\frown x) \to \text{len}(y)=0)$

(c) $\forall x, y, r, s\,(\text{term}(x) \wedge \text{term}(y) \wedge \text{term}(r) \wedge \text{term}(s)$

$$\to \left\{ \begin{array}{l} \text{if } \ulcorner(x*y)\urcorner \subseteq_p \ulcorner(r*'s)\urcorner, \text{ where } *, *' \text{ are} \\ \text{either } + \text{ or } \cdot, \text{ then:} \\ \qquad \text{either } \ulcorner(x*y)\urcorner \subseteq_p r \\ \qquad \text{or } \quad \ulcorner(x*y)\urcorner \subseteq_p s \\ \qquad \text{or } x=r \wedge y=s \wedge *=*'). \end{array} \right\}$$

Proof. (a) We use induction on z in

$$\forall x, y(\text{len}(x) \leqslant z \wedge \text{len}(y) \leqslant z \wedge \text{term}(x) \wedge \text{term}(x^\frown y) \to \text{len}(y)=0).$$

Suppose that this is true for some given z (it is obviously true for $z=0$) and suppose $\text{term}(x) \wedge \text{term}(x^\frown y)$ with $\text{len}(x)$ and $\text{len}(y)$ both $\leqslant z+1$. If $\text{len}(x)=1$ then $x=\ulcorner 0\urcorner$, $\ulcorner 1\urcorner$, or $\ulcorner \mathsf{v}_j\urcorner$ for some j, and so $\text{len}(x^\frown y)=1$ (since the only Gödel-numbers of terms with first symbol 0, 1, or v_j are those of length 1), hence $\text{len}(y)=0$. Otherwise $\text{len}(x)>1$ so $x=\ulcorner(r*s)\urcorner$ and $x^\frown y=\ulcorner(p*'q)\urcorner$ where r, s, p, q are all terms and $*, *'$ are $+$ or $\cdot$. If $\text{len}(r) \leqslant \text{len}(p)$, then $p=r^\frown w$ for some w with $\text{len}(r)$, $\text{len}(w) \leqslant z$, hence $\text{len}(w)=0$ by the induction hypothesis. Similarly, if $\text{len}(p) \leqslant \text{len}(r)$ then $r=p^\frown w$ for some w, and $\text{len}(w)=0$. Thus $\text{len}(p)=\text{len}(r)$ and hence $p=r$. This in turn implies $*=*'$ and $q=s^\frown u$ for some u. By the induction hypothesis again $\text{len}(u)=0$, hence $s=p$ and $x=x^\frown y$, thus $\text{len}(y)=0$, as required.

(b) is proved in the same way as (a).

(c) First fix terms r, s and $*' = +$ or $\cdot$. We prove by induction on z that

$$\begin{aligned}\forall u, v, w(\operatorname{term}(v) \wedge \operatorname{len}(v) \leqslant z \wedge u ^\frown v ^\frown w = \ulcorner(r*'s)\urcorner \\ \rightarrow (\operatorname{len}(u) = \operatorname{len}(w) = 0 \vee \\ \operatorname{len}(u) \geqslant \operatorname{len}(r) + 2 \vee \\ \operatorname{len}(w) \geqslant \operatorname{len}(s) + 2))\end{aligned} \tag{$\bullet$}$$

for all z, hence

$$\forall v\, (\operatorname{term}(v) \wedge v \subseteq_p \ulcorner(r*'s)\urcorner \rightarrow v \subseteq_p r \vee v \subseteq_p s \vee v = \ulcorner(r*'s)\urcorner.$$

($\bullet$) is obviously true for $z = 0$, 1, so assuming it is true for $z \geqslant 1$ consider u, v, w with $\operatorname{term}(v) \wedge \operatorname{len}(v) = z + 1 \wedge u ^\frown v ^\frown w = \ulcorner(r*'s)\urcorner$. Then $v = \ulcorner(x*y)\urcorner$ for some terms x, y and $* = +$ or $\cdot$, and $\operatorname{len}(x)$, $\operatorname{len}(y) \leqslant z$. Since

$$u ^\frown \ulcorner(\urcorner ^\frown x ^\frown \ulcorner * \urcorner ^\frown y ^\frown \ulcorner)\urcorner ^\frown w = \ulcorner(r*'s)\urcorner,$$

the inductive assumption applied to both x, y gives

(1) $\operatorname{len}(u ^\frown \ulcorner(\urcorner) \geqslant \operatorname{len}(r) + 2$

or (2) $\operatorname{len}(\ulcorner)\urcorner ^\frown w) \geqslant \operatorname{len}(s) + 2$

or (3) $\operatorname{len}(\ulcorner * \urcorner ^\frown y ^\frown \ulcorner)\urcorner ^\frown w) \geqslant \operatorname{len}(s) + 2$ and $\operatorname{len}\, (u ^\frown \ulcorner(\urcorner ^\frown x ^\frown \ulcorner * \urcorner) \geqslant \operatorname{len}(r) + 2$.

However, $\operatorname{len}(u ^\frown \ulcorner(\urcorner)$ cannot equal $\operatorname{len}(r) + 2$, for if it did $x ^\frown t = s$ for some t, hence $x = s$ by (a) above, which is impossible. Similarly $\operatorname{len}(\ulcorner)\urcorner ^\frown w) \neq \operatorname{len}(s) + 2$. Thus (1) and (2) above imply $\operatorname{len}(u) \geqslant \operatorname{len}(r) + 2$ and $\operatorname{len}(w) \geqslant \operatorname{len}(s) + 2$ respectively. We will show that in case (3) we must have $* = *'$, $x = r$ and $y = s$ with $\operatorname{len}(u) = \operatorname{len}(w) = 0$. But, in case (3), we must have

$$\operatorname{len}(\ulcorner * \urcorner ^\frown y ^\frown \ulcorner)\urcorner ^\frown w) = \operatorname{len}(s) + 2 \quad \text{and} \quad \operatorname{len}(u ^\frown \ulcorner(\urcorner ^\frown x ^\frown \ulcorner * \urcorner) = \operatorname{len}(r) + 2$$

since otherwise $\operatorname{len}(u ^\frown v ^\frown w) > \operatorname{len}(\ulcorner(r*'s)\urcorner)$. It follows then that $* = *'$ and $t_1 ^\frown x = r$, $s = y ^\frown t_2$ for some t_1, t_2, hence $r = x$ and $s = y$ by (a) above, hence $v = \ulcorner(r*'s)\urcorner$, hence $\operatorname{len}(u) = \operatorname{len}(w) = 0$, as required, completing the proof. □

LEMMA 9.3. $PA \vdash \forall x \exists s$ (s determines $\operatorname{term}(x)$).

Proof. Fix x. We show by induction on y that

$$\forall y \leqslant \operatorname{len}(x) \exists s (s \text{ determines } \operatorname{term}(x \restriction y)),$$

which suffices since 's determines $\operatorname{term}(x)$' only depends on the sequence coded by x, and not on the particular code x (so s determines $\operatorname{term}(x \restriction \operatorname{len}(x)) \rightarrow s$ determines $\operatorname{term}(x)$, even though x and $x \restriction \operatorname{len}(x)$ may be different!).

It is clear from the definitions that $s=[\,]$ determines term$(x \upharpoonright 0)$. For the induction step suppose s determines term$(x \upharpoonright y)$, $y<\mathrm{len}(x)$, $x \upharpoonright (y+1)=(x \upharpoonright y)^\frown[u]$. If $[u]$ is $\ulcorner 0 \urcorner$, $\ulcorner 1 \urcorner$, or $\ulcorner \mathsf{v}_j \urcorner$ for some j, then $s^\frown[u]$ clearly determines term$(x \upharpoonright (y+1))$ as no term $v \subseteq_{\mathrm{p}} x \upharpoonright (y+1)$ can have length $\geqslant 2$ and last symbol 0, 1 or v_j. If $[u]=\ulcorner) \urcorner$, then suppose

$$x \upharpoonright y = r^\frown \ulcorner (\urcorner ^\frown [s]_i {}^\frown \ulcorner * \urcorner ^\frown [s]_j$$

where $*=+$ or $\cdot$ and i, $j<\mathrm{len}(s)$. Then $s'=s^\frown \ulcorner ([s]_i*[s]_j) \urcorner$ determines term$(x \upharpoonright (y+1))$, since clearly termseq(s′) holds, and the maximality property of s' follows from unique readability, and the maximality property of s. Finally if $[u]=\ulcorner) \urcorner$ and $x \upharpoonright y$ is not of the above form, or if u is anything else, then s itself determines term$(x \upharpoonright (y+1))$, as can easily be checked. □

LEMMA 9.4. *PA* proves the following:

$$\forall x, s, t(s \text{ determines term}(x) \wedge t \text{ determines term}(x)$$
$$\rightarrow \forall i<\mathrm{len}(s) \exists j<\mathrm{len}(t)\,[s]_i=[t]_j).$$

Proof. Let s, t determine term(x). We show by induction on y that

$$\forall i<y \exists j<\mathrm{len}(t)\,[s]_i=[t]_j \qquad (\dagger)$$

for all $y \leqslant \mathrm{len}(s)$. (†) is trivial for $y=0$. Suppose (†) holds for some $y<\mathrm{len}(s)$; since $[s]_y \subseteq_{\mathrm{p}} x$, it suffices to show that termseq$(t^\frown[[s]_y])$, for then $[s]_y=[t]_j$ for some j, by the maximality property of t. If $[s]_y=\ulcorner 0 \urcorner$, $\ulcorner 1 \urcorner$, or $\ulcorner \mathsf{v}_k \urcorner$ for some k, termseq$(t^\frown[[s]_y])$ is obvious, and if not, $[s]_y=\ulcorner([s]_{y_1}*[s]_{y_2}) \urcorner$ for some y_1, $y_2<y$ and $*=+$ or $\cdot$. But then by the induction hypothesis there exists j_1, $j_2<\mathrm{len}(t)$ such that $[s]_{y_1}=[t]_{j_2}$ and $[s]_{y_2}=[t]_{j_2}$ so once again termseq$(t^\frown[[s]_y])$, as required. □

We can now put these lemmas together to prove the required properties of the formula term(x).

PROPOSITION 9.5. term(x) is equivalent in *PA* to a Π_1 formula, hence it is $\Delta_1(PA)$. Moreover *PA* proves:

$$\forall t[\mathrm{term}(t) \leftrightarrow (t=\ulcorner 0 \urcorner \vee t=\ulcorner 1 \urcorner \vee \exists j \leqslant t(t=\ulcorner \mathsf{v}_j \urcorner)$$
$$\vee \exists r, s \subseteq_{\mathrm{p}} t(\mathrm{term}(r) \wedge \mathrm{term}(s) \wedge (t=\ulcorner(r+s)\urcorner \vee t=\ulcorner(r\cdot s)\urcorner)))]$$

Proof. We claim term(x) is equivalent to the Π_1 formula

$$\forall s((s \text{ determines term}(x)) \rightarrow \exists i<\mathrm{len}(s)\,([s]_i=x)). \qquad (\ddagger)$$

Suppose (‡) holds. Then by Lemma 9.2 there is some s determining term(x). By (‡) there is $i<\mathrm{len}(s)$ with $[s]_i = x$ and hence termseq($(s\restriction i)^\frown[x]$), so term($x$) holds. Conversely, if term(x) holds we want to show there exists some s determining term(x) with the property $\exists i<\mathrm{len}(s)([s]_i = x)$, for then all such s have this property by Lemma 9.3. Our assumption, term(x), gives us some t with $\mathrm{len}(t)\geqslant 1\wedge[t]_{\mathrm{len}(t)-1}=x\wedge$ termseq(t). But then by induction on y up to len(t) in

$$\exists s\, \mathrm{termseq}(s)\wedge\forall i<\mathrm{len}(s)([s]_i\subseteq_{\mathrm{p}} x)\wedge\forall i<y([t]_i\subseteq_{\mathrm{p}} x\rightarrow\exists j<\mathrm{len}(s)[t]_i=[s]_j)$$

there is s satisfying

$$\mathrm{termseq}(s)\wedge\forall i<\mathrm{len}(s)([s]_i\subseteq_p x)\wedge\exists j<\mathrm{len}(s)([s]_j=x).$$

We claim s determines term(x), i.e., that s is 'maximal'. This is proved by induction on y up to len(s) in

$$\left.\begin{array}{c}\forall u\forall i<y((\mathrm{termseq}(s^\frown[u])\wedge u\subseteq_p[s]_i)\\ \rightarrow\exists j\leqslant i(u=[s]_j)).\end{array}\right\}\qquad(\circ)$$

Thus s is maximal, since $[s]_j=x$ for some j. We leave the full details to the reader to check; the reason why the induction step for ($\circ$) works is because if $u\subseteq_{\mathrm{p}}[s]_i$ with $u\neq[s]_i$, then $u\subseteq_{\mathrm{p}}[s]_i=[([s]_j * [s]_k)]$ for some $j, k<i$ and $* = +$ or $\cdot$, hence $u\subseteq_{\mathrm{p}}[s]_j$ or $u\subseteq_{\mathrm{p}}[s]_k$, by Lemma 9.4(b).

The 'moreover' part follows directly from the definition of term(x). □

Next, we shall use the same approach to define other syntactic notions in *PA*, such as the Gödel-number of a *formula*, a Σ_n *formula*, or a Π_n *formula*. *formula*.

Definition: formseq(s) is the formula

$$\forall i<\mathrm{len}(s)\left\{\begin{array}{l}\exists u, v\leqslant s(\mathrm{term}(u)\wedge\mathrm{term}(v)\wedge\\ \qquad\qquad([s]_i=\ulcorner(u=v)\urcorner\vee[s]_i=\ulcorner(u<v)\urcorner))\\ \vee\exists j,k<i([s]_i=\ulcorner([s]_j\vee[s]_k)\urcorner)\\ \vee\exists j,k<i([s]_i=\ulcorner([s]_j\wedge[s]_k)\urcorner)\\ \vee\exists j<i([s]_i=\ulcorner\neg[s]_j\urcorner)\\ \vee\exists j<i\exists k\leqslant s([s]_i=\ulcorner\exists \mathrm{v}_k[s]_j\urcorner\vee[s]_i=\ulcorner\forall \mathrm{v}_k[s]_j\urcorner)\end{array}\right\}$$

form(x) is the formula $\exists s$ formseq($s^\frown[x]$).

s determines form(x) is the formula

$$\forall i<\mathrm{len}(s)([s]_i \subseteq_{\mathrm p} x) \wedge \mathrm{formseq}(s)$$
$$\wedge \forall u(u \subseteq_{\mathrm p} x \wedge \mathrm{formseq}(s^\frown[u]) \rightarrow \exists j<\mathrm{len}(s)[s]_j = u).$$

The idea is that form(x) should hold just when x is the Gödel-number of a formula. The following properties of these formulas are proved in a similar way to their analogues for term(x).

Lemma 9.6 (Unique readability of formulas). *PA* proves the following:
(a) $\forall x, y(\mathrm{form}(x) \wedge \mathrm{form}(x^\frown y) \rightarrow \mathrm{len}(y)=0)$;
$\forall x, y(\mathrm{form}(x) \wedge \mathrm{form}(y^\frown x) \rightarrow \mathrm{len}(y)=0)$;

(b) $$\forall x, r, s\left((\mathrm{form}(x) \wedge \mathrm{form}(r) \wedge \mathrm{form}(s)) \rightarrow \left\{\begin{array}{l}\text{if } x \subseteq_{\mathrm p} \ulcorner(r * s)\urcorner (\text{where } * = \wedge \text{ or } \vee) \\ \text{then } x \subseteq_{\mathrm p} r \vee x \subseteq_{\mathrm p} s \vee x = \ulcorner(r*s)\urcorner.\end{array}\right\}\right);$$

(c) $$\forall x, y, r, s\left((\mathrm{form}(x) \wedge \mathrm{form}(y) \wedge \mathrm{form}(r) \wedge \mathrm{form}(s)) \rightarrow \left\{\begin{array}{l}\ulcorner(x*y)\urcorner = \ulcorner(r*'s)\urcorner (\text{where } *, *' \text{ are } \wedge \text{ or } \vee) \\ \rightarrow x=r \wedge y=s \wedge *=*'\end{array}\right\}\right).$$ □

Lemma 9.7. *PA* proves:

(a) $\forall x \exists s(s \text{ determines form}(x))$;

(b) $$\forall x, s, t((s \text{ determines form}(x) \wedge t \text{ determines form}(x)) \rightarrow \forall i<\mathrm{len}(s) \exists j<\mathrm{len}(t)([s]_i=[t]_j)).$$ □

And hence

Proposition 9.8. form(x) is $\Delta_1(PA)$. Moreover, *PA* proves

$$\begin{aligned}\forall x\{\mathrm{form}(x) \leftrightarrow [&\exists t, s(\mathrm{term}(t) \wedge \mathrm{term}(s) \wedge (x=\ulcorner(t=s)\urcorner \vee x=\ulcorner(t<s)\urcorner)) \\ &\vee \exists y, z(\mathrm{form}(y) \wedge \mathrm{form}(z) \wedge (x=\ulcorner(y \wedge z)\urcorner \vee x=\ulcorner(y \vee z)\urcorner)) \\ &\vee \exists y(\mathrm{form}(y) \wedge (x=\ulcorner \neg y\urcorner \\ &\qquad \vee \exists k \leqslant x(x=\ulcorner \exists \mathrm{v}_k y\urcorner \vee x=\ulcorner \forall \mathrm{v}_k y\urcorner)))]\}.\end{aligned}$$ □

Proposition 9.8 will be very important in Chapter 15. More important for this chapter are the definitions of $\mathrm{form}_{\Sigma_n}(x)$ and $\mathrm{form}_{\Pi_n}(x)$.

DEFINITION. $\mathrm{formseq}_{\Delta_0}(s)$ is the formula

$$\forall i<\mathrm{len}(s)\left\{\begin{array}{l} \quad\exists u,v\leqslant s(\mathrm{term}(u)\wedge\mathrm{term}(v) \\ \qquad\qquad\wedge([s]_i=\ulcorner(u=v)\urcorner\vee[s]_i=\ulcorner(u<v)\urcorner)) \\ \vee\exists j,k<i([s]_i=\ulcorner([s]_j\vee[s]_k)\urcorner\vee[s]_i=\ulcorner([s]_j\vee[s]_k)\urcorner) \\ \vee\exists j<i([s]_i=\ulcorner\neg[s]_j\urcorner) \\ \vee\exists j<i\exists k,u\leqslant s(\mathrm{term}(u)\wedge[s]_i=\ulcorner\exists \mathrm{v}_k((\mathrm{v}_k<u)\wedge[s]_j)\urcorner) \\ \vee\exists j<i\exists k,u\leqslant s(\mathrm{term}(u)\wedge[s]_i=\ulcorner\forall \mathrm{v}_k(\neg(\mathrm{v}_k<u)\vee[s]_j)\urcorner) \end{array}\right\}$$

$\mathrm{form}_{\Delta_0}(x)$ is the formula $\exists s\ \mathrm{formseq}_{\Delta_0}(s^\frown[x])$. $\mathrm{form}_{\Sigma_0}(x)$ and $\mathrm{form}_{\Pi_0}(x)$ are both alternative names for the formula $\mathrm{form}_{\Delta_0}(x)$.

For each $n\in\mathbb{N}$ $\mathrm{formseq}_{\Sigma_{n+1}}(s)$ is the formula

$$\forall i<\mathrm{len}(s)\left\{\begin{array}{l}(\mathrm{form}_{\Pi_n}([s]_i)\wedge i=0)\vee \\ (i>0\wedge\exists k\leqslant s([s]_i=\ulcorner\exists \mathrm{v}_k[s]_{i-1}\urcorner))\end{array}\right\}$$

and $\mathrm{formseq}_{\Pi_{n+1}}$ is the formula

$$\forall i<\mathrm{len}(s)\left\{\begin{array}{l}(\mathrm{form}_{\Sigma_n}([s]_i)\wedge i=0\vee \\ (i>0\wedge\exists k\leqslant s([s]_i=\ulcorner\forall \mathrm{v}_k[s]_{i-1}\urcorner))\end{array}\right\}$$

Finally, $\mathrm{form}_{\Sigma_{n+1}}(x)$ and $\mathrm{form}_{\Pi_{n+1}}(x)$ are $\exists s\ \mathrm{formseq}_{\Sigma_{n+1}}(s^\frown[x])$ and $\forall s\ \mathrm{formseq}_{\Pi_{n+1}}(s^\frown[x])$ respectively.

PROPOSITION 9.9. For each $n\in\mathbb{N}$, $\mathrm{form}_{\Sigma_n}(x)$ and $\mathrm{form}_{\Pi_n}(x)$ are $\Delta_1(PA)$, and moreover PA proves the following sentences:

$$\forall x(\mathrm{form}_{\Sigma_n}(x)\vee\mathrm{form}_{\Pi_n}(x)\rightarrow\mathrm{form}(x))$$
$$\forall x(\mathrm{form}_{\Sigma_n}(x)\rightarrow\mathrm{form}_{\Pi_{n+1}}(x)\wedge\mathrm{form}_{\Sigma_{n+1}}(x))$$
$$\forall x,k(\mathrm{form}_{\Sigma_n}(x)\rightarrow\mathrm{form}_{\Sigma_n}(\ulcorner\exists \mathrm{v}_k x\urcorner))$$
$$\forall x,k(\mathrm{form}_{\Pi_n}(x)\rightarrow\mathrm{form}_{\Pi_n}(\ulcorner\forall \mathrm{v}_k x\urcorner)). \qquad \square$$

Exercises for Section 9.2

9.4 Using the provably recursive functions $F(x,y)$ and $G(x)$ in Exercises 9.1, 9.2, show that

$$PA\vdash\forall t[\mathrm{term}(t)\rightarrow\exists s[\mathrm{termseq}(s)\wedge[s]_{\mathrm{len}(t)-1}=t\wedge s\leqslant F(G(t),\mathrm{len}(t))]].$$

9.5 Define the provably recursive relation free(t,i) (meaning 'v_i is free in t') and the provably recursive function subst(t,s,i) (= 'the result on substituting the term s

for v_i in t') in PA, such that PA proves that, for all i, j, k, all terms r, s, and all terms or formulas t, u, v,

$$\mathrm{free}(\ulcorner \mathsf{v}_j \urcorner, i) \leftrightarrow i = j$$

$$\neg \mathrm{free}(\ulcorner 0 \urcorner, i) \wedge \neg \mathrm{free}(\ulcorner 1 \urcorner, i)$$

$$\mathrm{free}(\ulcorner (u * v) \urcorner, i) \leftrightarrow \mathrm{free}(u, i) \vee \mathrm{free}(v, i) \, (\text{where } * \text{ is } +, \cdot, <, \wedge \text{ or } \vee)$$

$$\mathrm{free}(\ulcorner \neg u \urcorner, i) \leftrightarrow \mathrm{free}(u, i)$$

$$\mathrm{free}(\ulcorner \mathsf{Q} \mathsf{v}_k u \urcorner, i) \leftrightarrow \neg i = k \wedge \mathrm{free}(u, i) \, (\text{where } \mathsf{Q} = \exists \text{ or } \forall)$$

and

$$\mathrm{subst}(\ulcorner \mathsf{v}_j \urcorner, s, i) = \begin{cases} \ulcorner \mathsf{v}_j \urcorner & \text{if } j \neq i \\ s & \text{otherwise} \end{cases}$$

$$\mathrm{subst}(\ulcorner c \urcorner, s, i) = c \qquad (c = 0 \text{ or } 1)$$

$$\mathrm{subst}(\ulcorner (u * v) \urcorner, s, i) = \ulcorner (\mathrm{subst}(u, s, i) * \mathrm{subst}(v, s, i)) \urcorner$$

$$(* = +, \cdot, =, <, \wedge, \text{or } \vee)$$

$$\mathrm{subst}(\ulcorner \neg u \urcorner, s, i) = \ulcorner \neg \mathrm{subst}(u, s, i) \urcorner$$

$$\mathrm{subst}(\ulcorner \mathsf{Q} \mathsf{v}_k u \urcorner, s, i) = \begin{cases} \ulcorner \mathsf{Q} \mathsf{v}_k \ \mathrm{subst}(u, s, i) \urcorner & \text{if } k \neq i \\ \ulcorner \mathsf{Q} \mathsf{v}_k u \urcorner & \text{otherwise} \end{cases}$$

$$(\mathsf{Q} = \forall \text{ or } \exists).$$

Show also that PA proves:

$$\forall s, t, i (\mathrm{term}(t) \wedge \mathrm{term}(s) \rightarrow \mathrm{term}(\mathrm{subst}(t, s, i)))$$

$$\forall s, t, i (\mathrm{form}(t) \wedge \mathrm{term}(s) \rightarrow \mathrm{form}(\mathrm{subst}(t, s, i)))$$

and

$$\forall s, t, i (\neg \mathrm{free}(t, i) \rightarrow t = \mathrm{subst}(t, s, i)).$$

9.6 A *sequent* is a non-empty list of formulas $\theta_1, \theta_2, \ldots, \theta_k$ of $\mathscr{L}_A$. A system of *natural deduction* for PA is defined in which *proofs* are nonempty sequences $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$ of sequents such that each Γ_j is

either σ for some nonlogical axiom σ of PA^- or $I_0\varphi$ for some formula φ of $\mathscr{L}_A$ or one of the '*equality axioms*'

(a) $(t = t)$
(b) $(t = s), (s = t)$
(c) $\neg(t = s), \neg(s = r), (t = r)$

for some terms t, s, r of $\mathscr{L}_A$

or φ, $\neg\varphi$ for some atomic formula φ of $\mathscr{L}_A$
or Γ_i is 'derived' from Γ_j, Γ_k (j, $k<i$) by one of the following rules of deduction (in these rules Γ, Δ denote possibly empty sequences of $\mathscr{L}_A$-formulas; θ, ψ, φ $\mathscr{L}_A$-formulas; s any term of $\mathscr{L}_A$: v any variable; and

$$\frac{\Gamma}{\Delta} \quad \text{or} \quad \frac{\Gamma_1 \quad \Gamma_2}{\Delta}$$

means 'from the sequent(s) Γ (or Γ_1, Γ_2) deduce the sequent Δ')

(a) *structural inferences:*

$$\frac{\Gamma, \varphi, \theta, \Delta}{\Gamma, \theta, \varphi, \Delta} \quad \frac{\Gamma, \varphi, \varphi}{\Gamma, \varphi} \quad \frac{\Gamma}{\Gamma, \varphi}$$

(b) *inferences for* $\wedge$:

$$\frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi\wedge\psi} \quad \frac{\Gamma, \neg\varphi}{\Gamma, \neg(\varphi\wedge\psi)} \quad \frac{\Gamma, \neg\psi}{\Gamma, \neg(\varphi\wedge\psi)}$$

(c) *inferences for* $\vee$:

$$\frac{\Gamma, \varphi}{\Gamma, \varphi\vee\psi} \quad \frac{\Gamma, \psi}{\Gamma, \varphi\vee\psi} \quad \frac{\Gamma, \neg\varphi \quad \Gamma, \neg\psi}{\Gamma, \neg(\varphi\vee\psi)}$$

(d) *inference for* $\neg$:

$$\frac{\Gamma, \varphi}{\Gamma, \neg\neg\varphi}$$

(e) *inferences for* $\exists$ *and* $\forall$:

$$\frac{\Gamma, \varphi}{\Gamma, \forall v\varphi} \quad \frac{\Gamma, \neg\varphi}{\Gamma, \neg\exists v\varphi}$$

(provided v does not occur free in any formula in Γ) and

$$\frac{\Gamma, \varphi(s)}{\Gamma, \exists v\varphi(v)} \quad \frac{\Gamma, \neg\varphi(s)}{\Gamma, \neg\forall v\varphi(v)}$$

(f) *cut rule*

$$\frac{\Gamma, \varphi \quad \Gamma, \neg\varphi}{\Gamma}.$$

A proof $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$ is *a proof of the $\mathscr{L}_A$-formula θ* iff Γ_l is just the sequent with one element θ.

Formalize this notion inside PA, obtaining a provably recursive formula proof(p, x) for 'p is a proof in PA of the $\mathscr{L}_A$-formula with Gödel-number x', and a Σ_1 formula provable(x) for 'x is provable in PA'. Interpreting $(\theta \to \psi)$ as $(\neg\theta \vee \psi)$, verify that provable(x) is a provability predicate for PA, as defined in Exercise 3.8.

9.3 SEMANTICS

Having shown (we hope) that PA can adequately handle syntax, we turn to semantics. The goal is to give, inside PA, some restricted version of Tarski's definition of truth. The first step is to show how to evaluate terms.

DEFINITION. valseq(y, s, t) is the formula

$$\text{termseq}(s) \wedge \text{len}(t) = \text{len}(s) \wedge$$
$$\forall i < \text{len}(s) \left\{ \begin{array}{l} ([s]_i = \ulcorner 0 \urcorner \wedge [t]_i = 0) \vee \\ ([s]_i = \ulcorner 1 \urcorner \wedge [t]_i = 1) \vee \\ \exists j \leqslant s([s]_i = \ulcorner \mathrm{v}_j \urcorner \wedge [t]_i = [y]_j) \vee \\ \exists j, k < i([s]_i = \ulcorner([s]_j + [s]_k)\urcorner \wedge [t]_i = [t]_j + [t]_k) \vee \\ \exists j, k < i([s]_i = \ulcorner([s]_j \cdot [s]_k)\urcorner \wedge [t]_i = [t]_j \cdot [t]_k) \end{array} \right\}$$

val$(x, y) = z$ is the formula

$$\exists s, t\, \text{valseq}(y, s ^\frown [x], t ^\frown [z]) \vee (\neg \text{term}(x) \wedge z = 0).$$

The idea is that val$(x, y) = z$ holds when x is the Gödel-number of a term, and z is the value of that term when the variable v_0 is given value $[y]_0$, v_1 is given value $[y]_1, \ldots$ and so on. Recall that we defined $[y]_i = 0$ when $i \geqslant \text{len}(y)$, so we would expect val$(x, y)$ to be a well-defined function.

PROPOSITION 9.10. val(x, y) is a provably recursive function in PA, that is the formula val$(x, y) = z$ is $\Sigma_1(PA)$ and

$$PA \vdash \forall x, y \exists! z\, \text{val}(x, y) = z.$$

Moreover, PA proves:

$$\forall y(\text{val}(\ulcorner 0 \urcorner, y) = 0 \wedge \text{val}(\ulcorner 1 \urcorner, y) = 1)$$
$$\forall y, i(\text{val}(\ulcorner \mathrm{v}_i \urcorner, y) = [y]_i)$$
$$\forall x, y, z(\text{val}(\ulcorner(x + y)\urcorner, z) = \text{val}(x, z) + \text{val}(y, z))$$
$$\forall x, y, z(\text{val}(\ulcorner(x \cdot y)\urcorner, z) = \text{val}(x, z) \cdot \text{val}(y, z)).$$

Proof. Clearly $\mathrm{val}(x, y)=z$ is $\Sigma_1(PA)$, since $\mathrm{term}(x)$ is $\Delta_1(PA)$, and also

$$PA \vdash \forall x, y, z(\neg \mathrm{term}(x) \to (\mathrm{val}(x, y)=z \leftrightarrow z=0)).$$

Suppose we are give x, y with $\mathrm{term}(x)$ holding. Then $\mathrm{termseq}(s)$ for some s with $\mathrm{len}(s) \geq 1$ and $[s]_{\mathrm{len}(s)-1}=x$. To show that $\mathrm{val}(x, y)$ exists we must show that there exists t with $\mathrm{valseq}(y, s, t)$, for then clearly $\mathrm{val}(x, y)=[t]_{\mathrm{len}(t)-1}$.

We prove $\exists t\, \mathrm{valseq}(y, s, t)$ by induction on w up to $\mathrm{len}(s)$ in

$$\exists t(\mathrm{len}(t)=w \wedge \mathrm{valseq}(y, s \restriction w, t)).$$

For $w=0$ this is trivial (since we may take $t=[\,]$). Suppose that $\mathrm{len}(t)=w$ $\mathrm{valseq}(y, s \restriction w, t)$ holds for some $w<\mathrm{len}(s)$. Then $[s]_w$ is either $\ulcorner 0 \urcorner$, $\ulcorner 1 \urcorner$, $\ulcorner \mathrm{v}_k \urcorner$ or $\ulcorner([s]_i * [s]_j)\urcorner$ ($*=+$ or $\cdot$) for some $k \leq s$ and some $i, j<w$. But then putting $t'=t^\frown[0]$, $t^\frown[1]$, $t^\frown[[y]_k]$, or $t^\frown[[t]_i * [t]_j]$ respectively we have $\mathrm{len}(t')=\mathrm{len}(t)+1=w+1$ and $\mathrm{valseq}(y, s \restriction (w+1), t')$, and so the induction step is proved, and hence

$$PA \vdash \forall x, y \exists z(\mathrm{val}(x, y)=z).$$

For the more difficult *uniqueness* property, suppose s, t and s', t' satisfy

$$[s]_{\mathrm{len}(s)-1}=x \wedge \mathrm{valseq}(y, s, t)$$

and

$$[s']_{\mathrm{len}(s')-1}=x \wedge \mathrm{valseq}(y, s', t').$$

We need to show that $[t]_{\mathrm{len}(t)-1}=[t']_{\mathrm{len}(t')-1}$. This is done by showing (using induction on w up to $\mathrm{len}(t)$) that for all such $w \leq \mathrm{len}(t)$

$$\forall i<w \forall j<\mathrm{len}(t')\,([s]_i=[s']_j \to [t]_i=[t']_j) \qquad (\bullet)$$

for then, applying this to

$$i=\mathrm{len}(t)-1=\mathrm{len}(s)-1 \qquad \text{and} \qquad j=\mathrm{len}(s')-1=\mathrm{len}(t')-1$$

we have

$$[t]_{\mathrm{len}(t)-1} = [t']_{\mathrm{len}(t')-1}$$

since $[s]_{\mathrm{len}(s)-1} = [s']_{\mathrm{len}(s')-1} = x$.

(●) is obvious for $w=0$. Suppose that it is true for some $w < \mathrm{len}(t)$ and that there exists $j < \mathrm{len}(t')$ with $[s]_w = [s']_j$. Since termseq(s) holds, $[s]_w = \ulcorner 0 \urcorner$, $\ulcorner 1 \urcorner$, $\ulcorner \mathsf{v}_k \urcorner$, or $\ulcorner([s]_l * [s]_m)\urcorner$ ($* = +$ or $\cdot$) for some $k \leq s$, and some $l, m < w$. Similarly, $[s']_j = \ulcorner 0 \urcorner$, $\ulcorner 1 \urcorner$, $\ulcorner \mathsf{v}_{k'} \urcorner$ or $\ulcorner([s']_{l'} *' [s]_{m'})\urcorner$ for some $k' \leq s'$, some l', $m' < j$ and $*' = +$ or $\cdot$. As $[s]_w = [s']_j$ these various cases must coincide, by the unique readability of terms, i.e., either $[s]_w = [s']_j = \ulcorner 0 \urcorner$ or $[s]_w = [s']_j = \ulcorner 1 \urcorner$ or $[s]_w = [s']_j = \ulcorner \mathsf{v}_k \urcorner = \ulcorner \mathsf{v}_{k'} \urcorner$ or $[s]_w = \ulcorner([s]_l * [s]_m)\urcorner = [s']_j = \ulcorner([s']_{l'} *' [s']_{m'})\urcorner$. In the first two of these cases we have $[t]_w = [t']_j = 0$ or $[t]_w = [t']_j = 1$ respectively, and in the third of these cases we have $k = k'$ so that $[t]_w = [t']_j = [y]_k$. Finally, if

$$([s]_l * [s]_m) = [s]_w = [s']_j = ([s']_{l'} *' [s']_{m'})$$

then Lemma 9.2 (unique readability) implies that $[s]_l = [s']_{l'}$, $[s]_m = [s']_{m'}$ and $* = *'$, hence by the induction hypothesis (●), $[t]_l = [t']_{l'}$, $[t]_m = [t']_{m'}$ and so

$$[t]_w = [t]_l * [t]_m = [t']_{l'} *' [t']_{m'} = [t']_j$$

establishing the induction step, and hence proving the proposition. □

As a corollary to Proposition 9.10, notice that for any $\mathscr{L}_A$-term $t(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_k)$ (with only the free-variables shown), if $n = \ulcorner t(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_k)\urcorner$ then

$$PA \vdash \forall a_0, a_1, \ldots, a_k, b \forall z (z = t(a_0, \ldots, a_k)$$

$$\leftrightarrow z = \mathrm{val}(n, [a_0, \ldots, a_k] ^\frown b)). \tag{*}$$

This is proved by induction on the number of + s and · s in $t(\mathsf{v}_0, \ldots, \mathsf{v}_k)$, using the fact that, since term(x) is $\Delta_1(PA)$,

$$\mathbb{N} \vDash \mathrm{term}(m) \Leftrightarrow PA \vDash \mathrm{term}(m)$$

for each $m \in \mathbb{N}$.

We are at last in a position to define $\mathrm{Sat}_{\Delta_0}(x, y)$, the 'truth definition' for Δ_0 formulas.

DEFINITION. $\mathrm{satseq}_{\Delta_0}(s, t)$ is the formula

$$
\begin{aligned}
&\mathrm{formseq}_{\Delta_0}(s) \wedge \\
&\quad \forall l < \mathrm{len}(t) \exists i, z, w \leq t\big[[t]_l = \langle i, z, w\rangle \wedge i < \mathrm{len}(s) \wedge w \leq 1 \wedge \\
&\qquad \{\exists u, u' \leq s(\mathrm{term}(u) \wedge \mathrm{term}(u') \wedge [s]_i = \ulcorner(u = u')\urcorner \wedge \\
&\qquad\quad (w = 1 \leftrightarrow \mathrm{val}(u, z) = \mathrm{val}(u', z))) \\
&\qquad \vee \exists u, u' \leq s(\mathrm{term}(u) \wedge \mathrm{term}(u') \wedge [s]_i = \ulcorner(u < u')\urcorner \wedge \\
&\qquad\quad (w = 1 \leftrightarrow \mathrm{val}(u, z) < \mathrm{val}(u', z))) \\
&\qquad \vee \exists j, k < i([s]_i = \ulcorner([s]_j \wedge [s]_k)\urcorner \wedge \\
&\qquad\quad \exists l_1, l_2 < l \exists w_1, w_2 \leq 1([t]_{l_1} = \langle j, z, w_1\rangle \wedge [t]_{l_2} = \langle k, z, w_2\rangle \\
&\qquad\quad \wedge (w = 1 \leftrightarrow w_1 = 1 \wedge w_2 = 1))) \\
&\qquad \vee \exists j, k < i([s]_i = \ulcorner([s]_j \vee [s]_k)\urcorner \wedge \\
&\qquad\quad \exists l_1, l_2 < l \exists w_1, w_2 \leq 1([t]_{l_1} = \langle j, z, w_1\rangle \wedge [t]_{l_2} = \langle k, z, w_2\rangle \\
&\qquad\quad \wedge (w = 1 \leftrightarrow w_1 = 1 \vee w_2 = 1))) \\
&\qquad \vee \exists j < i([s]_i = \ulcorner\neg [s]_j\urcorner \wedge \\
&\qquad\quad \exists l_1 < l \exists w_1 \leq 1([t]_{l_1} = \langle j, z, w_1\rangle \wedge (w = 1 \leftrightarrow w_1 = 0))) \\
&\qquad \vee \exists j < i \exists k, u \leq s(\mathrm{term}(u) \wedge [s]_i = \ulcorner\exists \mathsf{v}_k((\mathsf{v}_k < u) \wedge [s]_j)\urcorner \wedge \\
&\qquad\quad \forall r < \mathrm{val}(u, z) \exists l_1 < l \exists w_1 \leq 1([t]_{l_1} = \langle j, z[r/k], w_1\rangle) \wedge \\
&\qquad\quad \wedge (w = 1 \leftrightarrow \exists r < \mathrm{val}(u, z) \exists l_1 < l([t]_{l_1} = \langle j, z[r/k], 1\rangle))) \\
&\qquad \vee \exists j < i \exists k, u \leq s(\mathrm{term}(u) \wedge [s]_i = \ulcorner\forall \mathsf{v}_k(\neg(\mathsf{v}_k < u) \vee [s]_j)\urcorner \wedge \\
&\qquad\quad \forall r < \mathrm{val}(u, z) \exists l_1 < l \exists w_1 \leq 1([t]_{l_1} = \langle j, z[r/k], w_1\rangle) \\
&\qquad\quad \wedge (w = 1 \leftrightarrow \exists r < \mathrm{val}(u, z) \exists l_1 < l([t]_{l_1} = \langle j, z[r/k], 1\rangle)))\}\big]
\end{aligned}
$$

$\mathrm{Sat}_{\Delta_0}(x, y)$ is the formula

$$\exists s, t[\mathrm{satseq}_{\Delta_0}(s ^\frown [x], z) \wedge \exists l < \mathrm{len}(t)([t]_l = \langle \mathrm{len}(s), y, 1\rangle)].$$

EXPLANATION. In the formula $\mathrm{satseq}_{\Delta_0}(s, t)$, t is considered as a code for a sequence of triples $\langle i, z, w\rangle$ where $i < \mathrm{len}(s)$ is an index to the sequence coded by s, and w is a 'truth value', $w = 0$ ('false') or 1 ('true'), of the formula $[s]_i$ when $\mathsf{v}_0, \mathsf{v}_1, \ldots$ are interpreted by $[z]_0, [z]_1, \ldots$. The various disjuncts inside the formula in curly brackets correspond to the different ways of building up Δ_0 formulas.

Since $\mathrm{val}(u, z)$, $\mathrm{term}(u)$ and $\mathrm{formseq}_{\Delta_0}(s)$ are all provably recursive in PA we see that $\mathrm{satseq}_{\Delta_0}(s, t)$ is $\Delta_1(PA)$, and hence $\mathrm{Sat}_{\Delta_0}(x, y)$ is $\Sigma_1(PA)$. The bulk of the work following is involved in showing $\mathrm{Sat}_{\Delta_0}(x, y)$ is actually $\Delta_1(PA)$. For this we need the

DEFINITION. 's, t determines $\mathrm{Sat}_{\Delta_0}(x, y)$' denotes the formula

$$\mathrm{satseq}_{\Delta_0}(s, t) \wedge \exists l < \mathrm{len}(t) \exists i < \mathrm{len}(s) \exists w \leqslant 1([s]_i = x \wedge [t]_l = \langle i, y, w\rangle).$$

Our strategy for showing $\mathrm{Sat}_{\Delta_0}(x, y)$ to be $\Delta_1(PA)$ is the same as for $\mathrm{term}(x)$ above: we will show that PA proves $\exists s, t$ (s, t determines $\mathrm{Sat}_{\Delta_0}(x, y)$) for each x, y s.t. $\mathrm{form}_{\Delta_0}(x)$, and that, for any two pairs s, t and s', t' determining $\mathrm{Sat}_{\Delta_0}(x, y)$, the sequences t, t' 'agree' on the truth of each subformula of x.

LEMMA 9.11. *PA proves the sentence*

$$\forall x, y(\mathrm{form}_{\Delta_0}(x) \to \exists s, t\,(s, t \text{ determines } \mathrm{Sat}_{\Delta_0}(x, y))).$$

Proof. Let x, y be arbitrary such that $\mathrm{form}_{\Delta_0}(x)$ holds, and let s satisfy $\mathrm{formseq}_{\Delta_0}(s)$ with $[s]_{\mathrm{len}(s)-1} = x$. We would like to build up a suitable t by concantenation, using an induction on the length of s, but for the cases where formulas are built using bounded quantifiers we will need a second induction argument.

Let $\theta(i)$ be the formula

$$\forall m \leqslant i \forall y \exists t\{\mathrm{satseq}_{\Delta_0}(s \restriction (m+1), t) \wedge \mathrm{len}(t) > 0 \wedge$$
$$\exists l < \mathrm{len}(t) \exists w \leqslant 1([t]_l = \langle m, y, w\rangle)\}.$$

We will show $\theta(i)$ holds for all $i < \mathrm{len}(s)$ by induction on i.

Let $i < \mathrm{len}(s)$, and if $i > 0$ assume inductively that $\theta(i)$ holds. There are several cases to consider.

(1) If $[s]_i = \ulcorner(u = u')\urcorner$ with $\mathrm{term}(u) \wedge \mathrm{term}(u')$ then $t = [\langle i, y, w\rangle]$ satisfies $\mathrm{satseq}_{\Delta_0}(s \restriction (i+1), t)$, where $w = 1$ if $\mathrm{val}(u, y) = \mathrm{val}(u', y)$, and $w = 0$ otherwise, and hence t satisfies the subformula inside $\{\}$ in $\theta(i)$. The case of $[s]_i = \ulcorner(u < u')\urcorner$ is treated similarly, and these are the only possibilities when $i = 0$.

(2) If $[s]_i = \ulcorner([s]_j \wedge [s]_k)\urcorner$ for some $j, k < i$ let t, t', l, l' and w, w' satisfy

$$\text{satseq}_{\Delta_0}(s \restriction (j+1), t) \wedge [t]_l = \langle j, y, w\rangle$$
$$\text{satseq}_{\Delta_0}(s \restriction (k+1), t') \wedge [t']_{l'} = \langle y, y, w'\rangle$$

where $w, w' \leq 1$, $l < \text{len}(t)$ and $l' < \text{len}(t')$, and all these numbers are found using the induction hypothesis $\theta(i-1)$. Then $t'' = t^\frown t'^\frown [\langle i, y, w''\rangle]$ satisfies $\text{satseq}_{\Delta_0}(s \restriction (i+1), t'')$ where $w'' = 1$ if $w = w' = 1$ and $w'' = 0$ otherwise. The cases when $[s]_i = \ulcorner([s]_j \vee [s]_k)\urcorner$ and $[s]_i = \ulcorner\neg[s]_j\urcorner$ are treated similarly.

(3) If

$$[s]_i = \ulcorner\exists \mathsf{v}_k((\mathsf{v}_k < u) \wedge [s]_j)\urcorner \qquad \text{or} \qquad [s]_j = \ulcorner\forall \mathsf{v}_k(\neg(\mathsf{v}_k < u) \vee [s]_j)\urcorner,$$

we use a second induction argument together with the assumption $\theta(i-1)$ to produce the required t. Let $U = \text{val}(u, y)$. The idea is to prove by induction on p up to U that

$$\begin{aligned}&\exists t[\text{satseq}_{\Delta_0}(s \restriction (j+1), t) \wedge \text{len}(t) > 0 \\ &\quad \wedge \forall q < p \exists l < \text{len}(t) \exists w \leq 1([t]_l = \langle j, y[q/k], w\rangle)].\end{aligned} \tag{*}$$

This second induction is very easy. For $p = 0$ take any t such that $\text{satseq}_{\Delta_0}(s \restriction (j+1), t)$ with $[t]_l = \langle j, y[0/k], w\rangle$ for some w, l, using the hypothesis $\theta(i-1)$. The induction step is proved simply by concatenating some previously obtained t with a t' satisfying $\text{satseq}_{\Delta_0}(s \restriction (j+1), t')$ and $[t']_l = \langle j, y[p+1/k], w\rangle$ for some l, w. Having obtained t satisfying the formula in square brackets of (*) for $p = U$, it is easy to check that $\text{satseq}_{\Delta_0}(s \restriction (i+1), t^\frown[\langle i, y, w\rangle])$ holds for some suitably chosen $w = 0$ or 1.

This concludes the proof of Lemma 9.11. □

LEMMA 9.12. *PA* proves the sentence

$$\begin{aligned}\forall s, t, s', t', i, i', l, l', w, w', y[&\text{satseq}_{\Delta_0}(s, t) \wedge \text{satseq}_{\Delta_0}(s', t') \wedge \\ &[s]_i = [s']_{i'} \wedge [t]_l = \langle i, y, w\rangle \wedge \\ &[t']_{l'} = \langle i', y, w'\rangle \rightarrow w = w'].\end{aligned}$$

Proof. By induction on $\max(l, l')$.

If $l = l' = 0$, $[t]_l = \langle i, y, w\rangle$, $[t']_{l'} = \langle i', y, w'\rangle$ then $[s]_i$ and $[s']_{i'}$ must be atomic, i.e. of the form $\ulcorner(u = u')\urcorner$ or $\ulcorner(u < u')\urcorner$ for terms u, u'. Moreover if $[s]_i = [s']_{i'}$, then $[s]_i = \ulcorner(u = u')\urcorner = [s']_{i'}$ or $[s]_i = \ulcorner(u < u')\urcorner = [s']_{i'}$, where u, u' are uniquely determined by $[s]_i$ or $[s']_{i'}$, by the unique readability of terms and formulas. Thus

$$w=1 \Leftrightarrow \mathrm{val}(u, y)=\mathrm{val}(u', y) \Leftrightarrow w'=1$$

in the first case, and

$$w=1 \Leftrightarrow \mathrm{val}(u, y)<\mathrm{val}(u', y) \Leftrightarrow w'=1$$

in the second.

The argument for the induction step splits into several cases, and we shall do two of them here, leaving the others to the reader. If $[t]_l=\langle i, y, w\rangle$, $[t']_{l'}=\langle i', y, w'\rangle$, and $[s]_i=[s']_{i'}$ then $[s]_i$ is built up in finitely many possible ways corresponding to $\wedge$, $\vee$, $\neg$, $\exists$ or $\forall$. Suppose $[s]_i=\ulcorner(r_1\wedge r_2)\urcorner$ where $r_1=[s]_{i_1}$ and $r_2=[s]_{i_2}$ for some i_1, $i_2<i$. Then also $[s']_{i'}=\ulcorner(r_1\wedge r_2)\urcorner$ where $r_1=[s']_{i_1'}$, $r_2=[s']_{i_2'}$ for some i_1', $i_2'<i'$, by the unique readability of formulas. Moreover there are l_1, $l_2<l$ and l_1', $l_2'<l'$, and w_1, w_2, w_1', $w_2'\leq 1$ such that $[t]_{l_1}=\langle i_1, y, w_1\rangle$, $[t]_{l_2}=\langle i_2, y, w_2\rangle$, $[t']_{l_1'}=\langle i_1', y, w_1'\rangle$, $[t']_{l_2'}=\langle i_2', y, w_2'\rangle$. Then by the induction hypothesis $w_1=w_1'$, and $w_2=w_2'$; and $w=1$ iff $w_1=1$ and $w_2=1$ iff $w_1'=1$ and $w_2'=1$ iff $w'=1$, so $w=w'$.

Suppose now that $[s]_i=\ulcorner\exists \mathsf{v}_k((\mathsf{v}_k<u)\wedge[s]_{i_1})\urcorner$ for some $i_1<i$. Then by unique readability, $[s']_{i'}=\ulcorner\exists \mathsf{v}_k((\mathsf{v}_k<u)\wedge[s']_{i_1'})\urcorner$ where $[s]_{i_1}=[s']_{i_1'}$, for some $i_1'<i'$. Let $U=\mathrm{val}(u, y)$. It follows from the definition of $\mathrm{satseq}_{\Delta_0}(s, t)$ that

$$w=1 \Leftrightarrow \exists z<U \exists l_1<l(\langle i_1, y[z/k], 1\rangle=[t]_{l_1})$$

and

$$w'=1 \Leftrightarrow \exists z<U \exists l_1'<l'(\langle i_1', y[z/k], 1\rangle=[t']_{l_1'}).$$

But then it follows that $w=1$ iff $w'=1$, by the induction hypothesis and the definition of $\mathrm{satseq}_{\Delta_0}(s, t)$, and hence $w=w'$.

All the other cases are proved in a similar way. □

We have now done all the hard work. All that is required is to collect the facts together into a theorem.

THEOREM 9.13. The formula $\mathrm{Sat}_{\Delta_0}(x, y)$ is $\Delta_1(PA)$, and PA proves the following properties of it:

$$\mathrm{Sat}_{\Delta_0}(\ulcorner(r=s)\urcorner, y) \leftrightarrow \mathrm{val}(r, y)=\mathrm{val}(s, y)$$

$$\mathrm{Sat}_{\Delta_0}(\ulcorner(r<s)\urcorner, y) \leftrightarrow \mathrm{val}(r, y)<\mathrm{val}(s, y)$$

$$\mathrm{Sat}_{\Delta_0}(\ulcorner(u\wedge v)\urcorner, y) \leftrightarrow \mathrm{Sat}_{\Delta_0}(u, y)\wedge \mathrm{Sat}_{\Delta_0}(v, y)$$

$$\mathrm{Sat}_{\Delta_0}(\ulcorner(u\vee v)\urcorner, y) \leftrightarrow \mathrm{Sat}_{\Delta_0}(u, y)\vee \mathrm{Sat}_{\Delta_0}(v, y)$$

$$\mathrm{Sat}_{\Delta_0}(\ulcorner\neg u\urcorner, y) \leftrightarrow \neg\mathrm{Sat}_{\Delta_0}(u, y)$$

$$\mathrm{Sat}_{\Delta_0}(\ulcorner\exists \mathsf{v}_i((\mathsf{v}_i<r)\wedge u)\urcorner, y) \leftrightarrow \exists x<\mathrm{val}(r, y)\, \mathrm{Sat}_{\Delta_0}(u, y[x/i])$$

$$\mathrm{Sat}_{\Delta_0}(\ulcorner \forall \mathsf{v}_i(\neg(\mathsf{v}_i < r) \vee u)\urcorner, y) \leftrightarrow \forall x < \mathrm{val}(r, y)\, \mathrm{Sat}_{\Delta_0}(u, y[x/i])$$

for all y, i, r, s, u, v.

Proof. We have already observed that $\mathrm{Sat}_{\Delta_0}(x, y)$ is $\Sigma_1(PA)$. It is also $\Pi_1(PA)$ since it is equivalent to

$$\begin{aligned}&\mathrm{form}_{\Delta_0}(x) \wedge \\ &\quad \forall s, t(s, t \text{ determines } \mathrm{Sat}_{\Delta_0}(x, y) \rightarrow \\ &\qquad \exists i < \mathrm{len}(s) \exists l < \mathrm{len}(t)\, ([t]_l = \langle i, y, 1\rangle \wedge [s]_i = x))\end{aligned}$$

by an argument similar to that in the proof of Proposition 9.5, using Lemmas 9.11 and 9.12. The properties indicated follow immediately from the Σ_1 form of $\mathrm{Sat}_{\Delta_0}(x, y)$. □

In particular it follows that

$$\begin{aligned}&PA \vdash \forall x_0, \ldots, x_{n-1}, y[\theta(\mathsf{v}_0, \ldots, \mathsf{v}_{n-1}) \leftrightarrow \\ &\qquad \mathrm{Sat}_{\Delta_0}(\ulcorner \theta(\mathsf{v}_0, \ldots, \mathsf{v}_{n-1})\urcorner, [x_0, \ldots, x_{n-1}] ^\frown y)]\end{aligned}$$

for each Δ_0 formula $\theta(\mathsf{v}_0, \ldots, \mathsf{v}_{n-1})$ with only the free-variables shown.

The last thing to do in this section is to define the Σ_n and Π_n formulas $\mathrm{Sat}_{\Sigma_n}(x, y)$ and $\mathrm{Sat}_{\Pi_n}(x, y)$.

Definition. $\mathrm{Sat}_{\Sigma_0}(x, y)$ and $\mathrm{Sat}_{\Pi_0}(x, y)$ are alternative names for the $\Delta_1(PA)$ formula $\mathrm{Sat}_{\Delta_0}(x, y)$.

$\mathrm{Sat}_{\Sigma_{n+1}}(x, y)$ is defined to be the $\Sigma_{n+1}(PA)$ formula

$$\begin{aligned}&\exists s, t[\mathrm{len}(t) = \mathrm{len}(s) > 0 \wedge \mathrm{formseq}_{\Sigma_{n+1}}(s) \wedge [s]_{\mathrm{len}(s)-1} = x \wedge \\ &\quad [t]_{\mathrm{len}(t)-1} = y \wedge \forall i < \mathrm{len}(s)\,(i > 0 \rightarrow \\ &\quad \exists k \leqslant s \exists z \leqslant t([s]_i = \ulcorner \exists \mathsf{v}_k [s]_{i-1}\urcorner \wedge [t]_{i-1} = [t]_i[z/k])) \wedge \mathrm{Sat}_{\Pi_n}([s]_0, [t]_0)]\end{aligned}$$

and $\mathrm{Sat}_{\Pi_{n+1}}(x, y)$ is the $\Pi_{n+1}(PA)$ formula

$$\begin{aligned}&\mathrm{form}_{\Pi_{n+1}}(x) \wedge \forall s, t[(\mathrm{len}(t) = \mathrm{len}(s) > 0 \wedge \mathrm{formseq}_{\Pi_{n+1}}(s) \wedge \\ &\quad [s]_{\mathrm{len}(s)-1} = x \wedge [t]_{\mathrm{len}(t)-1} = y \wedge \\ &\quad \forall i < \mathrm{len}(s)\,(i > 0 \rightarrow \exists k \leqslant s \exists z \leqslant t([s]_i = \ulcorner \forall \mathsf{v}_k [s]_{i-1}\urcorner \wedge [t]_{i-1} = [t]_i[z/k]))) \\ &\quad \wedge \mathrm{Sat}_{\Sigma_n}([s]_0, [t]_0)].\end{aligned}$$

THEOREM 9.14. For each $n \geq 1$, $\mathrm{Sat}_{\Sigma_n}(x, y)$ is $\Sigma_n(PA)$, $\mathrm{Sat}_{\Pi_n}(x, y)$ is $\Pi_n(PA)$, and PA proves the following sentences:

$$\forall x[\mathrm{form}_{\Sigma_{n-1}}(x) \rightarrow \forall y(\mathrm{Sat}_{\Sigma_n}(x, y) \leftrightarrow \mathrm{Sat}_{\Sigma_{n-1}}(x, y)) \wedge$$
$$\forall y(\mathrm{Sat}_{\Pi_n}(x, y) \leftrightarrow \mathrm{Sat}_{\Sigma_{n-1}}(x, y))]$$
$$\forall x[\mathrm{form}_{\Pi_{n-1}}(x) \rightarrow \forall y(\mathrm{Sat}_{\Sigma_n}(x, y) \leftrightarrow \mathrm{Sat}_{\Pi_{n-1}}(x, y)) \wedge$$
$$\forall y(\mathrm{Sat}_{\Pi_n}(x, y) \leftrightarrow \mathrm{Sat}_{\Pi_{n-1}}(x, y))]$$
$$\forall x, y, k(\mathrm{Sat}_{\Sigma_n}(\ulcorner \exists \mathsf{v}_k x \urcorner, y) \leftrightarrow \exists z\, \mathrm{Sat}_{\Sigma_n}(x, y[z/k]))$$

and

$$\forall x, y, k(\mathrm{Sat}_{\Pi_n}(\ulcorner \forall \mathsf{v}_k x \urcorner, y) \leftrightarrow \forall z\, \mathrm{Sat}_{\Pi_n}(x, y[z/k]))$$

Proof. By an easy induction on $n \in \mathbb{N}$. □

As a corollary, we have the following important property of $\mathrm{Sat}_{\Sigma_n}(x, y)$ and $\mathrm{Sat}_{\Pi_n}(x, y)$: PA proves

$$\forall x_0, \ldots, x_{k-1}, y(\mathrm{Sat}_{\Sigma_n}(\ulcorner \theta(\mathsf{v}_0, \ldots, \mathsf{v}_{k-1}) \urcorner,$$
$$[x_0, \ldots, x_{k-1}] \frown y) \leftrightarrow \theta(x_0, \ldots, x_{k-1}))$$

and

$$\forall x_0, \ldots, x_{k-1}, y(\mathrm{Sat}_{\Pi_n}(\ulcorner \psi(\mathsf{v}_0, \ldots, \mathsf{v}_{k-1}) \urcorner,$$
$$[x_0, \ldots, x_{k-1}] \frown y) \leftrightarrow \psi(x_0, \ldots, x_{k-1}))$$

for all $\theta \in \Sigma_n$ and all $\psi \in \Pi_n$ with only the free-variables shown.

We close this chapter with one very simple application, the theorem that the Σ_n/Π_n classes of formulas forms a hierarchy.

THEOREM 9.15. (Due to Kleene for $M = \mathbb{N}$). Let $M \vDash PA$ be arbitrary. Then for each $n \geq 1$ the formula $\neg \mathrm{Sat}_{\Sigma_n}(x, [x])$ is Π_n but not $\Sigma_n(M)$. Hence all the inclusions in Fig. 4 on p. 80 are proper.

Proof. If $\theta(x)$ is Σ_n and

$$M \vDash \forall x(\theta(x) \leftrightarrow \neg \mathrm{Sat}_{\Sigma_n}(x, [x]))$$

then

$$M \vDash \theta(\ulcorner \theta(\mathsf{v}_0) \urcorner) \leftrightarrow \neg \mathrm{Sat}_{\Sigma_n}(\ulcorner \theta(\mathsf{v}_0) \urcorner, [\ulcorner \theta(\mathsf{v}_0) \urcorner]) \leftrightarrow \neg \theta(\ulcorner \theta(\mathsf{v}_0) \urcorner)$$

contradicting the above remarks. Thus $\neg\mathrm{Sat}_{\Sigma_n}(x, [x])$ is $\Pi_n(M)$ but not $\Sigma_n(M)$, $\mathrm{Sat}_{\Sigma_n}(x, [x])$ is $\Sigma_n(M)$ but not $\Pi_n(M)$, and neither of these formulas is $\Delta_n(M)$. □

Exercises for Section 9.3

9.7 Let $S \subseteq \mathbb{N}$. Show that there is a formula $\theta(x)$ that is $\Delta_{n+1}(\mathbb{N})$ such that $S = \{n \in \mathbb{N} \mid \mathbb{N} \models \theta(n)\}$ if and only if S is recursive in $\{(k, m) \mid \mathbb{N} \models \mathrm{Sat}_{\Sigma_n}(k, m)\}$.
9.8 Let T be any consistent extension of PA. Show that $\mathrm{Sat}_{\Delta_0}(x, y)$ is not equivalent in T to any Δ_0 formula $\theta(x, y)$.
9.9 Prove in PA that $\forall x, y(\mathrm{val}(x, y) \leqslant H(x, y))$ where $H(x, y)$ is the provably recursive function

$$(\max(y, 2))^{[\mathrm{len}(x)/4]+1}$$

9.10$^+$ Let x, a be given such that $\mathrm{form}_{\Delta_0}(x)$ and let s satisfy

$$\mathrm{formseq}_{\Delta_0}(s) \wedge [s]_{\mathrm{len}(s)-1} = x \wedge \forall i < \mathrm{len}(s)([s]_i \subseteq_p x).$$

Show that

$$\psi(0) \wedge \forall i < \mathrm{len}(s) - 1(\psi(i) \rightarrow \psi(i+1)),$$

where $\psi(i)$ is the formula

$$\forall m \leqslant i \forall y < B(i-m, s, a) \exists t \{\mathrm{satseq}_{\Delta_0}(s \restriction (m+1), t) \wedge \mathrm{len}(t) > 0 \\ \wedge \exists l < \mathrm{len}(t) \exists w \leqslant 1([t]_l = \langle m, y, w\rangle)\}$$

and $B(m, s, a)$ is the function defined by

$$B(0, s, a) = a;$$

$$B(m+1, s, a) = \max\left\{ y[\mathrm{val}(t, y)/j] \,\middle|\, \begin{array}{l} y < B(m, s, a), t \subseteq_p x, \mathrm{term}(t) \\ \text{and } j < x \end{array} \right\}.$$

Hence deduce that all of the results of this chapter (including Theorems 9.13 and 9.14) are provable from a finite fragment of $I\Sigma_1$.
9.11 Let $M \models PA$ be arbitrary. Extend Theorem 9.15 by showing that $\neg\mathrm{Sat}(x, [x])$ is not equivalent to a Σ_n formula $\psi(x)$ even if ψ contains parameters $\bar{a} \in M$.
(*Hint:* Define $\mathrm{clterm}(a) = \ulcorner(\cdots(1+1)+\cdots+1)\urcorner$ (a 1s) in M and given $\psi(\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_n)$, consider $\psi(\mathsf{v}_0, \mathrm{clterm}(a_1), \ldots, \mathrm{clterm}(a_n))$ obtained using the subst function of Exercise 9.5.)
9.12$^+$ Let

$$\Pi_n - \mathrm{Th}(\mathbb{N}) = \{\sigma \in \Pi_n \mid \mathbb{N} \models \sigma, \sigma \text{ an } \mathscr{L}_\mathrm{A}\text{-sentence}\}$$

and

$$\Sigma_n - \mathrm{Th}(\mathbb{N}) = \{\sigma \in \Sigma_n \mid \mathbb{N} \models \sigma, \sigma \text{ an } \mathscr{L}_{\mathrm{A}}\text{-sentence}\}.$$

Suppose $n \geqslant 1$.

(a) Show that $\Pi_n - \mathrm{Th}(\mathbb{N}) \vdash \Sigma_{n+1} - \mathrm{Th}(\mathbb{N})$.

(b) Show that there is a $\Delta_{n+1}(\mathbb{N})$ formula $\theta(x)$ such that for all $n \in \mathbb{N}$

$$\mathbb{N} \models \theta(n) \Leftrightarrow n = \ulcorner \sigma \urcorner \text{ for some } \sigma \in \Pi_n - \mathrm{Th}(\mathbb{N}).$$

(c) Let $\theta(x_1, \ldots, x_k, y)$ be a Σ_{n+1} formula such that $\mathbb{N} \models \forall \bar{x} \exists ! y \theta(\bar{x}, y)$. Show that the function f: $\mathbb{N}^k \to \mathbb{N}$ defined by θ is Σ_{n+1}-represented in $\Pi_n - \mathrm{Th}(\mathbb{N})$.

(d) Deduce that for all consistent theories T extending $\Pi_n - \mathrm{Th}(\mathbb{N})$ such that there is a $\Delta_{n+1}(\mathbb{N})$ formula $\psi(x)$ with

$$\mathbb{N} \models \psi(m) \Leftrightarrow m = \ulcorner \tau \urcorner \text{ for some } \tau \in T$$

for each $m \in \mathbb{N}$, there is a Π_{n+1} sentence σ such that $T + \sigma$ and $T + \neg\sigma$ are both consistent. Use (a) to show that $\mathbb{N} \models \sigma$.

(*Hint*: Follow the argument in Section 3.2. You may find Exercise 9.7 helpful.)

10

Subsystems of PA

I cannot bring a world quite round,
Although I patch it as I can.

I sing a hero's head, large eye
And bearded bronze, but not a man,

Although I patch him as I can
And reach through him almost to man.

If to serenade almost to man
Is to miss, by that, things as they are,

Say that it is the serenade
Of a man that plays a blue guitar.

Wallace Stevens
From: *The man with the blue guitar*

In this chapter we return to considering the subsystems $I\Sigma_n$, $B\Sigma_n$ of PA defined in Section 7.1. (The reader should consult the diagram in Exercise 7.3 as a guide to the various implications between these theories.) The two main results of this chapter show that these subsystems form a hierarchy, and therefore PA is not finitely axiomatized.

10.1 Σ_n-DEFINABLE ELEMENTS

Let $M \vDash PA^-$ and $A \subseteq M$. We shall consider the substructures $K^n(M; A)$ of M, which are defined in a similar way to $K(M; A)$ of Chapter 8, except that only elements defined by Σ_n formulas are in $K^n(M; A)$.

DEFINITION. Let $M \vDash PA^-$, let $n \geqslant 1$ and let $A \subseteq M$. $K^n(M; A)$ in the substructure of M consisting of all $b \in M$ such that

$$M \vDash \theta(b, \bar{a}) \wedge \forall x(\theta(x, \bar{a}) \rightarrow x = b)$$

for some $\theta(x, \bar{y}) \in \Sigma_n$ and some $\bar{a} \in A$. (Such b are said to be Σ_n-definable in M over A.)

It is clear that $K^n(M; A)$ is a *substructure* of M: for example if $b, c \in K^n(M; A)$ are defined by $\eta(y, \bar{a})$ and $\xi(z, \bar{a})$ respectively, where

$\eta, \xi \in \Sigma_n$ and $n \geq 1$, then $b+c$ is defined by the Σ_n formula

$$\exists y, z(\eta(y, \bar{a}) \wedge \xi(z, \bar{a}) \wedge x = y + z).$$

Just as in Chapter 8 we denote $K^n(M; \{\bar{a}\})$ and $K^n(M; \emptyset)$ more simply by $K^n(M; \bar{a})$ and $K^n(M)$ respectively.

The next theorem is the analogue of Theorem 8.1 for the structures $K^n(M; A)$.

THEOREM 10.1. Let $n \geq k \geq 1$ and suppose $A \subseteq M \vDash I\Sigma_{k-1}$. Then $A \subseteq K^n(M; A) \prec_{\Sigma_k} M$.

Proof. $A \subseteq K^n(M; A)$ since each $a \in A$ is defined by the formula $x = a$.

Recall that, for each $l \geq 2$ the functions $\langle x_1, \ldots, x_l \rangle$ have Δ_0 graph and are one-to-one correspondences between M^l and M for all models $M \vDash I\Delta_0$. Thus, if $x, \bar{y} \in M$ and $x = \langle y_1, \ldots, y_l \rangle \in K^n(M; A)$, then each y_i is in $K^n(M; A)$, since if x is defined by $\zeta(x, \bar{a})$ with $\zeta \in \Sigma_n$ and $\bar{a} \in A$, then y_i is defined by

$$\exists x, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_l(\zeta(x, \bar{a}) \wedge x = \langle y_1, \ldots, y_i, \ldots, y_l \rangle)$$

which is Σ_n. This reduces the problem of showing $K^n(M; A) \prec_{\Sigma_k} M$ to showing that

$$\text{if} \quad \bar{b} \in K^n(M; A) \quad \text{and} \quad \varphi(x, \bar{b}) \in \Pi_{k-1} \quad \text{then}$$
$$M \vDash \exists x \varphi(x, \bar{b}) \Rightarrow \exists c \in K^n(M; A) \text{ s.t. } M \vDash \varphi(c, \bar{b})$$

by the Tarski–Vaught test (Exercise 7.4) using the function $\langle \bar{y} \rangle$ to replace a Π_{k-1} formula $\varphi(\bar{x}, \bar{b})$ with $\forall \bar{y} \leq x(x = \langle \bar{y} \rangle \rightarrow \varphi(\bar{y}, b))$.

Now suppose $M \vDash \exists x \varphi(x, \bar{b})$, $\varphi \in \Pi_{k-1}$, $\bar{b} \in K^n(M; A)$ and $\bar{b} = (b_1, \ldots, b_j)$ with each b_i defined by the Σ_n-formula $\eta_i(v_i, \bar{a})$ ($\bar{a} \in A$). Since $M \vDash I\Sigma_{k-1}$ and $I\Sigma_{k-1} \vdash L\Pi_{k-1}$ (Exercise 7.3), by the least number principle for $\varphi(x, \bar{b})$

$$M \vDash \exists! z[\varphi(z, \bar{b}) \wedge \forall u < z \neg \varphi(u, \bar{b})].$$

Now $\neg \varphi(u, \bar{b})$ is Σ_{k-1} and, if $k \geq 2$, $I\Sigma_{k-1} \vdash B\Sigma_{k-1}$ (Proposition 7.2) so $\forall u < z \neg \varphi(u, \bar{b})$ is equivalent in M to a Σ_{k-1} formula by Proposition 7.1. If $n = 1$ then $\forall u < z \neg \varphi(u, \bar{b})$ is Δ_0 since φ is. Thus in both cases

$$M \vDash \exists! z \exists v_1, \ldots, v_j[\textstyle\bigwedge_{i=1}^{j} \eta_i(v_i, \bar{a}) \wedge \varphi(z, \bar{v}) \wedge \forall u < z \neg \varphi(u, \bar{v})]$$

and the formula inside square brackets here is a conjunction of Σ_n, Π_{k-1}, and Σ_{k-1} formulas, so is Σ_n. If $\bar{v}, z$ satisfy this formula then $v_i = b_i$ for all i

and so the unique$_j$ z satisfying $\exists v_1, \ldots, v_j[\bigwedge_{i=1}^{j} \eta_i(v_i, \bar{a}) \wedge \varphi(z, \bar{v}) \wedge \forall u < z \neg\varphi(u, \bar{v})]$ is in $K^n(M; A)$ and $M \vDash \varphi(z, \bar{b})$, as required. □

Notice that, even if $A = \varnothing$ and $n = 1$, $K^n(M: A)$ may be nonstandard. This will happen for example when $M \vDash PA + \exists x \chi(x)$, where χ is Δ_0 and $\mathbb{N} \vDash \forall x \neg\chi(x)$. (Such M exist by the incompleteness theorem.) Then $K^1(M) \supseteq_e \mathbb{N}$ and contains the least $x \in M$ such that $M \vDash \chi(x)$. But as $\mathbb{N} <_{\Delta_0} M$ we cannot have $x = n \in \mathbb{N}$, since otherwise $\mathbb{N} \vDash \chi(n)$.

Now if $K^n(M; A)$ is nonstandard, $n \geqslant 1$, and A is finite ($A = \{\bar{a}\}$ say), then then $K^n(M; A)$ will fail to satisfy PA. This is because for all $c \in K^n(M; A)$ there exists $e \in \mathbb{N}$ such that

$$M \vDash \mathrm{Sat}_{\Sigma_n}(e, [\bar{a}, c]) \wedge \forall y(\mathrm{Sat}_{\Sigma_n}(e, [\bar{a}, y]) \rightarrow y = c).$$

Since this formula is the conjunction of a Σ_n formula and a Π_n formula, Theorem 10.1 gives us

$$K^n(M; A) \vDash \mathrm{Sat}_{\Sigma_n}(e, [\bar{a}, c]) \wedge \forall y(\mathrm{Sat}_{\Sigma_n}(e, [\bar{a}, y]) \rightarrow y = c).$$

Thus for any nonstandard $d \in K^n(M; A)$

$$K^n(M; A) \vDash \forall c \exists e < d[\mathrm{Sat}_{\Sigma_n}(e, [\bar{a}, c]) \wedge \forall y(\mathrm{Sat}_{\Sigma_n}(e, [\bar{a}, y]) \rightarrow y = c)].$$

However if $K^n(M; A) \vDash PA$ there would be a least such $d \in K^n(M; A)$, d_0 say, satisfying the above formula. But this would imply $d_0 \in \mathbb{N}$, since it must be less than or equal to any nonstandard $d \in K^n(M; A)$, and so $K^n(M; A)$ would only have finitely many elements, which is impossible. Thus

THEOREM 10.2. No consistent extension of PA is finitely axiomatized.

Proof. If $T \supseteq PA$ is finitely axiomatized, then for some $n \in \mathbb{N}$ all axioms of T are Π_n. Consider any nonstandard $M \vDash T$ and take $a > \mathbb{N}$ in M. Let $K = K^n(M; a) <_{\Sigma_n} M$. Then $K \vDash T$ since $M \vDash T$ and each axiom of T is Π_n, hence $K \vDash PA$, which is impossible since K is nonstandard. □

Theorem 10.2 is due (essentially) to Ryll-Nardzewski (1952). We shall now discuss, in rather more detail, which induction axioms hold and which fail in the models $K^n(M; A)$.

PROPOSITION 10.3. If $M \vDash I\Sigma_k$, $A \subseteq M$ and $n > k \geqslant 0$. Then $K^n(M; A) \vDash I\Sigma_k$.

Proof. Let $\bar{b} \in K^n(M; A)$ and consider a Σ_k formula $\theta(x, \bar{b})$. If

$$K^n(M; A) \vDash \theta(0, \bar{b}) \wedge \forall x(\theta(x, \bar{b}) \rightarrow \theta(x+1, \bar{b}))$$

then, since this is equivalent to a Π_{k+1} formula and $K^n(M; A) <_{\Sigma_{k+1}} M$ by

Theorem 10.1,

$$M \models \theta(0, \bar{b}) \wedge \forall x(\theta(x, \bar{b}) \to \theta(x+1, \bar{b})),$$

hence $M \models \forall x\theta(x, \bar{b})$ since $M \models I\Sigma_k$, and $\forall x\theta(x, \bar{b})$ is also Π_{k+1} hence $K^n(M; A) \models \forall x\theta(x, \bar{b})$, by Theorem 10.1 again, as required. □

The next result shows that in general Propositon 10.3 cannot be improved. It has the consequence that $I\Sigma_{n-1} \nvdash B\Sigma_n$ for all $n \geq 1$, a result due to Parsons (1970). The proof we give here is due to Paris and Kirby (1978).

THEOREM 10.4. Let $M \models PA$, $A \subseteq M$ a finite subset of M, and let $n \geq 1$ with $K^n(M; A)$ nonstandard. Then $K^n(M; A) \nvDash B\Sigma_n$.

Proof. Using the pairing function, every $b \in K^n(M; A)$ is definable in M by a formula of the form

$$\exists y \gamma(x, y, \bar{a}),$$

where γ is Π_{n-1}, $A = \{\bar{a}\}$ and y is a single variable. (For if b is defined by $\exists \bar{z}\delta(x, \bar{z}, \bar{a})$ then b is also defined by $\exists y(\forall \bar{z}(y = \langle \bar{z} \rangle \to \delta(x, \bar{z}, \bar{a}))$ which is of the appropriate form.) Thus for all $b \in K^n(M; A)$ there is $e \in \mathbb{N}$ (namely $e = \ulcorner \gamma(v_0, v_1, \bar{v}) \urcorner$ for a suitable γ) such that

$$M \models \exists u[\exists u_0, u_1(u = \langle u_0, u_1 \rangle \wedge u_0 = b \wedge \mathrm{Sat}_{\Pi_{n-1}}(e, [u_0, u_1, \bar{a}])$$
$$\wedge \forall z < u \forall z_0, z_1 \leq z(z = \langle z_0, z \rangle \to \neg \mathrm{Sat}_{\Pi_{n-1}}(e, [z_0, z_1, \bar{a}])))]$$

Since $\mathrm{Sat}_{\Pi_{n-1}}$ is Π_{n-1}, $\neg\mathrm{Sat}_{\Pi_{n-1}}$ is Σ_{n-1}, and Σ_{n-1} is closed under bonded quantification in PA, the formula in square brackets above is equivalent (in PA) to a Σ_n formula $\lambda(\bar{a}, b, e, u)$. Thus, if $t \in K^n(M; A)$ is nonstandard, for all $b \in K^n(M; A)$ we have

$$M \models \exists e < t \exists u \lambda(\bar{a}, b, e, u)$$

and hence (since this formula is Σ_n) by Theorem 10.1 we have

$$K^n(M; A) \models \exists e < t \exists u \lambda(\bar{a}, b, e, u).$$

But b here was arbitrary (as long as it was a member of $K^n(M; A)$) so we deduce that

$$K^n(M; A) \models \forall b < t + 1 \exists e < t \exists u \lambda(\bar{a}, b, e, u). \quad (*)$$

But the formula on the right-hand side of (*) is a Σ_n formula with a bounded universal quantifier preceding it, so if $K^n(M; A) \models B\Sigma_n$ this formula would be equivalent in both $K^n(M; A)$ and M to a Σ_n formula

$\chi(t, \bar{a})$, by Proposition 7.1. But then we would have $K^n(M; A) \vDash \chi(t, \bar{a}) \Rightarrow M \vDash \chi(t, \bar{a})$ by Theorem 10.1, hence

$$M \vDash \forall b < t + 1 \exists e < t \exists u \lambda(\bar{a}, b, e, u).$$

This would contradict the pigeonhole principle in M (Exercise 5.12), since clearly b is uniquely determined by any e satisfying $\exists u \lambda(\bar{a}, b, e, u)$, b being u_0 where $u = \langle u_0, u_1 \rangle$ and u is the least element of M satisfying $\mathrm{Sat}_{\Pi_{n-1}}(e, [u_0, u_1, \bar{a}])$. □

Exercises for Section 10.1

10.1 Let $M \vDash PA$ and $n \geq 1$. Show that $\mathbb{N} <_{\Sigma_n} M$ iff $K^n(M) = \mathbb{N}$.
10.2 Let $n \geq 1$.
(a) Show that both $I\Sigma_n$ and $B\Sigma_n$ can be axiomatized by sets of Π_{n+2} sentences of $\mathscr{L}_A$.
(b) Show that neither $I\Sigma_n$ nor $B\Sigma_n$ are axiomatized by a set of Σ_{n+2} sentences. (*Hint:* Take $M \vDash \mathrm{Th}(\mathbb{N})$ nonstandard with $a > \mathbb{N}$ in M and show $K^n(M; a) \vDash \Sigma_{n+2} - \mathrm{Th}(\mathbb{N})$.)
10.3 Let $B\Sigma_n^-$ denote the theory axiomatized by $I\Delta_0$ together with

$$\forall \bar{a} \forall x \exists y \theta(x, y, \bar{a}) \rightarrow \forall \bar{a} \forall t \exists z \forall x < t \exists y < z \theta(x, y, \bar{a})$$

for all $\theta(x, y, a)$ in Σ_n.
(a) Show that $B\Sigma_n^- \vdash I\Sigma_{n-1}$.
(b) Show that $B\Sigma_n^-$ is axiomatized by a set of Σ_{n+2} sentences.
(c) Let $M \vDash PA^-$. Show that $K^n(M) \vDash B\Sigma_n$ iff $K^n(M) \vDash B\Sigma_n^-$.
(d) Deduce that $I\Sigma_{n-1} \nvdash B\Sigma_n^-$ for all $n \geq 1$.
10.4 Assume that all the properties of $\mathrm{Sat}_{\Sigma_k}(x, y)$ described in Chapter 9 are provable in $I\Sigma_1$ (see Exercise 9.10). Show that $I\Sigma_n$ and $B\Sigma_{n+1}$ are finitely axiomatizable for all $n \geq 1$.
(*Hint:* Consider the sentence

$$\begin{aligned}\forall a, b[&\mathrm{Sat}_{\Sigma_n}(a, [0, b]) \wedge \forall x(\mathrm{Sat}_{\Sigma_n}(a, [x, b]) \\ &\rightarrow \mathrm{Sat}_{\Sigma_n}(a, [x+1, b])) \rightarrow \forall x \mathrm{Sat}_{\Sigma_n}(a, [x, b])]\end{aligned}$$

and something similar for $B\Sigma_{n+1}$. It is unknown whether $I\Delta_0$ or $B\Sigma_1$ are finitely axiomatizable.)

10.2 Σ_n-ELEMENTARY INITIAL SEGMENTS

We now turn our attention to the implication $I\Sigma_n \Rightarrow B\Sigma_n$. Our aim is to show that this does not reverse, that is we will construct a model of $B\Sigma_n$ that does not satisfy $I\Sigma_n$. The idea is to construct a suitable Σ_{n-1}-elementary initial segment of a model of PA.

PROPOSITION 10.5. Let $n \geqslant 1$, $M \models I\Sigma_{n-1}$, and let $I \subseteq_e M$ be a proper Σ_{n-1}-elementary initial segment of M closed under $+$ and $\cdot$. Then $I \models B\Sigma_n$.

Proof. Note that $I \models I\Delta_0$ since $I \prec_{\Delta_0} M \models I\Delta_0$ and $I\Delta_0$ is Π_1-axiomatized. (See Exercise 6.5 (b)). Suppose $a \in I$ and $I \models \forall x < a \exists \bar{y} \theta(x, \bar{y})$ with $\theta \in \Pi_{n-1}$. Then for every $x \in M$ with $x < a$, x is an element of I (since $I \subseteq_e M$) and so there is $\bar{y} \in I$ such that $\theta(x, \bar{y})$ is true in I, and hence $\theta(x, \bar{y})$ is also true in M (since $\theta(x, \bar{y})$ is Π_{n-1} and $I \prec_{\Sigma_{n-1}} M$). Therefore for all $b \in M \setminus I$, $M \models \forall x < a \exists \bar{y} < b$ $\theta(x, \bar{y})$. This formula is equivalent (in $I\Sigma_{n-1}$) to a Π_{n-1} formula, since it is formed by applying bounded quantifiers to a Π_{n-1} formula. Thus we may use the least number principle, $L\Pi_{n-1}$, to deduce that there is a least b in M such that $M \models \forall x < a \exists \bar{y} < b \theta(x, \bar{y})$. Clearly this least such b is in I, so for any $x < a$ there is $\bar{y} < b$ such that $M \models \theta(x, \bar{y})$. But this $\bar{y}$ is in I since $b \in I \subseteq_e M$, and as $I \prec_{\Sigma_{n-1}} M$, $I \models \theta(x, \bar{y})$, hence $I \models \forall x < a \exists \bar{y} < b \theta(x, \bar{y})$, as required. □

We need a way of constructing Σ_{n-1}-elementary initial segments of a model M.

DEFINITION. Let $M \models PA^-$ with $A \subseteq M$ and $n \geqslant 1$. $I^n(M; A)$ is the initial segment

$$\{x \in M \mid M \models x \leqslant b \text{ for some } b \in K^n(M; A)\}$$

of M. $I^n(m)$ and $I^n(M; \bar{a})$ denote $I^n(M, \varnothing)$ and $I^n(M; \{\bar{a}\})$ respectively. Notice that $I^n(M; A)$ is an $\mathscr{L}_A$-structure, since it is closed under $+$ and $\cdot$. For example if $x \leqslant b \in K^n(M; A)$ and $y \leqslant c \in K^n(M; A)$ then $x + y \leqslant b + c$, $x \cdot y \leqslant b \cdot c$ and $b + c \in K^n(M; A)$.

We will prove that if $M \models I\Sigma_{n-1}$, then $I^n(M; A) \prec_{\Sigma_{n-1}} M$. The surprising thing about this is that we only need $M \models I\Sigma_{n-1}$ for this to work. Essentially, the reason is in the following lemma.

LEMMA 10.6. Let $n \geqslant 1$. $I\Sigma_n$ proves the following instances of the Π_n-collection scheme

$$\forall \bar{a}, t \exists s \forall x < t (\forall \bar{y} \neg \varphi(x, \bar{y}, \bar{a}) \vee \exists \bar{z} < s \varphi(x, \bar{z}, \bar{a}))$$

for all $\varphi \in \Pi_{n-1}$.

Proof. The above instances are provable in Π_n-collection by considering

$$\forall x < t \exists \bar{z} [\forall \bar{y} \neg \varphi(x, \bar{y}, \bar{a}) \vee \varphi(x, \bar{z}, \bar{a})],$$

the formula inside square brackets being Π_n. We must prove it in $I\Sigma_n$.

In a model $M \models I\Sigma_n$, let $\bar{a} \in M$ and $\varphi(x, \bar{y}, \bar{a}) \in \Pi_{n-1}$. Let $\delta(w, x)$ be the $\Delta_1(PA)$ formula

$$\exists y, v \leqslant w (y < 2^x \wedge w = y + 2^x + 2^{x+1} \cdot v),$$

so $\delta(w, x)$ expresses 'the xth digit of the binary expansion of w is 1'. All the obvious properties of $\delta(w,x)$ are provable in $I\Sigma_1$, including the equivalence of its Π_1 and Σ_1 forms (so $\delta(w,x)$ is in fact $\Delta_1(I\Sigma_1)$) and all the other properties we will need here, as the reader may check. By considering $w=2^t-1$ (which has all its binary digits equal to 1) we have

$$M \vDash \exists w[\forall x<t(\exists \bar{z}\varphi(x, \bar{z}, \bar{a}) \rightarrow \delta(w, x))].$$

The formula in square brackets here is Π_n, so using $L\Pi_n$ in M there is a least such w, w_0 say. For this least such w, we have

$$M \vDash \forall x<t(\exists \bar{z}\varphi(x, \bar{z}, \bar{a}) \leftrightarrow \delta(w_0, x))$$

since if $\delta(w_0, x) \wedge \neg\exists \bar{z}\varphi(x, \bar{z}, \bar{a})$ then $w_0=y+2^x+2^{x+1}\cdot v$ for some $y<2^x$ and some $v \leqslant w$, and $w_0'=y+2^{x+1}\cdot v<w_0$ would also satisfy

$$\forall x<t(\exists \bar{z}\varphi(x, \bar{z}, \bar{a}) \rightarrow \delta(w_0', x)),$$

contradicting our choice of w_0.

But then $M \vDash \forall x<t\exists \bar{z}(\neg\delta(w_0, x) \vee \varphi(x, \bar{z}, \bar{a}))$, so by applying the Π_{n-1} collection scheme to this, there is $s \in M$ such that

$$M \vDash \forall x<t\exists \bar{z}<s(\neg\delta(w_0, x) \vee \varphi(x, \bar{z}, \bar{a}))$$

and so $M \vDash \forall x<t(\exists \bar{z}<s\varphi(x, \bar{z}, \bar{a}) \vee \forall \bar{y}\neg\varphi(x, \bar{y}, \bar{a}))$, as required. □

THEOREM 10.7. Let $n \geqslant 1$, $M \vDash I\Sigma_{n-1}$, and $A \subseteq M$. Then $I^n(M; A) \prec_{\Sigma_{n-1}} M$.

Proof. Since $I^n(M; A) \subseteq_e M$ and is closed under $+$, $\cdot$, $I^n(M; A) \prec_{\Delta_0} M$. Thus we may assume $n \geqslant 2$. By the Tarski–Vaught test, it is sufficient to show that whenever $\bar{b} \in I^n(M; A)$, $\theta(\bar{x}, \bar{b}) \in \Pi_{n-2}$ and $M \vDash \exists \bar{x}\theta(\bar{x}, \bar{b})$ there is $\bar{c} \in I^n(M; A)$ such that $M \vDash \theta(\bar{c}, \bar{b})$. By using the pairing function $\langle x, y\rangle = z$ and by considering the Π_{n-2} formula $\forall \bar{u}, \bar{v}(\langle \bar{u}\rangle = x \wedge \langle \bar{v}\rangle = b \rightarrow \theta(\bar{u}, \bar{v}))$ if necessary, we may assume that x, b are single variables in $\theta(x, b)$.

Now suppose $\theta(x, b) \in \Pi_{n-2}$, $M \vDash \exists x\theta(x, b)$ and $b<c$ where c is Σ_n-defined in M by the formula $\eta(x, \bar{a})$ ($\bar{a} \in A$). By Lemma 10.6 there is $d \in M$ such that

$$M \vDash \forall y<c(\exists x<d\theta(x, y) \vee \forall z\neg\theta(z, y)).$$

This formula is Π_{n-1}, since $\exists x<d\theta(x, y)$ is equivalent in M to a Π_{n-2} formula; and so there is a least such d in M, which satisfies

$$\begin{aligned} M \vDash \exists c[&\eta(c, \bar{a}) \wedge \forall y<c(\exists x<d\theta(x, y) \vee \forall z\neg\theta(z, y)) \\ &\wedge \forall v<d\exists y<c(\forall x<v\neg\theta(x, y) \wedge \exists z\theta(z, y))]. \end{aligned}$$

This last formula is a Σ_n definition of d in M, so $d \in K^n(M; A)$. But $M \vDash \exists x < d\theta(x, b)$, so there is $x \in I^n(M; A)$ satisfying $M \vDash \theta(x, b)$, as required. □

The following result about the suhstructures $I^n(M; A)$ is also useful.

THEOREM 10.8. Let $n \geqslant 1$, $M \vDash B\Sigma_n$ and $A \subseteq M$. Then

$$I^n(M; A) \vDash \Pi_{n+1} - \mathrm{Th}(M, a)_{a \in A},$$

the set of all Π_{n+1} formulas $\theta(\bar{a})$ with parameters $\bar{a} \in A$ that are true in M.

Proof. Let $\theta(\bar{a})$ be $\forall \bar{x} \exists \bar{y} \psi(\bar{x}, \bar{y}, \bar{a})$ with $\psi \in \Pi_{n-1}$, let $b \in K^n(M; A)$ be arbitrary, and defined by the Σ_n formula $\zeta(x, \bar{a})$. We show that $I^n(M; A) \vDash \forall \bar{x} < b \exists \bar{y} \psi(\bar{x}, \bar{y}, \bar{a})$, which suffices since $K^n(M; A) \subseteq_{cf} I^n(M; A)$.

Since $M \vDash B\Sigma_n$ there is $c \in M$ such that $M \vDash \forall \bar{x} < b \exists \bar{y} < c\psi(\bar{x}, \bar{y}, \bar{a})$, and since $\forall \bar{x} < b \exists \bar{y} < c\psi$ is equivalent in M to a Π_{n-1} formula, there is a least such c. This c is defined in M by a Σ_n formula equivalent to

$$\exists b[\zeta(b, \bar{a}) \wedge \forall \bar{x} < b \exists \bar{y} < c\psi(\bar{x}, \bar{y}, \bar{a}) \wedge \forall u < c \exists \bar{x} < b \forall \bar{y} < u \neg \psi(\bar{x}, \bar{y}, \bar{a})]$$

Thus $c \in I^n(M; A)$. But $\forall \bar{x} < b \exists \bar{y} < c\psi(\bar{x}, \bar{y}, \bar{a})$ is equivalent to the same Π_{n-1} formula in both M and $I^n(M; A)$, since $I^n(M; A) \vDash B\Sigma_n$ by Theorem 10.7 and Proposition 10.5, hence

$$I^n(M; A) \vDash \forall \bar{x} < b \exists \bar{y} < c\psi(\bar{x}, \bar{y}, \bar{a}),$$

as $I^n(M; A) \prec_{\Sigma_{n-1}} M$, as required. □

A very useful corollary that can be drawn from Theorem 10.7 is the following result, due (independently) to Jeff Paris and Harvey Friedman.

COROLLARY 10.9 (Paris, Friedman). For all $n \geqslant 0$, $B\Sigma_{n+1}$ is Π_{n+2}-conservative over $I\Sigma_n$; that is, for all Π_{n+2} sentences σ,

$$I\Sigma_n \vdash \sigma \quad \text{iff} \quad B\Sigma_{n+1} \vdash \sigma.$$

Proof. Left-to-right is obvious, as $B\Sigma_{n+1} \vdash I\Sigma_n$. For the converse, suppose that σ is $\forall \bar{x} \exists \bar{y} \psi(\bar{x}, \bar{y})$ with $\psi \in \Pi_n$ and $I\Sigma_n \nvdash \sigma$. We must show that $B\Sigma_{n+1} \nvdash \sigma$. Let $\bar{a} \in M \vDash I\Sigma_n + \forall \bar{y} \neg \psi(\bar{a}, \bar{y})$ and let $M' \succ M$ be such that $I^{n+1}(M'; \bar{a}) \neq M'$. To do this, let M' satisfy

$$\{\theta(\bar{b}) \mid \bar{b} \in M \vDash \theta(\bar{b}), \theta \text{ an } \mathscr{L}_A\text{-formula}\}$$
$$\cup \{\exists! x\eta(x, \bar{a}) \rightarrow \exists x < c\eta(x, \bar{a}) \mid \eta \in \Sigma_{n+1}\}$$

in the language $\mathscr{L}_A$ expanded by adding constants b for each $b \in M$ and a further new constant c. The reduct of M' to $\mathscr{L}_A$ clearly has the required property, as the element of M' realizing the constant c is in M' but not in $I^{n+1}(M', \bar{a})$.) Then by Theorem 10.7, $\bar{a} \in I^{n+1}(M'; \bar{a}) \prec_{\Sigma_n} M'$, so by Proposition 10.5 and our choice of M', $I^{n+1}(M'; \bar{a}) \vDash B\Sigma_{n+1}$. But $M' \vDash \forall \bar{y} \neg \psi(\bar{a}, \bar{y})$, so $M' \vDash \neg \psi(\bar{a}, \bar{b})$ for all $\bar{b} \in I^{n+1}(M'; \bar{a})$, so $I^{n+1}(M'; \bar{a}) \vDash \forall \bar{y} \neg \psi(\bar{a}, \bar{y})$ as $I^{n+1}(M'; \bar{a}) \prec_{\Sigma_n} M'$ and $\neg\psi$ is Σ_n. Thus $I^{n+1}(M'; \bar{a})$ satisfies $B\Sigma_{n+1} + \neg\sigma$ and hence $B\Sigma_{n+1} \nvdash \sigma$. □

Corollary 10.9 shows that $I\Sigma_n$ and $B\Sigma_{n+1}$ are 'closer together' than one might think. There is no non-trivial conservation result between $B\Sigma_{n+1}$ and $I\Sigma_{n+1}$ since for example it can be shown that $I\Sigma_{n+1} \vdash \text{con}(I\Sigma_n)$ but $B\Sigma_{n+1} \nvdash \text{con}(I\Sigma_n)$, where $\text{con}(I\Sigma_n)$ is a natural Π_1 sentence expressing '$I\Sigma_n$ is consistent'.

We conclude this chapter with the promised theorem that $B\Sigma_n \nvdash I\Sigma_n$. This theorem was first proved by Paris and Kirby (1978) and, independently, Lessan (1978). The proof here is Lessan's.

THEOREM 10.10. Let $M \vDash PA$, $A \subseteq M$ finite, $n \geqslant 1$, and suppose that $I^n(M; A)$ is non-standard. Then $I^n(M; A) \vDash B\Sigma_n$ but $I^n(M; A) \nvDash I\Sigma_n$.

Proof. We have already seen that $I^n(M; A) \vDash B\Sigma_n$. Let $A = \{\bar{a}\}$ and suppose $b \in I^n(M; A)$ is nonstandard. Our strategy is to find $w \in M$ with $w < b$ and a Σ_n formula $\theta(x, w)$ such that, for all $x \in I^n(M; A)$,

$$x \in \mathbb{N} \Leftrightarrow I^n(M; A) \vDash \theta(x, w)$$

for then $I^n(M; A) \nvDash I_x\theta$, as $I^n(M; A) \neq \mathbb{N}$.

We find w by using overspill. The following is obviously true for all $x \in \mathbb{N}$:

$$\begin{aligned}
M \vDash \exists w < b\{&\text{len}(w) \geqslant x \wedge [w]_0 = \ulcorner(\text{v}_0 = 0)\urcorner \\
&\wedge \forall i < \text{len}(w)\,([w]_i \leqslant \max(\ulcorner(\text{v}_0 = 0)\urcorner, i) \wedge \exists y\, \text{Sat}_{\Pi_{n-1}}([w]_i, [y, \bar{a}])) \\
&\wedge \forall i < \text{len}(w)\,(i > 0 \to ([w]_i = [w]_{i-1} \wedge (\neg \exists z\, \text{Sat}_{\Pi_{n-1}}(i, [z, \bar{a}]) \\
&\qquad \vee \forall y(\text{Sat}_{\Pi_{n-1}}([w]_{i-1}, [y, \bar{a}]) \to \exists z \leqslant y\, \text{Sat}_{\Pi_{n-1}}(i, [z, \bar{a}])))) \\
&\quad \vee ([w]_i = i \wedge \exists z\, \text{Sat}_{\Pi_{n-1}}(i, [z, \bar{a}]) \\
&\qquad \wedge \forall z(\text{Sat}_{\Pi_{n-1}}(i, [z, \bar{a}]) \to \exists y \leqslant z\, \text{Sat}_{\Pi_{n-1}}([w]_{i-1}, [z, \bar{a}]))))\}
\end{aligned}$$

(In words, w codes the sequence $[w]_0, [w]_1, \ldots, [w]_{x-1}$ such that $[w]_0 = \ulcorner(\text{v}_0 = 0)\urcorner$ and for each $0 < i < x$, *either* $[w]_i = i$ and this happens only when i is the Gödel number of a Π_{n-1} formula $\theta(\text{v}_0, \bar{\text{v}})$ such that the least $u \in M$ satisfying $M \vDash \theta(u, \bar{a})$ exists and is greater than the least $u \in M$ such that $M \vDash \text{Sat}_{\Pi_{n-1}}([w]_i, [u, \bar{a}])$ *or* $[w]_i = [w]_{i-1}$ (when i does not have the property described above).

Thus, by overspill, there is $w<b$ in M such that the formula above inside curly brackets $\{\ldots\}$ above is true for some $x>\mathbb{N}$. The idea is that w now codes a sequence of Π_{n-1} formulas which are all satisfied in M, the least y satisfying $[w]_i$ being at least as large as the least y satisfying $[w]_{i-1}$, and all standard Π_{n-1} formulas are considered at some (standard) stage i.

We claim that, with w chosen as in the previous paragraph, and $c\in I^n(M;a)$ arbitrary, $c\in\mathbb{N}$ holds iff

$$I^n(M;A)\vDash\exists y\,\mathrm{Sat}_{\Pi_{n-1}}([w]_c,[y,\bar{a}]).$$

To prove this, suppose first that $c\in\mathbb{N}$. Then

$$M\vDash\exists y\,\mathrm{Sat}_{\Pi_{n-1}}([w]_c,[y,\bar{a}]).$$

But $[w]_c\leqslant\max(c,\ulcorner(\mathsf{v}_0=0)\urcorner)$, so $[w]_c=k\in\mathbb{N}$, and as the formula '$[w]_c=k$' is Δ_0, $I^n(M;A)\vDash[w]_c=k$. Also

$$M\vDash\exists y\,\mathrm{Sat}_{\Pi_{n-1}}(k,[y,\bar{a}])$$

so by Theorem 10.8 $I^n(M;A)\vDash\exists y\,\mathrm{Sat}_{\Pi_{n-1}}(k,[y,\bar{a}])$ hence $I_n(M;A)\vDash\mathrm{Sat}_{\Pi_{n-1}}([w]_c,[y,\bar{a}])$. Conversely, if

$$c,y\in I^n(M;A)\vDash\mathrm{Sat}_{\Pi_{n-1}}([w]_c,[y,\bar{a}]),$$

then $M\vDash\mathrm{Sat}_{\Pi_{n-1}}([w]_c,[y,\bar{a}])$ since $M>_{\Sigma_{n-1}}I^n(M;A)$. But $y\in I^n(M;A)$, so $y<d$ for some $d\in M$ that is Σ_n-defined by a formula $\exists\bar{v}\theta(d,\bar{v},\bar{a})\,(\theta\in\Pi_{n-1})$. Let $\psi(\mathsf{v}_0,\bar{\mathsf{v}})$ be the Π_{n-1} formula $\forall r,\bar{s}(\langle r,\bar{s}\rangle=\mathsf{v}_0\rightarrow\theta(r,\bar{s},\bar{\mathsf{v}}))$ and let $i=\ulcorner\psi(\mathsf{v}_0,\bar{\mathsf{v}})\urcorner$. Clearly if $v\in M$ satisfies $\psi(v,\bar{a})$ then $v\geqslant d>y$. Thus, by induction on j in M, we have

$$M\vDash\forall j\geqslant i(j<\mathrm{len}(w)\rightarrow\forall u(\mathrm{Sat}_{\Pi_{n-1}}([w]_j,[u,\bar{a}])\rightarrow u>y))$$

which shows that $c<i$, and hence $c\in\mathbb{N}$. □

Exercises for Section 10.2

10.5 By considering Exercise 10.3 show that there is a Σ_{n+2} sentence σ such that $I\Sigma_n\nvdash\sigma$ but $B\Sigma_{n+1}\vdash\sigma$, and hence the '$\Pi_{n+2}$' in Corollary 10.9 cannot be altered to 'Σ_{n+2}' or 'Π_{n+3}'.

10.6 Prove that the weak form of the Π_n collection scheme in Lemma 10.6, together with $I\Delta_0$, is actually equivalent to $I\Sigma_n$.

(*Hint*: If $\exists b\exists\bar{x}\varphi(b,\bar{x})$ holds, with $\varphi\in\Pi_{n-1}$, in some model, use the collection scheme to find c such that

$$\forall u\leqslant b(\exists\bar{x}<c\varphi(u,\bar{x})\vee\forall x\neg\varphi(u,\bar{x}))$$

and (inductively assuming that the model satisfies $I\Sigma_{n-1}$) consider the least-number principle applied to $\exists \bar{x} < c\varphi(u, \bar{x})$.)

10.7 *Let* $M \models \mathrm{Th}(\mathbb{N})$ with $a > \mathbb{N}$ in M. Show that there is no Σ_n formula $\theta(x)$ with only one free-variable such that

$$x \in \mathbb{N} \qquad \text{iff} \qquad I^n(M; a) \models \theta(x)$$

for all $x \in I^n(M; a)$. Hence deduce the nse of the parameter w in the proof of Theorem 10.10 cannot be avoided.

10.8⁺ Prove $I\Sigma_{n+1} \vdash \mathrm{con}(I\Sigma_n)$ for all $n \geqslant 0$, as follows: in the natural deduction system of Exercise 9.6 replace the induction axioms $I_v\varphi$ (where φ is an $\mathscr{L}_A$-formula) by the induction rule

$$\frac{\Gamma, \neg\varphi(v), \varphi(v+1)}{\Gamma, \neg\varphi(0), \varphi(s)}$$

(v any variable not occurring free in Γ, s any $\mathscr{L}_A$-term, φ any Σ_n formula of $\mathscr{L}_A$).

(a) Show that each induction axiom $I_x\varphi$ of $I\Sigma_n$ is provable in this system. Hence deduce that the corresponding provability predicate $\mathrm{provable}_{I\Sigma_n}(x)$ is a provability predicate for $I\Sigma_n$ as defined in Exercise 3.8.

(b) The *cut eliminaton theorem* for this system says that if p is a proof of the sequent Γ then there is a second proof, q, of Γ such that in any instance of the cut rule

$$\frac{\Delta, \varphi \quad \Delta, \neg\varphi}{\Delta}$$

in q, the formula φ is *either* an axiom of PA^- *or* a formula in one of the equality axioms *or* a Σ_n formula $\varphi(s)$ occurring in the induction axiom above. State and prove this theorem in $I\Sigma_1$.

(*Hint:* Modify the proof in Schwichtenberg 1977.)

Deduce (in $I\Sigma_1$) that if there is a proof of $0=1$ from this system, then there is a proof of $0=1$ such that all the formulas occurring in it are Σ_n or Π_n.

(c) Let q be a proof of $0=1$, all formulas in q being Σ_n or Π_n. Using Π_{n+1} induction show that

$$\forall x\, \mathrm{Sat}_{\Pi_{n+1}}(\bigvee \Gamma, x)$$

for all sequents Γ in q, where $\bigvee \Gamma$ is some Π_{n+1} formula equivalent to the disjunction of all formulas in Γ. Deduce that $\neg\mathrm{provable}_{I\Sigma_n}(0=1)$ is provable in $I\Sigma_{n+1}$.

11

Saturation

The sea was wet as wet could be . . .

Lewis Carroll. *The Walrus and the Carpenter*

Recall from Chapter 8 that, given a theory T, a *type over* T is a set $p(\bar{x})$ of formulas in finitely many free-variables $\bar{x}$ such that $T \cup p(\bar{x})$ is consistent. If $M \models T$ then the type $p(\bar{x})$ is either *realized* or *omitted* by M, and there is always some model of T that realizes $p(\bar{x})$.

Chapter 8 showed that the models K_T (for complete extensions T of PA) omit as many types as possible. In this chapter we shall look at those types that are realized in nonstandard models of PA.

11.1 CODED SETS

This first section is devoted to an example of central importance, one that indicates why we should concentrate our attention on *recursive types* (i.e. types $p(\bar{x})$ whose set of Gödel numbers $\{\ulcorner\varphi(\bar{x})\urcorner \mid \varphi(\bar{x}) \in p(\bar{x})\}$ is recursive).

We met the idea of a number $a \in M \models PA$ coding a set $S \subseteq \mathbb{N}$ in Chapter 1. The following definition makes this notion precise. (Actually I have changed the method of coding to make use of the techniques developed in Chapter 5, but the idea remains the same.)

Definition. Let $M \models PA$ and let $a \in M$. Then *the set coded by a in M* is the set $S = \{n \in N \mid M \models (a)_n \neq 0\}$. We say M codes $S \subseteq \mathbb{N}$ iff there is $a \in M$ such that a codes S in M. The *standard system* of M, $SSy(M)$, is the set of all subsets of $\mathbb{N}$ that are coded in M, i.e.,

$$SSy(M) = \{S \subseteq \mathbb{N} \mid S = \{n \in \mathbb{N} \mid M \models (a)_n \neq 0\} \text{ for some } a \in M\}.$$

(Notice that we are using $(x)_y$ here rather than $[x]_y$, since we have no need to keep track of the length of sequences.)

Now let $M \models PA$ be nonstandard, let $S \subseteq \mathbb{N}$, and let $p(x)$ be the following set of formulas

$$\{(x)_i \neq 0 \mid i \in S\} \cup \{(x)_i = 0 \mid i \notin S\}.$$

Clearly M realizes $p(x)$ iff S is coded in M, so questions about $SSy(M)$ can be reduced to questions about which types are realized in M. If S is *recursive*, though, by Corollary 3.7 there is a Σ_1 formula $\theta(x)$ such that

$$n \in S \Rightarrow PA^- \vdash \theta(n)$$

and

$$n \notin S \Rightarrow PA^- \vdash \neg\theta(n)$$

so that $p(x)$ is realized in M just in case the type

$$q(x) = \{(x)_i \neq 0 \leftrightarrow \theta(i) \mid i \in \mathbb{N}\}$$

is realized in M. Now for each $n \in \mathbb{N}$ we have

$$M \models \exists x \forall i < n((x)_i \neq 0 \leftrightarrow \theta(i))$$

using Lemma 5.8. Thus by overspill

$$M \models \exists x \forall i < b((x)_i \neq 0 \leftrightarrow \theta(i))$$

for some $b > \mathbb{N}$ in M. Then there is $a \in M$ satisfying

$$M \models \forall i < b((a)_i \neq 0 \leftrightarrow \theta(i))$$

and clearly, since $b > \mathbb{N}$, a codes the set S. Thus we have proved

LEMMA 11.1. If $S \subseteq \mathbb{N}$ is recursive then S is coded in all nonstandard models of PA. □

There is also an interesting converse to Lemma 11.1.

LEMMA 11.2. For each non-recursive $S \subseteq \mathbb{N}$ there is a nonstandard model $M \models PA$ such that S is not coded in M.

Proof. Let $\sigma_0, \sigma_1, \ldots, \sigma_i, \ldots (i \in \mathbb{N})$ be an enumeration of all $\mathscr{L}_A$-sentences. We shall construct a complete extension $T \neq \mathrm{Th}(\mathbb{N})$ of PA such that S is not coded in K_T. T will be $PA \cup \{\tau_0, \tau_1, \ldots, \tau_i, \ldots\}$ for suitably chosen $\mathscr{L}_A$-sentences τ_i.

The first sentence, τ_0, is chosen so that $PA + \tau_0$ is consistent and $\mathbb{N} \not\models \tau_0$. (This will force $T \neq \mathrm{Th}(\mathbb{N})$, and hence $K_T \neq \mathbb{N}$. Such τ_0 exists by Corollary 3.10.)

Suppose $T_i = PA \cup \{\tau_0, \ldots, \tau_i\}$ has been found and is consistent. If σ_i is not of the form $\exists x \varphi(x)$ for some variable x and some formula $\varphi(x)$, then we let τ_{i+1} be either σ_i or $\neg\sigma_i$, chosen so that $T_i+\tau_{i+1}$ is consistent. Otherwise suppose σ_i is $\exists x\varphi(x)$. If $T_i+\neg\sigma_i$ is consistent let τ_{i+1} be $\neg\sigma_i$, and if $T_i+\neg\sigma_i$ is inconsistent then $T_i+\sigma_i$ must be consistent, as T_i is. In this last case suppose

$$T_i+\varphi(x)\vdash(x)_j=0 \qquad \text{for all } j\notin S$$

and

$$T_i+\varphi(x)\vdash(x)_j\neq 0 \qquad \text{for all } j\notin S.$$

then S would be recursive, since we could compute whether or not $j\in S$ by searching for the first proof of *either* $(x)_j=0$ *or* $(x)_j\neq 0$ from $T_i+\varphi(x)$, which is a recursively axiomatized theory, being a finite extension of PA. By hypothesis, S is not recursive, so either $T_i+\exists x(\varphi(x)\wedge(x)_j\neq 0)$ is consistent for some $j\notin S$, or $T_i+\exists x(\varphi(x)\wedge(x)_j=0)$ is consistent for some $j\in S$. Whichever it is, let τ_{i+1} be the new sentence found in this way.

At the end of this construction we will have obtained a complete consistent extension $T=PA+\{\tau_0,\tau_1,\ldots\}$ of PA. (T is complete because either $\tau_{i+1}\vdash\sigma_i$ or $\tau_{i+1}\vdash\neg\sigma_i$ for each $i\in\mathbb{N}$, and $\{\sigma_i \mid i\in\mathbb{N}\}$ contains all $\mathscr{L}_A$-sentences.) If $a\in K_T$ then a is defined by some $\mathscr{L}_A$-formula $\varphi(x)$, where $T\vdash\exists! x\varphi(x)$. But then $\exists x\varphi(x)$ is σ_i for some $i\in\mathbb{N}$, and since $T\vdash\exists x\varphi(x)$ we must have: either τ_{i+1} is $\exists x(\varphi(x)\wedge(x)_j\neq 0)$ for some $j\notin S$ *or* τ_{i+1} is $\exists x(\varphi(x)\wedge(x)_j=0)$ for some $j\in S$; and $K_T\vDash\tau_{i+1}$. Thus a (being the unique element of K_T satisfying $\varphi(x)$) cannot code the set S. This applies to all $a\in K_T$, and thus $S\notin SSy(K_T)$. □

It is worth mentioning that every nonstandard $M\vDash PA$ codes some non-recursive set, since by induction one can easily show that

$$PA\vdash\forall x\exists y\forall z<x((y)_z\neq 0\leftrightarrow \mathrm{Sat}_{\Pi_1}(z,[0])),$$

thus if $b\in M$ is nonstandard and $a\in M$ satisfies

$$M\vDash\forall z<b((a)_z\neq 0\leftrightarrow \mathrm{Sat}_{\Pi_1}(z,[0])),$$

then a codes the set of Gödel-numbers of Π_1 formulas $\theta(\bar v)$ which are true in M for $\bar v=0$, and this set is clearly non-recursive by Theorem 3.9.

Lemma 11.1 is about recursive *sets*. A recursive *function* $f:\mathbb{N}\to\mathbb{N}$ is coded in a nonstandard model $M\vDash PA$ iff there is $b\in M$ such that, for all $n,m\in\mathbb{N}$,

$$m=f(n)\Leftrightarrow M\vDash(b)_n=m.$$

The following variation of Lemma 11.1 will occasionally be useful.

LEMMA 11.3. Let $f\colon \mathbb{N}\rightarrow\mathbb{N}$ be recursive, and let $M\vDash PA$ be nonstandard. Then there is $b\in M$ such that $M\vDash(b)_n=f(n)$ for all $n\in\mathbb{N}$. If the image of f, $\mathrm{Im}(f)=\{f(n)\mid n\in\mathbb{N}\}$ is a recursive set, then there exists c in M so that $\mathrm{Im}(f)=\{n\in\mathbb{N}\mid M\vDash\exists x<c(b)_x=n\}$.

Proof. Given $f:\mathbb{N}\rightarrow\mathbb{N}$ recursive, by Theorem 3.3 there is a Δ_0 formula $\theta(x, y, z)$ such that, for all $n, m\in\mathbb{N}$,

$$\mathbb{N}\vDash\exists z\theta(n, m, z)\Leftrightarrow f(n)=m.$$

Hence by Corollary 2.9 and Lemma 5.9

$$M\vDash\exists b\forall x, y, z<i(\theta(x, y, z)\rightarrow(b)_x=y)$$

for all $i\in\mathbb{N}$. Hence by overspill there is $b\in M$ and $a\in M\setminus\mathbb{N}$ such that

$$M\vDash\forall x, y, z<a(\theta(x, y, z)\rightarrow(b)_x=y)$$

and in particular: $f(n)=m$ implies $\exists z\in\mathbb{N}\vDash\theta(n, m, z)$ implies $M\vDash(b)_n=m$, since $\mathbb{N}\subseteq_e M$ and θ is Δ_0; conversely if $n\in\mathbb{N}$ then $f(n)$ exists, and also $M\vDash(b)_n=c$ for some $c\in M$. Hence $c=f(n)$ by the above.

Now suppose $\mathrm{Im}(f)$ is recursive. Then by Corollary 3.5 there is a Δ_0 formula $\psi(y, w)$ such that for all $n\in\mathbb{N}$

$$n\in\mathrm{Im}(f)\Leftrightarrow\mathbb{N}\vDash\forall w\psi(n, w).$$

Clearly then, for all $i\in\mathbb{N}$,

$$M\vDash\forall x<i\forall w<i\psi((b)_x, w)$$

hence by overspill there is $c>\mathbb{N}$ such that

$$M\vDash\forall x<c\forall w<c\psi((b)_x, w).$$

Now if $n\in\mathrm{Im}(f)$, then $n=(b)_m$ for some $m\in\mathbb{N}$, hence $M\vDash\exists x<c(b)_x=n$ since $c>\mathbb{N}$. Conversely if $n\in\mathbb{N}$ and $M\vDash\exists x<c(b)_x=n$ then, by our choice of c, $M\vDash\forall w<c\psi((b)_x, w)$, hence in particular $M\vDash\psi(n, l)$ for all $l\in\mathbb{N}$, hence $\mathbb{N}\vDash\psi(n, l)$ for all $l\in\mathbb{N}$, since ψ is Δ_0 and $\mathbb{N}<_{\Delta_0}M$, hence $\mathbb{N}\vDash\forall w\psi(n, w)$, therefore $n\in\mathrm{Im}(f)$, as required. □

Note that, if $\mathrm{Im}(f)=\{n\in\mathbb{N}\mid M\vDash\exists x<c(b)_x=n\}$ for a model $M\vDash PA$, then $\mathrm{Im}(f)\in SSy(M)$, since by induction we can show that there is $a\in M$ such

that $M \vDash \forall y < c((a)_y \neq 0 \leftrightarrow \exists x < c(b)_x = y)$. Thus, by Theorem 0.11, Lemma 11.2 and the fact that not all r.e. sets are recursive, there are models $M \vDash PA$ and recursive functions $f: \mathbb{N} \to \mathbb{N}$ such that, for some $b \in M$, $M \vDash (b)_n = f(n)$ for all $n \in \mathbb{N}$, $\{(b)_x | x < c\} \cap \mathbb{N} \neq \mathrm{Im}(f)$ for all nonstandard $c \in M$.

We shall close this section with some remarks about the coding apparatus $(x)_y \neq 0$ that we have used.

The first remark is that, although we have so far only talked about coding subsets of $\mathbb{N}$, it is equally easy to consider coded subsets on $\mathbb{N}^k$ for arbitrary $k \in \mathbb{N}$ via the functions $\langle x_1, \ldots, x_k \rangle$ discussed in Chapter 3. Thus we say $a \in M \vDash PA$ codes $S \subseteq \mathbb{N}^k$ iff

$$S = \{(x_1, \ldots, x_k) \in \mathbb{N}^k | M \vDash \exists y(\langle x_1, \ldots, x_k \rangle = y \wedge (a)_y \neq 0)\}.$$

Notice that, since the formula $\langle x_1, \ldots, x_k \rangle = y$ is Δ_0, for all $\bar{x}, y \in \mathbb{N}$

$$\mathbb{N} \vDash \langle x_1, \ldots, x_k \rangle = y \Leftrightarrow M \vDash \langle x_1, \ldots, x_k \rangle = y,$$

so this definition is equivalent to saying that $a \in M$ codes $S \subseteq \mathbb{N}^k$ iff a codes the set $\{\langle x_1, \ldots, x_k \rangle \in \mathbb{N} | (x_1, \ldots, x_k) \in S\}$.

The second remark is that the notion of a model $M \vDash PA$ coding a set $S \subseteq \mathbb{N}$ depends very little on our choice of sequence-coding apparatus. In particular by Exercise 5.6 any function $f(x, y)$ satisfying Lemma 5.8 could be used in place of $(x)_y$, and the notion of the standard system of a model of PA would be exactly the same. There is however one particularly useful technique in common use that we describe now.

In PA we can define a function $\mathrm{pr}(x)$ = the xth prime, so $\mathrm{pr}(0) = 2$, $\mathrm{pr}(1) = 3$, $\mathrm{pr}(2) = 5$, and so on. In fact we can take '$\mathrm{pr}(x) = y$' to be a Σ_1 formula, so $\mathbb{N} \vDash \mathrm{pr}(n) = k \Rightarrow M \vDash \mathrm{pr}(n) = k$ for all $n, k \in \mathbb{N}$ and all $M \vDash PA$. Then we could redefine 'the set coded by a in M' to be

$$\{n \in \mathbb{N} | M \vDash \mathrm{pr}(n) | a\}$$

(See Exercises 11.2, 11.3). This idea will turn out to be very convenient in Section 11.3 below.

Exercises for Section 11.1

11.1 Show that, according to our definition, $SSy(\mathbb{N})$ is the collection of all finite subsets of $\mathbb{N}$.
(*Warning*: Some authors use a slightly different definition of $SSy(M)$ which coincides with ours in all cases except the anomolous case when $M \cong \mathbb{N}$. See Theorem 11.6.)
11.2 Define $\mathrm{pr}(x)$ = 'the xth prime' in PA so that $\mathrm{pr}(x) = y$ is Σ_1 and $PA \vdash \forall x \exists ! y\, \mathrm{pr}(x) = y$.

11.3 Show that for all $M \models PA$

$$SSy(M) = \{S \subseteq \mathbb{N} \mid S = \{n \in \mathbb{N} \mid M \models \mathrm{pr}(n) \mid a\} \text{ some } a \in M\}$$

11.4 (a) Let $\delta(w, x)$ be the $\Delta_1(PA)$ formula 'the xth digit of the binary expansion of w is 1' of the proof of Lemma 10.6, and write '$x \in w$' for $\delta(w, x)$. Show that

$$SSy(M) = \{S \subseteq \mathbb{N} \mid S = \{n \in \mathbb{N} \mid M \models n \in a\} \text{ for some } a \in M\}$$

for each model $M \models PA$.
(b) Let $M \models PA$ and let (V, ε) be the model for the language of set theory with domain of V = domain of M and binary relation ε given by $(V, \varepsilon) \models x\varepsilon y \Leftrightarrow M \models x \in y$. Show that (V, ε) satisfies all axioms of ZFC except the axiom of infinity.
(c) Let (V, ε) be a model of all axioms of ZFC except infinity, which is false in (V, ε). Interpreting $x\varepsilon y$ as 'the xth digit of the binary expansion of y is 1' define $0, 1, +, \cdot, <$ on (V, ε) so that $(V, 0, 1, +, \cdot, <) \models PA$. Check that the operations in (b) and (c) here are inverses to each other.
11.5 Verify that

$$\Pi_1 - \mathrm{Th}(M) = \{\ulcorner\sigma\urcorner \mid \sigma \text{ is a } \Pi_1 \text{ sentence and } M \models \sigma\}$$

is coded in all nonstandard $M \models PA$.

11.2 Σ_n-RECURSIVE SATURATION

We now return to considering types. We have already discussed the notion of a type over a theory; it will be useful to have a notion of a type over a model too.

DEFINITION. Let $M \models PA$. A *type over* M is a set $p(\bar{x})$ of formulas $\varphi(\bar{x}, \bar{a})$ *of* $\mathscr{L}_A \cup \{\bar{a}\}$ (where $\bar{x}$ is a fixed finite tuple of *free-variables* and $\bar{a}$ is a fixed finite tuple of *parameters*, i.e. $\bar{a} \in M$ and each a_i in $\bar{a}$ is considered as a constant symbol in the expanded language $\mathscr{L}_A \cup \{\bar{a}\}$ with its natural interpretation in M) such that $p(\bar{x})$ is *finitely satisfied* in M, i.e.,

$$M \models \exists \bar{x} \bigwedge_{i=1}^{k} \varphi_i(\bar{x}, \bar{a})$$

for each finite subset $\varphi_1(\bar{x}, \bar{a}), \ldots, \varphi_k(\bar{x}, \bar{a})$ of $p(\bar{x})$. $p(\bar{x})$ is a Σ_n type (respectively, a Π_n type) iff each $\varphi(\bar{x}, \bar{a}) \in p(\bar{x})$ is Σ_n (respectively Π_n). $p(\bar{x})$ is *recursive* iff the set

$$\{\ulcorner\varphi(\bar{x}, \bar{y})\urcorner \mid \varphi(\bar{x}, \bar{a}) \in p(\bar{x})\} \subseteq \mathbb{N}$$

is recursive, where $\bar{y}$ is some fixed tuple of variables from $\mathscr{L}_A$ disjoint with $\bar{x}$, and replacing the parameters $\bar{a}$ in $p(\bar{x})$. Finally, a type $p(\bar{x})$ over M is *complete* iff for all formulas $\varphi(\bar{x}, \bar{a})$ involving the same free-variables $\bar{x}$ as $p(\bar{x})$ and the same parameters $\bar{a}$ as $p(\bar{x})$,

$$\text{either} \quad M \vDash \forall \bar{x}(\theta_1(\bar{x}, \bar{a}) \wedge \ldots \wedge \theta_k(\bar{x}, \bar{a}) \to \varphi(\bar{x}, \bar{a}))$$

$$\text{or} \quad M \vDash \forall \bar{x}(\theta_1(\bar{x}, \bar{a}) \wedge \ldots \wedge \theta_k(\bar{x}, \bar{a}) \to \neg\varphi(\bar{x}, \bar{a}))$$

for some finite subset $\{\theta_1(\bar{x}, \bar{a}), \ldots, \theta_k(\bar{x}, \bar{a})\}$ *of* $p(\bar{x})$.

Notice that, when talking about a type over a model, we allow finitely many parameters $\bar{a}$ from the model to appear in formulas in the type. (In abstract model theory, infinitely many parameters are sometimes allowed, but for the purposes of this book we shall restict ourselves to only finitely many parameters.)

It is easy to check (using the compactness theorem) that $p(\bar{x})$ is a type over M (with parameters $\bar{a} \in M$) according to this new definition iff $p(\bar{x})$ is a type over $\mathrm{Th}(M, \bar{a})$ (the theory of the structure M expanded by adding constants for each element of $\bar{a}$) according to the definition in Section 8.1.

DEFINITION: A model $M \vDash PA^-$ is *Σ_n-recursively saturated* iff every recursive Σ_n type over M is realized in M. The model M is *recursively saturated* iff every recursive type over M is realized in M.

This definition of a model M being recursively saturated really has little to do with the language $\mathscr{L}_A$: all that is required is that the language of M has some suitable Gödel-numbering. Thus before proving the next result, that every model M has a recursively saturated elementary extension, we shall digress slightly and discuss recursive languages.

Let $\mathscr{L}$ be a language for the predicate calculus. Then $\mathscr{L}$ contains the symbols

$$\wedge \vee \neg = , ( ) \exists \forall \quad \text{and} \quad \mathsf{v}_i \quad (i \in \mathbb{N})$$

together with extra constant symbols c_i $(i \in I)$, relation symbols $R_j (j \in J)$, and function symbols $F_k (k \in K)$, where I, J, K are (possibly empy) index sets. Suppose also that, for each $j \in J$ and each $k \in K$, the symbol R_j has arity $m_j \in \mathbb{N}$ and the symbol F_k has arity $n_k \in \mathbb{N}$.

DEFINITION. The first-order language $\mathscr{L}$ above is *recursive* iff there is a 1–1 function ν from the set of all symbols s of $\mathscr{L}$ to $\mathbb{N}$ such that each of

$$\{\nu(c_i) \mid i \in I\} \subseteq \mathbb{N}$$

$$\{(\nu(R_j), m_j) \mid j \in J\} \subseteq \mathbb{N}^2$$

$$\{(\nu(F_k), n_k) \mid k \in K\} \subseteq \mathbb{N}^2, \quad \text{and}$$

$$\{\nu(\mathsf{v}_i) \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$$

are recursive sets. (Note that since ν is 1–1, this means that I, J, K must all be countable.) Given such a function ν, we can define the *Gödel-number* of a string of symbols $s_0 s_1 \ldots s_k$ of L as being

$$\ulcorner s_0 s_1 \cdots s_k \urcorner = [\nu(s_0), \nu(s_1), \ldots, \nu(s_k)],$$

where $[x_0, x_1, \ldots, x_k]$ is the function defined in Section 9.1.

Notice that if I, J, K are all finite sets then it is easy to construct such a function ν. Thus any language $\mathscr{L}$ with only finitely many nonlogical symbols is a recursive language. If $\mathscr{L}$ is a recursive language it is easy to construct recursive algorithms that decide whether 'x is the Gödel-number of an $\mathscr{L}$-term', '... of an $\mathscr{L}$-formula', etc. For example the set S of Gödel-numbers of $\mathscr{L}$-terms is defined to be the least set such that:

if $x \in \{\nu(v_i) \mid i \in \mathbb{N}\}$ then $[x] \in S$;
if $x \in \{\nu(c_i) \mid i \in I\}$ then $[x] \in S$;
if $(x, y) \in \{(\nu(F_k), n_k) \mid k \in K\}$ and $s_1, \ldots, s_y \in S$ then
$[x, \nu(()] ^\frown s_1 {}^\frown [\nu(,)] ^\frown \cdots {}^\frown [\nu(,)] ^\frown s_y {}^\frown [\nu())] \in S$.

Note too that (by relabeling variables if necessary) we may also assume that the function ν above for a recursive language $\mathscr{L}$ satisfies $\nu(\mathrm{v}_0) < \nu(\mathrm{v}_2) < \nu(\mathrm{v}_2) < \cdots$, so that, since $\{\nu(\mathrm{v}_i) \mid i \in \mathbb{N}\}$ is recursive, the function $i \mapsto \nu(\mathrm{v}_i)$ is an increasing recursive function given by

$$\nu(\mathrm{v}_i) \text{ is the } i\text{th element of } \{\nu(\mathrm{v}_i) \mid i \in \mathbb{N}\}.$$

Exercise 11.9 shows that, as far as recursive sets and recursive sequences of formulas, etc. are concerned (which is, in fact, all we will be interested in) the choice of the function ν is immaterial, thus it makes sense to talk about a language $\mathscr{L}$ being recursive, rather than the pair $(\mathscr{L}, \nu)$ being recursive.

The point of introducing all this terminology is that if M is an $\mathscr{L}$-structure, where $\mathscr{L}$ is a recursive language, then we may talk about recursive types over M, and whether or not M is recursively saturated. (In fact this notion is an extremely natural one, even if M is not a model of PA, as we shall see in Section 15.2.) The existence of recursively saturated models follows from

PROPOSITION 11.4 Let M be an infinite structure for a recursive language $\mathscr{L}$. Then there exists an $\mathscr{L}$-structure $M' \succ M$ with M' recursively saturated and $\mathrm{card}(M') = \mathrm{card}(M)$.

Proof. For each $m, n \in \mathbb{N}$ there are countably many sets $p(x_0, \ldots, x_m, y_0, \ldots, y_n)$ of formulas $\varphi(x_0, \ldots, x_m, y_0, \ldots, y_m)$ with at most the free-variables shown, and whose set of Gödel-numbers $\{\ulcorner\varphi(\bar{x}, \bar{y})\urcorner \mid \varphi \in p(\bar{x}, \bar{y})\}$ is recursive. The collection of all such sets over all $n, m \in \mathbb{N}$ is a countable union of countable sets, so is also countable. Enumerate this collection by $p_0(\bar{x}, \bar{y}), p_1(\bar{x}, \bar{y}), \ldots, p_i(\bar{x}, \bar{y}), \ldots$ so that the free-variables in $p_i(\bar{x}, \bar{y})$ are $x_0, \ldots, x_{m_i}, y_0, \ldots, y_{n_i}$.

Now let $M_0 = M$. Since M is infinite, card $(M^{<\omega}) = \text{card}(M)$, where $M^{<\omega}$ denotes the set of tuples $\bar{a} = (a_0, \ldots, a_m) \in M$ of some finite length $m+1$. Associate with each pair $p_i(\bar{x}, \bar{y})$ and $\bar{a} = (a_0, \ldots, a_{n_i})$ $m_i + 1$ new constant symbols $c_{i,\bar{a},0}, c_{i,\bar{a},1}, \ldots, c_{i,\bar{a},m_i}$, and (in addition to these constant symbols) add to $\mathscr{L}$ further constants naming each element $a \in M_0$. The resulting language has cardinality card(M_0). We now consider the following theory T in this expanded language: T has axioms

(a) $\theta(\bar{a})$ for every $\mathscr{L}$-formula $\theta(\bar{x})$ and every $\bar{a} \in M_0$ such that $M_0 \models \theta(\bar{a})$;

(b) $\varphi(c_{i,\bar{a},0}, \ldots, c_{i,\bar{a},m_i}, \bar{a})$ for every $\varphi(\bar{x}, \bar{y}) \in p_i$, every $i \in \mathbb{N}$ and every $a_0, a_1, \ldots, a_{n_i} \in M$ such that $p(\bar{x}, a_0, \ldots, a_{n_i})$ is a type over M_0.

This theory is consistent by the definition of 'type over M_0' and the compactness theorem, so it has a model M_1 of cardinality at most card(M). By identifying each $a \in M_0$ and the element of M_1 realizing the constant for a we may assume $M_1 \succ M_0$ and clearly M_1 realizes every recursive type over M_0. Continuing in this way we obtain a chain $M_0 \prec M_1 \prec M_2 \prec \ldots$ of models of cardinality card(M) such that M_{j+1} realizes every recursive type over M_j for each j. But then $M' = \bigcup_{j \in \mathbb{N}} M_j$ is an elementary extension of M, by the lemma on elementary chains, and card$(M') \leqslant \aleph_0 \cdot \text{card}(M) = \text{card}(M)$. Moreover, if $q(\bar{x})$ is a recursive type over M' involving finitely many parameters $\bar{b}$, then $\bar{b} \in M_j$ for some $j \in \mathbb{N}$, hence $q(\bar{x})$ is realized in M_{j+1} and hence in M' as $M' \succ M_{j+1}$. Thus M' is recursively saturated, as required. □

Recursive saturation is a much more natural and useful notion than the unhappy mixture of model theory and recursion theory it appears to be at first sight. It is also an extremely *stable* notion, in the sense that many other natural conditions on a model M turn out to be equivalent to M being recursively saturated. Some examples will be provided in Chapter 15, but here I would like to illustrate this 'stability' by showing that one obtains the same notion even if we replace 'recursive type' by 'r.e. type' or 'primitive recursive type' in the definition of 'recursively saturated'.

To see this, let $p(\bar{x})$ be a type over a model M for the recursive language $\mathscr{L}$, and suppose

$$S = \{\ulcorner\varphi(\bar{x}, \bar{y})\urcorner \mid \varphi(\bar{x}, \bar{a}) \in p(\bar{x})\}$$

is an r.e. set. Then $S = \text{Im}(f)$ for some recursive function $f: \mathbb{N} \to \mathbb{N}$, by

Theorem 0.11. Now consider the type

$$q(\bar{x}) = \left\{ \overbrace{(\cdots(\varphi(\bar{x}, \bar{a}) \wedge \varphi(\bar{x}, \bar{a})) \wedge \cdots \wedge \varphi(\bar{x}, \bar{a}))}^{n+1 \text{ conjuncts}} \;\middle|\; \begin{matrix} \ulcorner\varphi(\bar{x}, \bar{y})\urcorner = f(n) \\ n \in \mathbb{N} \end{matrix} \right\}.$$

$q(\bar{x})$ is clearly equivalent (in the predicate calculus for $\mathscr{L}$) to the type $p(\bar{x})$; however, $q(\bar{x})$ is recursive (and not just r.e.) since '$\psi(\bar{x}, \bar{a}) \in q(\bar{x})$' can be decided by the following algorithm:

> compute the largest integer $n \in \mathbb{N}$ such that
>
> $$\ulcorner\psi(\bar{x}, \bar{y})\urcorner = \ulcorner(\cdots(\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{y})) \wedge \cdots \wedge \varphi(\bar{x}, \bar{y}))\urcorner$$
>
> ($n+1$ conjuncts) for some $\mathscr{L}$-formula $\varphi(\bar{x}, \bar{y})$; compute $f(n)$, and if $\ulcorner\varphi(\bar{x}, \bar{y})\urcorner = f(n)$ then $\psi(\bar{x}, \bar{a}) \in q(\bar{x})$—if not, then $\psi(\bar{x}, \bar{a}) \notin q(\bar{x})$.

(This observation that any r.e. set of formulas is equivalent to a recursive set is known as *Craig's trick*.) Similarly $S = \{\ulcorner\varphi(\bar{x}, \bar{y})\urcorner \mid \varphi(\bar{x}, \bar{a}) \in p(\bar{x})\}$ is equivalent to a primitive recursive set of formulas, by considering

$$r(\bar{x}) = \left\{ \overbrace{(\cdots(\varphi(\bar{x}, \bar{a}) \wedge \varphi(\bar{x}, \bar{a})) \wedge \cdots \wedge \varphi(\bar{x}, \bar{a}))}^{n+1 \text{ conjuncts}} \;\middle|\; \begin{matrix} \text{for some } m \leqslant n\ f(m) = \ulcorner\varphi(\bar{x}, \bar{y})\urcorner \\ \text{and the computation for } f(m) \\ \text{halts in } \leqslant n \text{ steps} \end{matrix} \right\}$$

for some natural notion of the number of steps a computation for $f(m)$ takes. Thus 'recursive saturation' turns out to be equivalent to 'r.e. saturation' and 'primitive recursive saturation'.

Proposition 11.4 provides us with many recursively saturated models, but not all models are recursively saturated. For example, if T is a complete consistent extension of PA, K_T is not recursively saturated since it fails to realize the following recursive type:

$$p(x) = \{\exists! y\theta(y) \rightarrow \neg\theta(x) \mid \theta(x) \text{ an } \mathscr{L}_A\text{-formula}\}.$$

However, we have

THEOREM 11.5. Let $M \models PA$ be nonstandard. Then M is Σ_n-recursively saturated for all $n \in \mathbb{N}$.

Proof. Suppose $p(\bar{x})$ is a recursive Σ_n-type over M with parameters $\bar{a} \in M$. Since

$$S = \{\ulcorner\varphi(\bar{x}, \bar{y})\urcorner \mid \varphi(\bar{x}, \bar{a}) \in p(\bar{x})\}$$

is recursive, S is coded in M by some $c \in M$, by Lemma 11.1. Since $p(\bar{x})$ is finitely satisfied in M,

$$M \models \exists \bar{x} \forall y < k((c)_y \neq 0 \rightarrow \mathrm{Sat}_{\Sigma_n}(y, [\bar{x}, \bar{a}]))$$

for all $k \in \mathbb{N}$. By overspill there is $d > \mathbb{N}$ in M such that

$$M \models \exists \bar{x} \forall y < d((c)_y \neq 0 \rightarrow \mathrm{Sat}_{\Sigma_n}(y, [\bar{x}, \bar{a}])).$$

Let $\bar{b} \in M \models \forall y < d((c)_y \neq 0 \rightarrow \mathrm{Sat}_{\Sigma_n}(y, [\bar{b}, \bar{a}]))$ and suppose $\varphi(\bar{x}, \bar{a}) \in p(\bar{x})$. Then putting $l = [\varphi(\bar{x}, \bar{y})] < d$, $M \models (c)_l \neq 0$, since c codes S. Thus $M \models \mathrm{Sat}_{\Sigma_n}(l, [\bar{b}, \bar{a}])$ and hence $M \models \varphi(\bar{b}, \bar{a})$. This shows that $\bar{b}$ realizes the type $p(\bar{x})$, as required. □

The example of K_T shows that the restriction to Σ_n-types is necessary for Theorem 11.5 to be true. Moreover, by Lemma 11.2, there are non-recursive Σ_1 types $\{(x)_i \neq 0 \mid i \in S\} \cup \{(x)_i = 0 \mid i \notin S\}$ which are omitted by K_T for suitably chosen $T \neq \mathrm{Th}(\mathbb{N})$.

We will give one important application of Theorem 11.5 here. (Other applications will be found in the exercises and throughout the rest of the book.) The application gives an alternative characterization of the standard system of nonstandard models, and helps to explain why the notion of standard system does not change for various possible choices of coding paraphernalia.

THEOREM 11.6. Let $M \models PA$ be nonstandard, and let $S \subseteq \mathbb{N}$. Then $S \in SSy(M)$ iff for some formula $\theta(x, \bar{y})$ of $\mathscr{L}_A$ and some $\bar{a} \in M$

$$S = \{n \in \mathbb{N} \mid M \models \theta(n, \bar{a})\}$$

Proof. If $S \in SSy(M)$, then $S = \{n \in \mathbb{N} \mid M \models (a)_n \neq 0\}$ for some $a \in M$. Conversely, if $\bar{a} \in M$ and $\theta(x, \bar{y})$ is an $\mathscr{L}_A$-formula, then $S = \{n \in \mathbb{N} \mid M \models \theta(n, \bar{a})\}$ is coded by any $b \in M$ realizing the following recursive type over M:

$$p(x) = \{((x)_n \neq 0 \leftrightarrow \theta(n, \bar{a})) \mid n \in \mathbb{N}\};$$

and if θ is Σ_k, then $p(x)$ is a Σ_{k+1} type, so is realized in M by Theorem 11.5. □

Theorem 11.6 does not hold for the standard model, since for many $\theta(x)$ the set $S = \{n \in \mathbb{N} \mid \mathbb{N} \models \theta(n)\}$ is infinite. Some authors use the property in

Theorem 11.6 to *define* $SSy(M)$, so these people would define $SSy(\mathbb{N})$ to be

$$\{\{n\in\mathbb{N}\,|\,\mathbb{N}\vDash\theta(n)\}\,|\,\theta \text{ an } \mathscr{L}_{\mathrm{A}}\text{-formula}\}.$$

This, potentially, could lead to confusion, but in practice $SSy(\mathbb{N})$ is hardly ever used.

Exercises for Section 11.2

11.6 Show that every nonstandard $M\vDash PA$ is Σ_n*-tall* for all n, i.e. whenever $\bar{a}\in M$ is a tuple of finite length, then $K^n(M;\bar{a})$ is not cofinal in M.
11.7⁺ A nonstandard model $M\vDash PA$ is *tall* iff for all $\bar{a}\in M$, a tuple of finite length, $K(M;\bar{a})$ is not cofinal in M. Otherwise M is *short*. M is *short-recursively saturated* iff every recursive type over M of the form

$$p(\bar{x})=\{x_i<b\,|\,1\leqslant i\leqslant k\}\cup\{\varphi_j(\bar{x},\bar{a})\,|\,j\in\mathbb{N}\},$$

where $\bar{a}, b\in M$ and $\bar{x}=(x_1,\ldots,x_k)$, is realized in M.
(a) Show that every recursively saturated $M\vDash PA$ is tall and short-recursively saturated.
(b) Show that, if $M\prec N\vDash PA$, $M\subseteq_e N$ and N is recursively saturated, then M is short-recursively saturated. Deduce from Gaifman's splitting theorem that every nonstandard $M\vDash PA$ has an elementary extension $K\supseteq_{cf}M$ that is short-recursively saturated. Hence construct a nonstandard model $M\vDash PA$ that is short-recursively saturated but not recursively saturated.
(c) Using the MacDowell–Specker theorem and a union of chains argument, show that any $M\vDash PA$ has an elementary end-extension K that is tall, but K need not be recursively saturated.
(d) Suppose $M\vDash PA$ is nonstandard, tall, and short-recursively saturated. Show that M is recursively saturated.
(e) Suppose $M\vDash PA$ is nonstandard, countable and short-recursively saturated. Show that M has a recursively saturated elementary end-extension.
11.8 Let $M\vDash PA^-$ be nonstandard and Σ_n-recursively saturated for all n. Show that M satisfies overspill for the standard cut $\mathbb{N}$, i.e., whenever $\bar{a}\in M$ and $M\vDash\theta(\bar{a},n)$ for all $n\in\mathbb{N}$ then $M\vDash\forall x<b\,\theta(\bar{a},x)$ for some $b>\mathbb{N}$.
11.9* (a) Let $\mathscr{L}$ be a first-order language with constant symbols $c_i (i\in I)$, in relation symbols R_j of arity $m_j (j\in J)$ and function symbols F_k of arity $n_k (k\in K)$. Let ν_1 and ν_2 both satisfy the conditions on p. 147, i.e. both are 1–1 functions from the symbols of $\mathscr{L}$ into $\mathbb{N}$ s.t.

$$\begin{array}{ll}
\{\nu_1(c_i)\,|\,i\in I\} & \{\nu_2(c_i)\,|\,i\in I\}\\
\{(\nu_1(R_j),m_j)\,|\,j\in J\} & \{(\nu_2(R_j),m_j)\,|\,j\in J\}\\
\{(\nu_1(F_k),n_k)\,|\,k\in K\} & \{(\nu_2(F_k),n_k)\,|\,k\in K\}\\
\{\nu_1(\mathrm{v}_i)\,|\,i\in\mathbb{N}\} & \{\nu_2(\mathrm{v}_i)\,|\,i\in\mathbb{N}\}
\end{array}$$

are all recursive, and also that

$$\nu_1(\mathsf{v}_i) < \nu_1(\mathsf{v}_{i+1}) \qquad \nu_2(\mathsf{v}_i) < \nu_2(\mathsf{v}_{i+1})$$

for each $i \in \mathbb{N}$. Define Gödel-numberings N_1 and N_2 on strings $s_1 s_2 \cdots s_l$ of symbols of $\mathscr{L}$ by

$$N_1(s_1 s_2 \ldots s_l) = [\nu_1(s_1), \nu_1(s_2), \ldots, \nu_1(s_l)]$$
$$N_2(s_1 s_2 \ldots s_l) = [\nu_2(s_1), \nu_2(s_2), \ldots, \nu_2(s_l)].$$

Given that T is any set of strings of symbols from $\mathscr{L}$, show that $N_2(T) = \{N_2(t) | t \in T\}$ is recursive in $N_1(T) = \{N_1(t) | t \in T\}$ and hence $N_1(T)$ is recursive iff $N_2(T)$ is recursive. Thus N_1 and N_2 'agree' about which sets of formulas, terms, etc. of L are recursive.

(b) Given that $\mathscr{L}$ (as in part (a) above) is a recursive language, show that there is an injection ν from symbols of $\mathscr{L}$ to $\mathbb{N}$ such that $\nu(\mathsf{v}_i) < \nu(\mathsf{v}_{i+1})$ for all i, and that

$$\{\nu(c_i) | i \in I\}$$
$$\{(\nu(R_j), m_j) | j \in J\}$$
$$\{(\nu(F_k), n_k) | k \in K\}$$

and

$$\{\nu(\mathsf{v}_i) | i \in \mathbb{N}\}$$

are all primitive recursive.

11.3 TENNENBAUM'S THEOREM

The central result of this section is that nonstandard models of PA are not recursive, a result first proved by Stanley Tennenbaum (1959). Tennenbaum's theorem may be regarded as the model-theoretic analogue of the Gödel–Rosser incompleteness theorem, as we shall see.

First we must say what it means for an $\mathscr{L}_{\mathrm{A}}$-structure to be recursive. Let M be an $\mathscr{L}_{\mathrm{A}}$-structure: M is *recursive* iff there are recursive functions $\oplus$, $\odot : \mathbb{N}^2 \to \mathbb{N}$, a binary recursive relation $\otimes \subseteq \mathbb{N}^2$ and natural numbers n_0, $n_1 \in \mathbb{N}$ such that

$$(\mathbb{N}, \oplus, \odot, \otimes, n_0, n_1) \cong M.$$

In particular, note that only countable $\mathscr{L}_{\mathrm{A}}$-structures can be recursive.

THEOREM 11.7 (Tennenbaum's theorem). Let $M \vDash PA$ be nonstandard. Then M is not recursive. In fact if $M \cong (\mathbb{N}, \oplus, \odot, \otimes, n_0, n_1)$ then neither $\oplus$ nor $\odot$ is recursive.

Proof. By the remark after Lemma 11.2, $SSy(M)$ contains a non-recursive set $S \subseteq \mathbb{N}$. (The proof of this used a direct appeal to the Gödel–Rosser incompleteness theorem.) Thus by Theorem 11.5 there is $a \in M$ such that $S = \{n \in \mathbb{N} \mid M \models \mathrm{pr}(n) \mid a\}$, where $\mathrm{pr}(n)$ is the nth prime. But if $M \cong (\mathbb{N}, \oplus, \odot, \otimes, n_0, n_1)$ with $\oplus$ recursive we could compute S as follows:

Let $n_a \in \mathbb{N}$ be the image of $a \in M$ under the isomorphism $M \cong (\mathbb{N}, \oplus, \odot, \otimes, n_0, n_1)$. To decide if $n \in S$, compute $p \in \mathbb{N} \models \mathrm{pr}(n) = p$, so p is the nth prime in $\mathbb{N}$, and search through $\mathbb{N}$ for a natural number k s.t.

$$\begin{array}{ll} \text{either} & n_a = \overbrace{k \oplus k \oplus \cdots \oplus k}^{p} \\ \text{or} & n_a = \overbrace{k \oplus k \oplus \cdots \oplus k}^{p} \oplus n_1 \\ \vdots & \\ \text{or} & n_a = \overbrace{k \oplus k \oplus \cdots \oplus k}^{p} \oplus \overbrace{n_1 \oplus n_1 \oplus \cdots \oplus n_1}^{p-1} \end{array}$$

(This search must be terminated by Euclidean division, Theorem 5.1, in M.) But then $n \in S \Leftrightarrow p$ divides n_a exactly, i.e.,

$$n \in S \Leftrightarrow n_a = k \oplus k \oplus \cdots \oplus k \quad (p \text{ times})$$

and $n \notin S$ otherwise. Thus S is recursive if $\oplus$ is.

To show $\odot$ is also non-recursive we argue in a similar way, finding $b \in M$ with image $n_b \in \mathbb{N}$ such that

$$S = \left\{ n \in \mathbb{N} \;\middle|\; M \models \exists x(\overbrace{x \cdot x \cdot x \cdot \cdots \cdot x}^{\mathrm{pr}(n)} = b) \right\}$$

(for example we may choose b to be 2^a, where $a \in M$ is as above). A similar algorithm to that above shows that if $\odot$ were recursive then S would be too. □

We showed that, for nonstandard $M \models PA$, $SSy(M)$ contains a non-recursive set S by using the satisfaction relation $\mathrm{Sat}_{\Pi_1}(x, y)$. Another, more direct, argument can be given that uses the notion of a pair of recursively inseparable sets.

If $A, B \subseteq \mathbb{N}$ are disjoint sets (i.e., $A \cap B = \varnothing$) we say that they are *recursively inseparable* iff for no recursive $C \subseteq \mathbb{N}$ is $C \supseteq A$ and $C \cap B = \varnothing$. (This notion is symmetric in A and B since C is recursive iff $\mathbb{N} \setminus C$ is recursive.) The existence of a disjoint pair of r.e. recursively inseparable sets is the recursion-theoretic analogue of the Gödel–Rosser incompleteness theorem. In fact our proof of the Gödel–Rosser theorem in Chapter 3 supplies us with such a pair, namely the r.e. sets

$$A = \{\ulcorner\sigma\urcorner \mid \sigma \text{ is an } \mathscr{L}_{\mathrm{A}}\text{-sentence and } PA^- \vdash \sigma\}$$

and

$$B = \{\ulcorner\sigma\urcorner \mid \sigma \text{ is an } \mathscr{L}_{\mathrm{A}}\text{-sentence and } PA^- \vdash \neg\sigma\}$$

A, B are clearly r.e. and disjoint, since PA^- is finitely axiomatized and consistent. If $C \supseteq A$ is recursive and $C \cap B = \varnothing$, then we may assume C is a set of Gödel–numbers of $\mathscr{L}_{\mathrm{A}}$-sentences (by replacing C with the recursive set $C' = C \cap \{\ulcorner\sigma\urcorner \mid \sigma \text{ is an } \mathscr{L}_{\mathrm{A}}\text{-sentence}\}$ if necessary). Now consider the set

$$C'' = \left\{ \ulcorner\overbrace{\neg\neg\cdots\neg}^{l}\sigma\urcorner \;\middle|\; \begin{array}{l} \sigma \text{ has no leading } \neg\text{s, } l \text{ is odd,} \\ \ulcorner\neg\sigma\urcorner \in C \text{ and } \ulcorner\sigma\urcorner \notin C \end{array} \right\}$$
$$\cup \left\{ \ulcorner\overbrace{\neg\neg\cdots\neg}^{k}\sigma\urcorner \;\middle|\; \begin{array}{l} \sigma \text{ has no leading } \neg\text{s, } k \text{ is even, and either} \\ \ulcorner\neg\sigma\urcorner \notin C \text{ or } \ulcorner\sigma\urcorner \in C. \end{array} \right\}$$

We leave it to the reader as an exercise to check that (a) C'' is simply consistent, i.e., for no $\mathscr{L}_{\mathrm{A}}$-sentence τ is both $[\tau]$ and $[\neg\tau]$ in C''; (b) for all $\mathscr{L}_{\mathrm{A}}$-sentences τ, either $[\tau] \in C''$ or $[\neg\tau] \in C''$; (c) $C'' \supseteq A$ and $C'' \cap B = \varnothing$; and (d) if C is recursive then so is C''. But then by Theorem 3.9, C'' cannot be recursive, so C is not recursive either.

The existence of disjoint, r.e., recursively inseparable sets A, B can be proved by more direct recursion-theoretic arguments too. See for example (Cutland, 1980, p. 133) or (Rogers, 1967, p. 170). We now show how these ideas can be used to give a more direct prooof of Theorem 11.7, applicable in a wider setting.

THEOREM 11.8. Let $M \vDash PA^-$ be nonstandard and suppose M satisfies overspill for Δ_0 formulas and the standard cut $\mathbb{N}$. (For example any Δ_0-recursively saturated model $M \vDash PA^-$, or any model of $I\Delta_0$ will have this property.) Then M is not recursive.

Proof. Let $A, B \subseteq \mathbb{N}$ be disjoint, r.e., and recursively inseparable. Since they are r.e. there are Δ_0 formulas $\alpha(x, \bar{y})$ and $\beta(x, \bar{z})$ such that, for all $n \in \mathbb{N}$,

$$n \in A \Leftrightarrow \mathbb{N} \vDash \exists \bar{y}\alpha(n, \bar{y})$$

and

$$n \in B \Leftrightarrow \mathbb{N} \vDash \exists \bar{z}\beta(n, \bar{z}).$$

Thus for all $k\in\mathbb{N}$,

$$M\vDash\forall x,\bar{y},\bar{z}\leqslant k\neg(\alpha(x,\bar{y})\wedge\beta(x,\bar{z})),\tag{*}$$

since this sentence is Δ_0 and true in $\mathbb{N}$, using Theorem 2.7. Applying overspill to (*), there is $a>\mathbb{N}$ in M such that

$$M\vDash\forall x,\bar{y},\bar{z}\leqslant a\neg(\alpha(x,\bar{y})\wedge\beta(x,\bar{z})).\tag{+}$$

Let C be the set $\{n\in\mathbb{N}\,|\,M\vDash\exists\bar{y}\leqslant a\alpha(n,\bar{y})\}$. Then $C\supseteq A$, for if $n\in A$ then $\mathbb{N}\vDash\alpha(n,\bar{m})$ for some $\bar{m}\in\mathbb{N}$, hence $M\vDash\exists\bar{y}\leqslant a\alpha(n,\bar{y})$; and $C\cap B=\varnothing$, for if $n\in B$ then $\mathbb{N}\vDash\beta(n,\bar{k})$ for some $\bar{k}\in\mathbb{N}$, hence $M\vDash\exists\bar{z}\leqslant a\beta(n,\bar{z})$, hence $M\vDash\neg\exists\bar{y}\leqslant a(\alpha(n,\bar{y}))$, by our choice of a. We will show that, if $M\cong(\mathbb{N},\oplus,\odot,\otimes,n_0,n_1)$ with $\oplus$ recursive, then both C and $D=\mathbb{N}\setminus C$ are r.e., hence C is recursive, a contradiction.

Using Lemma 3.6, let $\exists\bar{w}\theta(u,v,\bar{w})$ be a Σ_1 formula representing 'v is the uth prime' in PA^-, where θ is Δ_0. Then for each $k\in\mathbb{N}$

$$\begin{aligned}&M\vDash\exists s\leqslant a\forall u\leqslant k\exists v\leqslant a\exists\bar{w}\leqslant a\\&\qquad[\theta(u,v,\bar{w})\wedge(\exists t\leqslant s(v\cdot t=s)\leftrightarrow\exists\bar{y}\leqslant a\alpha(u,\bar{y}))]\end{aligned}$$

since a is nonstandard. (To see this, let $s=p_{n_0}\cdot p_{n_1}\cdot p_{n_2}\cdot\cdots\cdot p_{n_k}$ where $n_0,n_1,n_2,\ldots,n_k$ are the first $k+1$ elements of C.) So by Δ_0-overspill again there is $b>\mathbb{N}$ and $s\in M$ such that

$$M\vDash\forall u\leqslant b\exists v\leqslant a\exists\bar{w}\leqslant a(\theta(u,v,\bar{w})\wedge(\exists t\leqslant s(v\cdot t=s)\leftrightarrow\exists\bar{y}\leqslant a\alpha(u,\bar{y}))).$$

Thus $C=\{n\in\mathbb{N}\,|\,M\vDash\exists t(t+t+\cdots+t=s)(\mathrm{pr}(n)\,ts)\}$ and hence if $\oplus$ is recursive then C is r.e., since

$$C=\{n\in\mathbb{N}\,|\,\exists t\in\mathbb{N}\;t\oplus t\oplus\cdots\oplus t=n_s\,(\mathrm{pr}(n)\,ts)\}$$

for some suitable $n_s\in\mathbb{N}$.

$D=\mathbb{N}\setminus C$ is also r.e. by a similar argument, where we use '$\neg\exists\bar{y}\leqslant a\,\alpha(u,\bar{y})$' in place of '$\exists\bar{y}\leqslant a\alpha(u,\bar{y})$'. Hence C is recursive by Theorem 0.10, our required contradiction. □

(The reader may like to modify the proof of the last theorem to show that if $\mathbb{N}\neq M\vDash PA^-$ satisfies Δ_0-overspill and $M\cong(\mathbb{N},\oplus,\odot,\otimes,n_0,n_1)$, then $\odot$ cannot be recursive either.)

This theorem, that no nonstandard model of $I\Delta_0$ is recursive, is due to McAloon (1982). Since it is not known if $I\Delta_0$ can define $\mathrm{Sat}_{\Delta_0}(x,y)$ in any reasonable way, this proof of Tennenbaum's theorem seems the most natural approach.

Tennenbaum's theorem can be extended and modified in many ways, especially with the use of Result 7.8, and the exercises following take the reader in some of these directions. In Section 13.2 we will reconsider the relationship between the Gödel–Rosser theorem and Theorem 11.7: it turns out that there is a straightforward proof of Corollary 3.10 from Theorem 11.7 using a model-theoretic construction!

Exercises for Section 11.3

11.10 Let $M \vDash PA$ be nonstandard and countable. Define a recursive binary relation $\otimes \subseteq \mathbb{N}^2$ such that $M \upharpoonright < \cong (\mathbb{N}, \otimes)$.
11.11$^+$ Let $M \cong (\mathbb{N}, \oplus, \odot, \otimes, n_0, n_1) \vDash PA + \Pi_n - \mathrm{Th}(\mathbb{N})$ be nonstandard, where $n \geqslant 1$. Show that neither $x \oplus y = z$ nor $x \odot y = z$ is defined by a $\Delta_{n+1}(\mathbb{N})$ formula $\theta(x, y, z)$.
(*Hint:* Let $A, B \subseteq \mathbb{N}$ be defined by $\Sigma_{n+1}(\mathbb{N})$ formulas $\alpha(x), \beta(x)$ respectively such that for no $\Delta_{n+1}(\mathbb{N})$ formula $\psi(x)$ does $\mathbb{N} \vDash \forall x(\alpha(x) \rightarrow \psi(x)) \wedge \forall y \neg(\beta(y) \wedge \psi(y))$. A, B can be found by considering Exercise 9.12, for example. Now use the following overspill principle (which is true for all nonstandard $M \vDash PA + \Pi_n - \mathrm{Th}(\mathbb{N})$)

if $\mathbb{N} \vDash \theta(k)$ for all $k \in \mathbb{N}$, where θ is Σ_{n+1}, then

$M \vDash \forall x < b\theta(x)$ for some $b > \mathbb{N}$.)

11.12$^+$ Let $M \vDash PA^-$ be a nonstandard such that for all Σ_1 formulas $\theta(\bar{x})$ there is an $\exists_1$ formula $\psi(\bar{x})$ such that $M \vDash \forall \bar{x}(\theta(\bar{x}) \leftrightarrow \psi(\bar{x}))$. (For example, by Result 7.8, any model of PA has this property.) Show that M is not recursive.
(*Hint:* Suppose M is recursive.
(a) Show that if θ is Δ_0 and $\bar{a} \in M$, then $\{n \in \mathbb{N} \mid M \vDash \theta(n, \bar{a})\}$ is recursive.
(b) Given that $A = \{x \in \mathbb{N} \mid \mathbb{N} \vDash \exists \bar{y}\alpha(x, \bar{y})\}$ and $B = \{x \in \mathbb{N} \mid \mathbb{N} \vDash \exists \bar{z}\beta(x, \bar{z})\}$ are r.e., disjoint, and recursively inseparable (where α and β are Δ_0) consider the set

$$D = \{b \in M \mid M \vDash \forall x, \bar{y}, \bar{z} < b(\neg\alpha(x, \bar{y}) \vee \neg\beta(x, \bar{z}))\}.$$

Given that $b \in D$ is nonstandard, show that

$$C = \{n \in \mathbb{N} \mid M \vDash \exists \bar{y} < b\alpha(n, \bar{y})\}$$

is recursive and separates A, B.

(c) Given that no $b \in D$ is nonstandard, show that for any $\mathscr{L}_A$-formula $\psi(x)$ there is a Δ_0 formula $\varphi(x, y)$ such that, for any $c \in M \setminus \mathbb{N}$,

$$\forall n \in \mathbb{N}(\mathbb{N} \models \psi(n) \Leftrightarrow M \models \psi(n, c)).)$$

11.13$^+$ Let $M \models PA$ be nonstandard. Using Result 7.8 and Exercise 7.10 show that there are polynomials $p(\bar{x}), q(\bar{x}) \in \mathbb{N}[\bar{x}]$ such that for no bijection $f: \mathbb{N} \to M$ is the set

$$\{\bar{n} \in \mathbb{N} \mid M \models p(f(n_1), \ldots, f(n_k)) \neq q(f(n_1), \ldots, f(n_k))\}$$

defined by a Π_1 formula in $\mathbb{N}$.
(*Hint:* Let $S = \{\ulcorner\sigma\urcorner \mid \sigma$ is Π_1 or Σ_1 and $M \models \sigma\}$ be coded by $a \in M$ and let p, q satisfy

$$M \models \forall x((a)_x \neq 0 \leftrightarrow \forall \bar{y}(p(a, x, \bar{y}) \neq q(a, x, \bar{y}))).$$

Show also that if S is defined in $\mathbb{N}$ by a Π_1 formula, then S is recursive.)

12

Initial segments

I measure myself
Against a tall tree.
I find that I am much taller,
For I reach right up to the sun,
With my eye;
And I reach right up to the shore of the sea
With my ear.
Nevertheless, I dislike
The way the ants crawl
In and out of my shadow.

Wallace Stevens
From *Six Significant Landscapes*, Harmonium, 1931

The saturation properties of nonstandard models of PA discussed in the last chapter shed a great deal of light on the structure of these models. In this chapter we will describe one of their most celebrated consequences: Harvey Friedman's theorem that any countable nonstandard model of PA is isomorphic to a proper initial segment of itself. The proof of this theorem in turn follows from more general considerations about when a model $M \vDash PA$ can be embedded onto an initial segment of another model $N \vDash PA$.

The standard system of these models M and N play a central role in such embedding problems. To see this, first notice that, if $M \cong K \prec_{\Delta_0} N$, then $SSy(M) \subseteq SSy(N)$, since the formula $(x)_y = 0$ is Δ_0 and so for all $a \in K$ and $n \in \mathbb{N}$, $K \vDash (a)_n = 0 \Leftrightarrow N \vDash (a)_n = 0$, so the set coded by a in K is the same as the set coded by a in N, thus $SSy(M) = SSy(K) \subseteq SSy(N)$. In the special case when $\mathbb{N} \neq M \cong K \subseteq_e N$ then M and N actually have the same standard system, since if $A \subseteq \mathbb{N}$ is coded by $a \in N$, say, then for all nonstandard $b \in K$ and all $n \in \mathbb{N}$,

$$N \vDash \exists x < b \forall i < n((x)_i = 0 \leftrightarrow (a)_i = 0)$$

so by overspill there is $c < b$ and $d > \mathbb{N}$ in K such that

$$N \vDash \forall i < d((c)_i = 0 \leftrightarrow (a)_i = 0).$$

That is: c codes A in N and $c < b \in K$. But K is an initial segment of N, so $c \in K$ and as $K \prec_{\Delta_0} N$, c codes the same set A in K also, hence $A \in SSy(K) = SSy(M)$. Thus standard systems give straightforward necessary conditions for embeddings $h: M \xrightarrow{\cong} K \prec_{\Delta_0} N$ and $k: M \xrightarrow{\cong} K \subseteq_e N$ to exist, namely: $SSy(M) \subseteq SSy(N)$ and $SSy(M) = SSy(N)$, respectively. The results of this chapter show that these conditions, together with obvious conditions on the theories of the models involved, are also sufficient, at least if M, N are countable.

12.1 FRIEDMAN'S THEOREM

We aim to prove here a theorem characterizing exactly when a countable model $M \vDash PA$ can be embedded as an initial segment of a second model $N \vDash PA$. As a consequence, we will deduce Friedman's theorem that any countable nonstandard model of PA is isomorphic to a proper initial segment of itself. First, though, we need a pair of lemmas that indicate again the very close connections between types over a model and the model's standard system.

Lemma 12.1 ('Every Σ_n or Π_n type realized in a model is coded'). Let $M \vDash PA$ be nonstandard, let $\bar{a} \in M$ be a tuple of finite length from M and let $n \in \mathbb{N}$. Then the sets

$$S = \{\ulcorner \theta(\bar{x}) \urcorner \mid \theta(\bar{x}) \in \Sigma_n \quad \text{and} \quad M \vDash \theta(\bar{a})\}$$

and

$$T = \{\ulcorner \theta(\bar{x}) \urcorner \mid \theta(\bar{x}) \in \Pi_n \quad \text{and} \quad M \vDash \theta(\bar{a})\}$$

are coded in M.

Proof. Observe that

$$p(v) = \{(v)_k \neq 0 \leftrightarrow \theta(\bar{a}) \mid k \in \mathbb{N},\, \theta \in \Sigma_n \text{ and } \ulcorner \theta(\bar{x}) \urcorner = k\}$$
$$\cup \left\{ (v)_k = 0 \;\middle|\; \begin{array}{l} k \in \mathbb{N} \text{ is not the Gödel-number} \\ \text{any } \Sigma_n \text{ formula} \end{array} \right\}$$

is a recursive Σ_{n+1} type over M, so is realized in M, by Theorem 11.5. Clearly any $c \in M$ realizing $p(v)$ codes S. The Π_n case is similar. □

Lemma 12.2 ('Coded Σ_n or Π_n types are realized'). Let $M \vDash PA$ be nonstandard, $\bar{b} \in M$ a tuple of finite length, and let

$$p(\bar{x}) = \{\theta_i(\bar{x}, \bar{b}) \mid i \in \mathbb{N}\}$$

be a Σ_n type (or a Π_n type) over M such that $S = \{\ulcorner \theta_i(\bar{x}, \bar{y}) \urcorner \mid i \in \mathbb{N}\} \in SSy(M)$. Then $p(\bar{x})$ is realized in M.

Proof. Let $c \in M$ code S and consider the type $q(\bar{x})$ defined by

$$q(\bar{x}) = \{(c)_i \neq 0 \rightarrow \theta(\bar{x}, \bar{b}) \mid i \in \mathbb{N}, \theta(\bar{x}, \bar{y}) \in \Sigma_n, i = \ulcorner \theta(\bar{x}, \bar{y}) \urcorner\}.$$

$q(\bar{x})$ is finitely satisfied in M, since $p(\bar{x})$ is, and $q(\bar{x})$ consists only of Σ_n formulas and is clearly a recursive type. Thus by Theorem 11.5, $q(\bar{x})$ is realized by some $\bar{a} \in M$. But such $\bar{a}$ realizes $p(\bar{x})$ too, as for all $\theta(\bar{x}, \bar{b})$ in $p(\bar{x})$, $M \models (c)_i \neq 0$ where $i = \ulcorner \theta(\bar{x}, \bar{y}) \urcorner$ so $M \models \theta(\bar{a}, \bar{b})$. The Π_n case is similar. □

We are now in a position to prove our main embedding theorem.

THEOREM 12.3. Let M, N be countable nonstandard models of PA, let $a, b \in N$, $c \in M$ and $n \in \mathbb{N}$. Then the following are equivalent:
(a) there exists an embedding $h: M \rightarrow N$ with image $I \prec_{\Sigma_n} N$ which is an initial segment of N, $h(c) = a$ and $b \notin I$;
(b) $SSy(M) = SSy(N)$ and, for all Π_n formulas $\theta(\bar{x}, \bar{y})$, $M \models \exists \bar{x} \theta(\bar{x}, c) \Rightarrow N \models \exists \bar{x} < b \theta(\bar{x}, a)$.

Proof. (a)$\Rightarrow$(b) is the easy direction. We have already observed that if $h: M \xrightarrow{\cong} I \subseteq_e N$ then $SSy(M) = SSy(I) = SSy(N)$. If $\theta(\bar{x}, y)$ is Π_n then $M \models \exists \bar{x} \theta(\bar{x}, c) \Rightarrow I \models \exists \bar{x} \theta(\bar{x}, h(c)) \Rightarrow \exists \bar{d} \in I\, I \models \theta(\bar{d}, h(c))$, and since $b > I$, $h(c) = a$ and $I \prec_{\Sigma_n} N$ this implies $N \models \theta(\bar{d}, a)$ hence $N \models \exists \bar{x} < b \theta(\bar{x}, a)$.

The converse, (b)$\Rightarrow$(a), is proved by constructing two sequences $u_0, u_1, \ldots, u_i, \ldots \in M$ and $v_0, v_1, \ldots, v_i, \ldots \in N$, and then defining $h(u_i) = v_i$ for all i. These sequences will be defined inductively on i, and so at any stage of the construction we will have sequences of finite length $u_0, \ldots, u_i \in M$ and $v_0, \ldots, v_i \in N$.

What properties do we require our sequences u, v to have? Firstly, we must make sure that $h(x)$ is defined for all $x \in M$, that is

(1) For all $x \in M$ there is some $j \in \mathbb{N}$ such that $x = u_j$.

Secondly, we require that $h(c) = a$. Thus we shall insist that

(2) $u_0 = c$ and $v_0 = a$.

Next, we require that the image of h, $\{v_0, v_1, \ldots\}$ is an initial segment of N, not containing b.

(3) $v_i < b$ for all i; and

(4) for all $y \in N$ and all $i \in \mathbb{N}$, if $y < v_i$ then $y = v_j$ for some $j \in \mathbb{N}$.

Finally we require that $I = \{v_0, v_1, \ldots\}$ is an Σ_n-elementary substructure of N. The following condition achieves this, and will also force h to be an embedding from M to N, i.e. to be 1–1 and to preserve the $\mathscr{L}_A$-structure of these models. This condition is suggested by the argument in (a)$\Rightarrow$(b) above, and can easily be seen to be necessary by the same argument. We require

(5) for all $i \in \mathbb{N}$ and for all Π_n formulas $\theta(\bar{x}, y_0, \ldots, y_i)$

$$M \models \exists \bar{x} \theta(\bar{x}, u_0, \ldots, u_i) \Rightarrow N \models \exists \bar{x} < b \theta(\bar{x}, v_0, \ldots, v_i).$$

Notice that (5) is automatically true for $i=0$ by our choice $u_0=c$, $v_0=a$ and our assumption (b) above.

To construct our *u*s and *v*s satisfying (1)–(5) we fix enumerations $\{d_0, d_1, d_2, \ldots\}$ of M and $\{e_0, e_1, e_2, \ldots\}$ of N in which each element $y \in N$ occurs infinitely often in the sequence $e_0, e_1, \ldots$, and where the indices i, j to d_i, e_j are non-negative integers. (This is where the countability of M and N is used in our argument.) Thus equipped, we can now show how the *u*s and *v*s are found.

Suppose we have already obtained $u_0, u_1, \ldots, u_{2k}$ and $v_0, v_1, \ldots, v_{2k}$ satisfying (5), where $k \geq 0$. Aiming to satisfy (1), we put $u_{2k+1} = d_k$, the kth element in our enumeration of M, and let A be the set

$$\{\ulcorner \theta(y_0, y_1, \ldots, y_{2k+1}) \urcorner \mid \theta \in \Sigma_{n+1} \;\&\; M \models \theta(u_0, u_1, \ldots, u_{2k+1})\}$$

which is in $SSy(M)$ by Lemma 12.1. Now consider

$$\begin{aligned} p(x) = \{\exists \bar{z} < b \psi(\bar{z}, v_0, \ldots, v_{2k}, x) \mid \psi \text{ is } \Pi_n \\ \text{and } \ulcorner \exists \bar{z} \psi(\bar{z}, y_0, \ldots, y_{2k+1}) \urcorner \in A\}. \end{aligned}$$

We claim that $p(x)$ is a type over N and is coded in N. To check that $p(x)$ is finitely satisfied, let $\exists \bar{z} < b \psi_j(\bar{z}, v_0, \ldots, v_{2k}, x)$ be in $p(x)$ for $1 \leq j \leq l$. Then

$$M \models \exists \bar{z}_1, \ldots, \bar{z}_j \bigwedge_{j=1}^{l} \psi_j(\bar{z}_j, u_0, \ldots, u_{2k}, u_{2k+1})$$

so

$$M \models \exists \bar{z}_1, \ldots, \bar{z}_j, w \bigwedge_{j=1}^{l} \psi_j(\bar{z}_j, u_0, \ldots, u_{2k}, w)$$

hence, by (5),

$$N \models \exists \bar{z}_1, \ldots, \bar{z}_j, w < b \bigwedge_{j=1}^{l} \psi_j(\bar{z}_j, u_0, \ldots, v_{2k}, w).$$

To check that $p(x)$ is coded in N, notice that $A \in SSy(M) = SSy(N)$ so there is $r \in N$ coding A in N. Thus $p(x)$ is coded in N by any $s \in N$ realizing the recursive type

$$q(x) = \left\{ (x)_j \neq 0 \leftrightarrow (r)_i \neq 0 \;\middle|\; \begin{array}{l} j = \ulcorner \exists \bar{z} < w_{2k+1} \psi(\bar{z}, w_0, \ldots, w_{2k}, x) \urcorner \\ i = \ulcorner \exists \bar{z} \psi(\bar{z}, w_0, \ldots, w_{2k}, x) \urcorner \\ \psi \in \Pi_n \end{array} \right\}$$

$$\cup \left\{ (x)_j = 0 \;\middle|\; \begin{array}{l} j \neq \ulcorner \exists \bar{z} < w_{2k+1} \psi(\bar{z}, \bar{w}, x) \urcorner \\ \text{for all } \psi \in \Pi_n \end{array} \right\}.$$

Thus by Lemma 12.2 $p(x)$ is realized in N by some $v_{2k+1} \in N$. It is easy to check that (5) holds for $i = 2k+1$ and, by considering (5) applied to the formula $x = y_{2k+1}$, since $M \models \exists x\, x = u_{2k+1}$, $N \models \exists x < b\, x = v_{2k+1}$, so $v_{2k+1} < b$.

Now suppose we have $u_0, \ldots, u_{2k+1}$ and $v_0, \ldots, v_{2k+1}$ satisfying (5). Consider e_k, the kth element of our enumeration of N. If $e_k > \max(v_0, \ldots, v_{2k+1})$ then we set $u_{2k+2} = u_0$, $v_{2k+2} = v_0$, and it is easy to check that (5) holds for $i = 2k+2$. Otherwise $e_k \leqslant \max(v_0, \ldots, v_{2k+1})$ and we put $v_{2k+2} = e_k$. We must find $u_{2k+2} \in M$ satisfying (5) for $i = 2k+2$ and this choice of v_{2k+2}. Let

$$B = \{\ulcorner \theta(y_0, \ldots, y_{2k+2}, w)\urcorner \mid \theta \in \Pi_{n+1} \ \&\ N \models \theta(v_0, \ldots, v_{2k+2}, b)\}.$$

Then B is coded in N by Lemma 12.1. Now consider

$$p(x) = \left\{ \forall \bar{z} \psi(\bar{z}, u_0, \ldots, u_{2k+1}, x) \,\middle|\, \begin{array}{l} \psi \text{ is } \Sigma_n \text{ and} \\ \ulcorner \forall \bar{z} < w \psi(\bar{z}, y_0, \ldots, y_{2k+2})\urcorner \in B \end{array} \right\}.$$

We claim that $p(x)$ is coded and finitely satisfied in M. To check that $p(x)$ is finitely satisfied, suppose $\forall \bar{z} \psi_j(\bar{z}, u_0, \ldots, u_{2k+1}, x) \in p(x)$ for $1 \leqslant j \leqslant l$ and that

$$M \models \forall x \leqslant \max(u_0, \ldots, u_{2k+1}) \exists \bar{z} \bigvee_{j=1}^{l} \neg \psi_j(\bar{z}, u_0, \ldots, u_{2k+1}, x). \qquad (*)$$

Since each ψ_j is Π_{n+1} the formula in (*) is equivalent in $B\Sigma_{n+1}$ to the Σ_{n+1} formula

$$\exists t \forall x \leqslant \max(u_0, \ldots, u_{2k+1}) \exists \bar{z} < t \bigvee_{j=1}^{l} \neg \psi_j(\bar{z}, u_0, \ldots, u_{2k+1}, x).$$

Thus, using (5) and the fact that Π_n is closed under bounded quantification (using the appropriate collection axioms, which are true in both M and N) we get

$$N \models \exists t < b \forall x \leqslant \max(v_0, \ldots, v_{2k+1}) \exists \bar{z} < t \bigvee_{j=1}^{l} \neg \psi_j(\bar{z}, v_0, \ldots, v_{2k+1}, x)$$

hence

$$N \models \forall x \leqslant \max(v_0, \ldots, v_{2k+1}) \exists \bar{z} < b \bigvee_{j=1}^{l} \neg \psi_j(\bar{z}, v_0, \ldots, v_{2k+1}, x)$$

which contradicts (*), since $v_{2k+2} \leqslant \max(v_0, \ldots, v_{2k+1})$ and by choice of the ψ_j,

$$M \models \exists x \leqslant \max(u_0, \ldots, u_{2k+1}) \forall \bar{z} \bigwedge_{j=1}^{l} \psi_j(\hat{z}, u_0, \ldots, u_{2k+1}, x),$$

and hence $p(x)$ is finitely satisfied. $p(x)$ is clearly coded in M since $B \in SSy(N) = SSy(M)$ and so if $r \in M$ codes B then any $s \in M$ realizing the recursive type

$$q(x)=\left\{(x)_j\neq 0\leftrightarrow(r)_i\neq 0\,\middle|\,\begin{array}{l}\psi\in\Sigma_n,\quad j=\ulcorner\forall\bar z\psi(\bar z,y_0,\ldots,y_{2k+1},y_{2k+2})\urcorner\\ i=\ulcorner\forall\bar z<w\psi(\bar z,y_0,\ldots,y_{2k+1},y_{2k+2})\urcorner\end{array}\right\}$$

$$\cup\{(x)_j=0\mid j\neq\ulcorner\theta\urcorner\text{ for all }\theta\in\Pi_{n+1}\}$$

codes $p(x)$. Thus by Lemma 12.2, $p(x)$ is realized in N by some $u_{2k+2}\in M$. It is then easy to check that if $M\vDash\exists\bar x\theta(\bar x,u_0,\ldots,u_{2k+2})$ with $\theta\in\Sigma_{n+1}$ then

$$\forall\bar z\neg\theta(\bar z,u_0,\ldots,u_{2k+1},x)\notin p(x)$$

so $\forall\bar z<b\neg\theta(\bar z,v_0,\ldots,v_{2k+2})$ is false in N, i.e., $N\vDash\exists\bar z<b\theta(\bar z,v_0,\ldots,v_{2k+2})$, thus (5) holds for $i=2k+2$.

Continuing in this way, we construct u_i and v_i for all $i\in\mathbb{N}$. We define $h(u_i)=v_i$. h is a well-defined function from M to N since every $x\in M$ is d_k for some k, so $x=u_{2k}$, and if $M\vDash u_i=u_j$ then $N\vDash v_i=v_j$ by (5), since '$y_i=y_j$' is (trivially) Π_n. Similarly

$$M\vDash u_i<u_j\Rightarrow N\vDash v_i<v_j,$$

$$M\vDash u_i+u_j=u_k\Rightarrow N\vDash v_i+v_j=v_k,$$

and

$$M\vDash u_i\cdot u_j=u_k\Rightarrow N\vDash v_i\cdot v_j=v_k,$$

so h preserves $<$, $+$ and $\cdot$, so is an embedding. The image of h, $I=\{v_0,v_1,\ldots\}$ is an initial segment of N since if $x\in N$, $x<v_i$ for some i, then $x=e_k$ for infinitely many k and in particular $x=e_k$ for some $k>i$. But then $e_k\leqslant\max(v_0,\ldots,v_{2k+1})$ so by the construction $e_k=v_{2k+2}$. Finally, $I<_{\Sigma_n}N$ since, if $\theta(\bar x)$ is Σ_n with $\bar u\in M$ and $\bar v=h(\bar u)\in I$, then since $h:M\to I$ is an isomorphism

$$I\vDash\theta(\bar v)\Leftrightarrow M\vDash\theta(\bar u),$$

and $M\vDash\theta(\bar u)\Leftrightarrow N\vDash\theta(\bar v)$, by (5) applied twice to the formulas $\neg\theta(\bar x)\in\Pi_n$, and $\psi(\bar u,\bar z)$ where $\theta(\bar u)$ is $\exists\bar z\psi(\bar u,\bar z)$. $\square$

We can now prove Friedman's theorem:

THEOREM 12.4 (H. Friedman). Let $M\vDash PA$ be nonstandard and countable, let $a\in M$ and let $n\in\mathbb{N}$. Then there is a proper initial segment $I\subseteq_e M$ containing a such that $I\cong M$ and $I<_{\Sigma_n}M$.

Proof. We must find $b\in M$ such that, for all $\theta(\bar x,y)\in\Pi_n$,

$$M\vDash\exists\bar x\theta(\bar x,a)\Rightarrow M\vDash\exists\bar x<b\theta(\bar x,a),$$

i.e., b must satisfy

$$p(z)=\{\exists\bar x\theta(\bar x,a)\to\exists\bar x<z\theta(\bar x,a)\mid\theta\in\Pi_n\}.$$

$p(z)$ is clearly recursive and Σ_{n+2}. To see that it is finitely satisfied note that if $\theta_1(\bar{x}, a), \ldots, \theta_k(\bar{x}, a)$ are Π_n and $M \models \exists \bar{x}\theta_i(\bar{x}, a)$ for all $i \leq k$ then there are $\bar{b}_1, \ldots, \bar{b}_k \in M$ such that $M \models \theta_i(\bar{b}_i, a)$ for all $i \leq k$, and $b = \max(\bar{b}_1, \ldots, \bar{b}_k) + 1$ clearly satisfies $M \models \exists \bar{y} < b\theta_i(\bar{y}, a)$ for all i. Thus there is $b \in M$ realizing $p(z)$, by Theorem 11.5, so by Theorem 12.3 there is an embedding $h: M \xrightarrow{\cong} I \subseteq_e M$ with $h(a) = a$, $b \notin I$ and $I \prec_{\Sigma_n} M$ as required. $\square$

REMARKS. It is necessary, in Theorem 12.4, that M is countable. (We used this when we took enumerations of M and M in Theorem 12.3.) For example, in Chapter 8, Corollary 8.9, we showed that there are ω_1-like models of PA. Such models clearly have no proper initial segments isomorphic to themselves.

Nor can we expect in general to get initial segments I with $M \cong I \prec M$ and $M \neq I$, i.e., elementary for all formulas. For example if $M = K_T$ (where $T \neq \mathrm{Th}(\mathbb{N})$ is a complete extension of PA) then M has no proper elementary substructures, and so certainly has no proper elementary initial segments!

The statement of the theorem shows that the initial segment $I \subseteq_e M$ can be taken to be *arbitrarily large*. An easy mistake that can be made is in thinking that I can be *arbitrarily small* also. The fallacious argument is as follows. Since $I \subseteq_e M$ is isomorphic to M, there is a proper initial segment $I_1 \subseteq_e I$ isomorphic to I; $I_2 \subseteq_e I_1$ with $I_2 \cong I_1$; and so on. Then 'by continuing in this way we can find an initial segment I_j smaller than any given non-standard $b \in M$'. The mistake is of course in the phrase in inverted commas. There is no way of guaranteeing that the sequence I_j reaches $\mathbb{N}$ in its limit, and a counterexample is provided as follows. Let $\forall x\theta(x)$ be a true Π_1 sentence, where $\theta \in \Delta_0$, that is independent of PA. Let $M \models PA + \neg\forall x\theta(x)$ be countable and let $a \in M$ be least in M satisfying $\neg\theta(x)$. Then, if I is any initial segment of M with $a \notin I$ and I closed under $+$, $\cdot$, $I \models \forall x\theta(x)$ since $I \prec_{\Delta_0} M$. Hence I cannot be isomorphic to M.

Friedman's theorem is very remarkable. Until now we have not been able to find any non-standard initial segments $I \subseteq_e M$ satisfying PA: we now know that there are lots of them actually *isomorphic* to M!

Exercises for Section 12.1

12.1* Let $M = \{a_0, a_1, \ldots, a_i, \ldots\}$ be a countable model of PA^- and let N be another model of PA^- with elements $\{b_0, b_1, \ldots, b_i \ldots\} \subseteq N$, and possibly other elements besides these. Suppose that for all $i \in \mathbb{N}$ and all $\theta(x_0, \ldots, x_{i-1}) \in \Sigma_{n+1}$

$$M \models \theta(a_0, \ldots, a_{i-1}) \Rightarrow N \models \theta(b_0, \ldots, b_{i-1}).$$

Show that $h: M \to N$ given by $h(a_i) = b_i$ is well-defined and is an embedding of $\mathscr{L}_A$-structures with image $K = \{b_0, \ldots, b_i, \ldots\} \prec_{\Sigma_n} N$.

12.2* Let $N \vDash PA$ and $M \vDash B\Sigma_{n+1}$ be countable nonstandard models, and suppose M is short Π_{n+1}-recursively saturated, i.e. each recursive type of the form

$$p(x) = \{x_i < b \mid x_i \in \bar{x}\} \cup \{\theta_j(\bar{x}, \bar{c}) \mid j \in \mathbb{N}\}$$

(where $\theta_j \in \Pi_{n+1}$ for all j) over M is realized in M. Suppose further that $SSy(M) = SSy(N)$ and there are $a, b \in N$ and $c \in M$ such that

$$M \vDash \exists \bar{x} \theta(\bar{x}, c) \Rightarrow N \vDash \exists \bar{x} < b \theta(\bar{x}, a)$$

for all $\theta(\bar{x}, y) \in \Pi_n$. Show that there is an embedding $h : M \xrightarrow{\cong} I \subseteq_e N$ with $h(c) = a$, $b \notin I$ and $I \prec_{\Sigma_n} N$.

12.3 Write down necessary and sufficient conditions on countable models $M, N \vDash PA$ for there to be an embedding $h : M \xrightarrow{\cong} I \subseteq N$ with $I \prec_{\Sigma_n} N$.

12.4 Let $M \vDash PA$ be countable and nonstandard.

(a) Given that $M \nvDash \Pi_1 - \mathrm{Th}(\mathbb{N})$, show there is $a \in M \setminus \mathbb{N}$ such that no $I \subseteq_e M$ not containing a is isomorphic to M.

(b) Given that $M \vDash \Pi_1 - \mathrm{Th}(\mathbb{N})$ and $a \in M \setminus \mathbb{N}$, show that there is $I \subseteq_e M$ with $a \notin I \cong M$.

12.5 Let $M \vDash PA$ be uncountable. Show that there is a proper initial segment $I \subseteq_e M$ with $\mathbb{N} \neq I \vDash PA$.

(*Hint:* Use the Löwenheim–Skolem theorem to find $K \prec M$ countable and nonstandard. Let $\mathbb{N} \neq J \subseteq_e K$ be a proper initial segment of K and consider Theorem 7.6 and

$$I = \{x \in M \mid M \vDash x < a \text{ some } a \in J\}.)$$

12.2 VARIATIONS

In this section we shall consider some variations of the embedding theorem, Theorem 12.3, and some of its consequences.

THEOREM 12.5. Let M, N be countable nonstandard models of PA. The following are equivalent:

(a) there exists an embedding h: $M \xrightarrow{\cong} I \subseteq_e N$ with $I \prec_{\Sigma_n} N$;

(b) $SSy(M) = SSy(N)$ and $N \vDash \Sigma_{n+1} - \mathrm{Th}(M)$, the set of all Σ_{n+1} sentences true in M.

Proof. (a)$\Rightarrow$(b) is easy. For (b)$\Rightarrow$(a) let $b \in N$ realize the type

$$p(y) = \{\exists \bar{x} \theta(\bar{x}) \rightarrow \exists \bar{x} < y \theta(\bar{x}) \mid \theta \in \Pi_n\}.$$

Then by Theorem 12.3 (putting $a = c = 0$) there is an embedding h: $M \xrightarrow{\cong} I \subseteq_e N$ with $I \prec_{\Sigma_n} N$ if $N \vDash \Sigma_{n+1} - \mathrm{Th}(M)$ and $SSy(M) = SSy(N)$. □

THEOREM 12.6. Let $M, N \models PA$ be countable and nonstandard. Then the following are equivalent:

(a) for all $a \in N$ there is an embedding $h: M \xrightarrow{\cong} I \subseteq_e N$ with $a \in I \prec_{\Sigma_n} N$;
(b) $SSy(M) = SSy(N)$ and $N \models \Pi_{n+2} - \mathrm{Th}(M)$.

Proof. (a)$\Rightarrow$(b): If $M \models \forall \bar{x} \exists \bar{y} \theta(\bar{x}, \bar{y})$ with $\theta \in \Pi_n$ and $\bar{a} \in N$ is arbitrary, then by (a) there is $I \prec_{\Sigma_n} N$ containing $\bar{a}$ with $M \cong I$. Thus $I \models \exists \bar{y} \theta(\bar{a}, \bar{y})$, hence $N \models \theta(\bar{a}, \bar{b})$ for some $\bar{b} \in I$ since $I \prec_{\Sigma_n} N$ and θ is Π_n. Since $\bar{a} \in N$ was arbitrary, $N \models \forall \bar{x} \exists \bar{y} \theta(\bar{x}, \bar{y})$, hence $N \models \Pi_{n+2} - \mathrm{Th}(M)$.
(b)$\Rightarrow$(a): If $SSy(M) = SSy(N)$ and $N \models \Pi_{n+2} - \mathrm{Th}(M)$, let $a \in N$ and let

$$A = \{\ulcorner \theta(x) \urcorner \mid \theta \in \Pi_{n+1} \text{ and } N \models \theta(a)\}.$$

A is coded in N, by Lemma 12.1. Now consider

$$p(x) = \{\theta(x) \mid \ulcorner \theta(x) \urcorner \in A\}.$$

$p(x)$ is a type over M since if $\theta_1(x), \ldots, \theta_l(x)$ are in $p(x)$, then $N \models \exists x \bigwedge_{i=1}^{l} \theta_i(x)$. This is a Σ_{n+2} sentence, so $M \models \exists x \bigwedge_{i=1}^{l} \theta_i(x)$ (since the Π_{n+2} sentence $\forall x \bigvee_{i=1}^{l} \neg \theta_i(x)$ cannot be true in M as $N \models \Pi_{n+2} - \mathrm{Th}(M)$.) Also $p(x)$ is clearly coded in M since $A \in SSy(N) = SSy(M)$. Thus by Lemma 12.2 there is $c \in M$ realizing $p(x)$, and for this c

$$M \models \exists \bar{x} \psi(\bar{x}, c) \Rightarrow N \models \exists \bar{x} \psi(\bar{x}, a)$$

for all $\psi \in \Pi_n$. So by Theorem 12.3 (and an argument similar to that in the proof of Theorem 12.5) there is an embedding $h: M \xrightarrow{\cong} I \subseteq_e N$ with $I \prec_{\Sigma_n} N$ and $h(c) = a$, as required. □

The last result in this section gives perhaps the best indication so far of how complicated countable models of PA can be.

THEOREM 12.7. Let M, N be countable nonstandard models of PA, let $a, b \in N$, $n \in \mathbb{N}$, and suppose there is an initial segment $I \subseteq_e N$ with $a \in I < b$ and $M \cong I \prec_{\Sigma_n} N$. Then there are $2^{\aleph_0}$ such initial segments I.

Proof. Let $h: M \to I$ be an isomorphism and let $c = h^{-1}(a)$. We shall follow the idea of the proof of Theorem 12.3 to construct embeddings $M \xrightarrow{\cong} I \prec_{\Sigma_n} N$, but at each stage the constructions all divide into two, and thus we will eventually have $2^{\aleph_0}$ different embeddings $M \to N$. We must take care that all these different embeddings have different images $I \subseteq_e N$.

More precisely, let $\{d_0, d_1, \ldots\} = M$ and $\{e_0, e_1, \ldots\} = N$ be enumerations as in the proof of Theorem 12.3, such that each $y \in N$ is e_j for infinitely many $j \in \mathbb{N}$. We shall construct $\bar{u}_\sigma$, $\bar{v}_\sigma$ for all finite sequences σ of 0s and 1s (i.e., $\sigma : \{0, 1, \ldots, i-1\} \to \{0, 1\}$ for some $i \in \mathbb{N}$), where each $\bar{u}_\sigma$ is a triple $(u_\sigma, u'_\sigma, u''_\sigma)$ from M, each $\bar{v}_\sigma$ is $(v_\sigma, v'_\sigma, v''_\sigma) \in N$ and for any $f : \mathbb{N} \to \{0, 1\}$ the map

$$h_f : \bar{u}_{f \restriction i} \mapsto \bar{v}_{f \restriction i} \text{ for all } i \in \mathbb{N}$$

(where $f \restriction i$ denotes the restriction of f to $\{0, 1, \ldots, i-1\}$) is an embedding $M \overset{\cong}{\to} I_f \prec_{\Sigma_n} N$. We will also construct $b_\sigma \in N$ for each σ so that for all $i \in \mathbb{N}$ and all $f : \mathbb{N} \to \{0, 1\}$,

$$b_{f \restriction i} \notin \mathrm{Im}(h_f)$$

whereas

$$f(i) = 1 \Rightarrow b_{(f \restriction i), 0} \in \mathrm{Im}(h_f).$$

This will force all the $\mathrm{Im}(h_f) = I_f$ to be different, since if $f, g : \mathbb{N} \to \{0, 1\}$ are not equal let $i \in \mathbb{N}$ be least such that $f(i) \neq g(i)$ and suppose $f(i) = 1$, $g(i) = 0$. Then

$$b_{g \restriction (i+1)} = b_{f \restriction i, 0} \in \mathrm{Im}(h_f) \setminus \mathrm{Im}(h_g).$$

To construct these $\bar{u}_\sigma$, $\bar{v}_\sigma$ and b_σ we use induction on the length of σ. For $\sigma = \emptyset$, the empty sequence, put $u_\emptyset = (c, c, c)$, $v_\emptyset = (a, a, a)$ and $b_\emptyset = b$. (This ensures that $a \in \mathrm{Im}(f)$ and $b \notin \mathrm{Im}(f)$ for all $f : \mathbb{N} \to \{0, 1\}$.) Now inductively suppose we have found $\bar{u}_\tau$, $\bar{v}_\tau$ and b_τ for all 0, 1 sequences τ of length less than $i \in \mathbb{N}$ such that, for all $\sigma : \{0, 1, \ldots, i-1\} \to \{0, 1\}$,

$$(*) \begin{cases} \text{for all } \Pi_n \text{ formulas } \theta(\bar{x}, \bar{y}_0, \ldots, \bar{y}_i) \\ M \models \exists \bar{x} \theta(\bar{x}, \bar{u}_{\sigma \restriction 0}, \ldots, \bar{u}_{\sigma \restriction i}) \Rightarrow N \models \exists \bar{x} < b_\sigma \theta(\bar{x}, \bar{v}_{\sigma \restriction 0}, \ldots, \bar{v}_{\sigma \restriction i}). \end{cases}$$

It is easy to check using Theorem 12.3 that (*) is equivalent to the statement that

$$\bar{u}_{\sigma \restriction 0}, \ldots, u_{\sigma \restriction i} \mapsto \bar{v}_{\sigma \restriction 0}, \ldots, v_{\sigma \restriction i}$$

can be extended to a map $h : M \to N$ with image $I \prec_{\Sigma_n} N$ and $b_\sigma \notin I \subseteq_e N$. Note too that (*) is true for $i = 0$ and the empty sequence $\sigma = \emptyset$, by our choice of $b_\emptyset$ and $\bar{u}_\emptyset$, $\bar{v}_\emptyset$. Now fix some

$$\sigma : \{0, 1, \ldots, i-1\} \to \{0, 1\}$$

and

$$\bar{u}_{\sigma \restriction 0}, \ldots, \bar{u}_{\sigma \restriction i}, \bar{v}_{\sigma \restriction 0}, \ldots, \bar{v}_{\sigma \restriction i}$$

and b_σ. We show how to find $\bar{u}_{\sigma 0}$, $\bar{v}_{\sigma 0}$, $b_{\sigma 0}$, $\bar{u}_{\sigma 1}$, $\bar{v}_{\sigma 1}$, $b_{\sigma 1}$ satisfying (*), where i is replaced with $i+1$, and σ is replaced by either $\sigma 0$ or $\sigma 1$.

Write $\bar{u}$ for $u_{\sigma\restriction 0}, \ldots, \bar{u}_{\sigma\restriction i}$ and $\bar{v}$ for $\bar{v}_{\sigma\restriction 0}, \ldots, \bar{v}_{\sigma\restriction i}$. Consider the Σ_{n+1} type

$$p(y)=\{\exists\bar{x}<y\theta(\bar{x},\bar{v})\,|\,\theta\in\Pi_n, M\models\exists\bar{x}\theta(\bar{x},\bar{u})\}$$
$$\cup\{\exists\bar{x}<b_\sigma\theta(\bar{x},\bar{v},y)\,|\,\theta\in\Pi_n, M\models\forall y\exists\bar{x}\theta(\bar{x},\bar{u},y)\}$$

over N. This is finitely satisfied in N since if

$$\theta_1,\ldots,\theta_l\in\Pi_n, \psi_1,\ldots,\psi_k\in\Pi_n$$

and

$$M\models \textstyle\bigwedge_{j=1}^{l}\exists\bar{x}\theta_j(\bar{x},\bar{u})\wedge\bigwedge_{j=1}^{k}\forall y\exists\bar{x}\psi_j(\bar{x},\bar{u},y)$$

then there is $w\in M$ such that

$$M\models \textstyle\bigwedge_{j=1}^{l}\exists\bar{x}<w\theta_j(\bar{x},\bar{u})\wedge\bigwedge_{j=1}^{k}\exists\bar{x}\psi_j(\bar{x},\bar{u},w).$$

So applying (*) to the Σ_{n+1} formula

$$\exists w,\bar{x}_1,\ldots,\bar{x}_l,\bar{z}_1,\ldots,\bar{z}_k(\textstyle\bigwedge_{j=1}^{l}(\bar{x}_j<w\wedge\theta_j(\bar{x}_j,\bar{u}))\wedge\bigwedge_{j=1}^{k}\psi_j(\bar{z}_j,\bar{u},w))$$

there is $y\in N$ such that $N\models\exists\bar{x}<y\theta_j(\bar{x},\bar{v})$ for each $j\leq l$ and $N\models\exists\bar{x}<b_\sigma$ $\psi_j(\bar{x},\bar{u},y)$ for each $j\leq k$. $p(y)$ is also coded in N since both of

$$A=\{\ulcorner\theta(\bar{x},\bar{z})\urcorner\,|\,\theta\in\Pi_n \ \&\ M\models\exists\bar{x}\theta(\bar{x},\bar{u})\}$$

and

$$B=\{\ulcorner\theta(\bar{x},\bar{z},y)\urcorner\,|\,\theta\in\Pi_n \ \&\ M\models\forall y\exists\bar{x}\theta(\bar{x},\bar{u},y)\}$$

are in $SSy(M)$, hence in $SSy(N)$, by Lemma 12.1; and if $r,s\in N$ code A, B respectively, then any $t\in N$ realizing the recursive type

$$q(t)=\left\{(t)_i\neq 0\leftrightarrow(r)_j\neq 0\,\middle|\,\begin{matrix}i,j\in\mathbb{N},\ i=\ulcorner\exists\bar{x}<y\theta(\bar{x},\bar{z})\urcorner\\ \theta\in\Pi_n,\ j=\ulcorner\theta(\bar{x},\bar{z})\urcorner\end{matrix}\right\}$$

$$\cup\left\{(t)_i\neq 0\leftrightarrow(s)_j\neq 0\,\middle|\,\begin{matrix}i,j\in\mathbb{N},\ i=\ulcorner\exists\bar{x}<w\theta(\bar{x},\bar{z},y)\urcorner\\ \theta\in\Pi_n\text{ and } j=\ulcorner\theta(\bar{x},\bar{z},y)\urcorner\end{matrix}\right\}$$

$$\cup\{(t)_i=0\,|\,i\in\mathbb{N}\text{ not of the above two forms}\}$$

codes the type p. Thus $p(y)$ is realized in N, by Lemma 12.2. We define $b_{\sigma 1}=b_\sigma$ and $b_{\sigma 0}$ to be any element of N realizing $p(y)$. By the choice of the type $p(y)$, (*) holds even when b_σ is replaced by $b_{\sigma 0}$. Thus, putting $u_{\sigma 0}=c\in M$ and $v_{\sigma 0}=a\in N$,

$$(*_0)\begin{cases}\text{for all } \theta(\bar{x},\bar{y},z)\in\Pi_n\\ M\vDash\exists\bar{x}\theta(\bar{x},\bar{u},u_{\sigma 0})\Rightarrow N\vDash\exists\bar{x}<b_{\sigma 0}\theta(\bar{x},\bar{v},v_{\sigma 0})\end{cases}$$

holds for this choice of $u_{\sigma 0}$, $v_{\sigma 0}$ and $b_{\sigma 0}$. But also there is $u_1\in M$ realizing the type

$$r(y)=\{\forall\bar{x}\neg\theta(\bar{x},\bar{u},y)\,|\,N\vDash\forall\bar{x}<b_{\sigma 1}\neg\theta(\bar{x},\bar{v},b_{\sigma 0}),\,\theta\in\Pi_n\}$$

over M, since this is coded in N hence in M, is of bounded complexity, and is finitely satisfied in M by our choice of $b_{\sigma 0}$. Then, putting $v_{\sigma 1}=b_{\sigma 0}$, we obtain

$$(*_1)\begin{cases}\text{for all } \theta(\bar{x},\bar{y},z)\in\Pi_n\\ M\vDash\exists\bar{x}\theta(\bar{x},\bar{u},u_{\sigma 1})\Rightarrow N\vDash\exists\bar{x}<b_{\sigma 1}\theta(x,\bar{v},v_{\sigma 1}).\end{cases}$$

The rest of the proof is now as in the proof of Theorem 12.3. We put $u'_{\sigma 0}=u'_{\sigma 1}=d_i$, the ith element of the enumeration $\{d_0,d_1,\ldots\}$ of M (recall that i is the length of σ). Then $v'_{\sigma 0}$ and $v'_{\sigma 1}\in N$ are found by the same argument as in Theorem 12.3 satisfying conditions analogous to $(*_0)$ and $(*_1)$ respectively. Finally, if $e_i\leq\max(\bar{v})$ we put $v''_{\sigma 0}=v''_{\sigma 1}=e_i$ and find $u''_{\sigma 0}$, $u''_{\sigma 1}$ satisfying conditions analogous to $(*_0)$ and $(*_1)$, using the argument in the proof of Theorem 12.3. If $e_i>\max(\bar{v})$ then we may put $v''_{\sigma 0}=v''_{\sigma 1}=u''_{\sigma 0}=u''_{\sigma 1}=0$. When this is finished (*) will be satisfied for $\sigma 0$ and $\sigma 1$ in place of σ, and so the inductive construction can be continued.

The verification that the maps h_f have the required properties is identical to that in the proof of Theorem 12.3, and distinct f,g give rise to different initial segments I_f, I_g by the construction since $b_{\sigma 0}$ is in only one of I_f, I_g, where $\sigma=f\restriction i$ and i is maximal so that $f\restriction i=g\restriction i$. This completes the proof. □

COROLLARY 12.8. Let $M\vDash PA$ be countable and nonstandard, let $a,b\in M$ and $I\subseteq_e M$ with $I\cong M$ and $a\in I<b$. Then there are $2^{\aleph_0}$ initial segments $J\subseteq_e M$ with $a\in J<b$ and $M\cong J$. In particular for any $a\in M$ there are $2^{\aleph_0}$ initial segments $I\subseteq_e M$ isomorphic to M and containing a.

Proof. By Theorems 12.4 and 12.7. □

Exercises for Section 12.2

12.6 Find countable nonstandard $L, K \models PA$ such that L, K satisfy the same complete extension of PA and have the same standard system, but $L \not\cong K$.
(*Hint:* Let $K = K_T$ for some suitable T and let L be an elementary end-extension of K.)
12.7 Let $M, N \models PA$ with M (but not necessarily N) countable. Show that, for all $n \in \mathbb{N}$, there is an embedding $h : M \overset{\cong}{\rightarrow} K \prec_{\Sigma_n} N$ iff $SSy(M) \subseteq SSy(N)$ and $N \models \Sigma_{n+1} - \mathrm{Th}(M)$
12.8 Let $M \models PA$ be countable and nonstandard, let $a, b \in M$ and $n \in \mathbb{N}$, and suppose there is $I \prec_{\Sigma_n} M$ with $a \in I < b$ and $M \cong I \subseteq_e M$. Show that there is $J \prec_{\Sigma_n} M$ with $a \in J < b$, $J \subseteq_e M$ and an isomorphism $h : M \to J$ such that $h(a) = a$.

13

The standard system

So how is life with your new bloke?
⋮
Tell me, is he bright enough to find
that memo-pad you call a mind?

Craig Raine
From: 'An attempt at jealousy', *Rich*, 1984

If M is a nonstandard model of PA, its 'memo-pad' is its standard system of of sets $A \subseteq \mathbb{N}$ that are coded in M. In particular we can consider computations that consult this memo-pad to determine whether or not a given integer n is in some $A \in SSy(M)$. In this chapter we shall investigate the structure of these standard systems further: in particular we will find certain closure conditions that characterize them exactly. This will allow us to construct models with a given standard system, and in Chapter 14 we will use these ideas together with the embedding theorems of the last chapter to construct new interesting initial segments.

13.1 SCOTT SETS

To describe the structure of $SSy(M)$ for nonstandard models $M \vDash PA$ we will need a series of definitions. We ask for the reader's patience, but promise that the results will be worth the effort involved!

We define $2^{<\mathbb{N}}$ to be the set of finite sequences of 0s and 1s. That is

$$2^{<\mathbb{N}} = \{f \mid f: \{0, 1, \ldots, k-1\} \to \{0, 1\} \text{ for some } k \in \mathbb{N}\}.$$

We shall often write a function $f: \{0, 1, \ldots, k-1\} \to \{0, 1\}$ by $a_0 a_1 a_2 \ldots a_{k-1}$ where each a_i is either 0 or 1 and $a_i = f(i)$. Thus 0010 is the function $f: \{0, 1, 2, 3\} \to \{0, 1\}$ defined by $f(2) = 1$ and $f(i) = 0$ for all other arguments i. The *empty sequence* (that is the sequence consisting of no entries) will be denoted by $\varnothing$ and is identified with the empty function $f: \varnothing \to \{0, 1\}$. The *length* of $f \in 2^{<\mathbb{N}}$, $\mathrm{len}(f)$, is the number of elements in the domain of f. Thus $\mathrm{len}(0010) = 4$ and $\mathrm{len}(\varnothing) = 0$. If $f \in 2^{<\mathbb{N}}$, $k \in \mathbb{N}$ and $\mathrm{len}(f) \geqslant k$ then $f \restriction k$ denotes the sequence $g \in 2^{<\mathbb{N}}$ of length k defined by $g(i) = f(i)$ for all $i < k$. $f \restriction k$ is the *restriction of f to k*.

A (*binary*) *tree* is a set $T \subseteq 2^{<\mathbb{N}}$ such that whenever $f \in T$ and $k \leqslant \mathrm{len}(f)$ then $f \restriction k \in T$. Thus a tree is a set of 0, 1 sequences that is closed under

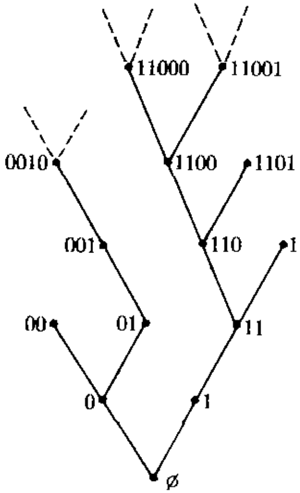

Fig. 5 A tree.

restrictions. A tree, T, in this sense corresponds to a tree in the graph-theoretical sense in which all nodes are labelled by sequences of 0s and 1s in T and an edge is drawn between two nodes $f, g \in T$ whenever $\mathrm{len}(f) = \mathrm{len}(g) + 1$ and $g = f \restriction \mathrm{len}(g)$. (See Fig. 5 for an example.) Note that two different trees $T, S \subseteq 2^{<\mathbb{N}}$ may be isomorphic in the graph-theoretic sense, since the same graph may be labelled in different ways.

A *path* $P \subseteq 2^{<\mathbb{N}}$ is a tree such that, whenever $f, g \in P$ with $\mathrm{len}(f) > \mathrm{len}(g)$, then $g = f \restriction \mathrm{len}(g)$. Thus a path is a tree with no branching.

A tree $T \subseteq 2^{<\mathbb{N}}$ is *infinite* iff T is infinite as a set. Regarding infinite trees, we recall the well-known and very important

LEMMA 13.1 (König's lemma). If T is an infinite tree then there is an infinite path $P \subseteq T$. □

The next step is to code each sequence $f \in 2^{<\mathbb{N}}$ by a natural number $\ulcorner f \urcorner \in \mathbb{N}$. The following method is particularly convenient.

$$\ulcorner f \urcorner = \begin{cases} \displaystyle\sum_{i=0}^{\mathrm{len}(f)} (1 + f(i)) \cdot 2^{\mathrm{len}(f) - i - 1} & (\text{if } \mathrm{len}((f) > 0) \\ 0 & (\text{if } f = \emptyset) \end{cases}$$

The function $\ulcorner \cdot \urcorner : 2^{<\mathbb{N}} \to \mathbb{N}$ is called *dyadic* coding, and is easily seen to be a bijection. The first few values of $\ulcorner \cdot \urcorner$ are given in Table 2.

With this coding given, definitions concerning sets $S \subseteq 2^{<\mathbb{N}}$ can be applied to sets $R \subseteq \mathbb{N}$ and vice versa. Thus $R \subseteq \mathbb{N}$ is a *tree* (*path*, etc.) iff

Table 2 Dyadic coding.

$f \in 2^{<\mathbb{N}}$	$\ulcorner f \urcorner$
$\varnothing$	0
0	1
1	2
00	3
01	4
10	5
11	6
000	7
001	8
010	9
011	10
100	11
$\vdots$	$\vdots$

$\{f \in 2^{<\mathbb{N}} \mid \ulcorner f \urcorner \in R\}$ is a tree (path, etc.), and a tree $T \subseteq 2^{<\mathbb{N}}$ is *recursive* (*r.e.*, etc.) iff $\{\ulcorner f \urcorner \mid f \in T\} \subseteq \mathbb{N}$ is recursive (r.e., etc.).

The key notion in this chapter is that of a *Scott set*, a beautiful combination of ideas from recursion theory and the theory of trees that first appeared in Scott (1962). Roughly, a Scott set is a collection, $\mathscr{X}$, of sets $A \subseteq \mathbb{N}$, which is closed under Boolean operations, relative recursion, and under a form of König's lemma. More precisely:

Definition: $\mathscr{X} \subseteq \mathcal{P}(\mathbb{N})$ is a *Scott set* iff $\mathscr{X}$ is non-empty and
(a) if $A, B \in \mathscr{X}$ then so are $A \cup B$, $A \cap B$, and $\mathbb{N} \setminus A$, $\mathbb{N} \setminus B$;
(b) if $A \in \mathscr{X}$ and $B \subseteq \mathbb{N}$ is recursive in A then $B \in \mathscr{X}$;
(c) if $T \in \mathscr{X}$ is an infinite tree, then there is an infinite path $P \subseteq T$ in $\mathscr{X}$.

Remarks. (1) Suppose $\mathscr{X}$ is a Scott set, $A_1, A_2, \ldots, A_n \in \mathscr{X}$, and B is recursive in $A_1, A_2, \ldots, A_n$. Then each $A_i' = \{n \cdot x + i \mid x \in A_i\}$ is recursive in A_i, so is in $\mathscr{X}$ by part (b) of the definition, and so $A = A_1' \cup A_2' \cup \cdots \cup A_n' \in \mathscr{X}$ by part (a). But clearly B is recursive in A, so B is also in $\mathscr{X}$. This shows that $\mathscr{X}$ is closed under relative recursion for any finite number of oracles $A_1, \ldots, A_n \in \mathscr{X}$.

(2) The definition of 'Scott set' does not depend on our choice of coding function $\ulcorner \cdot \urcorner$ (provided we chose it to be recursive in the obvious sense). For if $c\colon 2^{<\mathbb{N}} \to \mathbb{N}$ is any bijection such that the map $F\colon \mathbb{N} \to \mathbb{N}$ defined by $F(n) = c(f)$ for the unique f with $\ulcorner f \urcorner = n$ is recursive, $\mathscr{X}$ is a Scott set in the above sense, and $A \in \mathscr{X}$ satisfies

$$T = \{c^{-1}(n) \mid n \in A\} \text{ is an infinite tree,}$$

then

$$B = \{\ulcorner c^{-1}(n) \urcorner \mid n \in A\} = \{F^{-1}(n) \mid n \in A\}$$

is recursive in A, so $B \in \mathscr{X}$. Thus by part (c) of the definition above, there is $C \in \mathscr{X}$ such that $C \subseteq B$ and

$$P = \{f \in 2^{<\mathbb{N}} \mid \ulcorner f \urcorner \in C\} \subseteq T \qquad \text{is an infinite path.}$$

But then $D = \{F(n) \mid n \in C\} \in \mathscr{X}$, being recursive in C, and D is an infinite path of A in the sense of the coding function c.

The key result of this section is that the standard systems of countable nonstandard models of PA are exactly the countable Scott sets. One half of this result is provided by the following theorem.

THEOREM 13.2. Let $M \vDash PA$ be nonstandard. Then $SSy(M)$ is a Scott set.

Proof. We must verify (a), (b), and (c) hold for $\mathscr{X} = SSy(M)$ in the definition of a Scott set.

(a) If $A, B \in SSy(M)$ are coded by a, b respectively, then $A \cup B$ is coded by any $c \in M$ realizing the recursive type

$$p(x) = \{(x)_i \neq 0 \leftrightarrow ((a)_i \neq 0 \vee (b)_i \neq 0) \mid i \in \mathbb{N}\}$$

over M. Such c exist by Theorem 11.5. Similarly $\mathbb{N} \setminus A$ is coded by any element of M realizing $\{(x)_i \neq 0 \leftrightarrow (a)_i = 0 \mid i \in \mathbb{N}\}$, and $A \cap B$ is coded by any element of M realizing

$$\{(x)_i \neq 0 \leftrightarrow ((a)_i \neq 0 \wedge (b)_i \neq 0) \mid i \in \mathbb{N}\}.$$

(b) Suppose B is recursive in $A \in SSy(M)$ and A is coded by $a \in M$. By Corollary 3.5^A, there are Δ_0^A formulas in the language $\mathscr{L}_A(A)$ (which is $\mathscr{L}_A$ expanded by adding a new unary relation symbol A) $\varphi(x, \bar{y})$ and $\psi(x, \bar{z})$ such that, when A is interpreted in the natural way in $\mathbb{N}$,

$$n \in B \Leftrightarrow \mathbb{N} \vDash \exists \bar{y} \varphi(n, \bar{y}) \Leftrightarrow \mathbb{N} \vDash \forall \bar{z} \psi(n, \bar{z})$$

for all $n \in \mathbb{N}$. Now let $\hat{\varphi}$ and $\hat{\psi}$ be obtained from φ and ψ respectively by replacing each occurrence of $A(v)$ with $(a)_v \neq 0$. Since $(a)_v \neq 0$ *is* Δ_0 these formulas $\hat{\varphi}, \hat{\psi}$ are also Δ_0. Then we have, for all $n \in \mathbb{N}$,

$$\begin{aligned} n \in B &\Leftrightarrow \exists \bar{k} \in \mathbb{N} \;\; M \vDash \hat{\varphi}(n, \bar{k}) \\ &\Leftrightarrow \forall \bar{l} \in \mathbb{N} \;\; M \vDash \hat{\psi}(n, \bar{l}) \end{aligned} \qquad (+)$$

since all quantifiers in $\hat{\varphi}(x, \bar{y})$, $\hat{\psi}(x, \bar{z})$ except for those involved in the subformulas $(a)_v \neq 0$ are of the form $\mathrm{Q}v < t(x, \bar{y})$ or $\mathrm{Q}v < t(x, \bar{z})$ respectively, where t is an $\mathscr{L}_\mathrm{A}$-term *not involving* a. But then this means that for all $n \in \mathbb{N}$

$$M \vDash \forall x, \bar{y}, \bar{z} < n(\hat{\varphi}(x, \bar{y}) \to \hat{\psi}(x, \bar{z})),$$

so by overspill there is $b > \mathbb{N}$ in M such that

$$M \vDash \forall x, \bar{y}, \bar{z} < b(\hat{\varphi}(x, \bar{y}) \to \hat{\psi}(x, \bar{z})) \tag{$*$}$$

and hence, by Theorem 11.~~5~~6, the set

$$C = \{n \in \mathbb{N} \mid M \vDash n < b \wedge \exists \bar{y} < b\hat{\varphi}(n, \bar{y})\}$$

is coded in M. But $C = B$, for if $n \in B$ then $M \vDash \hat{\varphi}(n, \bar{k})$ for some $\bar{k} \in \mathbb{N} < b$, by $(+)$, hence $n \in C$; and conversely if $n \in C$ then for each $\bar{l} \in \mathbb{N}$ $M \vDash \hat{\psi}(n, \bar{l})$ by $(*)$, hence $n \in B$ by $(+)$ again.

(c) First notice that there are recursive functions $\lambda: \mathbb{N} \to \mathbb{N}$ and $\rho: \mathbb{N}^2 \to \mathbb{N}$ such that, for all $f \in 2^{<\mathbb{N}}$ and all $n \leqslant \mathrm{len}(f)$, $\lambda(\ulcorner f \urcorner) = \mathrm{len}(f)$ and $\rho(\ulcorner f \urcorner, n) = \ulcorner f \restriction n \urcorner$. (In fact, to be explicit, $\lambda(x) = [\log_2(x+1)]$ and

$$\rho(x, y) = \left\lfloor \frac{x+1}{2^{\lambda(x)-y}} \right\rfloor - 1.)$$

Since λ and ρ are recursive, they are represented in PA^- by Σ_1 formulas (which we shall also write as $\lambda(x) = y$ and $\rho(x, y) = z$), using Lemma 3.6.

Now suppose that $T \in SSy(M)$ is an infinite tree coded by $t \in M$. By overspill there is $a > \mathbb{N}$ in M such that:

(1) $M \vDash \forall x, y < a[\exists r, s(\lambda(x) = r \wedge \rho(x, y) = s \wedge y \leqslant s \wedge r \leqslant x)]$;

(2) $M \vDash \forall u, v, w, x, y < a$

$$[(u \leqslant v \wedge \rho(w, u) = x \wedge \rho(w, v) = y) \to \rho(y, u) = x];$$

(3) $M \vDash \forall v, w, x, y < a[\lambda(x) = w \wedge v < w \wedge \rho(x, v) = y \wedge (t)_x \neq 0 \to (t)_y \neq 0]$.

This is because the formulas in square brackets above are true in M for all $u, v, w, x, y \in \mathbb{N}$. ((1) express the fact that λ, ρ are functions; (2) expresses the identity $f \restriction k = (f \restriction l) \restriction k$ for all $k \leqslant l \leqslant \mathrm{len}(f)$; and (3) expresses the fact that t codes a tree.) Since T is infinite it contains arbitrarily long sequences, that is for each $n \in \mathbb{N}$

$$M \vDash \exists w < a \exists z < a(\lambda(w) = z \wedge z \geqslant n \wedge (t)_w \neq 0)$$

so, by overspill again, there is $c, d < a$ and $b > \mathbb{N}$ such that

(4) $M \models \lambda(c) = d \wedge d \geq b \wedge (t)_c \neq 0$.

Now let P be the set

$$P = \{n \in \mathbb{N} \mid M \models \exists z < d(\rho(c, z) = n)\},$$

so P is coded in M, by Theorem 11.$\overset{6}{\cancel{5}}$. We claim that $P \subseteq T$ is an infinite path. To see that $P \subseteq T$ note that if $M \models \exists z < d(\rho(c, z) = n)$ then $M \models (t)_n \neq 0$ by (3), thus $n \in T$. To check that P is a path note that if $n_1, n_2 \in \mathbb{N}$, $M \models \rho(c, z_1) = n_1 \wedge \rho(c, z_2) = n_2$ for some $z_1, z_2 < d$, then without loss of generality we may assume $z_1 \leq z_2$. But then (1) implies that $n_1 \geq z_1$, so z_1 is standard, and (3) implies that $M \models \rho(n_2, z_1) = n_1$, so $\mathbb{N} \models \rho(n_2, z_1) = n_1$ (since the formula $\rho(x, y) = z$ represents ρ in PA^-), thus n_1 is a restriction of n_2, and so P is a path. □

We now aim towards proving a converse to Theorem 13.2 that all countable Scott sets are realized as the standard system of some model of PA. The next theorem is the main stepping-stone toward this converse, but it is of interest in its own right too; it shows that Scott sets form a useful link between recursion theory and the predicate calculus. If T is a theory in a recursive language $\mathscr{L}$ and $\mathscr{X}$ is a Scott set we write $T \in \mathscr{X}$ to mean that, for some axomatization $\{\tau_i \mid i \in \mathbb{N}\}$ of T, $\{\ulcorner\tau_i\urcorner \mid i \in \mathbb{N}\} \in \mathscr{X}$. Similarly if $p(\bar{x})$ is a type over the theory T we write $p(\bar{x}) \in \mathscr{X}$ to mean $\{\ulcorner\varphi_i(\bar{x})\urcorner \mid i \in \mathbb{N}\} \in \mathscr{X}$ for some set of formulas $\{\varphi_i(\bar{x}) \mid i \in \mathbb{N}\}$ equivalent to $p(\bar{x})$ over T. With this notation we have

THEOREM 13.3. *Let $\mathscr{X}$ be a Scott set and let $T \in \mathscr{X}$ be a consistent theory in a recursive language $\mathscr{L}$. Then T has a complete consistent extension $T^* \in \mathscr{X}$.*

Proof. Let $\{\tau_i \mid i \in \mathbb{N}\}$ be an axiomatizaton of T such that $\ulcorner T\urcorner = \{\ulcorner\tau_i\urcorner \mid i \in \mathbb{N}\} \in \mathscr{X}$. Say a finite set of $\mathscr{L}$-sentences $\varphi_1, \ldots, \varphi_k$ is *n-consistent* iff there is no proof of $\exists x \neg(x = x)$ from $\{\tau_i \mid i \in \mathbb{N}\} \cup \{\varphi_1, \ldots, \varphi_k\}$ with Gödel-number of this proof less than n.

Clearly this notion of $\{\varphi_1, \ldots, \varphi_k\}$ being n-consistent is recursive in T, for given n and $\varphi_1, \ldots, \varphi_k$ (coded in some suitable way, by $[n, \ulcorner\varphi_1\urcorner, \ldots, \ulcorner\varphi_k\urcorner]$ say, where $[x_0, \ldots, x_k]$ is as in Chapter 9) there is a recursive procedure (using an oracle for $\ulcorner T\urcorner$) that decides if $\varphi_1, \ldots, \varphi_k$ is n-consistent: one simply enumerates all proofs from $T + \varphi_1 + \cdots + \varphi_k$ with Gödel-number $< n$ to see if any have $\exists x \neg(x = x)$ as a conclusion. There are only finitely many such proofs, so they can all be found using the oracle $\ulcorner T\urcorner$.

The reader should note that $T + \varphi_1 + \cdots + \varphi_k$ is consistent if and only if it is n-consistent for all n. However, a set $\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ that is n-

consistent for some n need not be consistent with T: the only proofs of $\exists x \neg(x=x)$ may have Gödel-number greater than n.

We define a tree $S \in \mathscr{X}$ as follows. Let $\{\theta_0, \theta_1, \theta_2, \ldots\}$ be a recursive enumeration of all $\mathscr{L}$-sentences. We define

$$S = \left\{ \ulcorner a_0 a_1 \cdots a_{k-1} \urcorner \,\middle|\, \begin{array}{l} k \in \mathbb{N},\ a_i = 0 \text{ or } 1 \text{ for each } i, \text{ and} \\ \{\theta_0^{(a_0)}, \theta_1^{(a_1)}, \ldots, \theta_{k-1}^{(a_{k-1})}\} \text{ is } k\text{-consistent} \end{array} \right\}$$

where $\theta_i^{(a)}$ is θ_i if $a=1$, and $\neg\theta_i$ if $a=0$. (The function $\ulcorner a_0 a_1 \cdots a_{k-1} \urcorner$ coding sequences $a_0 a_1 \cdots a_{k-1} \in 2^{<\mathbb{N}}$ should not be confused with our Gödel-numbering of $\mathscr{L}$.) By the remarks above, S is clearly recursive in $\ulcorner T \urcorner$ and hence $S \in \mathscr{X}$. S is a tree, since if $\{\theta_0^{(a_0)}, \theta_1^{(a_1)}, \ldots, \theta_k^{(a_k)}\}$ is $k+1$-consistent, then $\{\theta_0^{(a_0)}, \theta_1^{(a_1)}, \ldots, \theta_{k-1}^{(a_{k-1})}\}$ is k-consistent.

Now, since T is consistent, the tree S is infinite. It follows that $\mathscr{X}$ contains an infinite path $P \subseteq S$, since $\mathscr{X}$ is Scott. We define σ_i to be $\theta_i^{(a_i)}$ where $a_i = f(i)$ for $\ulcorner f \urcorner =$ the unique element of P of length $i+1$. Clearly $\{\sigma_i \mid i \in \mathbb{N}\}$ is recursive in P. We must show it has the required properties, that is: it is complete, consistent, and extends T. But $T + \sigma_0 + \cdots + \sigma_{k-1}$ is consistent for each k, since for each $n \geqslant k$ $T + \sigma_0 + \cdots + \sigma_{n-1}$ has no proof of the inconsistency $\exists x \neg(x=x)$ with Gödel-number $<n$, thus $T + \sigma_0 + \cdots + \sigma_{k-1} \nvdash \exists x \neg(x=x)$. Also each sentence τ_i in the axiomatization of T occurs as θ_j for some j. But $T + \sigma_0 + \cdots + \sigma_j$ is consistent as we have just seen, so it must be the case that σ_j is θ_j and not $\neg\theta_j$. Similarly $\{\sigma_0, \sigma_1, \ldots\}$ is complete since it contains either θ_j or $\neg\theta_j$ for each $\mathscr{L}$-sentence θ_j. □

As a corollary to Theorem 13.3 we see that every Scott set $\mathscr{X}$ contains non-recursive sets, for $T = PA$ clearly satisfies the hypotheses of the theorem, so $\mathscr{X}$ contains a complete consistent extension of PA. Such extensions cannot be recursive, by the Gödel–Rosser incompleteness theorem.

Theorem 13.3 is extremely powerful. We can interpret it as saying that the usual Henkin-style proof of the completeness theorem 'relativizes' to a Scott set; this remark is explained by the following theorem and its proof. Recall from Tennenbaum's theorem that recursively axiomatized theories (such as certain finite extensions of PA) need not have recursive models. Theorem 13.4 tells us that if $\mathscr{X}$ is a Scott set and $T \in \mathscr{X}$ is a consistent theory then, not only does T have a model in which all relations and functions are recursive in some oracles from $\mathscr{X}$, but in fact T has a model in which the whole satisfaction relation (defined by Tarski's definition of truth) is recursive in some oracle from $\mathscr{X}$.

THEOREM 13.4. Let $\mathscr{X}$ be a Scott set, let T be a consistent theory in a recursive language $\mathscr{L}$ such that $T \in \mathscr{X}$ and T has no finite models. Then there is a model $M \models T$ with domain $\mathbb{N}$ such that the set

$$\{\langle \ulcorner\varphi(\mathsf{v}_0, \ldots, \mathsf{v}_k)\urcorner, [a_0, \ldots, a_k]\rangle \mid k, \bar{a} \in \mathbb{N}, \varphi \in \mathscr{L}, M \models \varphi(\bar{a})\}$$

is in $\mathscr{X}$ (where $\langle u, v\rangle$ is the pairing function on $\mathbb{N}$ and $[a_0, \ldots, a_k]$ is the function on $\mathbb{N}$ as described in Section 9.1).

Proof. Define a sequence of expansions of the language $\mathscr{L}$, $\mathscr{L}_0$, $\mathscr{L}_1$, $\mathscr{L}_2$, $\ldots$, as follows. $\mathscr{L}_0$ is just $\mathscr{L}$. $\mathscr{L}_{i+1}$ is $\mathscr{L}_i$ together with new constant symbols c_φ for every formula $\varphi(\mathsf{v}_0)$ of $\mathscr{L}_i$ involving only one free-variable v_0. $\mathscr{L}^*$ is the language $\bigcup_{i \in \mathbb{N}} \mathscr{L}_i$. $\mathscr{L}^*$ is a recursive language and has a Gödel-numbering $\ulcorner\cdot\urcorner$ such that the set of Gödel-numbers

$$\{\ulcorner\varphi\urcorner \mid \varphi \text{ is an } \mathscr{L}\text{-formula}\}$$

is a recursive subset of

$$\{\ulcorner\theta\urcorner \mid \theta \text{ is an } \mathscr{L}^*\text{-formula}\},$$

and similarly for $\mathscr{L}$-terms, $\mathscr{L}$-sentences, etc. (This is expressed by saying that $\mathscr{L}^*$ is a *recursive extension* of the language $\mathscr{L}$. See Exercise 13.8.)

We let T^* be the $\mathscr{L}^*$-theory with axioms consisting of

(a) axioms of T, and

(b) $\exists \mathsf{v}_0 \varphi(\mathsf{v}_0) \to \varphi(c_\varphi)$, for every new constant c_φ in $\mathscr{L}^*$ not in $\mathscr{L}$, and every $\mathscr{L}^*$-formula $\varphi(\mathsf{v}_0)$ with only v_0 free.

Since the axiomatization of T is in $\mathscr{X}$, $T^* \in \mathscr{X}$ also. By Theorem 13.3, T^* has a complete consistent extension $S \in \mathscr{X}$.

Now let $C = \{c_0, c_1, \ldots\} = \{c_\varphi \mid \varphi \in \mathscr{L}^* \text{ has only } \mathsf{v}_0 \text{ free}\}$ be a recursive enumeration of all the constants of $\mathscr{L}^*$ that were added to $\mathscr{L}$. We define our model M to have domain $C/\!\sim$ where $c_i \sim c_j \Leftrightarrow \ulcorner c_i = c_j\urcorner \in S$; $\sim$ is clearly an equivalence relation since S is complete. We identify $C/\!\sim$ with $\mathbb{N}$ by defining $f: \mathbb{N} \to C$ by $f(0) = c_0$ and

$$f(n+1) = \text{the first } c_i \in C \text{ (i.e., of smallest index } i\text{) such that none of the following is true: } c_i \sim f(0),\ c_i \sim f(1),\ \ldots,\ c_i \sim f(n).$$

Thus f is a function that picks out exactly one representative from each equivalence class of $C/\!\sim$, and moreover f is recursive in S. The model M is made into an $\mathscr{L}$-structure by defining

$$R(n_1, \ldots, n_k) \Leftrightarrow R(f(n_1), \ldots, f(n_k)) \in S$$

for each relation symbol R of $\mathscr{L}$ and each $n_1, \ldots, n_k \in \mathbb{N}$, and

$$F(n_1, \ldots, n_k) = n_{k+1} \Leftrightarrow F(f(n_1, \ldots, f(n_k)) = f(n_{k+1}) \in S$$

for each function symbol F of $\mathscr{L}$, and each $n_1, \ldots, n_{k+1} \in \mathbb{N}$, and also

$$n = c \Leftrightarrow f(n_1) = c \in S$$

for each constant symbol c of $\mathscr{L}$. These are all well-defined, since for example if $n_1, \ldots, n_k \in \mathbb{N}$, $S \vdash \exists \mathsf{v}_0 F(f(n_1), \ldots, f(n_k)) = \mathsf{v}_0$ so $F(f(n_1), \ldots, f(n_k)) = c_\varphi \in S$ where φ is the formula $F(f(n_1), \ldots, f(n_k)) = \mathsf{v}_0$. But if $c_i \sim c_\varphi$ then $c_i = c_\varphi \in S$, so $S \vdash F(f(n_1), \ldots, f(n_k)) = c_i$ hence $F(f(n_1), \ldots, f(n_k)) = c_i \in S$ since S is complete. This shows that there is $n_{k+1} \in \mathbb{N}$ such that $F(f(n_1), \ldots, f(n_k)) = f(n_{k+1})$ is in S; but such n_{k+1} must be unique since if $F(f(n_1), \ldots, f(n_k) = f(l) \in S$, then $S \vdash f(l) = f(n_{k+1})$ so $f(l) \sim f(n_{k+1})$ so $l = n_{k+1}$. A similar argument shows that for every constant symbol c of $\mathscr{L}$ there is a unique $n \in \mathbb{N}$ such that $M \vDash n = c$.

We complete the proof by showing that for any formula $\varphi(\mathsf{v}_0, \ldots \mathsf{v}_k)$ and any $n_0, \ldots, n_k \in \mathbb{N}$,

$$M \vDash \varphi(n_0, \ldots, n_k) \Leftrightarrow \varphi(f(n_0), \ldots, f(n_k)) \in S.$$

(This suffices since this relation is clearly recursive in S, so the set

$$\{\langle \ulcorner \varphi(\bar{\mathsf{v}}) \urcorner, [\bar{a}] \rangle \mid k, \bar{a} \in \mathbb{N}, \varphi \in \mathscr{L}, M \vDash \varphi(\bar{a})\}$$

is recursive in S hence is in $\mathscr{X}$, and as $S \vdash T$ it also shows that $M \vDash T$.) This is proved by induction on the complexity of φ. For atomic formulas, it follows directly from the definitions above and the equality axioms of the predicate calculus. We prove the induction step for the existential quantifier. Clearly

$$\begin{aligned} & M \vDash \exists x \varphi(n_0, \ldots, n_k, x) \\ & \Leftrightarrow M \vDash \varphi(n_0, \ldots, n_k, n_{k+1}) \text{ for some } n_{k+1} \in \mathbb{N} \\ & \Leftrightarrow \varphi(f(n_0), \ldots, f(n_k), f(n_{k+1})) \in S \text{ for some } n_{k+1} \in \mathbb{N}. \end{aligned}$$

But if $\varphi(f(n_0), \ldots, f(n_{k+1})) \in S$, then we have

$$\exists x \varphi(f(n_0), \ldots, f(n_k), x) \in S,$$

and conversely if

$$\exists x \varphi(f(n_0), \ldots, f(n_k), x) \in S,$$

then

$$\exists \mathsf{v}_0 \varphi(f(n_0), \ldots, f(n_k), \mathsf{v}_0) \in S,$$

so

$$\varphi(f(n_0), \ldots, f(n_k), c_\psi) \in S,$$

where ψ is the formula $\varphi(f(n_0), \ldots, f(n_k), v_0)$, and $c_\psi \sim f(n_{k+1})$ for some $n_{k+1} \in \mathbb{N}$, so

$$\varphi(f(n_0), \ldots, f(n_k), f(n_{k+1})) \in S,$$

for some $n_{k+1} \in \mathbb{N}$, as required. □

We now apply Theorem 13.3 to obtain Scott's theorem that the countable Scott sets are exactly the standard systems of nonstandard models of PA. In fact we will do rather better than this in two important ways. Firstly, we will actually construct *recursively saturated* models $M \models PA$ with a given countable Scott set $\mathscr{X}$ as $SSy(M)$, and secondly the theorem extends to arbitrary theories T in a recursive language.

DEFINITION. If M is a structure for some recursively saturated language $\mathscr{L}$, and $\mathscr{X}$ is a Scott set, we say M is *$\mathscr{X}$-saturated* iff for every complete type $p(\bar{x}, \bar{a})$ over M

$$\{\ulcorner\varphi(\bar{x}, \bar{y})\urcorner \mid \varphi(\bar{x}, \bar{a}) \in p(\bar{x}, \bar{a})\} \in \mathscr{X} \Leftrightarrow p \text{ is realized in } M.$$

This notion is due to George Wilmers (1975) who first studied it with relation to recursive saturation.

LEMMA 13.5. If $\mathscr{X}$ is a Scott set and M is an $\mathscr{X}$-saturated structure for a recursive language $\mathscr{L}$, then M is recursively saturated.

Proof. Let $p(\bar{x}, \bar{a})$ be a recursive type over M, but not necessarily complete. We must show that p is realized in M.

Add constants $\bar{c}$ to $\mathscr{L}$, one constant c_i for each variable x_i in $p(\bar{x}, \bar{a})$, and also add constants naming the parameters $\bar{a}$. The resulting language $\mathscr{L}'$ is clearly still recursive, and the theory

$$T' = \{\varphi(\bar{c}, \bar{a}) \mid \varphi(\bar{x}, \bar{a}) \in p(\bar{x}, \bar{a})\}$$
$$\cup \{\psi(\bar{a}) \mid M \models \psi(\bar{a}), \psi \text{ an } \mathscr{L}\text{-formula}\}$$

is in $\mathscr{X}$, being recursive in

$$\{\ulcorner\psi(\bar{x})\urcorner \mid M \models \psi(\bar{a}), \ \psi \text{ an } \mathscr{L}\text{-formula}\}$$

which is in $\mathscr{X}$ since M is $\mathscr{X}$-saturated. (In fact $\mathscr{L}'$ is a recursive extension of the language $\mathscr{L}$ in the sense of Exercise 13.8.) T' is a consistent theory, since $p(\bar{x}, \bar{a})$ is finitely satisfied in M. But then by Theorem 13.3 T' has a complete extension $S \in \mathscr{X}$. We claim that

$$q(\bar{x}, \bar{a}) = \{\varphi(\bar{x}, \bar{a}) \mid \varphi(\bar{c}, \bar{a}) \in S\}$$

is a complete type over M, and it will follow that q (and hence p) is realized in M, since clearly $q(\bar{x}, \bar{a}) \in \mathscr{X}$.

We need only show that q is finitely satisfied in M, since it is obviously complete. But if $\varphi_1(\bar{x}, \bar{a}), \ldots, \varphi_k(\bar{x}, \bar{a})$ are in $q(\bar{x}, \bar{a})$, then $\exists\bar{x} \bigwedge_{i=1}^{k} \varphi_i(\bar{x}, \bar{a})$ is in q, by completeness, and so $M \models \exists\bar{x} \bigwedge_{i=1}^{k} \varphi_i(\bar{x}, \bar{a})$, since otherwise T' would be inconsistent. $\square$

A partial converse to Lemma 13.5, that any countable recursively saturated structure is $\mathscr{X}$-saturated for some Scott set $\mathscr{X}$, will he proved in Chapter 15. Notice that the proof of Lemma 13.5 shows that *any* type $p(\bar{x}, \bar{a}) \in \mathscr{X}$ (and not just a complete type) over an $\mathscr{X}$-saturated structure M is realized in M. In particular if M is an $\mathscr{X}$-saturated model of PA then $SSy(M) = \mathscr{X}$, since if $A \subseteq \mathbb{N}$ is in $\mathscr{X}$ then the type

$$p(x) = \{(x)_n \neq 0 \mid n \in A\} \cup \{(x)_n = 0 \mid n \notin A\}$$

is recursive in A, so is in $\mathscr{X}$, and hence is rcalized in M. Conversely if $a \in M$ codes the set $A \subseteq \mathbb{N}$, then

$$A = \{n \in \mathbb{N} \mid M \models (a)_n \neq 0\}$$

and the type realized by a in M, $tp_M(a) = \{\psi(x) \mid M \models \psi(a)\}$ is in $\mathscr{X}$ since M is $\mathscr{X}$-saturated. Thus A is recursive in $tp_M(a)$, so $A \in \mathscr{X}$. This means that our promised theorem, that any countahle Scott set $\mathscr{X}$ is $SSy(M)$ for some non-standard $M \models PA$ follows from the following theorem.

THEOREM 13.6 (Wilmers, 1975). Let $\mathscr{X}$ be a countahle Scott set, T a consistent theory in a recursive language $\mathscr{L}$, and suppose $T \in \mathscr{X}$. Then there is a countable $\mathscr{X}$-saturated structure $M \models T$.

Proof. The idea is to mimic the proof of Proposition 11.4, using Theorem 13.3 and the argument in Theorem 13.4 in place of the direct appeal to the completeness theorem. (Wilmer's original proof was much shorter, but uses machinery that will not be developed here until Chapter 15.)

We shall use notation similar to that in the proof of Theorem 13.4. In particular if $\mathscr{L}$ is any recursive language $\mathscr{L}^*$ is the recursive language resulting from adding sufficiently many constants c_φ so that $c_\varphi \in \mathscr{L}^*$ for all $\mathscr{L}^*$-formulas $\varphi(\mathsf{v}_0)$ with only v_0 free. We regard the Gödel-numbers of formulas of $\mathscr{L}$ as a recursive subset of the Gödel-numbers of formulas of $\mathscr{L}^*$.

Using Theorem 13.3, we will find a sequence of consistent theories T_0, $T_1, T_2, \ldots$ in recursive languages $\mathscr{L}_0, \mathscr{L}_1, \mathscr{L}_2, \ldots$ such that $T_i \in X$ for each i, $\mathscr{L}_i \subseteq \mathscr{L}_{i+1}$ for each i (in fact $\mathscr{L}_{i+1}$ will be a recursive extension of $\mathscr{L}_i$ as defined in Exercise 13.8) and $T_{i+1} \vdash T_i$ for each i. In the case of $\mathscr{L}_0$, T_0, we put $\mathscr{L}_0 = \mathscr{L}$ and let $T_0 \in \mathscr{X}$ be any complete consistent extension of T, found using Theorem 13.3.

Let $\{p_i(x_0, \ldots, x_{m_i}, y_0, \ldots, y_{n_i}) \mid i \in \mathbb{N}\}$ enumerate all sets of formulas p_i in the finitely many free-variables shown, such that $\{\ulcorner\varphi\urcorner \mid \varphi \in p_i\} \in \mathscr{X}$. (Since $\mathscr{X}$ is countable we can enumerate all these sets in this way.) We define $\mathscr{L}_i$ inductively; each $\mathscr{L}_{i+1}$ will be of the form

$$\mathscr{L}_{i+1} = (\mathscr{L}_i \cup \{c_{j,\bar{a},0}, \ldots, c_{j,\bar{a},m_j}\})^* \tag{$\bullet$}$$

for some new constants $c_{j,\bar{a},0}, \ldots, c_{j,\bar{a},m_j}$ not already in $\mathscr{L}_i$, indexed by some $j \in \mathbb{N}$, some constants $\bar{a} \in \mathscr{L}_i$, and an integer between 0 and m_j. $\mathscr{L}_\infty$ is $\bigcup_{i \in \mathbb{N}} \mathscr{L}_i$. We arrange this so that for every $j \in \mathbb{N}$ and every set $\{a_0, \ldots, a_{n_j}\}$ of constants in $\mathscr{L}_\infty$ there is $i \in \mathbb{N}$ such that $a_0, \ldots, a_{n_j} \in \mathscr{L}_i$ and $\mathscr{L}_{i+1}$ is the expansion of $\mathscr{L}_i$ as shown in ($\bullet$). We may also assume that each $\mathscr{L}_{i+1}$ is a recursive language in which the formulas of $\mathscr{L}_i$ forms a recursive subset.

This done, we find the sequence T_i of the theories as follows. As already mentioned, $T_0 \in \mathscr{X}$ is any complete consistent $\mathscr{L}_0$-theory extending T. Given $T_i \in \mathscr{X}$, T_{i+1} is any complete consistent $\mathscr{L}_{i+1}$-theory in $\mathscr{X}$ that extends T_i together with

$$\{\varphi(c_{j,\bar{a},0}, \ldots, c_{j,\bar{a},m_j}, a_0, \ldots, a_{n_j}) \mid \varphi(\bar{x}, \bar{y}) \in p_j\}$$

and

$$\{\exists v_0 \theta(v_0) \to \theta(c_\theta) \mid \theta \in \mathscr{L}_{i+1}\}$$

if this is consistent, or T_i together with

$$\{\exists v_0 \theta(v_0) \to \theta(c_\theta) \mid \theta \in \mathscr{L}_{i+1}\}$$

otherwise. Again, T_{i+1} exists by Theorem 13.3.

Having found T_i for all i in this way, let $T_\infty = \bigcup_{i \in \mathbb{N}} T_i$, let $K \models T$ be arbitrary, and let M be the substructure of K consisting of all elements of K that realize some constant of $\mathscr{L}_\infty$. Then by an easy induction on complexity of formulas similar to that given in Theorem 13.4, $M \prec K$, and in particular $M \models T_i$ for each i. We must show that M is $\mathscr{X}$-saturated.

If $\bar{a} \in M$ and $p(\bar{x}, \bar{a}) \in \mathscr{X}$ is a type over M, we must show that p is realized in M. But $p(\bar{x}, \bar{y}) = p_j(\bar{x}, \bar{y})$ for some $j \in \mathbb{N}$, and as $M \models T_i$ (where the constants $\bar{c} = c_{j,\bar{a},0}, \ldots, c_{j,\bar{a},m_j}$ were added at the ith stage of the construction as in ($\bullet$) above) then

$$T_i + p_j(\bar{c}, \bar{a}) + \{\exists v_0 \theta(v_0) \to \theta(c_\theta) \mid \theta \in \mathscr{L}_{i+1}\}$$

is consistent, hence $\bar{c} \in M \models \varphi(\bar{c}, \bar{a})$ for each $\varphi(\bar{x}, \bar{y}) \in p(\bar{x}, \bar{y})$. On the other hand, if $p(\bar{x}, \bar{a})$ is a complete $\mathscr{L}$-type that is realized in M, by $\bar{c} \in M$, say, then $\bar{a}, \bar{c}$ (being a finite set of constants) occur in L_i for some $i \in \mathbb{N}$, so

$$p(\bar{x}, \bar{a}) = \{\varphi(\bar{x}, \bar{a}) \mid \varphi \text{ is an } \mathscr{L}\text{-formula and } T_i \vdash \varphi(\bar{c}, \bar{a})\}.$$

Since T_i is complete the set of Gödel-numbers of formulas in $p(\bar{x}, \bar{a})$ is recursive in $T_i \in \mathscr{X}$, and hence $p(\bar{x}, \bar{a}) \in \mathscr{X}$, since $\mathscr{X}$ is closed under relative recursion. □

Comparing Propositions 11.4 and 13.6 one might expect every nonstandard model $M \models PA$ to have an $SSy(M)$-saturated elementary extension $M' > M$. This turns out not to be the case, since any $\mathscr{X}$-saturated model $M \models PA$ must have its complete theory $\mathrm{Th}(M) = \{\sigma \mid M \models \sigma,\ \sigma$ an $\mathscr{L}_A$-sentence$\} \in \mathscr{X}$, by the definition of $\mathscr{X}$-saturation. Exercise 13.4 shows that $T \notin SSy(K_T)$ for any complete extension T of PA, thus no elementary extension of K_T can be $SSy(K_T)$-saturated. Exercise 13.5 supplies necessary and sufficient conditions for such M' to exist.

The material in this section has a slightly complicated history. The notion of a Scott set is due to Scott (1962) who proved Theorem 13.2 and its converse, that every countable Scott set $\mathscr{X}$ is $SSy(M)$ for some model M of PA. (Scott, however, presented his results in terms of algebras of sets 'binumerable' in some complete extension of PA, and he did not call them 'Scott sets'.) This work lay unnoticed for some years until Harvey Friedman, realizing the importance of standard systems with respect to initial segments, rediscovered it in Friedman (1973). $\mathscr{X}$-saturated structures were considered by Wilmers whilst studying nonstandard models of set theory (see Exercise 15.4); Lemma 13.5 and its consequence that any countable Scott set is the standard system of a recursively saturated model of PA were discovered slightly later; this arose out of the work by Wilmers already mentioned, and also work by Barwise, Schlipf, and Ressayre on recursive saturation.

Exercises for Section 13.1

13.1 Go through the proof of Theorem 13.2 carefully and verify that if $M \models PA^-$ satisfies Δ_0-overspill, then $SSy(M)$ is a Scott set.
(*Hint:* Where the proof of Theorem 13.2 seems to require Σ_1-overspill, bound the unbounded existential quantifiers with a suitably chosen nonstandard parameter from M.)
13.2 Let $\mathscr{X} \subseteq \mathscr{P}(\mathbb{N})$ be non-empty and satisfy parts (a) and (b) of the definition of a Scott set. Suppose also that $\mathscr{X}$ satisfies the conclusion to Theorem 13.3. Show that $\mathscr{X}$ is a Scott set.
(*Hint:* If $T \in \mathscr{X}$ is a tree, consider the theory

$$PA \cup \{(c)_n = \rho((c)_{n+1}, n) \mid n \in \mathbb{N}\} \cup \{\textstyle\bigvee\!\!\!\bigvee_{k \in T,\, \mathrm{len}(k)=n} (c)_n = k \mid n \in \mathbb{N}\}.)$$

13.3 Let $\mathscr{X}$ be a countable Scott set. Find an $\mathscr{X}$-saturated $M \models PA$ and a type $p(x) \notin \mathscr{X}$ such that $p(x)$ is realized in M. (Obviously p cannot be complete.)

13.4 Let T be a complete consistent extension of PA, $T \neq \text{Th}(\mathbb{N})$, and let

$$S = \{\ulcorner\sigma\urcorner \mid T \vdash \sigma,\ \sigma \text{ an } \mathscr{L}_A\text{-sentence}\}.$$

Show that $S \notin SSy(K_T)$.
(*Hint:* For each $\mathscr{L}_A$-formula $\eta(x)$ find σ_η such that

$$PA^{\dot{-}} \vdash \sigma_\eta \leftrightarrow \exists x(\eta(x) \wedge (x)_{\ulcorner\sigma_\eta\urcorner} = 0).$$

Given that $T \vdash \exists! x\eta(x)$, show that the unique $a \in K_T$ satisfying $\eta(a)$ does not code S by considering σ_{η}.)
13.5 Let $M \vDash PA$ be countable, nonstandard, let $\mathscr{X} = SSy(M)$, and suppose that $M' > M$ with M' $\mathscr{X}$-saturated. Show that for each $\bar{a} \in M$

$$\{\ulcorner\varphi(\bar{v})\urcorner \mid M \vDash \varphi(\bar{a})\} \in \mathscr{X}. \tag{*}$$

Now suppose (*) holds for each $\bar{a} \in M$. Show that there is an $\mathscr{X}$-saturated $M' > M$.
(*Hint:* For the second part, repeat the proof of Theorem 13.6, but at each stage also add a new constant naming some element of M.)
13.6 Let T be a complete extension of PA, $T \neq \text{Th}(\mathbb{N})$. Show that

$$SSy(K_T) = \{S \subseteq \mathbb{N} \mid S = \{n \in \mathbb{N} \mid T \vdash \theta(n)\} \text{ for some } \theta \in \mathscr{L}_A\}.$$

13.7 Let $M \vDash PA$ be nonstandard. Show that the following are equivalent:
(a) $\Pi_1 - \text{Th}(\mathbb{N}) \in SSy(M)$;
(b) there is a nonstandard $I \subseteq_e M$ satisfying $\Pi_1 - \text{Th}(\mathbb{N})$;
(c) $\exists a \in M \setminus \mathbb{N}$ such that, for all $b \in M \vDash b < a$, b *is* Δ_0-definable in M *iff* $b \in \mathbb{N}$.
13.8 Let $\mathscr{L}_2$ be a first-order language involving: constant symbols $c_i (i \in I_2)$; relation symbols R_j of arity $m_j (j \in J_2)$; and function symbols F_k of arity $n_k (k \in K_2)$. Let $I_1 \subseteq I_2$, $J_1 \subseteq J_2$, $K_1 \subseteq K_2$ and let $\mathscr{L}_1$ be the sublanguage of $\mathscr{L}_2$ involving $c_i (i \in I_1)$, $R_j (j \in J_1)$ and $F_k (k \in K_1)$. $\mathscr{L}_2$ is a *recursive extension of* $\mathscr{L}_1$ iff there is a 1–1 function ν from $\mathscr{L}_2$-symbols to $\mathbb{N}$ such that

$$\begin{array}{ll} \{\nu(c_i) \mid i \in I_1\} & \{\nu(c_i) \mid i \in I_2\} \\ \{(\nu(R_j), m_j) \mid j \in J_1\} & \{(\nu(R_j), m_j) \mid j \in J_2\} \\ \{(\nu(F_k), n_k) \mid k \in K_1\} & \{(\nu(F_k), n_k) \mid k \in K_2\} \\ \{\nu(\mathrm{v}_i) \mid i \in \mathbb{N}\} & \end{array}$$

are all recursive, and $\nu(\mathrm{v}_{i+1}) > \nu(\mathrm{v}_i)$ for each $i \in \mathbb{N}$.

(a) Assume $\mathscr{L}_1$ and $\mathscr{L}_2$ are as above. Then clearly both $\mathscr{L}_1$ and $\mathscr{L}_1$ are recursive. If $\mathscr{L}_1$ already has a suitable Gödel-numbering defined on it, use Exercise 11.9 to show that for all sets T of strings of symbols from $\mathscr{L}_1$, the set $\{\ulcorner t\urcorner \mid t \in T\}$ (Gödel-numbering in the new sense) is recursive in $\{\ulcorner t\urcorner \mid t \in T\}$ (Gödel-numbering in the old sense).

(b) Show that any extension of a recursive language $\mathscr{L}$ by at most finitely many new nonlogical symhols is a recursive extension.

(c) Given that $\mathscr{L}$ is a recursive language, show that the language $\mathscr{L}^*$ of Theorem 13.4 is a recursive extension of $\mathscr{L}$.

13.2 THE ARITHMETIZED COMPLETENESS THEOREM

The starting point for this section is the following observation.

PROPOSITION 13.7. The set of *arithmetic sets*,

$$\{S \subseteq \mathbb{N} \mid S = \{n \in \mathbb{N} \mid \mathbb{N} \vDash \theta(n)\} \text{ for some } \mathscr{L}_A\text{-formula } \theta(x)\}$$

is a Scott set.

Proof. By Theorem 8.8 there is a proper conservative extension $M \succ \mathbb{N}$. $SSy(M)$ is a Scott set, by Theorem 13.2. But $SSy(M)$ is exactly the set of arithmetic sets, since if $S = \{n \in \mathbb{N} \mid \mathbb{N} \vDash \theta(n)\}$ then $\mathbb{N} \vDash \theta(n) \Leftrightarrow M \vDash \theta(n)$, as $M \succ \mathbb{N}$, hence $S \in SSy(M)$ by Theorem 11.5. Conversely if $S \in SSy(M)$, $S = \{n \in \mathbb{N} \mid M \vDash (a)_n \neq 0\}$ say, then S is the intersection with $\mathbb{N}$ of a definable subset of M, hence is definable in $\mathbb{N}$ since the extension $M \succ \mathbb{N}$ is conservative. Thus S is arithmetic. □

We shall say that a theory T in a recursive language $\mathscr{L}$ is *arithmetic* iff T has an axiomatization $\{\tau_i \mid i \in \mathbb{N}\}$ such that $\{\ulcorner \tau_i \urcorner \mid i \in \mathbb{N}\}$ is an arithmetic set. An $\mathscr{L}$-structure M is *arithmetic* iff $M \cong (\mathbb{N}, \bar{R}, \bar{f}, \bar{c})$ for some relations $\bar{R}$, functions $\bar{f}$, and constants $\bar{c}$ on $\mathbb{N}$ such that the satisfaction relation

$$S = \left\{ \langle \ulcorner \varphi(\mathrm{v}_0, \ldots, \mathrm{v}_k) \urcorner, [a_0, \ldots, a_k] \rangle \,\middle|\, \begin{array}{l} \varphi \in \mathscr{L}, k \in \mathbb{N}, a_0, \ldots, a_k \in \mathbb{N}, \\ \text{and } (\mathbb{N}, \bar{R}, \bar{f}, \bar{c}) \vDash \varphi(a_0, \ldots, a_k) \end{array} \right\}$$

is an arithmetic set. (This notion can be extended to finite models M too, by saying that M is arithmetic iff $M \cong (A, \bar{R}, \bar{f}, \bar{c})$ for some finite set $A \subseteq \mathbb{N}$, but these cases are uninteresting because every finite model is of this form with the satisfaction relation being recursive.)

COROLLARY 13.8. If T is a consistent arithmetic theory over a recursive language $\mathscr{L}$, then T has an arithmetic model.

Proof. By Proposition 13.7 and Theorem 13.4. □

The *arithmetized completeness theorem* is the assertion that Corollary 13.8 can be formalized and proved within PA. We shall indicate a proof of this theorem by re-proving Propositon 13.7 without reference to Theorem 8.8. An energetic reader will then be able to construct a proof of the arithmetized completeness theorem by formalizing these arguments inside PA. (The formalization is straightforward but lengthy, and constraints on the size of this book mean that I must omit it here. I write this parenthetical remark with many crocodile tears!)

If $S \subseteq \mathbb{N}$ then S is $\Delta_n^{\mathbb{N}}$ iff there are $\psi(x) \in \Pi_n$ and $\theta(x) \in \Sigma_n$ such that for all

$k \in \mathbb{N}$,

$$k \in S \Leftrightarrow \mathbb{N} \vDash \psi(k) \Leftrightarrow \mathbb{N} \vDash \theta(k).$$

In particular, by Corollary 3.5, a set is $\Delta_1^{\mathbb{N}}$ iff it is recursive, and also a set is arithmetic iff it is $\Delta_n^{\mathbb{N}}$ for some $n \in \mathbb{N}$.

LEMMA 13.9. For each $n \geqslant 1$, $\Delta_n^{\mathbb{N}}$ is closed under the Boolean operations of complement, intersection, and union, and also under relative recursion.

Proof. $\Delta_n^{\mathbb{N}}$ is closed under complement, since the negation of a Σ_n formula is Π_n and vice versa. $\Delta_n^{\mathbb{N}}$ is closed under $\cap$, $\cup$ since both Σ_n and Π_n are closed under $\wedge, \vee$.

If B is recursive in $A \in \Delta_n^{\mathbb{N}}$ there are Δ_0 formulas φ, ψ in the language $\mathscr{L}_{\mathrm{A}}(A)$ ($=\mathscr{L}_{\mathrm{A}}$ together with a new unary relation symbol for A) such that

$$k \in B \Leftrightarrow \mathbb{N} \vDash \exists \bar{y} \varphi(n, \bar{y}) \Leftrightarrow \mathbb{N} \vDash \forall \bar{z} \psi(n, \bar{z}).$$

By replacing $A(v)$ in φ and ψ by either the Σ_n formula equivalent to '$v \in A$' or the Π_n formula equivalent to '$v \in A$' we obtain $\hat{\varphi} \in \Sigma_n$ and $\hat{\psi} \in \Pi_n$ such that

$$k \in B \Leftrightarrow \mathbb{N} \vDash \exists \bar{y} \hat{\varphi}(n, \bar{y}) \Leftrightarrow \mathbb{N} \vDash \forall \bar{z} \hat{\psi}(n, \bar{z}). \qquad \square$$

THEOREM 13.10. If $n > 0$ and $T \subseteq \mathbb{N}$ is $\Delta_n^{\mathbb{N}}$ and is an infinite tree, then there is an infinite path $P \subseteq T$ such that P is $\Delta_{n+1}^{\mathbb{N}}$.

Proof. Let $T \in \Delta_n^{\mathbb{N}}$ be an infinite tree with $\theta(x) \in \Pi_n$ such that

$$T = \{k \in \mathbb{N} \mid \mathbb{N} \vDash \theta(k)\}.$$

Consider the following formula: $\psi(y) =$

$$\forall l \geqslant \lambda(y) \exists z < 2^{l+1} - 1[\lambda(z) = l \wedge \theta(z) \wedge \rho(z, \lambda(y)) = y].$$

(Here λ and ρ are the recursive functions in the proof of Theorem 13.2 representing 'length' and 'restriction' respectively.) $\psi(y)$ represents 'the tree T is infinite above the node y'. $\psi(y)$ is equivalent to a Π_n formula since ρ and λ are recursive functions with Σ_1 graphs, so $\psi(y)$ is equivalent to

$$\begin{aligned}&\forall u, v, l\{l \geqslant u \wedge u = \lambda(y) \wedge v = 2^{l+1} - 1\\ &\qquad \rightarrow \exists z < v \forall r, s[r = \lambda(z) \wedge s = \rho(z, u) \rightarrow \theta(z) \wedge r = l \wedge s = y]\}\end{aligned}$$

and the bounded quantifier $\exists z < v$ can be 'pushed inside' the unbounded quantifiers in the formula in square brackets, using the collection axioms.

We claim that the formula $\varphi(z)=$

$$\psi(z)\wedge\forall y<z(\lambda(y)=\lambda(z)\to\neg\psi(y))$$

defines an infinite path $P\subseteq T$. $\varphi(z)$ is clearly $\Delta^{\mathbb{N}}_{n+1}$ since ψ is Π_n, and also if $\mathbb{N}\models\varphi(k)$ then $\mathbb{N}\models\psi(k)$, so (by considering $l=\lambda(k)$) $\mathbb{N}\models\theta(k)$, hence $P\subseteq T$. Also P is infinite, since T is, and contains at most one element of any given length. Thus it suffices to show that P is closed under restrictions. But if $x\in P$ has length $l>0$ then $\rho(x,l-1)\in T$, and if y is in P with length $l-1$ then the tree T is infinite above node y, so in particular there is $z\in T$ with $\lambda(z)=l$, T infinite above z, and $z=$'$y0$' or '$y1$'. So $z=2y+1$ or $2y+2$, whereas $x=2\cdot\rho(x,l-1)+1$ or $x=2\cdot\rho(x,l-1)+2$; and assuming $\rho(x,l-1)>y$ this gives $z<x$, contradicting the assertion that x is the least element such that $\lambda(x)=l$ and $\psi(x)$. Hence in fact $\rho(x,l-1)=y$, so p is closed under restricton. □

Lemma 13.9 and Theorem 13.10 re-prove Proposition 13.7. In fact we get extra information for our labours. Say a theory T in a recursive language is $\Delta^{\mathbb{N}}_n$ iff it has an axiomatization $\{\tau_i|i\in\mathbb{N}\}$ such that $\{\ulcorner\tau_i\urcorner|i\in\mathbb{N}\}$ is $\Delta^{\mathbb{N}}_n$, and say a model M is $\Delta^{\mathbb{N}}_n$ iff $M\cong(\mathbb{N},\bar{R},\bar{f},\bar{c})$ where the satisfaction relation S on $(\mathbb{N},\bar{R},\bar{f},\bar{c})$ is $\Delta^{\mathbb{N}}_n$. Then we get:

Corollary 13.11. If $n\geqslant 1$ and T is a consistent $\Delta^{\mathbb{N}}_n$ theory in a recursive language $\mathscr{L}$, then T has a complete consistent $\Delta^{\mathbb{N}}_{n+1}$ extension and a $\Delta^{\mathbb{N}}_{n+1}$ model.

Proof. The proofs of Theorems 13.3 and 13.4 only involve one appeal to the closure of the Scott set $\mathscr{X}$ under König's lemma, thus there is only an increase in level in the $\Delta^{\mathbb{N}}_n$ hierarchy of one. □

There is also a useful variation of Corollary 13.11 for complete theories T.

Proposition 13.12. Let $n\geqslant 1$ and let T be a complete consistent $\Delta^{\mathbb{N}}_n$ theory in a recursive language $\mathscr{L}$. Then T has a $\Delta^{\mathbb{N}}_n$ model.

Proof. If the language $\mathscr{L}^*$ is constructed from $\mathscr{L}$ as in the proof of Theorem 13.4, we construct a complete $\mathscr{L}^*$-theory T^* extending T with $T^*\in\Delta^{\mathbb{N}}_n$. The construction of the model and verification that its satisfaction relation is $\Delta^{\mathbb{N}}_n$ is then achieved exactly as in Theorem 13.4.

The idea is to construct a tree as in Theorem 13.3. If $\{\varphi_1,\ldots,\varphi_k\}$ is a finite set of sentences of $\mathscr{L}^*$, we say $\varphi_1,\ldots,\varphi_k$ *is* *-consistent iff

$$T+\varphi_1+\cdots+\varphi_k+\{\exists v_0\theta(v_0)\to\theta(c_\theta)|c_\theta \text{ occurs in } \varphi_1,\ldots,\varphi_k\}$$

is consistent. The point here is that, using the fact that T is complete, this notion is recursive in T. This is because if

$$\varphi_1(c_{\theta_1}, \ldots, c_{\theta_l}), \ldots, \varphi_k(c_{\theta_1}, \ldots, c_{\theta_l})$$

are $\mathscr{L}^*$-sentences involving new constants $c_{\theta_1}, \ldots, c_{\theta_l}$ as shown, then $\{\varphi_1, \ldots, \varphi_k\}$ is $*$-consistent iff

$$T \nvdash \neg\exists x_1, \ldots, x_l[\bigwedge\!\!\!\bigwedge_{i=1}^{k}\varphi_i(x_1, \ldots, x_l) \wedge \bigwedge\!\!\!\bigwedge_{j=1}^{l}(\exists \mathsf{v}_0\theta_j(\mathsf{v}_0) \rightarrow \theta_j(x_j))],$$

which holds if and only if

$$T \vdash \exists x_1, \ldots, x_l[\bigwedge\!\!\!\bigwedge_{i=1}^{k}\varphi_i(x_1, \ldots, x_l) \wedge \bigwedge\!\!\!\bigwedge_{j=1}^{l}(\exists \mathsf{v}_0\theta_j(\mathsf{v}_0) \rightarrow \theta_j(x_j))].$$

It is also obvious that if $\varphi_1, \ldots, \varphi_k$, is $*$-consistent and σ is any $\mathscr{L}^*$-sentence, then either $\{\varphi_1, \ldots, \varphi_k, \sigma\}$ is $*$-consistent, or $\{\varphi_1, \ldots, \varphi_k, \neg\sigma\}$ is $*$-consistent. Thus any $*$-consistent set can be extended to a complete extension T^* of

$$T + \{\exists \mathsf{v}_0\theta(\mathsf{v}_0) \rightarrow \theta(c_\theta) \mid \theta \in L^*\}.$$

But T^* can be taken to be $\Delta_n^{\mathbb{N}}$ by taking some fixed recursive enumeration $\sigma_0, \sigma_1, \ldots$ of $\mathscr{L}^*$-sentences and defining

$$\tau_{i+1} = \begin{cases} \sigma_{i+1} & \text{if } \{\tau_0, \ldots, \tau_i, \tau_i, \sigma_{i+1}\} \text{ is } *\text{-consistent} \\ \neg\sigma_{i+1} & \text{otherwise.} \end{cases}$$

So τ_i can be computed using an oracle for (an axiomatization of) T, thus $T^* = \{\tau_i \mid i \in \mathbb{N}\}$ is recursive in T, and a complete $\mathscr{L}^*$-extension of the required theory, so $T^* \in \Delta_n^{\mathbb{N}}$ by Lemma 13.9. □

This discussion brings us back to considering the incompleteness theorem (Chapter 3) and Tennenbaum's theorem (Section 11.3). Proposition 13.12 together with Tennenbaum's theorem can be used to reprove the Gödel–Rosser incompleteness theorem for PA^-. (This justifies the remark in Section 11.3 that Tennenbaum's theorem can be considered as the model-theoretic analogue of the Gödel–Rosser theorem.) The argument goes like this. If T were a recursive complete consistent extension of PA^-, then T would have a recursive model M, by Proposition 13.12. Since $M \vDash PA^-$ we may regard the standard model as an initial segment $\mathbb{N} \subseteq_e M$. But $\mathbb{N}$ cannot be defined by a Σ_1 formula $\exists\bar{y}\theta(x, \bar{y}, \bar{a})$ in M (where $\theta \in \Delta_0$), for if

$$\forall n \in M(n \in \mathbb{N} \Leftrightarrow M \vDash \exists\bar{y}\theta(n, \bar{y}, \bar{a})),$$

then the truth of any Σ_1 sentence $\exists\bar{z}\psi(\bar{z})$ (where $\psi \in \Delta_0$) could be recursively decided by using the recursive satisfaction relation of M to verify if

$$M \models \exists z_1, \ldots, z_k[\bigwedge_{i=1}^k \exists \bar{y}\theta(z_i, \bar{y}, \bar{a}) \wedge \psi(\bar{z})].$$

This is impossible, since as we have seen there is no uniform recursive procedure to decide the Σ_1 sentences $\neg\mathrm{Sat}_{\Pi_1}(n, m)$. Thus M is nonstandard and has Σ_1-overspill, so M is non-recursive by Tennenbaum's theorem (Theorem 11.6), a contradiction.

In fact, the argument in the last paragraph can easily be modified to show that no consistent recursively axiomatized extension of PA^- can decide all Σ_1 and Π_1 sentences. See Exercise 13.10 for more details.

To return to the arithmetized completeness theorem, let $\mathscr{L}$ be a recursive language with some fixed Gödel-numbering, $\ulcorner\cdot\urcorner$, such that the functions $\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner \mapsto \ulcorner\varphi \wedge \psi\urcorner$, $i, \ulcorner\varphi\urcorner \mapsto \ulcorner\exists \mathrm{v}_i\varphi\urcorner$ etc. are defined by $\mathscr{L}_{\mathrm{A}}$-formulas (and that these functions are provably total in PA). Then there is an $\mathscr{L}_{\mathrm{A}}$-formula $\mathrm{proof}_{\mathscr{L}}(v_0, v_1)$ which represents 'v_0 is the Gödel-number of a proof in the predicate calculus for $\mathscr{L}$ of the $\mathscr{L}$-sentence with Gödel-number v_1'. (See Exercise 9.6.) Now suppose $M \models PA$ and $\theta(x)$ is an $\mathscr{L}_{\mathrm{A}}$-formula (possibly involving parameters from M) so that

$$M \models \forall x(\theta(x) \rightarrow \mathrm{sent}_{\mathscr{L}}(x))$$

where $\mathrm{sent}_{\mathscr{L}}(x)$ is some $\mathscr{L}_{\mathrm{A}}$-formula expressing 'x is the Gödel-number of a sentence of $\mathscr{L}$'. Then we define the $\mathscr{L}_{\mathrm{A}}$-formula $\mathrm{Con}(\theta)$ to be

$$\forall w(\forall i < \mathrm{len}(w)\theta([w]_i) \rightarrow \forall v \neg \mathrm{proof}_{\mathscr{L}}(v, \ulcorner \bigwedge w \rightarrow \perp \urcorner))$$

where $\ulcorner \bigwedge w \rightarrow \perp \urcorner$ denotes the Gödel-number of the $\mathscr{L}$-sentence $([w]_0 \wedge ([w]_1 \wedge \cdots \wedge [w]_{\mathrm{len}(w)-1}) \cdots)) \rightarrow \perp$ and $\perp$ is some convenient inconsistent $\mathscr{L}$-sentence, such as $\exists \mathrm{v}_0 \neg(\mathrm{v}_0 = \mathrm{v}_0)$.

Next, suppose that N is an $\mathscr{L}$-structure with domain a subset of the domain of M. We say N is *strongly interpreted* in M iff there are $\mathscr{L}_{\mathrm{A}}$-formulas $\mathrm{dom}_N(x)$, $\mathrm{val}_N(x, y) = z$, and $\mathrm{Sat}_N(x, y)$ such that

1. $\forall x \in M(x \in N \Leftrightarrow M \models \mathrm{dom}_N(x))$
2. For all $t, a \in M$, if $M \models$ 't is the Gödel-number of an $\mathscr{L}$-term', and $M \models \forall i < \mathrm{len}(a)\ \mathrm{dom}_N([a]_i)$, then

$$M \models \exists! z(\mathrm{dom}_N(z) \wedge \mathrm{val}_N(t, a) = z),$$

   and for all $z \in M$:
   (a) if $t = \ulcorner c \urcorner$ for some constant symbol c of $\mathscr{L}$, $M \models \mathrm{val}_N(t, a) = z \Leftrightarrow N \models z = c$;
   (b) if $t = \mathrm{v}_i$ where $i \in M$, then $M \models \mathrm{val}_N(t, a) = z \Leftrightarrow M \models [a]_i = z$;

(c) if $M \vDash t = \ulcorner F(s_1, \ldots, s_l) \urcorner$ where $s_1, \ldots, s_l \in M$, $M \vDash$ 's_i is the Gödel-number of an $\mathscr{L}$-term' each i, and F is a l-ary function symbol of $\mathscr{L}$, then $M \vDash \mathrm{val}_N(t, a) = z \Leftrightarrow \exists b_1, \ldots, b_l \in N$ s.t. $M \vDash \bigwedge\!\!\!\bigwedge_{i=1}^{l} \mathrm{val}_N(s_i, a) = b_i$ and $N \vDash F(b_1, \ldots, b_l) = z$.

3. For all $p, a \in M$, if $M \vDash$ 'p is the Gödel-number of an $\mathscr{L}$-formula' then:
   (a) if $M \vDash p = \ulcorner R(s_1, \ldots, s_l) \urcorner$ where $s_1, \ldots, s_l$ are $\mathscr{L}$-terms and R is a relation symbol of $\mathscr{L}$ (possibly $=$) then $M \vDash \mathrm{Sat}_N(p, a) \Leftrightarrow \exists b_1, \ldots, b_l \in N$ $M \vDash \bigwedge\!\!\!\bigwedge_{i=1}^{l} \mathrm{val}_N(s_i, a) = b_i$ and $N \vDash R(b_1, \ldots, b_l)$;
   (b) if $M \vDash p = \ulcorner (p * r) \urcorner$ for some $\mathscr{L}$-formulas $p, q \in M$, where $* = \wedge$ or $\vee$ then $M \vDash \mathrm{Sat}_N(p, a) \Leftrightarrow M \vDash \mathrm{Sat}_N(q, a) * \mathrm{Sat}_N(r, a)$;
   (c) if $M \vDash p = \ulcorner \neg q \urcorner$ for some $\mathscr{L}$-formula $q \in M$ then $M \vDash \mathrm{Sat}_N(p, a) \Leftrightarrow M \vDash \neg \mathrm{Sat}_N(q, a)$;
   (d) if $M \vDash p = \ulcorner \mathsf{Q} v_i q \urcorner$ for some $i \in M$ and some $\mathscr{L}$-formula $q \in M$, where Q is $\forall$ or $\exists$, then

$$M \vDash \mathrm{Sat}_N(p, a) \Leftrightarrow M \vDash \mathsf{Q} b \left(\mathrm{dom}_N(b) \begin{pmatrix} \rightarrow \\ \wedge \end{pmatrix} \mathrm{Sat}_N(q, a[b/i]) \right)$$

   (here, take $\rightarrow$ if $\mathsf{Q} = \forall$, and take $\wedge$ otherwise).

Thus M strongly interprets N iff the truth of any $\mathscr{L}$-formula or value of any $\mathscr{L}$-term is given by an $\mathscr{L}_{\mathrm{A}}$-formula over M. Thus Corollary 13.8 says that, if T is an arithmetic theory and $\mathbb{N} \vDash \mathrm{Con}(T)$, then there is a model of T that is strongly interpreted in $\mathbb{N}$. As already mentioned the arithmetized completeness theorem is the formalization of this theorem in PA.

THEOREM 13.13 (The arithmetized completeness theorem). Let $M \vDash PA$, let L, θ, $\mathrm{proof}_{\mathscr{L}}$, $\mathrm{Con}(\theta)$ all be as above. Then if $M \vDash \mathrm{Con}(\theta)$ there is an $\mathscr{L}$-structure N with domain $\subseteq M$ such that $N \vDash \sigma$ for every (standard) $\mathscr{L}$-sentence σ where $M \vDash \theta(\ulcorner \sigma \urcorner)$, and N is strongly interpreted in M.* If, moreover,

$$M \vDash \forall a \theta(\ulcorner \psi_a \urcorner)$$

or

$$M \vDash \forall a \exists w, p(\forall i < \mathrm{len}(w) \theta([w]_i) \wedge \mathrm{proof}_{\mathscr{L}}(p, \ulcorner \bigwedge\!\!\!\bigwedge w \rightarrow \psi_a \urcorner))$$

where ψ_a is some (canonically chosen) $\mathscr{L}$-sentence expressing 'there are at least a elements', then the domain of M can be taken equal to the domain of N. □

* In fact: if $\mathrm{Sat}_N(x, y)$ is the formula in part 3 of the definition of 'N strongly interpreted in M' we will have $M \vDash \forall x, a(\theta(x) \rightarrow \mathrm{Sat}_N(x, a))$, and it follows from this that $N \vDash \sigma$ for every standard $\mathscr{L}$-sentence σ such that $M \vDash \theta(\ulcorner \sigma \urcorner)$.

Exercises for Section 13.2

13.9 Show that there is a Scott set $\mathscr{X}$ such that all $A \in \mathscr{X}$ are $\Delta_2^{\mathbb{N}}$.
(*Hint:* Let $\mathscr{X} = SSy(K_T)$ for a $\Delta_2^{\mathbb{N}}$ complete extension of PA.)
13.10 (a) Let $T \supseteq PA^-$ have a recursive axiomatization, and suppose T decides all Σ_1 and Π_1 sentences. Show that the proof of Theorem 13.12 can be modified to give a recursive $M \models \{\sigma \in \Pi_2 | T \vdash \sigma\}$ in which

$$\Sigma_1 - tp_M(\bar{a}) = \{\varphi(\bar{x}) | M \models \varphi(\bar{a}),\ \varphi \in \Sigma_1\}$$

is recursive for each tuple $\bar{a} \in M$ of finite length.
(b) Use (a) to give a new proof of Corollary 3.11 based on the argument following the proof of Theorem 13.12.
13.11$^+$ (a) Define a provably recursive function in PA, $F(a) = \ulcorner \psi_a \urcorner$, whose values are Gödel-numbers of $\mathscr{L}_A$-sentences expressing 'there are at least a elements'. (Take for ψ_a the sentence

$$\exists v_0, v_1, \ldots, v_{a-1} \bigwedge_{i \neq j} \neg(v_i = v_j)$$

bracketed in some canonical way.)
(b) If $\text{proof}_{\mathscr{L}_A}(x, y)$ denotes 'x is the Gödel-number of a proof of y in the predicate calculus for $\mathscr{L}_A$', show that

$$PA \vdash \forall a \exists p\ \text{proof}_{\mathscr{L}_A}(p, \ulcorner \sigma \to \psi_a \urcorner),$$

where σ is a suitable (finite) conjunction of axioms of PA^-.
(c) Convince yourself that, were your life to depend on it, you could give a proof of Theorem 13.13 with all details.
13.12* Let M, N be $\mathscr{L}_A$-structures, $M \models PA$, $N \models PA^-$ and suppose N is strongly interpreted in M. Define an embedding f of M into N by

$$f(0^M) = 0^N;$$
$$f(x +^M 1^M) = f(x) +^N 1^N.$$

Show that f is definable in M and embeds M onto an initial segment of N. Use Tarski's theorem on the undefinability of truth to deduce that the embedding f is not elementary.
13.13 Define a formula $\theta(x, y)$ expressing 'x is the Gödel-number of an axiom of $I\Sigma_y$'. Hence define a formula $\psi(a)$ representing '$I\Sigma_a$ is consistent'. Let $M \models PA$ be nonstandard. Use Exercise 10.8 and overspill to find nonstandard $a \in M$ with $M \models \psi(a)$. Deduce that M has a proper end-extension $N \models PA$ that is strongly interpreted in M.

14

Indicators

Life, you know, is rather like opening a tin of sardines: we're all of us looking for the key.

Alan Bennett
'Take a pew' from *Beyond the fringe*

Suppose M is a nonstandard model of PA and we are interested in finding initial segments $I \subseteq_e M$ having a certain property Q. An *indicator* for Q is a function $Y: M^2 \to M$ such that, for each $a < b$ in M, $Y(a, b) > \mathbb{N}$ iff there is an initial segment I between a and b and having the property Q.

This notion was devised by Paris and Kirby in the 1970s to study the structure of models of PA, but it soon became clear that indicators were the key to several particularly interesting independence results. Paris and Harrington later used these ideas to exhibit the first examples of mathematically interesting statements that are true in $\mathbb{N}$ but unprovable in PA, and we shall present an introduction to this material in Section 14.3 below.

Although the Paris–Harrington independence results were first found using indicators, indicators have become less popular recently as other methods have been developed to prove these results. (For example, Buchholz and Wainer (1987) contains a clear account of the sort of proof-theoretic arguments that can be used.) Harry Simmons pointed out to me once that indicators are not used any more—that at least was the *gist* of what he said—however, it seems to me that they are still a useful tool that provides a *uniform* treatment of many constructions of interesting initial segments of models of PA, and essential if one asks certain questions about the cardinality of the set of initial segments with a certain property.

14.1 WHAT ARE INDICATORS?

Definition. Let $M \models PA$ be nonstandard and let Q be a property of initial segments I of M. (For example, Q might be the property that $I \models PA$.) A function $Y: M^2 \to M$ is an *indicator* for Q in M iff for every $a, b \in M$

$$Y(a, b) > \mathbb{N} \Leftrightarrow \text{there is } I \subseteq_e M \text{ with } a \in I < b \text{ and } I \text{ has the property } Q.$$

EXAMPLES. Let $M \models PA$ be nonstandard.

(a) The function $Y(a,b) = b \dot{-} a$ is an indicator for *cuts* $I \subseteq_e M$ (i.e., initial segments, I, that are closed under the successor function $x \mapsto x+1$ but not necessarily closed under $+$ or $\cdot$). For if $Y(a,b) > \mathbb{N}$ then $a < b$, $a+1 < b$, $a+2 < b, \ldots$ and so we can take

$$I = \{x \in M \mid x < a+n \text{ for some } n \in \mathbb{N}\}.$$

(b) The function $Y(a,b) = \lfloor \log_{a+2}(b) \rfloor$ (if $b > 0$, $Y(a,0) = 0$ otherwise) can be defined in PA. Y is an indicator for initial segments closed under $+$ and $\cdot$, for if $c = \lfloor \log_{a+2}(b) \rfloor > \mathbb{N}$, then

$$a \leqslant (a+2) < (a+2)^2 < (a+2)^3 < \cdots < (a+2)^c \leqslant b$$

so $I = \{x \in M \mid x < (a+2)^n \text{ for some } n \in \mathbb{N}\}$ is closed under $+$, $\cdot$ and $a \in I < b$. The converse is proved by an easy overspill argument.

As these examples show, indicators are often well-behaved and are defined by $\mathscr{L}_A$-formulas over *arbitrary* models of PA.

DEFINITION. An indicator $Y(x,y)$ is a *well-behaved indicator for Q in PA* iff

(a) $Y(x,y) = z$ is a Σ_1 $\mathscr{L}_A$-formula with only the free-variables x, y, z;
(b) $PA \vdash \forall x, y\, \exists! z\, Y(x,y) = z$;
(c) $PA \vdash \forall x, y\, Y(x,y) \leqslant y$;
(d) $PA \vdash \forall x, y, x', y' (x' \leqslant x \wedge y \leqslant y' \rightarrow Y(x,y) \leqslant Y(x',y'))$;

and

(e) for any nonstandard model $M \models PA$ and for any $a, b \in M$, $Y(a,b) > \mathbb{N}$ iff there is $I \subseteq_e M$ with $a \in I < b$ and I has property Q.

Obviously indicators (when they exist) will tell us a lot of useful information about the structure of nonstandard models of PA. They are also intimately connected with the notion of a *provably recursive function*.

Recall that $f(x)$ is a provably recursive function in an $\mathscr{L}_A$-theory T iff $f(x) = y$ is a Σ_1 formula and $T \vdash \forall \bar{x} \exists! y f(\bar{x}) = y$. Now suppose $Y(x,y)$ is a well-behaved indicator for the property '$I \models T$' of initial segments I. Associated with $Y(x,y)$ is the family of 'functions' $F_n(x) =$ the least $y \geqslant x$ such that $Y(x,y) \geqslant n$, where n here ranges over $\mathbb{N}$. (I write 'function' in inverted commas since in general PA may not be able to prove that $F_n(x)$ is defined for all x.) Then

PROPOSITION 14.1. Let T, $Y(x,y)$ and $F_n(x)$ be as in the last paragraph. If $M \models PA$ and $a \in I \models T$ for some proper initial segment $I \subseteq_e M$, then $M \models \exists c F_n(a) = c$ for each $n \in \mathbb{N}$. Hence $PA + T \vdash \forall x \exists! y F_n(x) = y$, and so each $F_n(x)$ is provably recursive in $PA + T$.

Proof. If $a \in I < b$ with $I \models T$ then $M \models Y(a, b) = d$ for some $d > \mathbb{N}$, since Y indicates initial segments satisfying T. Hence $M \models \exists x(Y(a, x) \geqslant n)$ and the least such $x \geqslant a$ is $F_n(a)$.

If $M \models PA + T$ then, by Theorem 8.6, there is a proper elementary end-extension $K > M$. Thus for each $a \in M$ and each $b \in K \setminus M$ $Y(a, b) > \mathbb{N}$, since M satisfies T. Hence $K \models \exists x(Y(a, x) \geqslant n)$ so $M \models \exists x(Y(a, x) \geqslant n)$ as $M < K$, and again the least such x is $F_n(a)$.

Finally, to check each $F_n(a)$ is provably recursive in $PA + T$ we must simply check that the formula '$F_n(x) = y$' is equivalent to a Σ_1 formula. But $F_n(x) = y$ is

$$Y(x, y) \geqslant n \wedge \forall z < y \exists w (Y(x, y) = w \wedge w < n)$$

which is $\Sigma_1(PA)$ since $Y(x, y) = z$ is Σ_1 and $\Sigma_1(PA)$ is closed under bounded quantification. □

The other side of the coin is that every function $f(\bar{x})$ that is provably recursive in T is 'majorized' by $F_n(x)$ for some $n \in \mathbb{N}$. In fact, a little more is true: notice that if $I \subseteq_e M \models PA$ is a proper initial segment of M closed under $+$ and $\cdot$, then $I \models B\Sigma_1$ (by Theorem 10.5) and also $I \models \Pi_1 - \mathrm{Th}(PA)$, where $\Pi_1 - \mathrm{Th}(PA)$ is the set of all Π_1 sentences that are provable in PA.

PROPOSITION 14.2. Let T, $Y(x, y)$, and $F_n(x)$ be as in Proposition 14.1. Suppose $f(\bar{x})$ is provably recursive in $\mathrm{T} + B\Sigma_1 + \Pi_1 - \mathrm{Th}(PA)$. Then for some $n \in \mathbb{N}$

$$PA \vdash \forall \bar{x}, y(y = F_n(\max(\bar{x})) \rightarrow \exists z \leqslant y f(\bar{x}) = z).$$

Proof. If not, then

$$PA + b = F_n(\max(\bar{a})) \wedge \forall z \leqslant b \neg f(\bar{a}) = z \tag{*}$$

is consistent for each $n \in \mathbb{N}$. Since $Y(x, y)$ is well-behaved, $F_{n+1}(x) \geqslant F_n(x)$ for all $n \in \mathbb{N}$ in any model of PA where these are defined. Hence, by compactness,

$$PA + \{\exists x \leqslant b(F_n(\max(\bar{a})) = x) | n \in \mathbb{N}\} + \forall z \leqslant b \neg f(\bar{a}) = z \tag{+}$$

is consistent. Let $\bar{a}, b \in M$ satisfy (+). Then $M \models Y(\max(\bar{a}), b) \geqslant n$ for each $n \in \mathbb{N}$, hence $M \models Y(\max(\bar{a}), b) = c$ for some $c > \mathbb{N}$. Since Y indicates initial segments satisfying T there is an initial segment $I \subseteq_e M$ with $\bar{a} \in I < b$ and $I \models T + B\Sigma_1 + \Pi_1 - \mathrm{Th}(PA)$. Thus $I \models f(\bar{a}) = d$ for some $d \in I$. In particular $d < b$, and as $f(\bar{x}) = y$ is a Σ_1 formula and $I \subseteq_e M$ we have $M \models f(\bar{a}) = d \wedge d < b$, contradicting our choice of M satisfying (+). □

Having shown the connections between indicators and provably recursive functions, we now show how these ideas give rise to independence results.

THEOREM 14.3. Let T be an $\mathscr{L}_A$-theory, let $Y(x, y)$ be a well-behaved indicator for initial segments satisfying T, and suppose that $T + B\Sigma_1 + \Pi_1 - \mathrm{Th}(PA)$ is consistent. Then

$$T + B\Sigma_1 + \Pi_1 - \mathrm{Th}(PA) \nvdash \forall x, z \exists y Y(x, y) \geqslant z.$$

Proof. Using the hypothesis that $T + B\Sigma_1 + \Pi_1 - \mathrm{Th}(PA)$ is consistent, let $M \vDash T + B\Sigma_1 + \Pi_1 - \mathrm{Th}(PA)$ be nonstandard. Let $a \in M \setminus \mathbb{N}$ be arbitrary and let $c < a$ be also nonstandard. Clearly we may suppose that $M \vDash \exists y Y(a, y) \geqslant c$, since otherwise we are finished.

We now use $M \vDash \Pi_1 - \mathrm{Th}(PA)$ to construct $K >_{\Delta_0} M$ with $K \vDash PA$. Clearly it is sufficient to show that

$$S = PA + \{\theta(\bar{d}) \mid \bar{d} \in M,\ \theta \in \Delta_0 \quad \text{and} \quad M \vDash \theta(\bar{d})\}$$

is a consistent theory in the language $\mathscr{L}_A$ expanded by adding a new constant symbol d for each $d \in M$. For then any model K of this theory will have a natural embedding $M \hookrightarrow K$ of M onto a Δ_0-elementary substructure of K. But if $\theta_1(\bar{d}), \ldots, \theta_k(\bar{d})$ are Δ_0 formulas, $\bar{d} \in M$ and $M \vDash \bigwedge\!\!\!\bigwedge_{i \leqslant k} \theta_i(\bar{d})$ with $PA \vdash \forall \bar{x} \bigvee\!\!\!\bigvee_{i \leqslant k} \neg \theta_i(\bar{x})$, then $M \nvDash \Pi_1 - \mathrm{Th}(PA)$, since $\forall \bar{x} \bigvee\!\!\!\bigvee_{i \leqslant k} \neg \theta_i(\bar{x})$ is equivalent to a Π_1 sentence. Thus, by compactness, S is consistent and has a model $K >_{\Delta_0} M$.

The formula $\exists y Y(a, y) \geqslant c$ is Σ_1 and is true in M, hence is true in K. Using the least-number principle in K there is a least $b \in K \vDash Y(a, b) \geqslant c$. Then, since $Y(x, y)$ indicates initial segments satisfying T and c was chosen to be nonstandard, there is $I \subseteq_e K$ with $c, a \in I$, $b \notin I$ and $I \vDash T$. But also $I \vDash \Pi_1 - \mathrm{Th}(PA) + B\Sigma_1$, since I is a proper initial segment of a model of PA.

We show that $I \nvDash \exists y Y(a, y) \geqslant c$, completing the proof. If $d \in I \vDash Y(a, d) \geqslant c$, then $d < b$ and $K \vDash Y(a, d) \geqslant c$, since $I \subseteq_e K$ and $Y(x, y) \geqslant z$ is Σ_1. This contradicts our choice of b being the least element of K satisfying $K \vDash Y(a, b) \geqslant c$. □

REMARK. In the case when T is a consistent extension of PA and $Y(x, y)$ is a well-behaved indicator for initial segments satisfying T, Proposition 14.1 and Theorem 14.3 show that

$$T \vdash \forall x \exists y Y(x, y) \geqslant n$$

for every $n \in \mathbb{N}$, but

$$T \nvdash \forall x, z \exists y Y(x, y) \geqslant z.$$

Exercises for Section 14.1

14.1 Let $Y(x,y)$ be a Σ_1 formula, $PA\vdash\forall x,y\exists!zY(x,y)=z$ and suppose Y is an indicator for initial segments having the property Q. By modifying Y so that (c) and (d) on p. 194 hold, show that there is a well-behaved indicator for Q in PA.
14.2 Suppose that Y is a well-behaved indicator for Q in PA, $a,b\in M\vDash PA$, and $Y(a,b)>\mathbb{N}$. Show that there is $c\in M$ with $a<c<b$ and both of $Y(a,c)$, $Y(c,b)$ nonstandard. Deduce that there are infinitely many $I\subseteq_e M$ with $a\in I<b$ having the property Q.
14.3 Let $n\in\mathbb{N}\setminus\{0\}$. Working in PA, define

$$F_n(x)=\max\{g_n(u,v)\,|\,u,v\leqslant x\}$$

where $g_n(u,v)$ is the least w such that $\mathrm{Sat}_{\Sigma_n}(u,[v,w])$, if such w exists, 0 otherwise. Also define

$$Y(x,y)=\max\left\{e\leqslant x\,\middle|\,\begin{array}{l}\exists w(\mathrm{len}(w)=e\wedge[w]_0=x\wedge\\ \qquad\forall i<e-1([w]_{i+1}=F_n([w]_i)\wedge[w]_{i+1}<y))\end{array}\right\}.$$

(a) Show that $PA\vdash\forall x,y(x\leqslant y\rightarrow F_n(x)\leqslant F_n(y))$.
(b) Show that, provably in PA, $F_n(x)=y$ and $Y(x,y)=z$ are equivalent to Σ_{n+1} formulas.
(c) Let $M\vDash PA$ and $I\subseteq_e M$. Show that $I<_{\Sigma_n}M$ holds if and only if

$$\forall a\in I\,\exists b\in I\; M\vDash F_n(a)=b.$$

(*Hint:* If $I<_{\Sigma_n}M$ is an initial segment, note that $I\vDash B\Sigma_{n+1}$ by Proposition 10.5. Use your definition of $F_n(x)=y$ to show that $B\Sigma_{n+1}\vdash\forall x\exists yF_n(x)=y$.)
(d) Hence deduce that $Y(x,y)$ is an indicator for Σ_n-elementary initial segments satisfying all the properties of a well-behaved indicator, *except* that '$Y(x,y)=z$' is not Σ_1.
14.4 In PA, define for each $n\in\mathbb{N}$,

$$Y_n(x,y)=\max\left\{e\leqslant y\,\middle|\,\begin{array}{l}\forall v\forall f<e(\mathrm{Sat}_{\Pi_n}(f,[x,v])\rightarrow\\ \qquad\qquad\exists w<y\,\mathrm{Sat}_{\Pi_n}(f,[x,w]))\end{array}\right\}$$

Show that $Y_n(x,y)$ is an indicator for initial segments $I<_{\Sigma_n}M$ that are isomorphic to M, in all countable models $M\vDash PA$. Show also that $Y_0(x,y)\geqslant z$ is $\Delta_2(PA)$, but not equivalent in PA to any Σ_1 formula.
(*Hint:* For the last part, let $M\vDash PA$ in which $\Pi_1-\mathrm{Th}(\mathbb{N})$ is not coded (see Exercise 13.7). Let $a\in M\setminus\mathbb{N}$ be Δ_0-definable, and $I\cong M$ a proper initial segment containing a. Then $M\vDash\exists yY(a,y)\geqslant c$ for some Δ_0-definable nonstandard $c\in M$. Let $b\in M$ be the least such y, and let $J\subseteq_e M$ with $J\cong M$ and $a,c\in J<b$. Then $J\vDash Y(a,d)\geqslant c$ for some $d\in J$ (by Δ_0-definability of a,c). If $Y(a,d)\geqslant c$ were Σ_1, then $M\vDash Y(a,d)\geqslant c$ which is impossible.)

14.5 Let $Y(x, y)$ be a well-behaved indicator for initial segments satisfying T, and suppose that σ is a Π_2 $\mathscr{L}_A$-sentence such that

$$\Pi_1 - \mathrm{Th}(PA) + B\Sigma_1 + T \vdash \sigma.$$

Show that

$$\Pi_1 - \mathrm{Th}(PA) + \{\forall x \exists y Y(x, y) \geqslant n \mid n \in \mathbb{N}\} \vdash \sigma.$$

(*Hint:* Let K be a recursively saturated model of

$$\Pi_1 - \mathrm{Th}(PA) + \{\forall x \exists y Y(x, y) \geqslant n \mid n \in \mathbb{N}\} + \neg\sigma.$$

Construct $M >_{\Delta_0} K$ with $M \vDash PA$ and an initial segment $I \subseteq_e M$ such that $I \vDash T + \neg\sigma$.)

14.2 RECURSIVELY AXIOMATIZED THEORIES HAVE INDICATORS

In the last section we investigated the properties of an indicator for initial segments $I \vDash T$ without showing that such indicators exist. This section is devoted to proving that they do exist for each recursively axiomatized theory T.

Propositons 14.1 and 14.2 (and Exercise 4.8) suggest that one approach to constructing an indicator $Y(x, y)$ is to consider the functions

$$f_n(x, y) = \max_\theta \max_{u \leqslant x} \begin{cases} \text{the least } v \leqslant y \text{ such that } \theta(u, v) \\ y + 1, \text{ if no such } v \text{ exists} \end{cases}$$

for each $n \in \mathbb{N}$, where '$\max_\theta$' is over all $\theta(u, v) \in \Delta_0$ such that there is a proof with Gödel-number $< n$ of $\forall u \exists v \theta(u, v)$ from $T + B\Sigma_1$, and to define

$$Y(x, y) = \max\{n \leqslant y \mid f_n(x, y) \leqslant y\}.$$

Of course, we must define all these functions in PA (and show they have all the right properties there). This is what we will do next.

Since T is recursive, the relation 'p is the Gödel-number of a proof from $T + B\Sigma_1$ of the sentence with Gödel number q' is recursive, and hence by Corollary 3.7 is represented by a formula $\exists \bar{w} \Psi(p, q, \bar{w})$ in PA^-, where Ψ is Δ_0. That is: if p is the Gödel-number of a proof of σ from $T + B\Sigma_1$ then

$$PA^- \vdash \exists \bar{w} \Psi(p, \ulcorner\sigma\urcorner, \bar{w})$$

and if not then

$$PA^- \vdash \neg \exists \bar{w} \Psi(p, \ulcorner\sigma\urcorner, \bar{w}).$$

Now define $f(n, x, y) \leqslant z$ to be the $\mathscr{L}_A$-formula

$$(z \geqslant y+1) \vee \forall p, q, r, \bar{w} < n(\Psi(p, q, \bar{w}) \wedge q = \ulcorner \forall v_0 \exists v_1 r \urcorner \wedge \mathrm{form}_{\Delta_0}(r) \rightarrow \forall u \leqslant x \exists v \leqslant z \, \mathrm{Sat}_{\Delta_0}(r, [u, v])).$$

So $f(n, x, y) \leqslant z$ is $\Delta_1(PA)$, since $\mathrm{Sat}_{\Delta_0}(x, y)$ is $\Delta_1(PA)$ and $\Delta_1(PA)$ is closed under bounded quantification. Moreover,

$$PA \vdash \forall n, x, y(f(n, x, y) \leqslant y+1)$$

and

$$PA \vdash \forall n, x, y, z, z'(z' \geqslant z \wedge f(n, x, y) \leqslant z \rightarrow f(n, x, y) \leqslant z').$$

Thus we may define $f(n, x, y) = z$ to be the formula

$$f(n, x, y) \leqslant z \wedge \forall u < z \neg (f(n, x, y) \leqslant u)$$

which is also $\Delta_1(PA)$, and $PA \vdash \forall n, x, y \exists ! z f(n, x, y) = z$, by applying the least-number principle to $f(n, x, y) \leqslant z$. Thus $f(n, x, y)$ is a provably recursive function in PA.

LEMMA 14.4. Provably in PA, $f(n, x, y)$ has the following properties:
(a) $f(0, x, y) = 0$;
(b) $n' \geqslant n \rightarrow f(n', x, y) \geqslant f(n, x, y)$;
(c) $x' \geqslant x \rightarrow f(n, x', y) \geqslant f(n, x, y)$;
(d) $y' \geqslant y \rightarrow f(n, x, y') \geqslant f(n, x, y)$;
(e) $f(n, x, y') > f(n, x, y) \rightarrow f(n, x, y) = y+1$.

Proof. (a) holds, since there are no $p, q, r, \bar{w} < 0$ with $\Psi(p, q, w)$ holding.

(b) If $n' \geqslant n$ then there are at least as many $p, q, r, \bar{w} < n'$ satisfying

$$\Psi(p, q, \bar{w}) \wedge q = \ulcorner \forall v_0 \exists v_1 r \urcorner \wedge \mathrm{form}_{\Delta_0}(r)$$

as there are such $p, q, r, \bar{w} < n$.

(c) holds for a similar reason, since if $x' \geqslant x$ then $u \leqslant x \rightarrow u \leqslant x'$.

For (d), notice that if $y' \geqslant y$ and $f(n, x, y') \leqslant z$ then either $z \geqslant y+1$ or $z \leqslant y$ and $f(n, x, y) \leqslant z$. This also shows that (e) holds, since the only way that we might have $f(n, x, y) \leqslant z$ but $\neg(f(n, x, y') \leqslant z)$ is if $z \geqslant y+1$. □

By Lemma 14.4(a) we can define in PA

$$Y(x, y) = \max\{n \mid n \leqslant y \,\&\, f(n, x, y) \leqslant y\}.$$

For example, we may take $Y(x, y) = z$ to be the $\Sigma_1(PA)$ formula

$$f(z, x, y) \leqslant y \wedge z \leqslant y \wedge (f(z+1, x, y) > y \vee z+1 > y)$$

which has the appropriate properties by Lemma 14.4(b).

LEMMA 14.5. $Y(x, y)$ satisfies the conditions (a)–(d) in the definition of a well-behaved indicator given on p. 194.

Proof. We have seen that $Y(x, y)$ has Σ_1 graph and is provably a function in PA. Thus (a) and (b) hold. But (by definition) $Y(x, y) \leqslant y$, so (c) holds, and if $x' \leqslant x \wedge y \leqslant y'$ with $Y(x, y) = z$ then $f(z, x', y) \leqslant f(z, x, y) \leqslant y$ by Lemma 14.4(b), and $f(z, x, y) = f(z, x, y')$ by Lemma 14.4(e). Hence $Y(x', y') \geqslant z$. □

LEMMA 14.6. If $M \models PA$ is nonstandard, $I \subseteq_e M$ an initial segment of M satisfying T, and $a, b \in M$ with $a \in I < b$, then $Y(a, b) > \mathbb{N}$.

Proof. It suffices to show that $Y(a, b) > n$ for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$; we must show that $M \models f(n, a, b) < b$. Let $\theta_1(\mathrm{v}_0, \mathrm{v}_1), \ldots, \theta_k(\mathrm{v}_0, \mathrm{v}_1)$ list all the Δ_0-formulas such that

$$\ulcorner \forall \mathrm{v}_0 \exists \mathrm{v}_1 \theta_i(\mathrm{v}_0, \mathrm{v}_1) \urcorner < n \qquad \text{and} \qquad M \models \Psi(p, \ulcorner \forall \mathrm{v}_0 \exists \mathrm{v}_1 \theta_i(\mathrm{v}_0, \mathrm{v}_1) \urcorner, \bar{w})$$

for some $p, \bar{w} < n$. Since there are finitely many $p < n$ there are only finitely many such θ_i, and by our choice of Ψ we have $T + B\Sigma_1 \vdash \forall \mathrm{v}_0 \exists \mathrm{v}_1 \theta_i(\mathrm{v}_0, \mathrm{v}_1)$ for each $i \leqslant k$.

Now $I \models T$ and, as I is a proper initial segment of M, $I \models B\Sigma_1$. Thus $I \models \forall u \exists v \theta_i(u, v)$ for each i, and in particular

$$I \models \forall u \leqslant a \exists v_1, v_2, \ldots, v_k \bigwedge_{i=1}^{k} \theta_i(u, v_i).$$

Hence, by $B\Sigma_1$ in I, there is $c \in I$ such that

$$I \models \forall u \leqslant a \exists v_1, v_2, \ldots, v_k \leqslant c \bigwedge_{i=1}^{k} \theta_i(u, v_i),$$

and so (since this formula is Δ_0 and $I \subseteq_e M$)

$$M \models \forall u \leqslant a \exists v_1, v_2, \ldots, v_k \leqslant c \bigwedge_{i=1}^{k} \theta_i(u, v_i).$$

Thus $M \models f(n, a, b) \leqslant c$, and since $c \in I < b$, $Y(a, b) \geqslant n$ as required. □

Lemmas 14.4, 14.5, and 14.6 show that $Y(x, y)$ has most of the properties required of a well-behaved indicator for initial segments satisfying T. The last (and most important property is supplied by the following theorem.

THEOREM 11.7. Let $M \models PA$ with $a, b \in M$ and $Y(a, b) > \mathbb{N}$. Then there is $I \subseteq_e M$ with $a \in I < b$ and $I \models T$.

Proof. Suppose first that M is countable, and let T' be the theory

$$T+B\Sigma_1+\{\forall\bar{x}\psi(\bar{x},c)\,|\,\psi\in\Delta_0\&M\vDash\forall\bar{x}\leqslant b\psi(\bar{x},a)\}$$

in the language $\mathscr{L}_{\mathrm{A}}$ expanded by adding a new constant symbol c. T' is consistent, since if not

$$T+B\Sigma_1\vdash\exists\bar{x}\bigvee\!\!\!\bigvee_{i=1}^{k}\neg\psi_i(\bar{x},c)$$

for some $\psi_1,\ldots,\psi_k\in\Delta_0$ such that $M\vDash\forall\bar{x}<b\psi_i(\bar{x},a)$. But then

$$T+B\Sigma_1\vdash\forall y\exists\bar{x}\bigvee\!\!\!\bigvee_{i=1}^{k}\neg\psi_i(\bar{x},y)$$

since a does not occur in T or $B\Sigma_1$, so there are $p,q,r,\bar{w},n\in\mathbb{N}$ with $p,q,r,\bar{w}<n$,

$$q=\ulcorner\forall\mathsf{v}_0\exists\mathsf{v}_1\exists\bar{z}\leqslant\mathsf{v}_1\bigvee\!\!\!\bigvee_{i=1}^{k}\neg\psi_i(\bar{z},\mathsf{v}_0)\urcorner,$$
$$r=\ulcorner\exists\bar{z}\leqslant\mathsf{v}_1\bigvee\!\!\!\bigvee_{i=1}^{k}\neg\psi_i(z,\mathsf{v}_0)\urcorner,$$

and $PA^-\vdash\Psi(p,q,\bar{w})$ (so p is the Gödel-number of a proof of

$$\forall\mathsf{v}_0\exists\mathsf{v}_1\exists\bar{z}\leqslant\mathsf{v}_1\bigvee\!\!\!\bigvee_{i=1}^{k}\neg\psi_i(\bar{z},\mathsf{v}_0),$$

from $T+B\Sigma_1$). Since $M\vDash f(n,a,b)\leqslant b$ this means that

$$M\vDash\forall u\leqslant a\exists v\leqslant b\,\mathrm{Sat}_{\Delta_0}(r,[u,v]),$$

and hence that

$$M\vDash\exists\bar{z}\leqslant b\bigvee\!\!\!\bigvee_{i=1}^{k}\neg\psi_i(a,\bar{z}),$$

a contradiction.

Note also that $\{\ulcorner\forall\bar{x}\psi(\bar{x},c)\urcorner\,|\,\psi\in\Delta_0\&M\vDash\forall\bar{x}\leqslant b\psi(\bar{x},a)\}$ is coded in M. (It is coded by any d realizing the Σ_1 type

$$\{(d)_i\neq0\leftrightarrow\forall\bar{x}\leqslant b\psi(\bar{x},a)\,|\,\psi\in\Delta_0,\,i=\ulcorner\forall\bar{x}\psi(\bar{x},c)\urcorner\}\cup$$
$$\{(d)_j=0\,|\,j\in\mathbb{N}\text{ not }\ulcorner\forall\bar{x}\psi(\bar{x},c)\urcorner\text{ for any }\psi\in\Delta_0\}.$$

This type is clearly recursive.) Putting $\mathscr{X}=SSy(M)$ (and remembering that we are assuming for the moment that M and hence $\mathscr{X}$ are countable) we conclude $T'\in\mathscr{X}$. Hence there is a countable $\mathscr{X}$-saturated model $K\vDash T$, by Theorem 13.6. In particular K is recursively saturated with $SSy(K)=SSy(M)$, by Lemma 13.5 and the argument in the paragraph preceding Theorem 13.6. Thus by Theorem 12.3 (or Exercise 12.2 if T is not an extension of PA) there is an embedding $h\colon K\to M$ sending K onto an initial segment $I \subseteq_e M$ such that $b \notin I$ and the element of K realizing the constant c is sent to $a \in M$. (To see that

$$K \models \exists \bar{x}\theta(\bar{x}, c) \Rightarrow M \models \exists \bar{x} < b\theta(\bar{x}, a)$$

for all Δ_0 formulas θ, note that if $K \models \exists \bar{x}\theta(\bar{x}, c)$ then $T' \nvdash \forall y \forall \bar{x} < y \neg\theta(\bar{x}, c)$, so $M \models \exists y \leqslant b \exists \bar{x} < y\theta(\bar{x}, a)$. The recursive saturation of K was only needed to ensure that K is short Π_1-recursively saturated, and hence can be embedded into M.)

Before we prove the case for uncountable M in general, we note a particular case when the theorem is easy: namely when T extends PA. In this case, by the Löwenheim—Skolem theorem there is a countable $M' \prec M$ containing a, b, and by the above argument there is $I' \subseteq_e M'$ with $a \in I' < b$ and $I' \models T$. But then

$$I = \{x \in M \mid x < y \text{ for some } y \in I'\} \subseteq_e M$$

is a cofinal extension of I'. So as $I' \prec_{\Delta_0} M' \prec M \succ_{\Delta_0} I$, $I' \prec_{\Delta_0} I$, hence by Theorem 7.7, $I' \prec I$, and thus $I \models T$ with $a \in I < b$.

In the general case, we again take $M' \prec M$ to be countable with $a, b \in M'$ and $I' \models T$ an initial segment of M' with $a \in I' < b$. We also take a Σ_1 formula $\exists \bar{y}\psi(t, i, \bar{y})$ representing (in PA^-) '$t = \ulcorner\tau_i\urcorner$' where $\{\tau_0, \tau_1, \ldots\}$ is some recursive axiomatization of T. We shall assume that each τ_i is in prenex normal form (i.e., all the quantifiers in τ_i appear on the 'outside' of τ_i), and we write '$x \in y$' for the $\mathscr{L}_A$-formula $\exists i < \text{len}(y)\ (x = [y]_i)$. Then using the PA axioms in M' together with the existence of $I' \subseteq_e M'$ with $a \in I' < b$ and $I' \models T$, it is straightforward to check that M' satisfies the following for all $e \in \mathbb{N}$:

$$\begin{aligned}
&\exists s, t[\text{len}(s) = \text{len}(t) = e \wedge [s]_0 = a \wedge \\
&\qquad \forall i < e - 1(([s]_i + 1)^2 < [s]_{i+1} < b) \wedge \\
&\qquad \forall i < e \forall x(x \in [t]_i \rightarrow \exists f, y(\text{form}(f) \wedge \langle f, y\rangle = x)) \wedge \\
&\qquad \forall i, x, \bar{y} < e(\psi(x, i, \bar{y}) \rightarrow \langle x, 0\rangle \in [t]_i) \wedge \\
&\qquad \forall i < e - 1 \forall x(x \in [t]_i \rightarrow x \in [t]_{i+1}) \wedge \\
&\qquad \forall i < e \forall f, y(\langle f, y\rangle \in [t]_i \wedge \text{form}_{\Delta_0}(f) \rightarrow \text{Sat}_{\Delta_0}(f, y)) \wedge \\
&\qquad \forall i < e - 1 \forall j, f, y(\langle \ulcorner\exists v_j f\urcorner, y\rangle \in [t]_i \rightarrow \exists w < [s]_{i+1} \langle f, y[w/j]\rangle \in [t]_{i+1}) \wedge \\
&\qquad \forall i < e - 1 \forall j, f, y(\langle \ulcorner\forall v_j f\urcorner, y\rangle \in [t]_i \rightarrow \forall w < [s]_{i+1} \langle f, y[w/j]\rangle \in [t]_{i+1})].
\end{aligned}$$

The idea is that s codes a sequence $a = [s]_0 < [s]_1 < \cdots$ and t codes a sequence $[t]_0, [t]_1, \ldots$ of sets of pairs $\langle f, y\rangle$ where f is the Gödel-number of an $\mathscr{L}_A$-formula thought of as 'true' in the initial segment $I = \{x \mid x < [s]_i$ for some $i \in \mathbb{N}\}$ when $v_0, v_1, \ldots$, are substituted by $[y]_0, [y]_1, \ldots$. In particular one of the clauses in the formula above says that this idea of 'true' agrees with the usual one for Δ_0 formulas as defined by $\mathrm{Sat}_{\Delta_0}(x, y)$.

By overspill, there is a nonstandard $e \in M'$ satisfying this formula, and since $M' < M$ it is clear that there are $s, t \in M$ such that the formula in square brackets above is true in M. We claim that, for such s,

$$I = \{x \in M \mid M \models [s]_i > x \text{ some } i \in \mathbb{N}\}$$

is an initial segment of M with $a \in I < b$ and $I \models T$. I is indeed an $\mathscr{L}_A$-structure since $[s]_i < ([s]_i + 1)^2 < [s]_{i+1}$ for all $i \in \mathbb{N}$; and $a \leqslant [s]_i < b$ for all i, so $a \in I < b$. To check that $I \models T$, note that, for all $i \in \mathbb{N}$, $\langle \ulcorner \tau_i \urcorner, 0\rangle \in [t]_j$ for some $j \in \mathbb{N}$. Thus it suffices to prove that

for all $i, j \in \mathbb{N}$ and all subformulas $\varphi(v_0, \ldots, v_k)$ of τ_i, if $\langle \ulcorner \varphi(v_0, \ldots, v_k)\urcorner, y\rangle \in [t]_j$ then $I \models \varphi([y]_0, \ldots, [y]_k)$.

This is proved by induction on the complexity of formulas. If φ is Δ_0 it follows by the 7th conjunct of the formula inside square brackets above, by the properties of $\mathrm{Sat}_{\Delta_0}(x, y)$, and the absoluteness of Δ_0 formulas between M and I. If φ is not Δ_0 it is $\exists v_l \theta$ or $\forall v_l \theta$ for some θ and some variable v_l, by our assumption that each τ_i is in prenex normal form. Thus, by the last two conjuncts of the formula inside square brackets, our definition of I, and the induction hypothesis:

$$\begin{aligned}\langle \varphi, y\rangle \in [t]_j &\Rightarrow \langle \theta, y[w/l]\rangle \in [t]_{j+1} \text{ for some } w < [s]_{j+1} \in I\\ &\Rightarrow I \models \varphi([y]_0, \ldots, [y]_k),\end{aligned}$$

when φ is $\exists v_l \theta$; and

$$\begin{aligned}\langle \varphi, y\rangle \in [t]_j &\Rightarrow \langle \theta, y[w/l]\rangle \in [t]_{m+1} \text{ for all } w < [s]_m \in I \text{ and all } m \geqslant j \text{ in } \mathbb{N}\\ &\Rightarrow I \models \varphi([y]_0, \ldots, [y]_k),\end{aligned}$$

when φ is $\forall v_l \theta$. □

COROLLARY 14.8. Any recursive $\mathscr{L}_A$-theory T has a well-behaved indicator, $Y(x, y)$, for initial segments satisfying T in PA. □

Corollary 14.8 can be modified in several ways. For example if a nonstandard model $M \models PA$ is given with $T \in SSy(M)$ (but not necessarily T recursive) then there is an indicator $Y(x, y)$ for initial segments of M satisfying T; however, $Y(x, y)$ will contain a parameter $a \in M$ coding T. Thus this indicator only applies to M, and not to an arbitrary model of PA.

Similarly, if we are given $n \in \mathbb{N}$ and we are interested in Σ_n-elementary initial segments satisfying a given recursive theory T, then there will be an indicator Y for such initial segments. This indicator Y will satisfy (b)–(e) in the definition on p. 194, but '$Y(x, y) = z$' will be Σ_{n+1} rather than Σ_1. Thus again $Y(x, y)$ fails to satisfy the technical condition of being 'well behaved'.

It is also worth remarking that the definition given above of the indicator $Y(x, y)$ for initial segments satisfying T only depended on the Π_2 sentences provable in $T + B\Sigma_1$, that is on

$$\Pi_2 - \mathrm{Th}(T + B\Sigma_1) = \{\sigma \in \Pi_2 \mid \sigma \text{ an } \mathscr{L}_\mathrm{A}\text{-sentence s.t. } T + B\Sigma_1 \vdash \sigma\},$$

and the proof of Lemma 14.6 only depended on the initial segment I with $a \in I < b$ satisfying $\Pi_2 - \mathrm{Th}(T + B\Sigma_1)$. Thus any two recursive theories T, T' with $\Pi_2 - \mathrm{Th}(T + B\Sigma_1) = \Pi_2 - \mathrm{Th}(T' + B\Sigma_1)$ have the same indicators. Put another way, this says that if $M \models PA$ and $a, b \in M$ then there exists $I \subseteq_e M$ with $a \in I < b$ and $I \models T$ if and only if there exists $J \subseteq_e M$ with $a \in J < b$ and $J \models T'$. (This is described by saying that initial segments satisfying T and initial segments satisfying T' are symbiotic—see Exercise 14.6.)

As one would expect, Corollary 14.8 has many interesting consequences from the structure of models of PA. For example,

COROLLARY 14.9. Let T be any recursively axiomatized $\mathscr{L}_\mathrm{A}$-theory that is true (i.e., $\mathbb{N} \models T$), and let $M \models PA$ with $a \in M$ non-standard. Then there is an initial segment $I \models T$ of M with $a \notin I \neq \mathbb{N}$.

Proof. Let $Y(x, y)$ be a well-behaved indicator for T. Then $M \models Y(n, a) > n$ for all $n \in \mathbb{N}$, since $n \in \mathbb{N} < a$, $\mathbb{N} \subseteq_e M$ and $\mathbb{N} \models T$. By overspill there is $c \in M \setminus \mathbb{N}$ with $M \models Y(c, a) > c$. So there is $I \subseteq_e M$ with $c \in I < a$ and $I \models T$. □

In particular any nonstandard model of PA has arbitrarily small non-standard initial segments satisfying PA.

One surprise in the proof of Theorem 14.7 was that if $a, b \in M \models PA$ with $Y(a, b) > \mathbb{N}$ and M is countable, then there is in fact a *recursively saturated* initial segment $I \subseteq_e M$ with $a \in I < b$ and $I \models T$. (This in in fact true even if M is uncountable—see Exercises 14.9 and 15.10—but we shall restrict our discussion to countable M here.) Also, by Theorem 12.7, there are $2^{\aleph_0}$ such initial segments I. Therefore we have

COROLLARY 14.10. Let $M \models PA$ be nonstandard and countable, and let T be a recursively axiomatized $\mathscr{L}_\mathrm{A}$-theory. Then the following are equivalent:

(a) there exists $I \subseteq_e M$ with $a \in I < b$ and $I \vDash \Pi_2 - \mathrm{Th}(T + B\Sigma_1)$;
(b) there exists $I \subseteq_e M$ with $a \in I < b$ and $I \vDash T$;
(c) there exists $2^{\aleph_0}$ different initial segments $I \vDash T$ with $a \in I < b$ and I recursively saturated. □

Other applications will be found in the exercises to follow.

Exercises for Section 14.2

14.6 Two classes P, Q of initial segments of a given model $M \vDash PA$ are *symbiotic* iff P, Q have the same indicators, i.e. iff for all $a, b \in M$

$$\exists I \in P(a \in I < b) \Leftrightarrow \exists J \in Q(a \in J < b).$$

Let P be the class of all $I \subseteq_e M$ satisfying PA such that I is recursively saturated, and let Q be the class of all $J \subseteq_e M$, $J \vDash PA$ and J not recursively saturated. Show that P and Q are symbiotic.

14.7 Let $M \vDash PA$ be nonstandard and suppose T is coded in M. Show that there is a formula $Y(x, y, w) = z$ in Σ_1 such that $M \vDash \forall x, y, w \exists! z Y(x, y, w) = z$ and whenever $c \in M$ codes T then the function $Y(x, y, c)$ is an indicator for initial segments of M satisfying T.

14.8 Let T be any recursively axiomatized $\mathscr{L}_A$-theory. Define a Σ_{n+1}-formula $Y(x, y) = z$ such that $PA \vdash \forall x, y \exists! z Y(x, y) = z$ such that Y is a well-behaved indicator (except that $Y(x, y) = z$ is not Σ_1) for Σ_n-elementary initial segments satisfying T.
(*Hint:* Use $B\Sigma_{n+1}$ in place of $B\Sigma_1$. You might find Exercise 14.3 helpful.)

14.9 Prove the Smorynski–Stavi Theorem (Smorynski and Stavi 1980) that if $M \subseteq_{cf} N$ are models of PA and M is recursively saturated, then N is recursively saturated too.
(*Hint:* If $p(x) = \{\varphi_i(x, a) \mid i \in \mathbb{N}\}$ is a recursive type over N, where $f(i) = \ulcorner \varphi_i(x, a) \urcorner$ is a recursive function with recursive image, consider the type

$$q(y) = \{\mathrm{len}(y) = b\} \cup \left\{ \begin{array}{l} \forall i < b(\varphi_{j-1}([y]_i, i) \rightarrow (\varphi_j([y]_i, i) \vee \\ \qquad\qquad \forall z \neg \bigwedge_{k \leq j} \varphi_k(z, i))) \end{array} \middle| \, j \in \mathbb{N} \right\}$$

where $\varphi_{-1}(x, i)$ is $x = x$ say.)

Deduce that Corollary 14.10 holds for all recursive extensions T of PA, even if M is not countable.

14.10 Use indicators to prove that every nonstandard $M \vDash PA$ has arbitrarily large recursively saturated initial segments $I \vDash PA$ and has recursively saturated end-extensions $J \vDash PA$.

Using Exercise 14.7, this may be improved to: for all nonstandard $M \vDash PA$, all $n \in \mathbb{N}$ and all $a \in M$, there are $I \subseteq_e M$ and $M \subseteq_e J$ such that I, J are recursively saturated, $I, J \vDash PA$, $a \in I$, and $I \prec_{\Sigma_n} M \prec_{\Sigma_n} J$. Show that '$\prec_{\Sigma_n}$' cannot in general be improved further to '$\prec$'.

14.3 THE PARIS–HARRINGTON THEOREM

In this section we shall exhibit 'mathematically interesting' (rather than just 'metamathematically interesting') indicators $Y(x, y)$ for initial segments satisfying PA. The chief consequence will be the celebrated Paris–Harrington theorem, that a certain combinatorial sentence of $\mathscr{L}_A$ is true but not provable in PA.

The set-up we need to consider involves a model $M \models PA$ with $a < b$ in M, and we wish to find a condition $Y(a, b) > \mathbb{N}$ that is true iff there is $I \subseteq_e M$ with $a \in I < b$ and $I \models PA$. The first simplification is to look for *coded initial segments* I of the form

$$I = \{x \in M \mid M \models x < [c]_i \text{ some } i \in \mathbb{N}\},$$

where $c \in M$, $\text{len}(c) = \delta > \mathbb{N}$,

$$a \leqslant [c]_0 < [c]_1 < \cdots < [c]_i < \cdots < [c]_{\delta-1} \leqslant b,$$

and $[c]_{i+1} > [c]_i^2$ for all $i < \delta - 1$. (The condition $[c]_{i+1} > [c]_i^2$ will make I an $\mathscr{L}_A$-structure.) To make I a model of PA we aim to choose c so that the following 'truth definition' holds.

Truth definiton for I:

For all Δ_0 formulas $\theta(x_1, \ldots, x_k, \bar{y})$, for $i \in \mathbb{N}$, and all $\bar{a} \in I$, if $\bar{a} < [c]_i$ then

$$I \models \exists x_1 \forall x_2 \exists x_3 \ldots \mathrm{Q} x_k \theta(x_1, \ldots, x_k, \bar{a})$$

holds (where Q is $\exists$ or $\forall$) just in case that

$$M \models \exists x_1 < [c]_{i+1} \forall x_2 < [c]_{i+2} \ldots \mathrm{Q} x_k < [c]_{i+k} \theta(x_1, \ldots, x_k, \bar{a}).$$

LEMMA 14.11. If I, as above, has this truth definition, then $I \models PA$.

Proof. Since $I \subseteq_e M$ is an $\mathscr{L}_A$-structure, $I \models PA^-$. If $\bar{a} \in I \models \exists x \psi(x, \bar{a})$, where ψ is an $\mathscr{L}_A$-formula, then ψ is equivalent to

$$\forall y_1 \exists y_2 \cdots \mathrm{Q} y_k \theta(x, y_1, \ldots, x_k, \bar{a})$$

for some $k \in \mathbb{N}$ and some $\theta \in \Delta_0$. Thus

$$I \models \exists x \forall y_1 \exists y_2 \cdots \mathrm{Q} y_k \theta(x, y_1, \ldots, y_k, \bar{a}).$$

Hence, if $\bar{a} < [c]_i$, say, then by the truth definition

$$M \models \exists x < [c]_{i+1} \forall y_1 < [c]_{i+2} \cdots \mathrm{Q} y_k < [c]_{i+k+1} \theta(x, y_1, \ldots, y_k, \bar{a}).$$

Using the least-number principle in M there is a least $x < [c]_{i+1}$ such that

$$M \vDash \forall y_1 < [c]_{i+2} \cdots Qy_k < [c]_{i+k+1} \theta(x, y_1, \ldots, y_k, \bar{a}).$$

This least such x is actually the least $x \in I$ such that $I \vDash \psi(x, \bar{a})$. To see this, note that, by the truth definition, I does satisfy $\psi(x, \bar{a})$, and that if $x' < x$ with $I \vDash \psi(x', \bar{a})$ then by the truth definition again,

$$M \vDash \forall y_1 < [c]_{i+2} \cdots Qy_y < [c]_{i+k+1} \theta(x', y_1, \ldots, y_k, \bar{a})$$

contradicting our choice of x. □

The truth definition for I is equivalent to saying that the elements of the sequence $[c]_0, [c]_1, \ldots$ are 'indiscernible' (i.e. cannot be distinguished from each other) in the following sense: if $n_1 < n_2 < n_3 < \cdots < n_k$ and $m_1 < m_2 < m_3 < \cdots < m_k$ are positive integers and $k \in \mathbb{N}$, and if θ is a Δ_0 formula, then for all $\bar{a} \in M$ such that $\bar{a} < \min([c]_{n_1-1}, [c]_{m_1-1})$

$$M \vDash \exists x_1 < [c]_{n_1} \forall x_2 < [c]_{n_2} \cdots Qx_k < [c]_{n_k} \theta(x_1, \ldots, x_k, \bar{a})$$

holds if and only if

$$M \vDash \exists x_1 < [c]_{m_1} \forall x_2 < [c]_{m_2} \cdots Qx_k < [c]_{m_k} \theta(x_1, \ldots, x_k, \bar{a}),$$

as these are both equivalent to

$$I \vDash \exists x_1 \forall x_2 \cdots Qx_k \theta(x_1, \ldots, x_k, \bar{a}).$$

(This is proved by an easy induction on k.) In model theory, indiscernible sequences like these are usually constructed using results from combinatorics, especially *Ramsey's theorem* (see below), and this is what we will do here. The end-result will be that an interesting variation of Ramsey's theorem independent of PA.

First, then, we must give some definitions concerning the combinatorial notions we will need.

Definitions. If X is a set and n is a natural number, $[X]^n$ denotes the collection of all suhsets of X of cardinality n. Usually X will be a subset of the natural numbers, and we will identify an element $S = \{x_1, x_2, \ldots, x_n\}$ of $[X]^n$ with an *increasing sequence* $x_1 < x_2 < \cdots < x_n$ of elements of X. Similarly, given a function $f: [X]^n \to \mathbb{N}$, we write $f(x_1, x_2, \ldots, x_n)$ for $f(\{x_1, x_2, \ldots, x_n\})$ with the understanding that $x_1 < x_2 < \cdots < x_n$. A natural number n will be identified with the set $\{0, 1, \ldots, n-1\}$ of its predecessors; $[m, n]$ denotes the set $\{m, m+1, \ldots, n-1, n\}$; and $\mathbb{N}$ will denote both the set $\{0, 1, 2, \ldots\}$ and its cardinality. If n, k, c are either $\mathbb{N}$ or elements of $\mathbb{N}$, $X \to (k)^n_c$ means that whenever $f: [X]^n \to c$ there is a subset $H \subseteq X$ with cardinality k such that f is constant on $[H]^n$. (The set H here is often described as being *homogeneous* for f.)

Ramsey (1930) proved the well-known

INFINITE RAMSEY THEOREM. $\forall n, c \in \mathbb{N}\ \mathbb{N} \rightarrow (\mathbb{N})^n_c$,

and derived from this the

FINITE RAMSEY THEOREM. $\forall n, c, k \in \mathbb{N}\ \exists m \in \mathbb{N}\ m \rightarrow (k)^n_c$.

We shall consider two variations of the finite Ramsey theorem. Notice that whilst the infinite Ramsey theorem is 'second-order' (it refers to all functions $[\mathbb{N}]^n \rightarrow c$), the finite Ramsey theorem can be expressed in the language of PA, using Gödel's lemma (Lemma 5.8) or some other suitable coding device to express notions such as increasing sequences of length n from $\{0, 1, \ldots, m\}$, and to quantify over all codes for functions $[m]^n \rightarrow c$. We shall assume from now on that these notions have been formalized in PA in some suitable way, and in particular that notions such as $a \rightarrow (b)^c_d$ are described by $\Delta_1(PA)$ formulas. (Note also that '$a \rightarrow (b)^c_d$' is in fact a primitive recursive predicate, stating that all possible codes for functions $[a]^c \rightarrow d$ have a certain property, so there is indeed a $\Delta_1(PA)$ formula for it, by Exercise 5.10.) We shall also assume that these notions behave in the expected way in models of PA, with respect to certain properties.

It turns out that the finite Ramsey theorem is actually provable in PA. Thus we will be interested in strengthenings of this result. The first of these is the Paris–Harrington principle (PH), which stipulates further that the homogeneous set should be 'relatively large'. Let $X \underset{*}{\rightarrow} (k)^n_c$ mean that whenever $f: [X]^n \rightarrow c$ there is $H \subseteq X$ that is homogeneous for f and $\mathrm{card}(H) \geqslant \max(k, \min(H))$.

PARIS–HARRINGTON PRINCIPLE (PH). $\forall n, c, k \in \mathbb{N}\ \exists m \in \mathbb{N}\ m \underset{*}{\rightarrow} (k)^n_c$.

The second variation was considered by Kanamori and McAloon (1987). Let $X \rightarrow (k)^n_{\mathrm{reg}}$ mean that whenever $f: [X]^n \rightarrow \mathbb{N}$ is *regressive*, that is $f(x_1, x_2, \ldots, x_n) < x_1$ for all $x_1 < x_2 < \cdots < x_n$ from X, then there is $H \subseteq X$ with cardinality k such that for all $x_1 < x_2 < \cdots < x_n$ from H, $f(x_1, x_2, \ldots, x_m)$ only depends on x_1, i.e., if $x_1 < y_2 < \cdots < y_n$ with $y_2, \ldots, y_n \in H$ also then $f(x_1, x_2, \ldots, x_n) = f(x_1, y_2, \ldots, y_n)$. (Kanamori and McAloon call such H *min-homogeneous* for f.)

KANAMORI–MCALOON PRINCIPLE (KM). $\forall n, k \in \mathbb{N}\ \exists m \in \mathbb{N}\ m \rightarrow (k)^n_{\mathrm{reg}}$.

The Paris–Harrington priniciple follows from the infinite Ramsey theorem (see Lemma 14.15). Kanamori and McAloon point out that (KM)

follows from a variation of the infinite Ramsey theorem due to Erdös and Rado. Obviously both (PH) and (KM) are *expressible* in the language of PA; we shall see that neither are *provable* in PA.

In fact we will first show how the Kanamori–McAloon principle may be used to obtain initial segments I satisfying the 'truth definition' above. We will then show how (KM) can be deduced from (PH), and finally show that $(PH_n) = \forall c, k \exists m\, m \underset{*}{\rightarrow} (k)^n_c$ is provable in PA for each standard natural number n. This will not only let us deduce that (PH) is true in $\mathbb{N}$ but unprovable in PA, but it will also give us indicators $Y_{PH}(a, b)$ and $Y_{KM}(a, b)$ for initial segments satisfying PA based on the principles (PH) and (KM) respectively.

The next lemma is the first step in this programme: it appeared first in Kanamori and McAloon (1987), and simplifes the direct argument from (PH) considerably.

Lemma 14.12. Let $M \vDash PA$ with $a, b, d, e \in M$ satisfying $\mathbb{N} < 2e+1 < a < b$, and suppose M satisfies

$$[a, b] \to (d)^{2e+1}_{\text{reg}} \qquad \text{and} \qquad d \to (3e+1)^{2e+1}_{e+2}.$$

then there exists $c \in M$ coding a sequence

$$a \leqslant [c]_0 < [c]_1 < \cdots < [c]_e \leqslant b$$

such that $[c]_{i+1} > [c]_i^2$ for all $i < \lfloor e/2 \rfloor$, and

$$I = \{x \in M \mid M \vDash x < [c]_i \text{ for some } i \in \mathbb{N}\}$$

has the 'truth definition' given above.

Proof. We argue in an informal way, leaving the reader to check that the arguments can be carried out in PA.

Let $\psi_i(v_0, v_1, \ldots, v_e)$ $(i \leqslant e)$ be the first $e+1$ Δ_0 formulas (according to the Gödel-numbering given in Chapter 9) in at most the free-variables shown. In particular the function $i \mapsto \psi_i(v_0, v_1, \ldots, v_e)$ is provably recursive in PA and, since e is nonstandard, every standard Δ_0 formula $\psi(v_0, v_1, \ldots, v_n)$ is $\psi_i(v_0, v_1, \ldots, v_e)$ for some $i < e$. Next we define two functions $f(x_0, \ldots, x_{2e})$ and $i(x_0, \ldots, x_{2e})$, which will both be coded in M. Given $a \leqslant x_0 < x_1 < \cdots < x_{2e} \leqslant b$, define $f(x_0, x_1, \ldots, x_{2e})$ to be the least $p < x_0$ such that

$$\exists i \leqslant e [\neg \mathrm{Sat}_{\Delta_0}(\ulcorner \psi_i \urcorner, [p, x_1, \ldots, x_e]) \leftrightarrow \mathrm{Sat}_{\Delta_0}(\ulcorner \psi_i \urcorner, [p, x_{e+1}, \ldots, x_{2e}])]$$

and $i(x_0, x_1, \ldots, x_{2e})$ to be the least $i \leqslant e$ such that

$$\neg\mathrm{Sat}_{\Delta_0}(\ulcorner\psi_i\urcorner, [p, x_1, \ldots, x_e]) \leftrightarrow \mathrm{Sat}_{\Delta_0}(\ulcorner\psi_i\urcorner, [p, x_{e+1}, \ldots, x_{2e}])$$

where $p = f(x_0, \ldots, x_{2e})$. If no such p exists we set $f(x_0, \ldots, x_e) = 0$ and $i(x_0, \ldots, x_e) = e+1$.

The reader may check that f and i are coded in M. Then by $[a, b] \to (d)^{2e+1}_{\mathrm{reg}}$ there is a set $H_0 \subseteq [a, b]$ that is coded in M, of cardinality d, and is min-homogeneous for f. By $d \to (3e+1)^{2e+1}_{e+2}$ there is $H_1 \subseteq H_0$ coded in M, of cardinality $3e+1$, and homogeneous for i. Let i_0 be the constant value that $i(x_0, \ldots, x_{2e})$ takes on $[H_1]^{2e+1}$. We show next that i_0 must be $e+1$. Suppose that $i_0 \leqslant e$, and that $x_0 < x_1 < \cdots < x_{3e}$ are the elements of H_1. Since $H_1 \subseteq H_0$ and H_0 is min-homogeneous for f, there is $p_0 < x_0$ such that $p_0 = f(x_0, y_1, \ldots, y_{2e})$ for any $x_0 < y_1 < y_2 < \cdots < y_{2e}$ from H_1. Thus by considering

$$f(x_0, x_1, \ldots, x_e, x_{e+1}, \ldots, x_{2e}),$$
$$f(x_0, x_1, \ldots, x_e, x_{2e+1}, \ldots, x_{3e}),$$

and

$$f(x_0, x_{e+1}, \ldots, x_{2e}, x_{2e+1}, \ldots, x_{3e}),$$

the three formulas

$$\mathrm{Sat}_{\Delta_0}(\ulcorner\psi_{i_0}\urcorner, [x_0, x_1, \ldots, x_e]),$$
$$\mathrm{Sat}_{\Delta_0}(\ulcorner\psi_{i_0}\urcorner, [x_0, x_{e+1}, \ldots, x_{2e}]),$$

and

$$\mathrm{Sat}_{\Delta_0}(\ulcorner\psi_{i_0}\urcorner, [x_0, x_{2e+1}, \ldots, x_{3e}]),$$

are inequivalent in pairs, which is impossible as there are only two truth values, *true* and *false*.

Now let $z_1 < z_2 < \cdots < z_e$ be the last e elements of H_1, and let $H = H_1 \setminus \{z_1, \ldots, z_e\}$. We claim that:

$$\left.\begin{array}{l}\text{whenever } c_0 < c_1 < \cdots < c_e,\ c_0 < d_1 < \cdots < d_e \text{ are elements} \\ \text{of } H,\ j \leqslant e \text{ and } p < c_0, \text{ then} \\ M \vDash \mathrm{Sat}_{\Delta_0}(\ulcorner\psi_j\urcorner, [p, c_1, \ldots, c_e]) \leftrightarrow \mathrm{Sat}_{\Delta_0}(\ulcorner\psi_j\urcorner, [p, d_1, \ldots, d_e]).\end{array}\right\} \qquad (*)$$

To see this, notice simply that

$$c_0 < c_1 < \cdots < c_e < z_1 < \cdots < z_e \quad \text{and} \quad c_0 < d_1 < \cdots < d_e < z_1 < \cdots < z_e,$$

so by the last paragraph

$$\begin{aligned}\mathrm{Sat}_{\Delta_0}(\ulcorner\psi_j\urcorner, [p, c_1, \ldots, c_e]) &\leftrightarrow \mathrm{Sat}_{\Delta_0}(\ulcorner\psi_j\urcorner, [p, z_1, \ldots, z_e]) \\ &\leftrightarrow \mathrm{Sat}_{\Delta_0}(\ulcorner\psi_j\urcorner, [p, d_1, \ldots, d_e]).\end{aligned}$$

Next, suppose $c_0 < c_1 < c_2 < c_3$ are elements of H. Then $c_0 + c_1 \leqslant c_2$ since if $c_0 + c_1 > c_2$ there would be $p < c_0$ with $p + c_1 = c_2$, so by (*) $p + c_1 = c_3$, which is impossible. By (*) again we also have $c_1 + c_2 \leqslant c_3$ From this we can also deduce that $c_0 \cdot c_1 < c_2$, for if $c_0 \cdot c_1 \geqslant c_2$ there would be $p < c_0$ with $p \cdot c_1 < c_2 \leqslant (p+1) \cdot c_1$. Then using (*),

$$(p+1) \cdot c_1 \geqslant c_3 \geqslant c_1 + c_2 > (p+1) \cdot c_1,$$

a contradiction. Hence $c_0 \cdot c_0 < c_2$ so, by (*), $c_0^2 < c_1$. Thus $c_0^2 \cdot c_0 < c_2$, so by (*) again $c_0^3 < c_1$, and so on. It follows that, for all $c < d$ in H and all $n \in \mathbb{N}$, $c^n < d$.

Now let $H = \{c_0, c_1, \ldots, c_{2e}\}$, where $c_0 < c_1 < \cdots < c_{2e}$, and let $c \in M$ code the subsequences of $c_0, c_1, \ldots, c_{2e}$ formed by elements c_j of even index j, i.e., $[c]_i = c_{2i}$ for all $i \leqslant e$. Then $[c]_i^2 = c_{2i}^2 < c_{2i+1} < [c]_{i+1}$ by the last paragraph, and if $\bar{a} < [c]_i$ then $\langle \bar{a} \rangle < [c]_i^n = c_{2i}^n < c_{2i+1}$ for some $n \in \mathbb{N}$, where $\langle \bar{x} \rangle$ denotes the coding function based on the pairing function $\langle x, y \rangle = (x+y+1)(x+y)/2 + y$. Thus, by (*) the sequence $[c]_0, \ldots, [c]_e$ has the required indiscernibility condition, and so the 'truth definition' for I holds, as required. □

Lemma 14.12 already tells us that (KM) is not provable in PA. The next lemma relates (KM) to (PH).

Lemma 14.13. $PA \vdash (PH) \rightarrow (KM)$. In fact, provably in PA, given a, k, e, and b satisfying

$$[a+e, b] \xrightarrow{*} (e+k)_3^{e+1}$$

we have

$$[a, b] \rightarrow (k)_{\text{reg}}^e.$$

Proof. Let $f : [[a, b]]^e \rightarrow b$ be regressive, and define $g' : [[a+e, b]]^{e+1} \rightarrow 3$ by

$$g'(x_0, \ldots, x_e) = \begin{cases} 0 \text{ if } f(x_0 - e, x_1 - e, \ldots, x_{e-1} - e) \\ \qquad = f(x_0 - e, x_2 - e, \ldots, x_e - e) \\ 1 \text{ if } f(x_0 - e, x_1 - e, \ldots, x_{e-1} - e) \\ \qquad < f(x_0 - e, x_2 - e, \ldots, x_e - e) \\ 2 \text{ if } f(x_0 - e, x_1 - e, \ldots, x_{e-1} - e) \\ \qquad > f(x_0 - e, x_2 - e, \ldots, x_e - e). \end{cases}$$

Then, using $[a+e,b] \xrightarrow{*} (e+k)_3^{e+1}$, there is $H' \subseteq [a+e,b]$ with $|H'| \geqslant \min(H')$, $|H'| \geqslant e+k$, and H' homogeneous for g'. Then clearly $H = \{h-e \mid h \in H'\}$ is homogeneous for

$$g(x_0,\ldots,x_e) = \begin{cases} 0 \text{ if } f(x_0,x_1,\ldots,x_{e-1}) \\ \qquad = f(x_0,,x_2,\ldots,x_e) \\ 1 \text{ if } f(x_0,x_1,\ldots,x_{e-1}) \\ \qquad < f(x_0,x_2,\ldots,x_e) \\ 2 \text{ if } f(x_0,x_1,\ldots,x_{e-1}) \\ \qquad > f(x_0,x_2,\ldots,x_e). \end{cases}$$

with $|H| = |H'| \geqslant \min(H') = \min(H)+e$. Writing H as $\{x_0,x_1,\ldots,x_{x_0+e-1}\}$ where $x_0 < x_1 < \cdots < x_{x_0+e-1}$, we have

$$g(x_0,x_1,\ldots,x_e) = g(x_0,x_2,\ldots,x_{e+1}) = \cdots = g(x_0,x_{x_0},\ldots,x_{x_0+e-1}),$$

and so g must have constant value 0 on $[H]^{e+1}$, since otherwise

$$f(x_0,x_1,\ldots,x_{e-1}) < f(x_0,x_2,\ldots,x_e) < \cdots < f(x_0,x_{x_0+1},\ldots,x_{x_0+e-1}),$$

or

$$f(x_0,x_1,\ldots,x_{e-1}) > f(x_0,x_2,\ldots,x_e) > \cdots > f(x_0,x_{x_0+1},\ldots,x_{x_0+e-1}),$$

and so $f(x_0,x_i,\ldots,x_{i+e-1}) \geqslant x_0$ for some i, contradicting our assumption that f is regressive. Now let H_0 be the first k elements of H, so $|H \setminus H_0| \geqslant e$, and let $z_1,z_2,\ldots,z_e$ be the last e elements of H, in ascending order. We claim that H_0 is min-homogeneous for f.

If $x_0 < x_1 < \cdots < x_{e-1}$ and $x_0 < y_1 < \cdots < y_{e-1}$ are arbitrary elements of H_0, then since g has constant value 0 on $[H]^{e+1}$,

$$\begin{aligned} f(x_0,x_1,\ldots,x_{e-1}) &= f(x_0,x_2,\ldots,x_{e-1},z_1) \\ &= f(x_0,x_3,\ldots,x_{e-1},z_1,z_2) \\ &\;\vdots \\ &= f(x_0,x_{e-1},z_1,\ldots,z_{e-2}) \\ &= f(x_0,z_1,\ldots,z_{e-1}) \\ &= f(x_0,y_{e-1},z_1\ldots,z_{e-2}) \\ &\;\vdots \\ &= f(x_0,y_2,\ldots,y_{e-1},z_1) \\ &= f(x_0,y_1,\ldots,y_{e-1}) \end{aligned}$$

as required. □

We now turn to considering provable versions of the principles (PH) and (KM). For each $n \in \mathbb{N}$, (PH_n) denotes the $\mathscr{L}_A$-sentence

$$\forall x, z \exists y [x, y] \underset{*}{\to} (2z)^n_z$$

and (KM_n) denotes

$$\forall x, z \exists y [x, y] \to (z)^n_{\text{reg}}.$$

Obviously, provably in PA,

$$(PH_{n+1}) \Rightarrow \forall x, z \exists y [x, y] \underset{*}{\to} (z)^{n+1}_3 \Rightarrow (KM_n),$$

by Lemma 14.13. We prove that $PA \vdash (PH_n)$ for each n. (The proof is based on Ramsey's original proof of the infinite Ramsey theorem and the reduction of (PH) to this theorem.)

LEMMA 14.14. Suppose $e \in M \models PA$, $n \neq 0$, $n \in \mathbb{N}$, $A \subseteq M$ a cofinal definable subset of M, and $F : [A]^n \to e$ a definable function. Then there is a definable subset $B \subseteq A$, cofinal in M, such that F is constant on $[B]^n$.

Proof. By induction on n.

The case $n = 1$ is proved using the collection axioms. We write '$z \in A$' and '$F(z) = y$' for the formulas defining these relations in M. If the lemma fails for $n = 1$, then M satisfies

$$\forall y < e \exists x \forall z > x (z \in A \to F(z) \neq y).$$

Then, by collection, M satisfies

$$\exists t \forall y < e \forall z > t (z \in A \to F(z) \neq y)$$

which clearly contradicts

$$\forall t \exists y < e \exists z > t (z \in A \wedge F(z) = y).$$

Now suppose that the result holds for $n - 1 \geqslant 1$. Working in M we define a function $G: M \to A$ as follows. Let $\theta(z, x, g)$ be the formula

'g codes a function from $\{0, 1, \ldots, d\}$ to A for some $d \geqslant z$, and x is least satisfying:

$$\begin{aligned} & x \in A \wedge x > g(z-1) \\ & \wedge \forall z_1 < z_2 < \cdots < z_n < z \\ & \qquad (F(g(z_1), \ldots, g(z_n)) = F(g(z_1), \ldots, g(z_{n-1}), x)) \\ & \wedge \forall w \exists y > w (y \in A \wedge \forall z_1 < \cdots < z_{n-1} < z \\ & \qquad (F(g(z_1), \ldots, g(z_{n-1}), x) = F(g(z_1), \ldots, g(z_{n-1}), y)))\text{'}. \end{aligned}$$

We then define $G(z)=x$ by the formula

> $\exists g$, the code of a function $\{0, 1, \ldots, z\} \to A$ such that, for all $v \leqslant z$
>
> $$\begin{array}{ll} g(v) = \text{the } (v+1)\text{st element on } A, & \text{if } v < n-1 \\ \theta(v, g(v), g), & \text{if } v \geqslant n-1 \end{array}$$
>
> and $G(z)=g(z)$.

We use induction to show that G is well-defined, i.e. $M \vDash \forall z \exists! x G(z)=x$. Clearly, if $g(0), g(1), \ldots, g(n-2)$ are the first $n-1$ elements of A, then by an argument using the collection axioms as in the last paragraph we have

$$\begin{aligned} M \vDash \exists x \forall w \exists y > w(F(g(0), \ldots, g(n-2), x) \\ = F(g(0), \ldots, g(n-2), y)) \end{aligned}$$

and $G(n-1)$ will be the least such x. If $z \geqslant n-1$ is given and $G(z)$ exists, then there is $g \in M$ such that for all $v \leqslant z$

$$\begin{array}{ll} g(v) = \text{the } (v+1)\text{st element on } A, & \text{if } v < n-1, \text{ and} \\ \theta(v, g(v), g) & \text{if } v \geqslant n-1. \end{array}$$

So, clearly

$$\begin{aligned} \forall w \exists x > w\{y \in A \wedge \forall z_1 < \cdots < z_{n-1} < z_n \leqslant z \\ (F(g(z_1), \ldots, g(z_n)) = F(g(z_1), \ldots, g(z_{n-1}), x))\} \end{aligned}$$

holds. Thus by the collection axioms again, of the cofinitely-many x satisfying the formula in curly brackets $\{\cdots\}$ above, there is some such x satisfying

$$\begin{aligned} \forall w \exists y > w\{y \in A \wedge \forall z_1 < \cdots < z_{n-1} < z+1 \\ (F(g(z_1), \ldots, g(z_{n-1}), x) = F(g(z_1), \ldots, g(z_{n-1}), y))\} \end{aligned}$$

since there are only M-finitely many possible sequences $z_1 < z_2 < \cdots < z_{n-1} < z+1$ and thus only M-finitely many possible values for $F(g(z_1), \ldots, g(z_{n-1}), x)$. The least such x is $G(z+1)$. Clearly $G(z+1)$ is uniquely determined, so therefore defines a function.

The function G defined above has the special property that for all $z_1 < z_2 < \cdots < z_{n+1}$ in M,

$$F(G(z_1), \ldots, G(z_n)) = F(G(z_1), \ldots, G(z_{n-1}), G(z_{n-1}+1))$$
$$= F(G(z_1), \ldots, G(z_{n-1}), G(z_{n+1})).$$

Using our main inductive hypothesis there is a set $C \subseteq M$ that is definable in M, cofinal in M and homogeneous for the function

$$F(G(x_1), \ldots, G(x_{n-1}), G(x_{n-1}+1))$$

defined on $[M]^{n-1}$. It follows then that $B \subseteq A$ defined by

$$x \in B \Leftrightarrow \exists y \in C(x = G(y))$$

is cofinal in M and homogeneous for F. □

REMARK. Notice that, in Lemma 14.14, the quantifier complexity of B (measured in terms of its position in the Σ_n/Π_n hierarchy) may be greater than that of A.

LEMMA 14.15. $PA \vdash (PH_n)$ for all $n \in \mathbb{N}$, and hence also $PA \vdash (KM_n)$ for all $n \in \mathbb{N}$.

Proof. Assume that, in some model $M \models PA$ and for some $n \in \mathbb{N}$, $\forall b \neg([a, b] \underset{*}{\rightarrow} (2c)^n_c)$. Then for each $b \in M$ we can define f_b = the least code for a function $[[a, b]]^n \rightarrow c$ such that there is no $H \subseteq [a, b]$ that is homogeneous for f_b with $|H| \geqslant \min(H)$, $2c$. Now define $F: [[a, \infty)]^{n+1} \rightarrow c$ (where $[a, \infty) = \{x \in M \mid M \models x \geqslant a\}$) by

$$F(x_1, \ldots, x_{n+1}) = f_{x_{n+1}}(x_1, \ldots, x_n).$$

By Lemma 14.14 there is $B \subseteq [a, \infty)$ cofinal and definable in M such that F is constant on $[B]^{n+1}$. By induction we can define: $h(0)$ = the least element of B, and $h(x+1)$ = the least element of B greater than $h(x)$. In particular consider $h_0 = h(2c + \min(B) + 1)$. The set $H = \{h(x) \mid x < 2c + \min(B)\}$ has $2c + \min(B) = 2c + \min(H)$ many elements, and f_{h_0} is constant on $[H]^n$, since for $x_1 < \cdots < x_n$ in H,

$$f_{h_0}(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, h_0).$$

Moreover it is easy to see that H is coded in M, contradicting our assumption. Hence $PA \vdash (PH_n)$ for all $n \in \mathbb{N}$, and by Lemma 14.13, $PA \vdash (KM_n)$ for all $n \in \mathbb{N}$ too. □

We now want to tidy these results up, and obtain our indicators and the independence results promised.

THEOREM 14.16. The functions

$$Y_{PH}(a,b) = (\mu e)([a,b] \underset{*}{\rightarrow} (2e)^e_3)$$

and

$$Y_{KM}(a,b) = (\mu e)([a,b] \rightarrow (2e)^e_{\text{reg}})$$

are well-behaved indicators for initial segments satisfying PA in models of PA.

Proof. We leave the (easy, but rather tedious) checking that these functions have Σ_1 graph and are well-behaved, and concentrtate on the property concerning the existence of initial segments (property (e) on p. 194).

Suppose $a, b \in M \vDash PA$ with $a \in I \subseteq_e M$, $I \vDash PA$ and $b \in M \setminus I$. Then, by Lemma 14.15,

$$I \vDash \exists c([a,c] \underset{*}{\rightarrow} (2n)^n_n)$$

for each $n \in \mathbb{N}$, and in particular

$$I \vDash \exists c([a,c] \underset{*}{\rightarrow} (2n)^n_3).$$

Since $[a,c] \underset{*}{\rightarrow} (2x)^x_3$ is equivalent to a Σ_1 formula in PA, $I \subseteq_e M$ and $b \in M \setminus I$,

$$M \vDash \exists c < b([a,c] \underset{*}{\rightarrow} (2n)^n_3)$$

for all $n \in \mathbb{N}$, hence by overspill

$$M \vDash \exists c < b([a,c] \underset{*}{\rightarrow} (2e)^e_3)$$

for some $e > \mathbb{N}$, and hence $Y_{PH}(a,b) > \mathbb{N}$. Similarly,

$$M \vDash \exists c < b([a+n,c] \underset{*}{\rightarrow} (3n)^{n+1}_3)$$

for each $n \in \mathbb{N}$, hence

$$M \models [a+e, b] \underset{*}{\to} (3e)_3^{e+1}$$

for some $e > \mathbb{N}$, hence

$$M \models [a, b] \to (2e)_{\mathrm{reg}}^{e}$$

by Lemma 14.13, and so $Y_{KM}(a, b) > \mathbb{N}$.

For the converses, suppose first that $a, b \in M \models PA$ with $[a, b] \underset{*}{\to} (2e)_3^e$ and $e > \mathbb{N}$. Obviously we may assume $a > \mathbb{N}$ and $e < \lfloor a/2 \rfloor$. By the finite Ramsey theroem in $\mathbb{N}$ and the fact that $x \to (y)_w^z$ is equivalent to a Σ_1 formula in PA,

$$M \models 2n+1 \leqslant e \wedge e \to (3n+1)_{n+2}^{2n+1}$$

for all $n \in \mathbb{N}$. Hence by overspill there is $f > \mathbb{N}$ such that

$$M \models 2f+1 \leqslant e \wedge e \to (3f+1)_{f+2}^{2f+1}$$

and since $2f+1 \leqslant e$ we have

$$M \models [a, b] \underset{*}{\to} (2f+1+e)_3^{2f+1},$$

so by Lemma 14.13

$$M \models [a-2f-1, b] \to (e)_{\mathrm{reg}}^{2f+1}.$$

Hence by Lemma 14.12 there is $I \subseteq_e M$ with $I \models PA$ and $a-2f-1 \in I < b$. But $2f+1 < \lfloor a/2 \rfloor$ so $a-2f-1 > \frac{1}{2}a$, hence $a \in I$ also.

Finally, suppose that $e, a, b \in M \models [a, b] \to (2e)_{\mathrm{reg}}^e$ with $e > \mathbb{N}$. Then by an overspill argument as before there is $f \in \mathbb{N}$ with $2f+1 < e$ and $M \models e \to (3f+1)_{f+2}^{2f+1}$. Hence $M \models [a, b] \to (e)_{\mathrm{reg}}^{2f+1}$ and by Lemma 14.12 again there is $I \subseteq_e M$ satisfying PA with $a \in I < b$. □

Corollary 14.17. The statements

$$\forall x, z \exists y([x, y] \underset{*}{\to} (2z)_3^z)$$

and

$$\forall x, z \exists y([x, y] \to (2z)_{\mathrm{reg}}^z)$$

are true, but unprovable in PA.

Proof. By Theorems 14.16 and 14.3. □

The results presented here are really only the tip of the iceberg. We have seen that

$$\forall x, y \exists z([x, y] \underset{*}{\to} (2z)_3^z)$$

is true but unprovable in PA. It is interesting to ask if this result can be improved. Harrington, Paris, and Quinsey (see Quinsey 1980) were able to show that

$$\forall x \exists y (y \underset{*}{\to} (x+1)_3^x)$$

is unprovable in PA, but it is still unknown whether '3' may be replaced by '2' here.

Another interesting line of research is that of classifying the strengths of the hierarchy of theories $I\Sigma_n$ by their indicators. It turns out that $Y_n(a, b) = (\mu e)\ ([a, b] \underset{*}{\to} (e)_3^{n+1})$ and $Y'_n(a, b) = (\mu e)\ ([a, b] \to (e)_{\text{reg}}^{n+1})$ are both indicators for initial segments satisfying $I\Sigma_n$, and it follows that, for each $n \geqslant 1$, $I\Sigma_{n+1}$ proves (PH_{n+1}) and (KM_{n+1}) but $I\Sigma_n$ does not. (See (Paris 1980) and (Kanamori and McAloon, 1987) respectively.)

Comparing these combinatorial independence results with Gödel's incompleteness theorems we see an obvious difference: Gödel's independent sentences are Π_1, whereas the sentences in Corollary 14.17 are Π_2. Indeed, since Π_1 sentences are preserved downwards to initial segments, it is difficult to see how independent Π_1 sentences could be obtained by constructing initial segments. However, there is one important connection between Gödel's result and the Paris–Harrington result: the Paris–Harrington statement (PH) (equivalently (KM)) is provably equivalent in PA to the so-called *1-consistency* of PA, that is the statement that 'there is no proof of $0=1$ from axioms of PA and Π_1 sentences σ such that $\mathrm{Sat}_{\Pi_1}([\sigma])$'. See McAloon (1980) for details. Similarly (PH_{n+1}) and (KM_{n+1}) turn out to equivalent (in $I\Sigma_n$) to the 1-consistency of $I\Sigma_n$.

Exercises for Section 14.3

(These exercises are intended as a short guide to some of the literature on combinatorial statements independent of PA. They are probably too hard to be attempted directly.)

14.11 *Hercules and the hydra.* (Kirby and Paris 1982). A *hydra* is a finite tree, considered as a finite collection of straight-line *segments* joining two *nodes*, with a fixed node called the *root*. A *head* of the hydra is a node which is adjacent to exactly one segment, together with this unique segment. (See Fig. 6.)

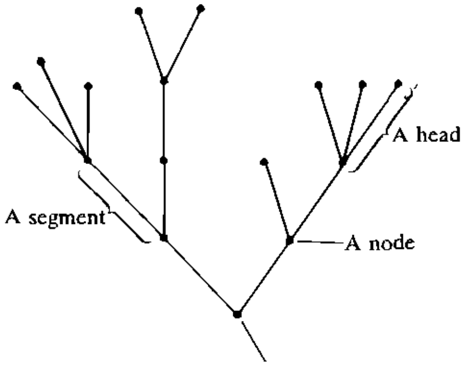

Fig. 6 A hydra.

In a battle between Hercules and a hydra, Hercules attempts to decapitate the hydra, reducing it to a single node. At stage n ($n \geqslant 1$), Hercules chops off one head from the hydra which then grows new heads in the following way: from the node that used to be attached to the head just chopped off, traverse one segment towards the root, and from this node just reached sprout n replicas of that part of the hydra above the segment just traversed.

For example, if at stage 3, Hercules removed the head indicated in Fig. 6, then the hydra would grow new heads becoming as shown in Fig. 7.

It is comparatively easy to show that Hercules can always win. In fact, if he continues to chop off heads for long enough, he can't help winning. Show that:

(a) for each hydra h there exists n such that whenever $h = h_0, h_1, \ldots, h_n$ are hydras, $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ are heads of these hydras (such that σ_i is a head of the hydra h_i, or $\sigma_i = h_i$ if h_i is the 'completely decapitated' hydra $\bullet$) and h_{i+1} is the result of Hercules chopping off head σ_i from the hydra h_i at stage $i+1$, then $h_i = \bullet$ for some $i \leqslant n$;

(b) the statement in (a) above is expressible in the language of arithmetic, $\mathscr{L}_A$, but not provable in PA.

(*Hint:* For (a), associate an ordinal $\alpha(h)$ with each hydra h and use induction on ordinals. See Fig. 8.)

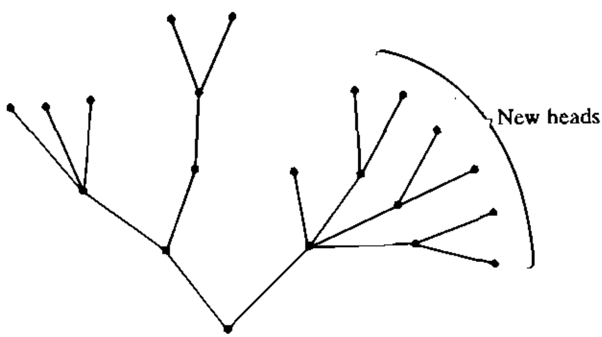

Fig. 7 The hydra of Fig. 6 after removing a head.

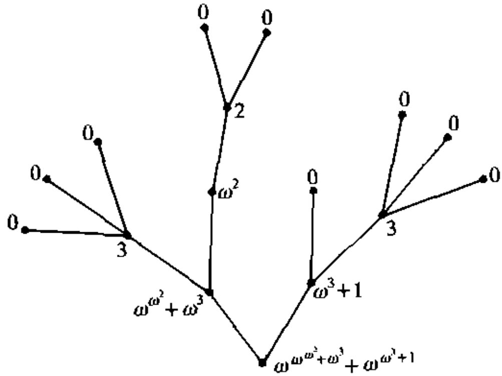

Fig. 8 Assigning ordinals to the nodes of a hydra.

14.12 *Goodstein sequences* (Goodstein, 1944; Kirby and Paris, 1982). See also Cichon (1983) and Buchholz and Wainer (1987). For each $m \geqslant 1$ and $n \geqslant 2$ we define the *pure base n representation of m* as follows.

First write m as a sum of powers of n (e.g. if $m=266$, $n=2$, we write $266=2^8+2^3+2^1$.) Now write each exponent as a sum of powers of n (for example, $266=2^{2^3}+2^{2^1}+2^1$) and so on until the representation stabilizes. The result is the pure base n representation of m. (For example, $266=2^{2^{2^1+1}}+2^{2^1+1}+2^1$ is the pure base 2 representation of 266.)

We now define $G_n(m)$ ($n \in \mathbb{N}$) as follows: $G_0(m)=m$. If $m=0$, $G_n(m)=0$, and if $m \neq 0$ then $G_n(m)$ is the result of writing m in its pure base n representation, replacing each n by $n+1$, and then subtracting one. The *Goodstein sequence for m starting at n* is the sequence

$$m_0=m, \quad m_1=G_n(m_0), \quad m_2=G_{n+1}(m_1), \quad m_3=G_{n+2}(m_2), \ldots .$$

For example the Goodstein sequence for 266 starting at 2 has its first terms equal to

$$266_0=2^{2^{2+1}}+2^{2+1}+2$$

$$266_1=3^{3^{3+1}}+3^{3+1}\approx 10^{38}$$

$$266_2=4^{4^{4+1}}+4^{4+1}\approx 10^{616}$$

$$266_3=5^{5^{5+1}}+5^{5+1}\approx 10^{10\,000}.$$

(a) Prove that for all $n, m \geqslant 1$ the Goodstein sequence for m starting at n eventually hits zero (Goodstein 1944).

(b) The statement 'for all m the Goodstein sequence for m starting at 2 eventually hits zero' is not provable in PA.

14.13 *The extended Grzegorczyk hierarchy*. (Löb and Wainer, 1970a, 1970b, 1971). See Buchholz and Wainer (1987) or Rose (1984) for good introductions.

ε_0 is the least ordinal such that $\omega^{\varepsilon_0}=\varepsilon_0$ (ordinal exponentiation). ε_0 may be defined by setting $\omega_n=\omega^{\omega_{n-1}}$ and $\omega_0=1$, and putting $\varepsilon_0=\bigcup_{n\in\omega}\omega_n$.

(a) Show that every $\alpha < \varepsilon_0$ can be written as

$$\alpha = \omega^\beta \cdot \gamma + \delta \; (\beta \leqslant \alpha, \gamma < \omega, \delta < \omega^\beta)$$

for some uniquely determined β, γ, δ. Hence deduce that each $0 < \alpha < \varepsilon_0$ has a unique *Cantor normal form*,

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_k}$$

where $\alpha > \beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_k$, $k = \mathbb{N}$. This process can be repeated, writing each non-zero β_i in Cantor normal form, and so on until the process stabilizes at the *pure Cantor normal form for α*. (This is obviously analogous to the pure base n representation of m in Exercise 14.12, and these notions are related: see the references by Buchholz and Wainer or by Cichon already given.)

(b) Define the *fundamental sequence* for each $\alpha < \varepsilon_0$, $\{\alpha\}(0)$, $\{\alpha\}(1), \ldots, \{\alpha\}(n), \ldots$ $(n \in \mathbb{N})$ by: $\{0\}(n) = 0$ for each n; and if $\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_k}$, $\alpha > \beta_1 \geqslant \cdots \geqslant \beta_k$, then

$$\{\alpha\}(n) = \omega^{\beta_1} + \cdots + \omega^{\beta_{k-1}} + \omega^\gamma \cdot n \quad \text{if } \beta = \gamma + 1 \text{ is a successor ordinal}$$

and

$$\{\alpha\}(n) = \omega^{\beta_1} + \cdots + \omega^{\beta_{k-1}} + \omega^{\{\beta_k\}(n)} \quad \text{if } \beta_k = 0 \text{ or } \beta_k \text{ is a limit ordinal.}$$

Show that, for all $\alpha < \varepsilon_0$, $\alpha = \bigcup_{n \in \mathbb{N}} \{\alpha\}(n)$.

(c) Define the *extended Grzegorczyk hierarchy* of functions $F_\alpha \colon \mathbb{N} \to \mathbb{N}$ $(\alpha < \varepsilon_0)$ by

$$F_0(x) = x + 1$$

$$F_{\alpha+1}(x) = F_\alpha^{(x)}(x)$$

$$F_\lambda(x) = F_{\{\lambda\}(x)}(x), \text{ if } \lambda \text{ is a limit ordinal,}$$

where $F^{(k)}(x) = F \circ F \circ \cdots \circ F(x)$ $(k\ F\text{s})$. Show that F_α is provably recursive in PA for all $\alpha < \varepsilon_0$, and that

$$Y(a, b) = \max\{c \mid F_{\omega_c}(a) \leqslant b\}$$

is an indicator for initial segments satisfying PA. Hence deduce $F_{\varepsilon_0}(x) = F_{\omega_x}(x)$ is recursive but not provably recursive in PA.

14.14 *Largeness properties* (Ketonen and Solovay 1981). A set $S \subseteq \mathbb{N}$ is 0-large iff $\mathrm{card}(S) \geqslant 1$. For ordinals $\alpha < \varepsilon_0$, define

$$S \text{ is } (\alpha+1)\text{-large iff } S \setminus \{\min(S)\} \text{ is } \alpha\text{-large}$$

and

$$S \text{ is } \lambda\text{-large iff } S \setminus \{\min(S)\} \text{ is } \{\lambda\}(n)\text{-large}$$

for limit ordinals λ, where $n = \min(S \setminus \{\min(S)\})$.

(a) Show that $F_\alpha(x) =$ the least y s.t. $[x, y] = \{x, x+1, x+2, \ldots, y\}$ is ω^α-large (where F_α is as in Exercise 14.13).
(b) Show that

$$Y(a, b) = \max\{c \mid [a, b] \text{ is } \omega_c\text{-large}\}$$

is an indicator for initial segments satisfying PA.
(*Hint:* See Ketonen and Solovay (1981) or Paris (1980).)

15

Recursive saturation

Many a true word is spoken through false teeth.

Ronnie Barker

In Chapter 9 we described a suitable Gödel-numbering $\ulcorner\cdot\urcorner$ of terms and formulas of $\mathscr{L}_{\mathrm{A}}$, and developed these ideas far enough to define the *satisfaction* relations $\mathrm{Sat}_{\Sigma_n}(\ulcorner\varphi\urcorner, a)$ and $\mathrm{Sat}_{\Pi_n}(\ulcorner\varphi\urcorner, a)$. These were used in Chapters 10–14 to prove many results on the structure of nonstandard models of PA, in particular via the notion of Σ_n-recursive saturation (which was a consequence of overspill and the properties of the $\mathrm{Sat}_{\Sigma_n}(\ulcorner\varphi\urcorner, a)$ relation). Tarski's theorem on the undefinability of truth (Exercise 3.7) shows that there is no analogous definition in $\mathscr{L}_{\mathrm{A}}$ of satisfaction for *all* $\mathscr{L}_{\mathrm{A}}$-formulas. In this chapter we shall examine conditions under which it is possible to add a new binary relation $S(\ulcorner\varphi\urcorner, a)$ to a model $M \models PA$ with the intended meaning 'the (possibly nonstandard) formula φ is true at $a = [a_0, a_1, \ldots, a_{\mathrm{len}(a)-1}]$.' The axioms we shall use for '(M, S) thinks φ is true' will be based on Tarski's definition of truth, and such a relation $S \subseteq M^2$ will be called a *satisfaction class*. It will turn out that, if M is countable, M has such a satisfaction class if and only if M is recursively saturated (as defined in Chapter 11) but there are some surprises.

There are many reasons for studying satisfaction classes: they are a useful vehicle for understanding the strengths and limitations of Tarski's definition of truth; the notion of a *full inductive* satisfaction class leads naturally to a new theory of arithmetic extending PA in which the consistency of PA and the Paris–Harrington statement can be proved; and there are a good number of naturally occurring examples arising from the arithmetized completeness theorem or from nonstandard models of set theory (see Exercise 15.4). Another excellent reason is that the countable recursively saturated structures form a particularly well-behaved class of structures. Often we are interested in models as a tool to understand their first-order theories so, by the downward Löwenheim–Skolem theorem and the existence theorem for recursively saturated models (Proposition 11.4), we may restrict our attention in such cases to countable recursively saturated models. This combination of countability and recursive saturation will be exploited many times in Section 15.2 below to prove some beautiful theorems on the structure of these models, one highlight being the theorem on *resplendency* (Theorem 15.7). Satisfaction classes provide a powerful technique to describe the model theory of such structures.

15.1 SATISFACTION CLASSES

In this section we shall use the Gödel-numbering $\ulcorner\cdot\urcorner$ and syntactic operations on terms and formulas defined in Chapter 9. In particular recall that there are $\Delta_1(PA)$ formulas term(x) and form(x) of $\mathscr{L}_A$ expressing 'x is the Gödel-number of a term' and 'x is the Gödel-number of a formula'. Our object here is to study the effect of adding a new relation symbol $S(\ulcorner\varphi\urcorner, a)$ representing satisfaction for all formulas φ of $\mathscr{L}_A$. In contrast with $\mathrm{Sat}_{\Sigma_n}(\ulcorner\varphi\urcorner, a)$, we will put no restriction on the complexity of the formula φ (as measured by its position in the Σ_n/Π_n hierarchy).

EXAMPLE. Let $M \vDash PA$ and let

$$S_0 = \left\{ (\ulcorner\theta(\mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_n)\urcorner, a) \;\middle|\; \begin{array}{l} n, \ulcorner\theta\urcorner \in \mathbb{N}, a \in M, \text{and} \\ M \vDash \theta([a]_0, [a]_1, \ldots, [a]_n) \end{array} \right\}.$$

So S_0 is the set of all pairs $(\ulcorner\theta\urcorner, a)$ where θ is an $\mathscr{L}_A$-formula satisfied in M by the sequence coded by a. S_0 is called the standard satisfaction class for M, and we may expand M to the structure (M, S_0) for the language $\mathscr{L}_S = \mathscr{L}_A \cup \{S\}$, where S is a new binary relation symbol. (The historical reason for calling S_0 a *class* rather than a *set* is that S_0 is not definable in M: see Proposition 15.3 below.)

If $M \vDash PA$ is nonstandard, then we have a good notion of an $\mathscr{L}_A$-formula *in the sense of* M, namely an element $x \in M$ such that $M \vDash \mathrm{form}(x)$. From now on we shall use Greek letters φ, ψ, θ, ... to range over (possibly nonstandard) formulas in this sense, and we will identify formulas with their Gödel-numbers, dropping $\ulcorner\cdot\urcorner$ from the notation (even when these symbols are used to denote the application of a function defined on formulas or terms). It will always be clear from the context which model M we have in mind, so we will also drop references to M when talking about formulas in the sense of M. Can we define a good notion of truth for such non-standard formulas?

DEFINITION. Let $M \vDash PA$ be nonstandard, and suppose $S \subseteq M^2$ is a binary relation on M such that $(x, y) \in S \Rightarrow M \vDash \mathrm{form}(x)$. Then S is a *full satisfaction class* for M just in case that whenever $\varphi \in M \vDash \mathrm{form}(\varphi)$ and $a \in M$, we have $(\varphi, a) \in S$ iff one of the following holds in (M, S):

$T_1(\varphi, a)$: $\exists t, s(\mathrm{term}(t) \wedge \mathrm{term}(s) \wedge \varphi = (t = s) \wedge \mathrm{val}(t, a) = \mathrm{val}(s, a))$

$T_2(\varphi, a)$: $\exists t, s(\mathrm{term}(t) \wedge \mathrm{term}(s) \wedge \varphi = (t < s) \wedge \mathrm{val}(t, a) < \mathrm{val}(s, a))$

$T_3(\varphi, a)$: $\exists \psi_1, \psi_2(\mathrm{form}(\psi_1) \wedge \mathrm{form}(\psi_2) \wedge$
$$\varphi = (\psi_1 \wedge \psi_2) \wedge S(\psi_1, a) \wedge S(\psi_2, a))$$

$T_4(\varphi, a)$: $\exists \psi_1, \psi_2(\mathrm{form}(\psi_1) \wedge \mathrm{form}(\psi_2) \wedge$
$$\varphi = (\psi_1 \vee \psi_2) \wedge (S(\psi_1, a) \vee S(\psi_2, a)))$$

$T_5(\varphi, a)$: $\exists\psi(\text{form}(\psi)\wedge\varphi=\neg\psi\wedge\neg S(\psi, a))$

$T_6(\varphi, a)$: $\exists i, \psi(\text{form}(\psi)\wedge\varphi=\exists \mathrm{v}_i\psi\wedge\exists bS(\psi, a[b/i]))$

$T_7(\varphi, a)$: $\exists i, \psi(\text{form}(\psi)\wedge\varphi=\forall \mathrm{v}_i\psi\wedge\forall bS(\psi, a[b/i]))$.

S is a *partial non standard satisfaction class* for M iff there is $\alpha\in M\setminus\mathbb{N}$ such that whenever $\varphi\in M\vDash\text{form}(\varphi)$ and $a\in M$, then*

$$(\varphi, a)\in S \quad\text{iff}\quad \varphi<\alpha \quad\text{and}\quad (M, S)\vDash \bigvee\!\!\!\bigvee_{i=1}^{7} T_i(\varphi, a).$$

A satisfaction class S for M (either full or partial) is said to be *inductive* iff the expanded structure (M, S) satisfies all induction axioms $I_x\varphi$ for all (standard) formulas φ in the language $\mathscr{L}_S=\mathscr{L}_A\cup\{S\}$.

Notice that the formulas $T_1(\varphi, a), \ldots, T_7(\varphi, a)$ are formulas of $\mathscr{L}_S$. (They are just the usual clauses for Tarski's definition of truth.) Thus the class of all $\mathscr{L}_S$-structures (M, S) such that $M\vDash PA$ and S is a full satisfaction class for M is axiomatized by a first-order theory in $\mathscr{L}_S$.

Examples. (a) Let S_0 be the standard satisfaction class on the standard model $\mathbb{N}$. Then S_0 is a full inductive satisfaction class for $\mathbb{N}$. Any elementary extension $(M, S)\succ(\mathbb{N}, S_0)$ where M is a nonstandard model of $\text{Th}(\mathbb{N})$, will then have the property that S is a full inductive satisfaction class for M.

(b) Let M, N be nonstandard models of PA, with N strongly interpreted in M. (See Section 13.2; the existence of such N for an arbitrary M follows from Exercise 13.13.) Then N has a partial nonstandard satisfaction class. To see this, suppose for simplicity that M and N have the same domains, and that $f: M\to M$ is the definable function in M embedding M onto an initial segment of N (see Exercise 13.12). Now for all $a, i\in M$ define $g(a, i)$ using induction on i so that $c=g(a, i)$ satisfies

$$M\vDash \text{len}(c)=i\wedge\forall j<i\forall x, y(x=[c]_j\wedge y=f(j)$$
$$\rightarrow \text{Sat}_N(\theta(\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2), [a, y, x]))$$

where $\theta(\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2)$ is the $\mathscr{L}_A$-formula '$\mathrm{v}_2=[\mathrm{v}_0]_{\mathrm{v}_1}$'; that is in M, c codes the sequence

$$c_0, c_1, \ldots, c_{i-1}$$

where, in N, a codes the sequence

$$b_0, b_1, \ldots, b_{\text{len}(a)-1},$$

and $c_j=b_{f(j)}$ for all $j<i$. It is now almost immediate from the definitions of 'strongly

* There is an annoying technical problem here: for this definition to have the intended meaning we must assume that the Gödel-numbering of formulas has the property that whenever $\ulcorner\varphi\urcorner<\alpha$ and ψ is a subformula of φ then $\ulcorner\psi\urcorner<\alpha$. (It is easy to devise such a Gödel-numbering. See for example Exercise 9.3.) *We assume from now on that our Gödel-numbering has this property.* An alternative (and more general) approach would be to allow a partial nonstandard satisfaction class to be defined on a definable set of formulas closed under subformulas and containing all the standard formulas. We will not, however, need this extra generality in the sequel.

interprets' and 'partial satisfaction class' that if $a \in M$ is nonstandard

$$S = \left\{ (f(\varphi), a) \,\middle|\, \begin{array}{l} M \vDash \mathrm{form}(\varphi) \wedge \varphi < a \text{ and } M \vDash \mathrm{Sat}_N(\varphi, g(a, i)) \\ \text{for suitably large } i \in M \text{ (e.g., } i = \mathrm{len}(\varphi)) \end{array} \right\}$$

is a partial nonstandard satisfaction class for N. This is because all the syntactic operations are provably recursive in PA, so for example

$$N \vDash x = (f(\varphi) \wedge f(\psi)) \Leftrightarrow N \vDash x = f(\theta)$$

for all $x \in N$ and all $\varphi, \psi, \theta \in M \vDash \mathrm{form}(\varphi) \wedge \mathrm{form}(\psi)$ and $M \vDash \theta = (\varphi \wedge \psi)$, and also because we can prove (by induction in M) that

$$M \vDash \mathrm{val}_N(t, g(a, i)) = b \leftrightarrow \mathrm{Sat}_N(\chi(\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2), [f(t), a, b])$$

for all $a, b, t \in M \vDash \mathrm{term}(t)$ and all suitably large $i \in M$ (e.g. $i = \mathrm{len}(t)$), where $\chi(t, a, b)$ is the $\mathscr{L}_{\mathrm{A}}$-formula '$\mathrm{val}(t, a) = b$'.

(c) A further class of examples is provided by those models $M \vDash PA$ that occur as an 'element' of some nonstandard model of set theory $(V, \in)$. (See Exercise 15.4.)

These examples are intended to show that satisfaction classes occur in very natural ways (even if, for technical reasons, (b) above was rather complicated!) The next proposition provides some further examples.

PROPOSITON 15.1. Let $M \vDash PA$ be nonstandard. Then there exists $N > M$ with $\mathrm{card}(N) = \mathrm{card}(M)$ and $S \subseteq N^2$ such that S is a partial nonstandard inductive satisfaction class for N.

Proof. Let $\bar{M}$ be the structure $(M, a)_{a \in M}$ obtained by adding constant symbols for each $a \in M$, and let T be the following theory:

$\mathrm{Th}(\bar{M})$ + Induction for all (standard) formulas of $\mathscr{L}_{\mathrm{S}}$ + the scheme

$$\forall y (S(\varphi, y) \leftrightarrow \bigvee\!\!\!\bigvee_{i=1}^{7} T_i(\varphi, y)) \qquad (*)$$

for each standard $\mathscr{L}_{\mathrm{A}}$-formula φ.

We claim that T is consistent.

If T' is a finite subtheory of T only involving formulas θ of Gödel-number less than $n \in \mathbb{N}$ in the scheme (*), then we may define $S' \subseteq M^2$ by

$$(x, a) \in S' \Leftrightarrow \bigvee\!\!\!\bigvee_{\theta < n} x = \theta \wedge \theta([a]_0, [a]_1, \ldots, [a]_{i_\theta})$$

where the disjunction is over all formulas θ of $\mathscr{L}_{\mathrm{A}}$ with Gödel-number less than n, and i_θ is the maximum index i of any variable v_i occurring free in θ. Obviously S' is definable in M by an $\mathscr{L}_{\mathrm{A}}$-formula, so (M, S') satisfies all induction axioms for all formulas of $\mathscr{L}_{\mathrm{S}}$. Hence $(M, S') \vDash T'$, since we are assuming that the set of all formulas of Gödel-number less than a given $n \in \mathbb{N}$ is closed under subformulas. Thus by compactness T is consistent.

Now let $(N, S'') \models T$ with card $(N) = \text{card}(M)$. Since $T \vdash \text{Th}(\tilde{M})$ we may assume that $N > M$. We prove that N contains $\alpha > \mathbb{N}$ such that the restriction of S'' to all (φ, a) with $\varphi < \alpha$ is a partial nonstandard satisfaction class. Clearly

$$(N, S'') \models \forall \varphi, y(\text{form}(\varphi) \wedge \varphi < n \rightarrow (\bigvee\!\!\bigvee_{i=1}^{7} T_i(\varphi, y) \leftrightarrow S''(\varphi, y)))$$

for all $n \in \mathbb{N}$. But this is an $\mathcal{L}_S$-formula, so overspill for this formula is valid in N (since (N, S'') satisfies induction for all $\mathcal{L}_S$-formulas). So there is $\alpha > \mathbb{N}$ with

$$(N, S'') \models \forall \varphi, y(\text{form}(\varphi) \wedge \varphi < \alpha \rightarrow (\bigvee\!\!\bigvee_{i=1}^{7} T_i(\varphi, y) \leftrightarrow S''(\varphi, y)))$$

and $S = \{(\varphi, a) \mid \varphi < \alpha \;\&\; (\varphi, a) \in S''\}$ is definable in (N, S''), hence (N, S) satisfies all induction axioms too. Thus S is a partial nonstandard inductive satisfaction class for N. □

Later, we will prove a similar theorem for full satisfaction classes. But first we shall describe a few properties of satisfaction classes and in particular the connection with recursive saturation.

Lemma 15.2. Let $M \models PA$ be nonstandard, $S \subseteq M^2$ a partial (or full) nonstandard satisfaction class, and $\varphi(\mathsf{v}_0, \ldots, \mathsf{v}_n)$ a standard $\mathcal{L}_A$-formula in which only $\mathsf{v}_0, \ldots, \mathsf{v}_n$ appear free. Then for all $a \in M$

$$(M, S) \models S(\varphi, a) \leftrightarrow \varphi([a]_0, \ldots, [a]_n).$$

Proof. An easy induction on the length of standard $\mathcal{L}_A$-formulas φ using the definition of a satisfaction class and the properties of the val function in Proposition 9.10. □

Thus the 'definition of truth' for formulas given via a satisfaction class at least agrees with the usual one for standard formulas φ. It follows from Lemma 15.2 that

Proposition 15.3. If $M \models PA$ and $S \subseteq M^2$ is a partial (or full) nonstandard satisfaction class, then S is not definable in M by an $\mathcal{L}_A$-formula.

Proof. If '$(x, y) \in S$' is definable by the formula $\varphi(x, y, \bar{a})$ where $\bar{a} \in M$ is a tuple of fixed parameters, then for any standard formula ψ

$$M \models \psi(\bar{a}) \Leftrightarrow M \models \forall w(w = [\bar{a}] \rightarrow \varphi(\psi, w, \bar{a}))$$

contradicting Tarski's theorem on the undefinability of truth (Exercise 3.7). □

Now for the connection with saturation.

PROPOSITION 15.4. If $M \models PA$ is nonstandard and $S \subseteq M^2$ is a partial nonstandard inductive satisfaction class on M, then M is recursively saturated.

Proof. This is very similar to the proof of Theorem 11.5. Suppose $p(\bar{x})$ is a recursive type over M with parameters $\bar{a} \in M$. Then

$$A = \{\varphi(\bar{x}, \bar{y}) \mid \varphi(\bar{x}, \bar{a}) \in p(\bar{x})\} \subseteq \mathbb{N}$$

is coded in M by some $c \in M$ (using Lemma 11.1). Since $p(\bar{x})$ is finitely satisfied in M we have, for each $k \in \mathbb{N}$,

$$(M, S) \models \exists \bar{x} \forall y < k((c)_y \neq 0 \rightarrow S(y, [\bar{x}, \bar{a}])),$$

so by overspill (using here that (M, S) satisfies induction for $\mathscr{L}_S$ formulas) there is $\beta > \mathbb{N}$ such that

$$(M, S) \models \exists \bar{x} \forall y < \beta((c)_y \neq 0 \rightarrow S(y, [\bar{b}, \bar{a}])).$$

Lemma 15.2 now shows that any $\bar{b} \in M$ satisfying $\forall y < \beta((c)_y \neq 0 \rightarrow S(y, [\bar{b}, \bar{a}]))$ satisfies the type $p(\bar{x})$, as required. □

Propositions 15.1 and 15.4 give an alternative proof of Proposition 11.4, at least for models of PA. 15.4 is central to this chapter: in the next section we shall consider the notion of *resplendency* and prove a converse to 15.4 for countable models. The first of the two main theorems of this section is Lachlan's remarkable result that 15.4 is true *even if the partial nonstandard satisfaction class S is not inductive*. It is completely unexpected that 15.4 can be proved without using overspill in the expanded language $\mathscr{L}_S$. (By way of comparison, note that $\mathrm{Sat}_{\Sigma_n}(x, y)$ has all the analogous Tarski-conditions provable in $I\Sigma_1$, yet models of $I\Sigma_1$ need not be Σ_n-recursively saturated for $n \geqslant 2$.) Here, then, is the surprising theorem and its proof.

THEOREM 15.5 (Lachlan 1981). Let $M \models PA$ be nonstandard and let $S \subseteq M^2$ be a partial nonstandard satisfaction class for M. Then M is recursively saturated.

Proof. Let $p(x)$ be a recursive finitely satisfied type over M that is not realized in M. (We will assume here that x is a single variable, using the pairing function if necessary to replace each $\varphi(\bar{x})$ in $p(\bar{x})$ by $\exists\bar{y}(\langle\bar{y}\rangle = x \wedge \varphi(\bar{y}))$. Finitely many parameters $\bar{a} \in M$ may occur in $p(x)$; these will be omitted from the notation that follows.) We write $p(x)$ as

$$\{\varphi_i(x) \mid i \in \mathbb{N}\}$$

and shall suppose that the function $f: i \mapsto \varphi_i(x)$ is recursive with recursive image. (For example if $p(x)$ is a recursive set, we can define $f(x)$ using the Church–Turing thesis by

$$\begin{aligned} f(i) = {} & \text{the least Gödel-number of a formula } \theta(x) \\ & \text{in } p(x) \text{ that is greater than } f(i-1) \text{ (if } i>0).) \end{aligned}$$

For each $i \in \mathbb{N}$ let

$$A_i = \{x \in M \mid M \vDash \varphi_i(x)\}.$$

Without loss of generality we may assume that $A_0 = M \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ and $A_i \neq A_{i+1}$ for each i; for otherwise we may replace φ_i by φ_i' $(i \in \mathbb{N})$ where

$$\varphi_0'(x) \quad \text{is} \quad x = x$$

and

$$\varphi_{i+1}'(x) \quad \text{is} \quad \varphi_i'(x) \wedge \varphi_i(x) \wedge \exists y < x \varphi_i'(y).$$

(Note that each $D_i = \{b \in M \mid M \vDash \bigwedge_{j \leqslant i} \varphi_j(b)\}$ must be an infinite set, since $p(x)$ is finitely satisfied, but not realized in M. Putting $A_i' = \{b \in M \mid M \vDash \varphi_i'(x)\}$ we see that the effect of the above definition is to make $A_0' = M$ and $A_{i+1}' = D_i \cap (A_i' \setminus \min(A_i'))$, so $M = A_0' \supseteq A_1' \supseteq A_2' \supseteq \cdots$ is a properly descending sequence with $A_{i+1}' \subseteq D_i$ for each i, and each set $D_i \setminus A_{i+1}'$ is *finite*. Thus each A_i' is non-empty with $\bigcap_{i \in \mathbb{N}} A_i' \subseteq \bigcap_{i \in \mathbb{N}} D_i = \varnothing$. Notice also that the function $i \mapsto \varphi_i'(x)$ is a recursive function with recursive image, since $i \mapsto \varphi_i(x)$ was. This justifies us assuming that $A_0 = M \supseteq A_1 \supseteq A_2 \supseteq \cdots$ is a proper descending sequence, where A_i is defined by $\varphi_i(x)$, $\bigcap_{i \in \mathbb{N}} A_i = \varnothing$, and the sequence of formulas $\{\varphi_i \mid i \in \mathbb{N}\}$ is recursive.) Putting $B_0 = \varnothing$ and $B_{i+1} = A_i \setminus A_{i+1}$ we obtain a partition $\{B_i \mid i \in \mathbb{N} \;\&\; i \geqslant 1\}$ of M defined by a recursive sequence of formulas $\{\theta_i \mid i \in \mathbb{N}\}$ where $\theta_{i+1}(x)$ is $(\varphi_i(x) \wedge \neg\varphi_{i+1}(x))$ and $\theta_0(x)$ is $\neg(x = x)$.

The idea now is to define a third sequence of subsets of M. $\{C_i \mid i \in \mathbb{N}\}$ as follows:

$$C_0 = \varnothing;$$

$$C_{i+1} = \begin{cases} B_1 & \text{if } C_i = \varnothing; \\ B_{j+1} & \text{if } j \text{ is least such that } B_j \cap C_i \neq \varnothing; \\ \varnothing & \text{if } C_i \neq \varnothing \text{ but no such } j \text{ exists.} \end{cases}$$

The reader may be wondering what's going on here, since it is easy to check that $C_i = B_i$ for each $i \in \mathbb{N}$, and the third clause '$C_{i+1} = \varnothing$' will never happen. But we will apply this definition to C_i for *nonstandard* $i \in M$, using the satisfaction class. Since $\{B_i | i \in \mathbb{N}, i \geqslant 1\}$ is a partition of M and $B_0 = \varnothing$ it will always be the case that $C_i = B_j$ for some $j \in \mathbb{N}$, *even if i happens to be nonstandard.* The definition of the C_is is arranged so that if $C_i = B_j$ then $C_{i+1} = B_{j+1}$, thus we will be able to define an increasing function f from a nonstandard initial segment of M to $\mathbb{N}$ by: $f(i) =$ the least j such that $C_i = B_j$. It will then follow that *if i is nonstandard* $f(i) > f(i-1) > f(i-2) > \cdots$, an infinite decreasing sequence of natural numbers, and we thus obtain our contradiction.

Obviously, the trick is to define the sets C_i correctly. First of all, using the fact that the sequence $\{\theta_i | i \in \mathbb{N}\}$ is given by a recursive function, by Lemma 11.3 there is $b \in M$ coding the sequence $\theta_0(\mathsf{v}_0), \theta_1(\mathsf{v}_0), \ldots$, so that $\theta_i(\mathsf{v}_0) = (b)_i$ for all $i \in \mathbb{N}$. We choose $\nu \in M \setminus \mathbb{N}$, a sufficiently small non-standard number, so that

$$M \vDash \forall i \leqslant \nu \operatorname{form}((b)_i) \qquad (+)$$

and that all the formulas that we consider later (including $(b)_i$ for $i \leqslant \nu$) are sufficiently small compared with the upper bound α for the formulas that are handled by the satisfaction class. (If the satisfaction class is a *full* satisfaction class, then $(+)$ is the only restriction on ν. Otherwise the precise meaning of 'small enough' will be explained soon.) For simplicity we shall write $(b)_i$ as θ_i even if $i \leqslant \nu$ is nonstandard, and we shall define B_i by

$$B_i = \{x \in M \,|\, (M, S) \vDash S(\theta_i, [x])\}$$

for all i. (The parameters, if there were any in the type $p(x)$, would be included in the square brackets here with 'x', replacing $[x]$ by $[x, \bar{a}]$, say.) Lemma 15.2 shows that this new definition of the B_is agrees with the old one in the cases when i is a standard integer.

We are now able to give the precise definition of the C_is: we set $C_i = \{x \in M \,|\, (M, S) \vDash S(\gamma_i, [x])\}$ where the γ_is are the following formulas:

$$\gamma_0(\mathsf{v}_0) \quad \text{is just} \quad \neg(\mathsf{v}_0 = \mathsf{v}_0)$$

(so $C_0 = \varnothing$), and, assuming that γ_i has been defined for $i < \nu$ we define γ_{i+1} by setting

$$\begin{aligned}
\delta_1 &= \neg\exists y \gamma_i(y) \\
\delta_2 &= \exists y(\gamma_i(y) \wedge \theta_1(y)) \\
&\ \vdots \\
\delta_{j+1} &= \exists y(\gamma_i(y) \wedge \theta_j(y)) \quad (1 \leqslant j < \nu)
\end{aligned}$$

and then putting $\gamma_{i+1}(\mathrm{v}_0)$ equal to

$$
\begin{aligned}
&(\delta_1 \wedge \theta_1(\mathrm{v}_0)) \vee \\
&(\neg\delta_1 \wedge ((\delta_2 \wedge \theta_2(\mathrm{v}_0)) \vee \\
&\qquad (\neg\delta_2 \wedge ((\delta_3 \wedge \theta_3(\mathrm{v}_0)) \vee \\
&\qquad\qquad (\neg\delta_3 \wedge (\cdots \\
&\qquad\qquad\qquad \cdots \\
&\qquad\qquad\qquad\qquad \cdots \\
&\qquad\qquad\qquad\qquad\qquad ((\delta_\nu \wedge \theta_\nu(\mathrm{v}_0)) \vee \\
&\qquad\qquad\qquad\qquad\qquad (\neg\delta_\nu \wedge \neg(\mathrm{v}_0 = \mathrm{v}_0))) \cdots)))).
\end{aligned}
$$

(The reader should compare this with the previous informal definition of the C_is given above.) More formally $\gamma_{i+1}(\mathrm{v}_0) = F(\gamma_i(\mathrm{v}_0), b, \nu)$, where F is some suitable primitive recursive function corresponding to the formula above. (F will therefore be provably recursive in PA.) Similarly $\gamma_i(\mathrm{v}_0) = G(i, b, \nu)$ for a second primitive recursive function G defined using F. (We now see precisely how to choose ν: ν is obtained using overspill so that

$$M \vDash \forall i \leqslant \nu(\mathrm{form}((b)_i) \wedge \mathrm{form}(G(i, b, \nu)) \wedge (b)_i < \alpha \wedge G(i, b, \nu) < \alpha)$$

where α is the hound on the size of formulas handled by the partial nonstandard satisfaction class. It is easy to check that this is true for all $\nu \in \mathbb{N}$ since $G(i, b, \nu)$ only depends on i, ν and $(b)_0, \ldots, (b)_\nu$.)

To complete the proof we must show that: (a) for all $i \leqslant \nu$, $C_i = B_j$ for some $j \in \mathbb{N}$; (b) if $C_i = B_j$ then $C_{i+1} = B_{j+1}$; and (c) $C_i \neq \varnothing$ for all $i > 0$.

To prove this, note first that if $C_i = \varnothing$ then $C_i = B_0$. Also if $C_i = \varnothing$ and $i < \nu$, then $C_{i+1} = B_1$ since

$$
\begin{aligned}
& & &(M, S) \vDash \neg\exists y S(\gamma_i, [y]), \\
&\text{so} & &(M, S) \vDash \forall x \neg S(\exists y \gamma_i(y), [x]), \quad \text{by } T_6 \\
&\text{so} & &(M, S) \vDash \forall x S(\neg\exists y \gamma_i(y), [x]), \quad \text{by } T_5 \\
&\text{i.e.} & &(M, S) \vDash S(\delta_1, [x]), \quad \text{for any } x \in M.
\end{aligned}
$$

Hence, for any $x \in M$, $x \in C_{i+1}$ holds iff

$$(M, S) \vDash S((\delta_1 \wedge \theta_1(\mathrm{v}_0)) \vee (\neg\delta_1 \wedge \psi(\mathrm{v}_0)), [x])$$

(for some suitable subformula $\psi(x)$ of γ_{i+1}), which holds

$$
\begin{aligned}
&\text{iff} \quad (M, S) \vDash S((\delta_1 \wedge \theta_1(\mathrm{v}_0)), [x]) \vee S((\neg\delta_1 \wedge \psi(\mathrm{v}_0)), [x]), \quad \text{by } T_4, \\
&\text{iff} \quad (M, S) \vDash (S(\delta_1, [x]) \wedge S(\theta_1, [x])) \vee (S(\neg\delta_1, [x]) \wedge S(\psi, [x])), \quad \text{by } T_3 \\
&\text{iff} \quad (M, S) \vDash (S(\delta_1, [x]) \wedge S(\theta_1, [x])) \vee (\neg S(\delta_1, [x]) \wedge S(\psi, [x])), \quad \text{by } T_5.
\end{aligned}
$$

But $(M, S) \models \forall x S(\delta_1, [x])$ so $x \in C_{i+1}$ iff $(M, S) \models S(\theta_1, [x])$, i.e. $C_{i+1} = B_1$, as claimed.

Now suppose $C_i \neq \emptyset$ and $i < \nu$. Since $\{B_j | j \geq 1, j \in \mathbb{N}\}$ is a partition of M there is $j \in \mathbb{N}$ such that $C_i \cap B_j \neq \emptyset$. Choose the least such j. We claim that $C_{i+1} = B_{j+1}$. This will complete the proof of (a), (b), and (c), since obviously $C_0 = B_0$ and, if $C_i = B_j \neq \emptyset$ with $j \in \mathbb{N}$, then

$$C_i \cap B_0 = C_i \cap B_1 = \cdots = C_i \cap B_{j-1} = \emptyset,$$

so (a) and (b) would be proved. Also, if $i > 0$ and $C_i = \emptyset$ then $C_{i-1} \neq \emptyset$ (since otherwise $C_i = B_1 \neq \emptyset$ by the above) and hence we would have $C_i = B_{j+1}$ where $j \in \mathbb{N}$ is least such that $C_{i-1} \cap B_j \neq \emptyset$. Hence we would have proved (c) too.

To prove the claim of the last paragraph, since $C_i \neq \emptyset$ and using T_5, T_6 we have

$$(M, S) \models \forall x S(\neg \delta_1, [x]),$$

and since $C_i \cap B_k = \emptyset$ for all $1 \leq k < j$, using T_3, T_5, and T_6 we have

$$(M, S) \models \forall x S(\neg \delta_{k+1}, [x])$$

for all such k. On the other hand $C_i \cap B_j \neq \emptyset$ so by T_3, T_5 we have

$$(M, S) \models \forall x S(\delta_{j+1}, [x]).$$

Thus for some suitable subformula $\psi(\mathsf{v}_0)$ of $\gamma_{i+1}(\mathsf{v}_0)$ we have for all $x \in M$

$$x \in C_{i+1} \quad \text{iff} \quad (M, S) \models S(\chi(\mathsf{v}_0), [x]),$$

where $\chi(\mathsf{v}_0)$ is the formula

$$\begin{aligned}
&((\delta_1 \wedge \theta_1(\mathsf{v}_0)) \vee \\
&\ (\neg\delta_1 \wedge ((\delta_2 \wedge \theta_2(\mathsf{v}_0)) \vee \\
&\qquad (\neg\delta_2 \wedge (\cdots \\
&\qquad\qquad (\cdots \\
&\qquad\qquad\qquad \cdots \\
&\qquad\qquad\qquad\quad \wedge((\delta_{j+1} \wedge \theta_{j+1}(\mathsf{v}_0)) \vee \\
&\qquad\qquad\qquad\qquad (\neg\delta_{j+1} \wedge \psi(\mathsf{v}_0)))\cdots).
\end{aligned}$$

By repeated applications of T_3, T_4, and T_5 this reduces to

$$x \in C_{i+1} \quad \text{iff} \quad (M, S) \models S(\theta_{j+1}(\mathsf{v}_0), [x]);$$

hence $C_{i+1}=B_{j+1}$, as required.

Finally, defining $f(i)=$ the unique $j\in\mathbb{N}$ such that $C_i=B_j$, we obtain an infinite descending sequence

$$f(i)>f(i-1)>f(i-2)>\cdots$$

of natural numbers, for each nonstandard i, since no C_i is empty for $i>0$, our contradiction. □

We turn now to *full* satisfaction classes. Our goal is to prove a strong converse to Theorem 15.5 for countable models of PA, namely that every countable recursively saturated $M\vDash PA$ has a full satisfaction class. It also follows from this that any $M\vDash PA$ (of any cardinality) has $N>M$ with $\text{card}(N)=\text{card}(M)$ and N having a full satisfaction class. Notice first that these results need not hold for full *inductive* satisfaction classes, since if $M\vDash PA$ has such a satisfaction class then $M\vDash\text{Con}(PA)$ (and recursively saturated models of $PA+\neg\text{Con}(PA)$ exist by Gödel's second incompleteness theorem (Exercise 3.8), the completeness theorem, and Proposition 11.4). That such an M satisfies $\text{Con}(PA)$ is proved by induction on x in the $\mathscr{L}_S$-formula

$$\forall p<x\forall\theta<x\,(\text{'}p\text{ is a proof of }\theta\text{ in }PA\text{'}\rightarrow\forall aS(\theta,a))$$

using some suitable formalization of any standard notion of 'proof in PA'. Thus (M,S) thinks that 'if θ is provable in PA then θ is true'. However, if $\neg\text{Con}(PA)$ then $M\vDash$ 'θ is provable in PA' and $M\vDash$ '$\neg\theta$ is provable in PA' for any θ, whereas, by $T_5(\neg\theta,[\,])$ it is impossible that both $(\theta,[\,])$ and $(\neg\theta,[\,])$ are in S.

THEOREM 15.6 (Kotlarski *et al.* 1981). Let $M\vDash PA$ be countable and recursively saturated. Then there is a full satisfaction class $S\subseteq M^2$ for M.

Proof. We first need some technical definitions.

$\mathscr{L}_A(M)$ denotes the first-order language $\mathscr{L}_A\cup M$, i.e., $\mathscr{L}_A$ together with a new constant symbol for each $a\in M$. ${}^*\mathscr{L}_A(M)$ denotes the set of $\mathscr{L}_A$-formulas *in the sense of* M, i.e., the set of $\theta\in M$ such that $M\vDash\text{form}(\theta)$. We can regard $\mathscr{L}_A(M)\subseteq{}^*\mathscr{L}_A(M)$ by identifying each $a\in M$ with its *canonical term* $\text{clterm}(a)=\ulcorner 0\urcorner$, if $a=0$, or $\ulcorner(\cdots(1+1)+\cdots+1)\urcorner$ (a 1s) otherwise. (Exercise: show that the function $\text{clterm}(x)$ is provably recursive in PA and hence well-defined in M.) We shall construct a maximal set S_∞ of

${}^*\mathscr{L}_{\mathrm{A}}(M)$-sentences consistent in a certain sense. By the construction of S_∞ it will be clear that

$$S=\left\{(\varphi(\mathsf{v}_0,\ldots,\mathsf{v}_i),[a_0,\ldots,a_i])\,\middle|\, \begin{array}{l} i\in M,\\ \varphi(\mathrm{clterm}(a_0),\ldots,\mathrm{clterm}(a_i))\in S_\infty \end{array}\right\}$$

is a full satisfaction class for M.

The consistency notion we shall use is based on the idea of a *finite approximation* to a ${}^*\mathscr{L}_{\mathrm{A}}(M)$-formula. Roughly, a finite approximation to $\varphi\in{}^*\mathscr{L}_{\mathrm{A}}(M)$ is a formula obtained from φ by replacing some subformulas $\psi(\mathsf{v}_0,\ldots,\mathsf{v}_j)$ of φ by $R(\mathsf{v}_0,\ldots,\mathsf{v}_j)$, where R is a new relation symbol, and some subterms $t(\mathsf{v}_1,\ldots,\mathsf{v}_j)$ by $f(\mathsf{v}_0,\ldots,\mathsf{v}_j)$, where f is a new function symbol, obtaining a formula $\hat{\varphi}$ of 'finite length'. More precisely, let $\mathscr{L}_{\mathrm{FA}}$ be the language consisting of relation symbols $R_{\chi(\bar{\mathsf{v}})}(\bar{\mathsf{v}})$ for every $\chi(\bar{\mathsf{v}})\in{}^*\mathscr{L}_A(M)$ with (actually-) finite number of free-variables $\bar{\mathsf{v}}$, and function symbols $f_{t(\bar{\mathsf{v}})}(\bar{\mathsf{v}})$ for each term $t(\bar{\mathsf{v}})$ of ${}^*\mathscr{L}_{\mathrm{A}}(M)$ with (again) only finitely many free-variables $\bar{\mathsf{v}}$. Here, the arity of $R_{\chi(\bar{\mathsf{v}})}$ or $f_{t(\bar{\mathsf{v}})}$ is equal to the number of free-variables $\bar{\mathsf{v}}$ in $\chi(\bar{\mathsf{v}})$ or $t(\bar{\mathsf{v}})$ respectively, and we allow 0-ary function symbols and relation symbols. As well as these nonlogical symbols, $\mathscr{L}_{\mathrm{FA}}$ has the usual symbols $(\;)\ \wedge\ \vee\ \exists\ \forall\ =$ and variables v_i for all $i\in M$.

We define the relation '$\hat{\theta}$ is a finite approximation to θ' to be the least relation between terms and formulas $\hat{\theta}$ of $\mathscr{L}_{\mathrm{FA}}$ and terms and formulas θ of ${}^*\mathscr{L}_{\mathrm{A}}(M)$ satisfying:

(a) v_i is a finite approximation to v_i, for all variables $\mathsf{v}_i (i\in M)$;

(b) f_0, f_1 are, respectively, finite approximations to the closed terms $0, 1$ of ${}^*\mathscr{L}_{\mathrm{A}}(M)$; more generally, if $t(\mathsf{v}_{i_0},\mathsf{v}_{i_1},\ldots,\mathsf{v}_{i_{l-1}})$ is a term of ${}^*\mathscr{L}_{\mathrm{A}}(M)$ with exactly the free-variables shown, $l\in\mathbb{N}$, and $i_0<i_1<\cdots<i_{l-1}\in M$, then $f_{t(\mathsf{v}_{i_0},\ldots,\mathsf{v}_{i_{l-1}})}(\mathsf{v}_{i_0},\mathsf{v}_{i_1},\ldots,\mathsf{v}_{i_{l-1}})$ is a finite approximation to $t(\mathsf{v}_{i_0},\ldots,\mathsf{v}_{i_{l-1}})$;

(c) if $\hat{t}_1,\hat{t}_2$ are terms of $\mathscr{L}_{\mathrm{FA}}$ and are finite approximations to t_1,t_2, then $f_{(\mathsf{v}_0+\mathsf{v}_1)}(\hat{t}_1,\hat{t}_2)$ is a finite approximation to (t_1+t_2), and $f_{(\mathsf{v}_0\cdot\mathsf{v}_1)}(\hat{t}_1,\hat{t}_2)$ is a finite approximation to $(t_1\cdot t_2)$;

(d) if $\hat{t}_1$, $\hat{t}_2$ are $\mathscr{L}_{\mathrm{FA}}$-terms and are finite approximations to the ${}^*\mathscr{L}_{\mathrm{A}}(M)$-terms t_1,t_2, then $(\hat{t}_1=\hat{t}_2)$ is a finite approximation to $(t_1=t_2)$ and $R_{(\mathsf{v}_0<\mathsf{v}_1)}(\hat{t}_1,\hat{t}_2)$ is a finite approximation to the formula $(t_1<t_2)$;

(e) if $\chi(\mathsf{v}_{i_0},\ldots,\mathsf{v}_{i_{l-1}})$ is an ${}^*\mathscr{L}_{\mathrm{A}}(M)$-formula with exactly the free-variables shown, $l\in\mathbb{N}$, and $i_0<i_1<\cdots<i_{l-1}\in M$, then $R_{\chi(\mathsf{v}_{i_0},\ldots,\mathsf{v}_{i_{l-1}})}(\mathsf{v}_{i_0},\ldots,\mathsf{v}_{i_{l-1}})$ is a finite approximation to $\chi(\mathsf{v}_{i_0},\ldots,\mathsf{v}_{i_{l-1}})$;

(f) if $\hat{\varphi}_1,\hat{\varphi}_2$ are $\mathscr{L}_{\mathrm{FA}}$-formulas and are finite approximations to the ${}^*\mathscr{L}_{\mathrm{A}}(M)$-formulas φ_1,φ_2 respectively, then $(\hat{\varphi}_1\wedge\hat{\varphi}_2)$ is a finite approximation to $(\varphi_1\wedge\varphi_2)$, $(\hat{\varphi}_1\vee\hat{\varphi}_2)$ is a finite approximation to $(\varphi_1\vee\varphi_2)$, $\neg\hat{\varphi}_1$ is a finite approximation to $\neg\varphi_1$, and $\exists\mathsf{v}_i\hat{\varphi}_1$, $\forall\mathsf{v}_i\hat{\varphi}_1$ are finite approximations to $\exists\mathsf{v}_i\varphi_1$, $\forall\mathsf{v}_i\varphi_1$, respectively, whenever $i\in M$.

Table 3 Refining a finite approximation.

if f or R is ...	replace f or R with ...
f_0	f_0 i.e. leave unchanged
f_1	f_1
$f_{\mathsf{v}_i}(t)$	t
$f_{(t_1+t_2)}(\cdots)$	$f_{(\mathsf{v}_0+\mathsf{v}_1)}(f_{t_1}(\cdots), f_{t_2}(\cdots))$
$f_{(t_1\cdot t_2)}(\cdots)$	$f_{(\mathsf{v}_0\cdot \mathsf{v}_1)}(f_{t_1}(\cdots), f_{t_2}(\cdots))$
$R_{(t_1=t_2)}(\cdots)$	$f_{t_1}(\cdots)=f_{t_2}(\cdots)$
$R_{(t_1<t_2)}(\cdots)$	$R_{(\mathsf{v}_0<\mathsf{v}_1)}(f_{t_1}(\cdots), f_{t_2}(\cdots))$
$R_{(\varphi_1\wedge\varphi_2)}(\cdots)$	$(R_{\varphi_1}(\cdots)\wedge R_{\varphi_2}(\cdots))$
$R_{(\varphi_1\vee\varphi_2)}(\cdots)$	$(R_{\varphi_1}(\cdots)\vee R_{\varphi_2}(\cdots))$
$R_{\neg\varphi}(\cdots)$	$\neg R_{\varphi}(\cdots)$
$R_{\exists \mathsf{v}_i\varphi}(\cdots)$	$\exists \mathsf{v}_i R_{\varphi}(\cdots, \mathsf{v}_i, \cdots)$
$R_{\forall \mathsf{v}_i\varphi}(\cdots)$	$\forall \mathsf{v}_i R_{\varphi}(\cdots, \mathsf{v}_i, \cdots)$

For example, the formula

$$((\cdots(\overbrace{\neg(\mathsf{v}_0=\mathsf{v}_0)\vee\neg(\mathsf{v}_0=\mathsf{v}_0))\vee\cdots)\vee\neg(\mathsf{v}_0=\mathsf{v}_0)}^{a}))$$

where there are a disjuncts, a being a non-standard element of M, has amongst its finite approximations,

$$((\cdots(R(\mathsf{v}_0)\vee\overbrace{\neg(\mathsf{v}_0=\mathsf{v}_0))\vee\cdots)\vee\neg(\mathsf{v}_0=\mathsf{v}_0)}^{n}))$$

for various $n\in\mathbb{N}$ and suitable relation symbols $R(\mathsf{v}_0)$.

It will also be useful to have a more 'concrete' characterization of finite approximations. To do this we define, for each $i\in\mathbb{N}$, each function symbol f of $\mathscr{L}_{\mathrm{FA}}$, and each relation symbol R of $\mathscr{L}_{\mathrm{FA}}$, operations $\mathscr{F}^i_f(\sigma)$ and $\mathscr{F}^i_R(\sigma)$ on terms or formulas σ of $\mathscr{L}_{\mathrm{FA}}$, such that $\mathscr{F}^i_f(\sigma)$, $\mathscr{F}^i_R(\sigma)=$ the result of replacing the ith occurrence in σ of the symbol f or R (respectively) according to Table 3.

In Table 3 the ellipsis $(\cdots)$ refers to a sequence of variables of $\mathscr{L}_{\mathrm{FA}}$: in the cases for $\forall$ and $\exists$ the variable v_i must be placed in its correct position so that other variables v_j with $j<i$ come before it and each v_j with $j>i$ come after it, as in part (e) of the definition above.

We also define $\mathscr{F}_f(\sigma)$ and $\mathscr{F}_R(\sigma)$ for each f, R to be $\mathscr{F}^0_f\circ\mathscr{F}^1_f\circ\cdots\circ\mathscr{F}^k_f(\sigma)$, $\mathscr{F}^0_R\circ\mathscr{F}^1_R\circ\cdots\circ\mathscr{F}^k_R(\sigma)$, respectively, for suitably large k so that *all* occurrences of f (respectively, R) are removed from σ.

With these definitions, if t is an $^*\mathscr{L}_{\mathrm{A}}(M)$-term with free-variables $\mathrm{v}_{i_0}, \ldots, \mathrm{v}_{i_{l-1}}$, $l \in \mathbb{N}$ and $i_0 < i_1 < \cdots < i_{l-1} \in M$, then $\hat{t}$ is a finite approximation to t iff t is the result of applying (actually-) finitely many operations $\mathscr{F}^i_f$ for various f, i to $f_t(\mathrm{v}_{i_0}, \ldots, \mathrm{v}_{i_{l-1}})$. Similarly, if θ is an $^*\mathscr{L}_{\mathrm{A}}(M)$ formula with free-variables $\mathrm{v}_{i_0}, \ldots, \mathrm{v}_{i_{l-1}}$ as above, then $\hat{\theta}$ is a finite approximation to θ iff $\hat{\theta}$ is the result of applying (actually-) finitely many operations $\mathscr{F}^i_f$, $\mathscr{F}^i_R$ for various f, R, i to $R_\theta(\mathrm{v}_{i_0}, \ldots, \mathrm{v}_{i_{l-1}})$.

If Γ is a set of $^*\mathscr{L}_{\mathrm{A}}(M)$ formula and φ is an $^*\mathscr{L}_{\mathrm{A}}(M)$-formula, then we write $\Gamma \vdash_{\mathrm{FA}} \varphi$ to mean that there are $\theta_1, \ldots, \theta_n \in \Gamma$ and finite approximations $\hat{\theta}_1, \ldots, \hat{\theta}_n, \hat{\varphi}$ to $\theta_1, \ldots, \theta_n, \varphi$, such that $\hat{\theta}_1 + \cdots + \hat{\theta}_n \vdash \hat{\varphi}$. Γ is *FA-consistent* iff it is not the case that $\Gamma \vdash_{\mathrm{FA}} 0=1$. Γ is *M-consistent* iff $\Gamma \cup \mathrm{Th}(M, a)_{a \in M} \cup {}^*\mathrm{At}(M)$ is *FA*-consistent, where $\mathrm{Th}(M, a)_{a \in M}$ is the set of all standard $\mathscr{L}_{\mathrm{A}}(M)$-sentences true in M (considered as $^*\mathscr{L}_{\mathrm{A}}(M)$ sentences via canonical terms) and $^*\mathrm{At}(M)$ is the set of all (possibly non-standard) atomic $^*\mathscr{L}_{\mathrm{A}}(M)$ sentences that are true in M, i.e.

$$^*\mathrm{At}(M) = \left\{ (t_1 = t_2) \,\middle|\, \begin{array}{l} t_1, t_2 \in M \vDash \text{‘}t_1 \text{ and } t_2 \text{ are closed} \\ \text{terms and } \mathrm{val}(t_1) = \mathrm{val}(t_2)\text{’} \end{array} \right\}$$

$$\cup \left\{ (t_1 < t_2) \,\middle|\, \begin{array}{l} t_1, t_2 \in M \vDash \text{‘}t_1 \text{ and } t_2 \text{ are closed} \\ \text{terms and } \mathrm{val}(t_1) < \mathrm{val}(t_2)\text{’} \end{array} \right\}.$$

We shall construct a maximal M-consistent set of $^*\mathscr{L}_A(M)$-sentences S_∞ using the recursive saturation of M to ensure that

$$\text{if } \exists v \varphi(v) \in S_\infty \text{ then } \varphi(\mathrm{clterm}(a)) \in S_\infty \text{ for some } a \in M. \qquad (+)$$

It will follow from this property and maximality that the set $S \subseteq M^2$ as above will be a full satisfaction class for M.

The notion of provability, $\vdash_{\mathrm{FA}}$, has many pleasant properties inherited from provability in $\mathscr{L}_{\mathrm{FA}}$, and we explain some of these next. There are two key lemmas for understanding the connections between $\vdash$ and $\vdash_{\mathrm{FA}}$.

LEMMA A. Let p be a proof in $\mathscr{L}_{\mathrm{FA}}$ of the $\mathscr{L}_{\mathrm{FA}}$-sentence σ from assumptions $\tau_1, \tau_2, \ldots, \tau_n$. Let $\mathscr{F}_f(p)$ be the result of replacing each formula θ in p by $\mathscr{F}_f(\theta)$, and similarly let $\mathscr{F}_R(p)$ be the result of replacing each θ in p by $\mathscr{F}_R(\theta)$, where f, R are arbitrary function or relation symbols of $\mathscr{L}_{\mathrm{FA}}$. Then $\mathscr{F}_f(p)$ ($\mathscr{F}_R(p)$) are (respectively) proofs in $\mathscr{L}_{\mathrm{FA}}$ of $\mathscr{F}_f(\sigma)$ ($\mathscr{F}_R(\sigma)$) from $\mathscr{F}_f(\tau_1), \ldots, \mathscr{F}_f(\tau_n)$ ($\mathscr{F}_R(\tau_1), \ldots, \mathscr{F}_R(\tau_n)$).

Proof. Induction on the length of the proof p (left to the reader). □

LEMMA B. Let $\theta_1, \ldots, \theta_k$ be $^*\mathscr{L}_{\mathrm{A}}(M)$-formulas with finitely many free-variables each, and $k \in \mathbb{N}$. Also let $\varphi_1, \ldots, \varphi_k$ and $\psi_1, \ldots, \psi_k$ be

$\mathscr{L}_{\mathrm{FA}}$-formulas so that both φ_i and ψ_i are finite approximations to θ_i for each i. Then there are finitely many operations $\mathscr{F}_1, \ldots, \mathscr{F}_l$, each being $\mathscr{F}_f$ or $\mathscr{F}_R$ for some symbol f or R of $\mathscr{L}_{\mathrm{FA}}$, such that for each $1 \leqslant i \leqslant k$

$$\chi_i = \mathscr{F}_1 \circ \mathscr{F}_2 \circ \cdots \circ \mathscr{F}_l(\varphi_i) = \mathscr{F}_1 \circ \mathscr{F} \circ \cdots \circ \mathscr{F}_l(\psi_i).$$

Proof (sketch). This relies heavily on the *unique readability* of terms and formulas being provable in PA, and hence true in M (see Lemmas 9.2 and 9.6). Unique readability here says that if $\chi \in {}^*\mathscr{L}_{\mathrm{A}}(M)$ is a formula, then it cannot be both a conjunction $(\varphi \wedge \psi)$ and a negation $\neg\theta$, or indeed any other pair of different syntactic constructs, and if, say, it is a conjunction, $(\varphi \wedge \psi)$, then φ and ψ are uniquely determined by χ.

Now suppose $\theta_1, \ldots, \theta_k$, $\varphi_1, \ldots, \varphi_k$ and $\psi_1, \ldots, \psi_k$ are as in the statement of the lemma. Then by the alternative characterization of 'finite approximation' given above,

$$\varphi_i = \mathscr{G}_{i,0} \circ \cdots \circ \mathscr{G}_{i,j_i}(R_{\theta_i}(\cdots))$$
$$\psi_i = \mathscr{H}_{i,0} \circ \cdots \circ \mathscr{H}_{i,l_i}(R_{\theta_i}(\cdots))$$

for each i, where the $\mathscr{G}_{i,j}$ and the $\mathscr{H}_{i,l}$ are all $\mathscr{F}_R^m$ or $\mathscr{F}_f^m$ for $m \in \mathbb{N}$ and symbols R, f of $\mathscr{L}_{\mathrm{FA}}$. But, if $\mathscr{F}_1, \ldots, \mathscr{F}_l$ include all $\mathscr{F}_R$ or $\mathscr{F}_f$ such that R or f occur as a subscript to one of the $\mathscr{F}_R^m$ or $\mathscr{F}_f^m$ amongst the $\mathscr{G}_{i,j}$ and $\mathscr{H}_{i,l}$, and, if $\mathscr{F}_1, \ldots, \mathscr{F}_l$ is ordered so that, if $\mathscr{F}_j = \mathscr{F}_{R_\lambda}$ and $\mathscr{F}_k = \mathscr{F}_{R_\mu}$ where λ, μ are subformulas of some θ_i, then if λ is a subformula of μ, $j < k$; similarly also for terms t, s when $\mathscr{F}_j = \mathscr{F}_{f_t}$ and $\mathscr{F}_k = \mathscr{F}_{f_s}$. Then, for all $i \leqslant l$,

$$\begin{aligned}\chi_i &= \mathscr{F}_1 \circ \mathscr{F}_2 \circ \cdots \circ \mathscr{F}_l(\varphi_i)\\ &= \mathscr{F}_1 \circ \mathscr{F}_2 \circ \cdots \circ \mathscr{F}_l(R_{\theta_i}(\cdots))\\ &= \mathscr{F}_1 \circ \mathscr{F}_2 \circ \cdots \circ \mathscr{F}_l(\psi_i)\end{aligned}$$

by unique readability, as required. □

For example, from Lemmas A and B we have the *deduction* theorem for $\vdash_{\mathrm{FA}}$: for all Γ, φ, θ, $\Gamma, \varphi \vdash_{\mathrm{FA}} \theta$ holds iff $\Gamma \vdash_{\mathrm{FA}} (\neg\varphi \vee \theta)$ does. Left-to-right here here is immediate, following from the left-to-right direction in the deduction theorem for the predicate calculus for $\mathscr{L}_{\mathrm{FA}}$. To see the right-to-left implication holds, suppose $\gamma_1, \ldots, \gamma_k \in \Gamma$, $\hat{\gamma}, \ldots, \hat{\gamma}_k$ are finite approximations to $\gamma_1, \ldots, \gamma_k$, ψ is a finite approximation to $(\neg\varphi \vee \theta)$, and p is a proof of ψ from $\hat{\gamma}_1, \ldots, \hat{\gamma}_k$ in the predicate calculus for $\mathscr{L}_{\mathrm{FA}}$. Then, if $\mathscr{F}$ is $\mathscr{F}_{R_{\neg\varphi}} \circ \mathscr{F}_{R_{(\neg\varphi \vee \theta)}}$, by Lemma A $\mathscr{F}(p)$ is a proof of $\mathscr{F}(\psi)$ from $\mathscr{F}(\hat{\gamma}_1), \ldots, \mathscr{F}(\hat{\gamma}_k)$. But also $\mathscr{F}(\psi)$ is $(\neg\hat{\varphi} \vee \hat{\theta})$ for some finite approximations $\hat{\varphi}$, $\hat{\theta}$ of φ, θ respectively, and $\mathscr{F}(\hat{\gamma}_1), \ldots, \mathscr{F}(\hat{\gamma}_k)$ are also finite approximations to $\gamma_1, \ldots, \gamma_k$. Therefore by the deduction theorem for $\mathscr{L}_{\mathrm{FA}}$ there is a proof of $\hat{\theta}$ from $\mathscr{F}(\hat{\gamma}_1), \ldots, \mathscr{F}(\hat{\gamma}_k)$, $\hat{\varphi}$, as required.

A slightly harder example on the same lines is the *cut rule* for $\vdash_{\mathrm{FA}}$, that is: if $\Gamma, \neg\varphi \vdash_{\mathrm{FA}} \theta$ and $\Gamma, \varphi \vdash_{\mathrm{FA}} \theta$ then $\Gamma \vdash_{\mathrm{FA}} \theta$. To see this, suppose $\gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_l \in \Gamma$ and there are finite approximations $\hat{\gamma}_1, \ldots, \hat{\gamma}_k, \hat{\delta}_1, \ldots, \hat{\delta}_l$ of $\gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_l$ respectively, ψ_1, ψ_2 of $\neg\varphi, \varphi$ respectively, and θ_1, θ_2 of θ, with proofs p, q showing that

$$\hat{\gamma}_1, \ldots, \hat{\gamma}_k, \psi_1 \vdash \theta_1$$

and

$$\hat{\delta}_1, \ldots, \hat{\delta}_l, \psi_2 \vdash \theta_2.$$

Then, by Lemma B, there is a composition $\mathscr{F}$ of suitable $\mathscr{F}_f$s and $\mathscr{F}_R$s such that $\mathscr{F}(\theta_1) = \mathscr{F}(\theta_2)$, $\mathscr{F}(\psi_1) = \neg\mathscr{F}(\psi_2)$, and whenever $\gamma_i = \delta_j$ then $\mathscr{F}(\hat{\gamma}_i) = \mathscr{F}(\hat{\delta}_j)$. Thus $\mathscr{F}(p)$, $\mathscr{F}(q)$ are proofs showing that

$$\mathscr{F}(\hat{\gamma}_1), \ldots, \mathscr{F}(\hat{\gamma}_k), \neg\mathscr{F}(\psi_2) \vdash \mathscr{F}(\theta_1)$$

and

$$\mathscr{F}(\hat{\delta}_1), \ldots, \mathscr{F}(\hat{\delta}_l), \mathscr{F}(\psi_2) \vdash \mathscr{F}(\theta_1)$$

hence

$$\mathscr{F}(\hat{\gamma}_1), \ldots, \mathscr{F}(\hat{\gamma}_k), \mathscr{F}(\hat{\delta}_1), \ldots \mathscr{F}(\hat{\delta}_l) \vdash \mathscr{F}(\theta_1),$$

so $\Gamma \vdash_{\mathrm{FA}} \theta$, as required.

Using this, we can now see that, if S_∞ is a maximally M-consistent set of ${}^*\mathscr{L}_{\mathrm{A}}(M)$-sentences satisfying $(+)$, then S (as defined on p. 234) is a full satisfaction class for M, since, for example:

(a) S_∞ contains exactly one of $\varphi, \neg\varphi$ for each ${}^*\mathscr{L}_{\mathrm{A}}(M)$-sentence φ, as if both $S_\infty \cup \{\varphi\}$ and $S_\infty \cup \{\neg\varphi\}$ are M-inconsistent then so is S_∞ by the cut rule. So by maximality one of $\varphi, \neg\varphi$ is in S_∞, and certainly if both $\varphi, \neg\varphi \in S_\infty$, then S_∞ is M-inconsistent.

(b) If $(\varphi \wedge \psi) \in S_\infty$, then $\vdash_{\mathrm{FA}} \neg\varphi \rightarrow \neg(\varphi \wedge \psi)$, so $S_\infty, \neg\varphi \vdash_{\mathrm{FA}} 0 = 1$, so $\neg\varphi \notin S_\infty$, so $\varphi \in S_\infty$.

(c) If $\varphi(\mathrm{clterm}(a)) \in S_\infty$ for some $a \in M$, then $S_\infty \vdash_{\mathrm{FA}} \exists v\varphi(v)$, so $\exists v\varphi(v) \in S_\infty$. Conversely if $\exists v\varphi(v) \in S_\infty$ then by $(+)$, $\varphi(\mathrm{clterm}(a)) \in S_\infty$ for some $a \in M$.

(d) If $t, s \in M$ are closed $\mathscr{L}_{\mathrm{A}}$-terms and $M \vDash \mathrm{val}(t) = \mathrm{val}(s)$ then $(t = s) \in {}^*\mathrm{At}(M)$, hence $(t = s) \in S_\infty$. Conversely, if $M \vDash \mathrm{val}(t) \neq \mathrm{val}(s)$ then

$$(t = s + \mathrm{clterm}(a) + 1))) \in {}^*\mathrm{At}(M)$$

for some $a \in M$, and it is easy to check from this that $\mathrm{Th}(M, a)_{a \in M} + {}^*\mathrm{At}(M) \vdash_{\mathrm{FA}} \neg(t=s)$, so $(t=s) \notin S_\infty$.

We shall now construct a maximally M-consistent S_∞ with the property (+). This will be done by constructing S_∞ as $\bigcup_{i \in \mathbb{N}} S_i$ where $\varnothing = S_0 \subseteq S_1 \subseteq \cdots$ is a suitable chain of finite M-consistent sets. The countability of M will be used to ensure that, for every ${}^*\mathscr{L}_{\mathrm{A}}(M)$ formula $\varphi(v)$ with one free-variable only, there is some $i \in \mathbb{N}$ so that *either* $\neg\exists v\varphi(v) \in S_i$ *or* $\varphi(\mathrm{clterm}(a)) \in S_i$ for some $a \in M$.

LEMMA C. $\varnothing$ is M-consistent.

Proof. If ${}^*\mathrm{At}(M) + \mathrm{Th}(M, a)_{a \in M} \vdash_{\mathrm{FA}} 0=1$ then there are

$$\varphi_1, \ldots, \varphi_n \in {}^*\mathrm{At}(M), \qquad \theta_1, \ldots, \theta_k \in \mathrm{Th}(M, a)_{a \in M}$$

and finite approximations $\hat{\varphi}, \ldots, \hat{\varphi}_n, \hat{\theta}_1, \ldots, \hat{\theta}_k$ to these sentences such that $\hat{\varphi}_1, \ldots, \hat{\varphi}_n, \hat{\theta}_1, \ldots, \hat{\theta}_k \vdash f_0 = f_1$. Let p be a proof of $f_0 = f_1$ from the $\hat{\varphi}_i$ and the $\hat{\theta}_j$; clearly we may assume p only involves function and relation symbols already present in $\hat{\varphi}_1, \ldots, \hat{\varphi}_n, \hat{\theta}_1, \ldots, \hat{\theta}_k$ and f_0, f_1. Each of these symbols is of the form $f_{t(\mathrm{v}_{i_1}, \ldots, \mathrm{v}_{i_l})}$ or $R_{\chi(\mathrm{v}_{i_1}, \ldots, \mathrm{v}_{i_l})}$ for a term t in ${}^*\mathscr{L}_{\mathrm{A}}(M)$ or a formula χ in $\mathscr{L}_{\mathrm{A}}(M)$. (Note that χ is actually a *standard* formula, but may contain parameters a from M in the form $\mathrm{clterm}(a)$.) Since χ is standard we can interpret $R_{\chi(\mathrm{v}_{i_1}, \ldots, \mathrm{v}_{i_l})}$ in M by setting

$$M \vDash R_{\chi(\mathrm{v}_{i_1}, \ldots, \mathrm{v}_{i_l})}(b_1, \ldots, b_l) \Leftrightarrow M \vDash \chi(b_1, \ldots, b_l),$$

and we can interpret each $f_{t(\mathrm{v}_{i_1}, \ldots, \mathrm{v}_{i_l})}$ in M using the val function by

$$M \vDash f_{t(\mathrm{v}_{i_1}, \ldots, \mathrm{v}_{i_l})}(b_1, \ldots, b_l) = b_{l+1} \Leftrightarrow M \vDash b_{l+1} = \mathrm{val}(t, c)$$

where $c = [\,][b_1/i_1][b_2/i_2] \cdots [b_l/i_l]$ is the sequence coded in M with b_j in the i_jth place, for all $j \leqslant l$, and 0s elsewhere.

With this interpretation and the usual properties of the val function, it is easy to check (using induction on the length of subproofs of p) that every statement in the proof p is true in M. Since the interpretation of $f_0 = f_1$ is clearly false in M, we have our contradiction. □

We now complete the proof by showing how all the S_is are constructed. Enumerate all ${}^*\mathscr{L}_{\mathrm{A}}(M)$-sentences as $\varphi_0, \varphi_1, \ldots, \varphi_i, \ldots$ $(i \in \mathbb{N})$, using the countability of M. Suppose S_i has been constructed and is finite and M-consistent. If φ_i is not of the form $\exists w\theta(w)$ for some variable w, then we define S_{i+1} to be either $S_i \cup \{\varphi_i\}$ or $S_i \cup \{\neg\varphi_i\}$. (One of these sets is M-consistent, by the cut rule.) Otherwise we must set S_{i+1} to be either $S_i \cup \{\neg\varphi_i\}$ or $S_i \cup \{\varphi_i, \theta(\mathrm{clterm}(a))\}$ for some $a \in M$. Obviously $S_i \vdash_{\mathrm{FA}} \theta(\mathrm{clterm}(a)) \to \exists w\theta(w)$ for each $a \in M$, so if $S_i \cup \{\theta(\mathrm{clterm}(a))\}$ is

M-consistent then so is $S_i \cup \{\varphi_i, \theta(\mathrm{clterm}(a))\}$. Suppose then that $S_i \cup \{\theta(\mathrm{clterm}(a))\}$ is M-inconsistent for each $a \in M$; we shall show that $S_i + S_i + \exists w\theta(w)$ is M-inconsistent, and hence (by the cut rule) $S_i \cup \{\neg\varphi_i\}$ is M-consistent.

Write $\chi(w)$ for $\theta(w) \to 0=1$. By the deduction theorem and our supposition, $S_i \vdash_{\mathrm{FA}} \chi(\mathrm{clterm}(a))$ for each $a \in M$. We must show that $S_i \vdash_{\mathrm{FA}} \forall x\chi(w)$, which suffices since $\forall w\chi(w) \vdash_{\mathrm{FA}} \exists w\theta(w) \to 0=1$. By hypothesis S_i is finite. Suppose $S_i = \sigma_1, \ldots, \sigma_n$. We observe that, for all valid *standard* proofs p in the predicate calculus for the recursive language $\mathscr{L}_{\mathrm{T}}$ which has k-ary function and relation symbols $f_{k,i}, R_{k,i}$ for all $k, i \in \mathbb{N}$ (including $k=0$), and for all standard $\mathscr{L}_{\mathrm{A}}$-formulas $\tau_1(\bar{x}), \ldots, \tau_m(\bar{x})$, the following is expressed by a standard $\mathscr{L}_{\mathrm{A}}$-formula

$$\zeta_{p,\tau_1,\tau_2,\ldots,\tau_m}(\sigma_1, \ldots \sigma_n, \chi(w), a)$$

with the parameters from M as shown:

'There exist $c, d_1, \ldots, d_i, e_1, \ldots, e_j, \rho_1, \ldots, \rho_k$ in M such that

(a) $\tau_l(c)$ holds, for each $1 \leqslant l \leqslant m$;
(b) $\mathrm{term}(d_l)$, for each $1 \leqslant l \leqslant i$;
(c) $\mathrm{form}(e_l)$, for each $1 \leqslant l \leqslant i$;
(d) ρ_l is a true atomic $\mathscr{L}_{\mathrm{A}}$-sentence (in the sense of M), for each $1 \leqslant l \leqslant k$;

and, if p is an $\mathscr{L}_{\mathrm{T}}$-proof of θ from axioms $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n$ and $\eta_1, \ldots \eta_k$ such that, reading the lth new function symbol in p as $f_{d_l}(\cdots)$ for $1 \leqslant l \leqslant i$, and reading the lth new relation symbol in p as $R_{e_l}(\cdots)$ $(1 \leqslant l \leqslant i)$, p becomes a valid proof in $\mathscr{L}_{\mathrm{FA}}$, and

(e) φ_l is a finite approximation to $\tau_l(\mathrm{clterm}(c))$ $(l \leqslant m)$;
(f) ψ_l is a finite appoximation to σ_l, $(l \leqslant n)$;
(g) η_l is a finite approximation to ρ_l $(l \leqslant k)$ and
(h) θ is a finite approximation to $\chi(\mathrm{clterm}(a))$.'

Here i, j, k are standard integers, and $\varphi_1, \ldots, \varphi_m$, $\psi_1, \ldots, \psi_n$ and $\eta_1, \ldots, \eta_k$ are standard $\mathscr{L}_{\mathrm{T}}$-formulas that depend on p only. The subscript 'T' in $\mathscr{L}_{\mathrm{T}}$ stands for 'template', since $\mathscr{L}_{\mathrm{T}}$-proofs and formulas are used as templates for $\mathscr{L}_{\mathrm{FA}}$-proofs and formulas, via some substitution of $\mathscr{L}_{\mathrm{FA}}$-functions and relations for $\mathscr{L}_{\mathrm{T}}$-functions and relations. Note too that the formula $\zeta_{p,\tau_1,\ldots,\tau_m}$ can be computed recursively from $p, \tau_1, \ldots, \tau_m$; thus the set of all such $\zeta_{p,\tau_1,\ldots,\tau_m}$ is recursive, since the set of all valid $\mathscr{L}_{\mathrm{T}}$-proofs p is recursive. It follows that '$S_i + {}^*\mathrm{At}(M) + \mathrm{Th}(M, a)_{a \in M} \vdash_{\mathrm{FA}} \chi(a)$' is expressed in M by

$$\bigvee_{p,\tau_1,\ldots,\tau_m} \left(\zeta_{p,\tau_1,\ldots,\tau_m}(\sigma_1, \ldots, \sigma_n, \chi(w), a) \right).$$

Using recursive saturation of M, we can show that there is a finite set F of such $\zeta_{p,\tau_1,\ldots,\tau_m}$ so that

$$M \models \forall a \bigvee_F \zeta_{p,\tau_1,\ldots,\tau_m}(\bar{\sigma}, \chi(w), a);$$

for if this were not the case then

$$q(x) = \left\{ \neg\zeta_{p,\tau_1,\ldots,\tau_m}(\bar{\sigma}, \chi(w), x) \,\middle|\, \begin{array}{l} m \in \mathbb{N}, p, \tau_1, \ldots, \tau_m \\ \text{as above} \end{array} \right\}$$

would be finitely satisfied in M, so there would be $b \in M$ realizing $q(x)$ in M, i.e.

$$S_i + {}^*\mathrm{At}(M) + \mathrm{Th}(M, a)_{a \in M} \nvdash_{\mathrm{FA}} \chi(b),$$

contradicting our hypothesis. Thus there are finitely many $\mathscr{L}_{\mathrm{T}}$-proofs $p_1, \ldots, p_r$ that act as templates for $\mathscr{L}_{\mathrm{FA}}$-proofs showing that $S_i + {}^*\mathrm{At}(M) + \mathrm{Th}(M, a)_{a \in M} \vdash_{\mathrm{FA}} \chi(a)$ for all $a \in M$. We shall write the collection of all $\tau(x)$s occurring in the subscript of some $\zeta_{p_i,\tau}$ as $\hat{\tau} = \tau_1(x), \ldots, \tau_m(x)$ and assume that each p_i is a proof of θ_i from axioms $\varphi_{i,1}, \ldots, \varphi_{i,m}$, $\psi_{i,1}, \ldots, \psi_{i,n}$, $\eta_{i,1}, \ldots, \eta_{i,k}$, and further assume that

$$M \models \bigwedge_{i=1}^{r} \exists a \zeta_{p_i,\tau}(\bar{\sigma}, \chi(w), a),$$

i.e., each p_i is a valid proof template for some $a \in M$.

We have to combine these proofs $p_1, \ldots, p_r$ to obtain

$$S_i + \mathrm{Th}(M, a)_{a \in M} + {}^*\mathrm{At}(M) \vdash_{\mathrm{FA}} \forall x \chi(x).$$

Our immediate problem is that, when one of the proofs p_i is used as a template for a proof of some finite approximation of $\chi(\mathrm{clterm}(a))$, the various function and relation symbols in p_i are replaced by f_t or R_ξ of $\mathscr{L}_{\mathrm{FA}}$ *where t or ξ may vary with a.* It will be important to understand the limitations on how t or ξ can vary.

In the case of $\psi_{i,1}, \ldots, \psi_{i,n}$, which are templates for finite approximations to $\sigma_1, \ldots, \sigma_n$, there is no ambiguity: by unique readability the $f_{j,k}$s and $R_{j,k}$s in $\psi_{i,1}, \ldots, \psi_{i,n}$ have a unique interpretation as f_ts and R_ξs for subterms t and subformulas ξ of $\sigma_1, \ldots, \sigma_n$. For $\varphi_{i,1}, \ldots, \varphi_{i,m}$, θ_i and $\eta_{i,1}, \ldots, \eta_{i,k}$ we are not so lucky, since these are templates for finite approximations to

$$\tau_1(\mathrm{clterm}(\bar{c})), \ldots \tau_m(\mathrm{clterm}(\bar{c})),\ \chi(\mathrm{clterm}(a)),\ \text{and}\ \rho_1, \ldots, \rho_k,$$

where the choice of $\bar{c}, \bar{\rho}$ may depend on a. However, since we are only substituting closed terms $\mathrm{clterm}(\bar{c})$ for variables in $\tau_l(\bar{x})$, each relation symbol $R_{j,k}$ in $\varphi_{i,k}$ will be interpreted as $R_{\xi(\bar{v},\mathrm{clterm}(\bar{c}))}$ for some subformula $\xi(\bar{v}, \bar{x})$ of $\tau_l(\bar{x})$ which is uniquely determined—the only variation possible is the choice of $\bar{c}$ here. The $\eta_{i,1}, \ldots, \eta_{i,k}$ will of course be atomic, corresponding to the atomic formulas $\bar{\rho}$ of $^*\mathrm{At}(M)$, where $\bar{\rho}$ may depend on a.

By the remarks in the last paragraph, then, we can emply a 'common approximation' type of argument as in Lemma B above, refining the subformulas of $\bar{\varphi}$, $\bar{\psi}$, $\bar{\theta}$ and $\bar{\eta}$ until either they agree, or we are down to the level of atomic subformulas. In the case of the ψs we will succeed in making $\psi_{i,l}=\psi_{j,l}(=\psi_l$, say) for all $i, j \leqslant r$ and all $l \leqslant n$, since there is no dependency on a. In the case of the φs and θs we will be able to find $\mathscr{L}_{\mathrm{T}}$-formulas $\varphi_1(\bar{y}), \ldots, \varphi_m(\bar{y})$ and $\theta(\bar{z})$, and also closed $\mathscr{L}_{\mathrm{T}}$-terms $\bar{t}_i$, $\bar{s}_i$ such that $\varphi_{i,l}=\varphi_l(\bar{t}_i)$ $(1 \leqslant i \leqslant r,\ 1 \leqslant l \leqslant m)$ and $\theta_i=\theta(\bar{s}_i)$ $(1 \leqslant i \leqslant r)$, where the $\bar{t}$s and $\bar{s}$s will be templates for finite approximations of $^*\mathscr{L}_{\mathrm{A}}(M)$-terms clterm$(c)$ and clterm(a) for various $a, c \in M$. Also, since the φs are templates for finite approximations of $\tau_l(\mathrm{clterm}(\bar{c}))$, where $\tau_l(\bar{x})$ is a standard $\mathscr{L}_{\mathrm{A}}$-formula, we may continue the refining process until the only relation symbol R of $\mathscr{L}_{\mathrm{T}}$ (other than $=$) occurring in the φs is some fixed symbol $R_<$, which will be replaced by $R_{(\mathrm{v}_0<\mathrm{v}_1)}$ in all cases. In the case of the ηs, we can apply the refining process so that each $\eta_{i,l}$ is either $R_<(t, s)$ or $(t=s)$, where t, s are some closed $\mathscr{L}_{\mathrm{T}}$-terms.

When we obtain these 'refined' proof-templates, we can identify certain relation symbols of $\mathscr{L}_{\mathrm{T}}$ with $R_<$, so that $R_<$ is the *only* relation symbol of $\mathscr{L}_{\mathrm{T}}$ in the proofs $p_1, \ldots, p_r$ that will be read as $R_{(\mathrm{v}_0<\mathrm{v}_1)}$. (Identifying symbols of $\mathscr{L}_{\mathrm{T}}$ does not change the fact that the p_is are valid $\mathscr{L}_{\mathrm{T}}$-proofs, and if we identify two symbols that will always be read as the same thing, we do not destroy the finite approximation property either.) Similarly, we can fix function symbols $f_{\mathrm{zero}}, f_{\mathrm{one}}, f_{\mathrm{plus}}, f_{\mathrm{times}}$ of $\mathscr{L}_{\mathrm{T}}$ that will be the only function symbols to be read as $f_0, f_1, f_{(\mathrm{v}_0+\mathrm{v}_1)}$, and $f_{(\mathrm{v}_0\cdot\mathrm{v}_1)}$ respectively. We shall think of $<$, $R_{(\mathrm{v}_0<\mathrm{v}_1)}$ and $R_<$ as being the 'same' symbols, and, similarly, we identify: $0, f_0, f_{\mathrm{zero}}$; $1, f_1, f_{\mathrm{one}}$; $+, f_{(\mathrm{v}_0+\mathrm{v}_1)}, f_{\mathrm{plus}}$; and $\cdot, f_{(\mathrm{v}_0\cdot\mathrm{v}_1)}, f_{\mathrm{times}}$. Thus we regard $\mathscr{L}_{\mathrm{T}}$ and $\mathscr{L}_{\mathrm{FA}}$ as being extensions of the first-order language of arithmetic, $\mathscr{L}_{\mathrm{A}}$.

With all these modifications to the p_is completed, we now re-examine the structure of the formulas $\theta_i=\theta(\bar{s}_i)$, ψ_l, $\varphi_{i,l}=\varphi_l(\bar{t}_i)$ and $\eta_{i,l}$. By the conventions in the last paragraph these formulas can be regarded as $\mathscr{L}_{\mathrm{A}}$-formulas with extra function and relation symbols $\bar{f}, \bar{R}$ from $\mathscr{L}_{\mathrm{T}}$. For example, each ψ_l $(1 \leqslant l \leqslant n)$ will contain: 0-ary function symbols $\bar{g}$ representing fixed closed terms in the corresponding σ_l; 1-ary and higher function symbols $\bar{f}$; and various relation symbols $\bar{R}$. We display these new symbols explicitly in our notation, writing ψ_l as $\psi_l(\bar{g}, \bar{f}, \bar{R})$. Similarly $\theta(\bar{x})$ is written as $\theta(\bar{x}, \bar{g}, \bar{f}, \bar{R})$ where the $\bar{g}$ are 0-ary function symbols representing fixed closed terms in $\chi(w)$, $\bar{f}$ are 1-ary and higher function symbols, and $\bar{R}$ are relation symbols. As the notation suggests, there may be some overlap between the $\bar{g}$s, the $\bar{f}$s and the $\bar{R}$s in ψ_l and $\theta(\bar{x})$. The closed terms $\bar{s}_i$ that are used to replace $\bar{x}$ in $\theta(\bar{x})$ will contain 0-ary function symbols $\bar{g}, \bar{h}$ from $\mathscr{L}_{\mathrm{T}}$. We write $\bar{s}_i$ as $\bar{s}_i(\bar{g}, \bar{h})$, thinking of this as a tuple of $\mathscr{L}_{\mathrm{A}}$-terms $\bar{s}_i$ in constants $\bar{g}, \bar{h}$. Since $\varphi_l(\bar{t}_i)$ represent $\tau_l(\mathrm{clterm}(\bar{c}))$ for some $\bar{c}$, and we got rid of all relation symbols R (other than that for $<$) in φ_l, $\varphi_l(\bar{y})$ is a standard $\mathscr{L}_{\mathrm{A}}$-formula, and the terms $\bar{t}_i$ will be of the form $\bar{t}_i(\bar{g}, \bar{h})$ where $\bar{g}, \bar{h}$ are 0-ary function symbols of $\mathscr{L}_{\mathrm{T}}$ as before. Finally, the $\eta_{i,l}$ will be sentences of the form $u(\bar{g}, \bar{h}) = u'(\bar{g}, \bar{h})$ or $u(\bar{g}, \bar{h}) < u'(\bar{g}, \bar{h})$, where u, u' are some $\mathscr{L}_{\mathrm{A}}$-terms. We write $\eta_{i,l}$ as $\eta_{i,l}(\bar{g}, \bar{h})$.

The important distincition between $\bar{g}$ and $\bar{h}$ in the above is that the $\bar{g}$ represent fixed closed terms occurring in the σ s or in $\chi(w)$, whereas the $\bar{h}$ represent closed terms that may vary with a. We shall use this information to deduce that

$$\text{for all } a \in M,\ \mathrm{Th}(M, a)_{a \in M} \vdash \bigwedge\!\!\!\bigwedge_{l=1}^{n} \psi_l(\bar{b}, \bar{f}, \bar{R}) \rightarrow \hat{\chi}(a, \bar{b}, \bar{f}, \bar{R}), \qquad (\circ)$$

where $\hat{\chi}(x, \bar{g}, \bar{f}, \bar{R})$ is the formula obtained from θ by replacing every subterm $s_i(\bar{g}, \bar{h})$ for w in $\chi(w)$ by x, $\bar{b} \in M$ is the tuple of unique values for the subterms of $\sigma_1, \ldots, \sigma_n, \chi(w)$ corresponding to $\bar{g}$, and provability in $(\circ)$ is the usual notion where each parameter $c \in M$ is identified with its own 0-ary function symbol $f_{\mathrm{clterm}(c)}$.

It is easy to see that, if $\bar{g}$ is $g_1, \ldots, g_s$ and each g_i represents the closed term $\check{g}_i$ of ${}^*\mathscr{L}_{\mathrm{A}}(M)$ in $\sigma_1, \ldots, \sigma_n$ and $\chi(w)$, then for some suitable interpretation of $\bar{f}$ and $\bar{R}$ as symbols of $\mathscr{L}_{\mathrm{FA}}$, $\psi_l(f_{\check{g}_1}, \ldots, f_{\check{g}_s}, \bar{f}, \bar{R})$ is a finite approximation to σ_l $(1 \leq l \leq n)$ and $\hat{\chi}(w, f_{\check{g}_1}, \ldots, f_{\check{g}_s}, \bar{f}, \bar{R})$ is a finite approximation to $\chi(w)$. We shall then use the recursive saturation of M together with $(\circ)$ to deduce that

$$\mathrm{Th}(M, a)_{a \in M} \vdash \bigwedge\!\!\!\bigwedge_{l=1}^{n} \psi_l(\bar{b}, \bar{f}, \bar{R}) \rightarrow \forall w \hat{\chi}(w, \bar{b}, \bar{f}, \bar{R}). \qquad (\bullet)$$

From this it easily follows that

$$\mathrm{Th}(M, a)_{a \in M} + \bigwedge\!\!\!\bigwedge_{j=1}^{s} b_j = f_{\check{g}_j} \vdash \bigwedge\!\!\!\bigwedge_{l=1}^{n} \psi_l(\bar{f}_{\check{g}}, \bar{f}, \bar{R}) \rightarrow \forall w \hat{\chi}(w, \bar{f}_{\check{g}}, \bar{f}, \bar{R});$$

hence $\mathrm{Th}(M, a)_{a \in M} + {}^*\mathrm{At}(M) \vdash_{\mathrm{FA}} \bigwedge\!\!\!\bigwedge_{l=1}^{n} \sigma_l \rightarrow \forall w \chi(w)$, since each '$b_j = f_{\check{g}_j}$' (or rather: '$f_{\mathrm{clterm}(b_j)} = f_{\check{g}_j}$') is a finite approximation of a true atomic sentence. Hence

$$\mathrm{Th}(M, a)_{a \in M} + {}^*\mathrm{At}(M) + S_i \vdash_{\mathrm{FA}} \forall w \chi(w)$$

by the deduction rule, as required.

To verify $(\circ)$, first fix $a \in M$. Then $M \vDash \zeta_{p_i, \tau}(\bar{\sigma}, \chi(w), a)$ for some $i \leq r$, and fix such an i. Thus there are also $\bar{c}, \bar{d}, \bar{e}, \bar{\rho} \in M$, as in the statement of $\zeta_{p_i, \tau}(\bar{\sigma}, \chi(w), a)$. Thus, for every $\tau_l(\bar{x})$ occurring as an assumption in p_i, $M \vDash \tau_l(\bar{c})$, and there are values $\bar{e}$ (read off as values of certain closed subterms $\bar{d}$ of $\bar{\rho}$, clterm(a) and clterm($\bar{c}$)) such that $a = s_i(\bar{b}, \bar{e})$, and $c_j = t_{ij}(\bar{b}, \bar{e})$ for suitable terms t_{ij} in $\bar{t}_i$ representing clterm(c_j). (Note that, since we defined $\bar{b}$ as the values of the appropriate closed terms, these values $\bar{b}$ are fixed.) Thus, interpreting $\bar{g}, \bar{h}$ in p_i as $f_{\mathrm{clterm}(x)}$ for suitable x in $\bar{b}$

or $\bar{e}$, the proof p_i shows

$$\bigwedge_{s\in s_i} a=s_i(\bar{b},\bar{e})+\bigwedge_{t_{ij}\in t_i} t_{ij}(\bar{b},\bar{e})=c_j+\bigwedge_{l=1}^{m} \tau_l(\bar{c})\vdash \bigwedge_{l=1}^{n} \psi_l(\bar{b},\bar{f},\bar{R})\rightarrow\hat{\chi}(a,\bar{b},\bar{f},\bar{R}).$$

But all the sentences on the left-hand-side of the '$\vdash$' symbol here are true in M, hence are in $\mathrm{Th}(M,a)_{a\in M}$, thus

$$\mathrm{Th}(M,a)_{a\in M}\vdash \bigwedge_{l=1}^{n} \psi_l(\bar{b},\bar{f},\bar{R})\rightarrow\hat{\chi}(a,\bar{b},\bar{f},\bar{R}).$$

Finally, to deduce (•) from (○), let $\xi(x,\bar{y},\bar{R},\bar{f})$ be the formula

$$\bigwedge_{l=1}^{n} \psi_l(\bar{y},\bar{f},\bar{R})\wedge\neg\hat{\chi}(x,\bar{y},\bar{f},\bar{R}).$$

Suppose to the contrary that $\mathrm{Th}(M,a)_{a\in M}+\exists x\xi(x,\bar{b},\bar{f},\bar{R})$ is consistent, and consider the following set of $\mathscr{L}_{\mathrm{A}}$-formulas:

$$\Phi(x)=\{\varphi(x,\bar{b})\in\mathscr{L}_{\mathrm{A}}\cup\{\bar{b}\}\mid \xi(x,\bar{y},\bar{f},\bar{R})\vdash\varphi(x,\bar{y})\}.$$

$\Phi(x)$ is an r.e. set of $\mathscr{L}_{\mathrm{A}}\cup\{\bar{b}\}$-formulas, as $\mathscr{L}_{\mathrm{A}}\cup\{\bar{b}\}$ is a recursive extension of $\mathscr{L}_{\mathrm{A}}$; so by Craig's trick, $\Phi(x)$ is equivalent to a recursive set of formulas. But also $\Phi(x)$ is finitely satisfied in M since we assumed that $\mathrm{Th}(M,a)_{a\in M}+\exists x\xi(x,\bar{c},\bar{f},\bar{R})$ is consistent. Thus, by recursive saturation, $\Phi(x)$ is realized by some $a\in M$. But, by (○),

$$\mathrm{Th}(M,a)_{a\in M}\vdash\neg\xi(a,\bar{b},\bar{f},\bar{R}),$$

that is,

$$\xi(a,\bar{b},\bar{f},\bar{R})\vdash\neg\lambda(a,\bar{b},\bar{c})$$

for some $\lambda(x,\bar{y},\bar{z})$ in $\mathscr{L}_{\mathrm{A}}$ and some $\bar{c}\in M$ such that $M\models\lambda(a,\bar{b},\bar{c})$. But then $\xi(x,\bar{y},\bar{f},\bar{R})\vdash\forall\bar{z}\neg\lambda(x,\bar{y},\bar{z})$, so a cannot satisfy the type $\Phi(x)$, a contradiction. Hence (•), as required. □

REMARKS. From this theorem it is easy to deduce an analogue to Proposition 15.1 for uncountable models and full satisfaction classes: every $M\models PA$ has an elementary extension $N\succ M$ with $\mathrm{card}(M)=\mathrm{card}(N)$ and N having a full satisfaction class. To see this, note that 'S is a full satisfaction class' is expressed by an $\mathscr{L}_S$-sentence σ, so such an N exists if and only if $\mathrm{Th}(M,a)_{a\in M}+\sigma$ is consistent. But by the downward Löwenheim–Skolem theorem, every $\bar{a}\in M$ of finite length is contained in some countable $K\prec M$, and by Proposition 11.4, K has a countable recursively saturated

$\bar{K} > K$. By Theorem 15.6 there is $S \subseteq \bar{K}^2$ such that $(\bar{K}, S) \vDash \sigma$. Thus, $\mathrm{Th}(M, \bar{a}) + \sigma$ is consistent, and hence, by compactness, $\mathrm{Th}(M, a)_{a \in M} + \sigma$ is consistent.

It is also worth noting that the proof of Theorem 15.6 gives necessary and sufficient conditions for a sentence φ of $^*\mathscr{L}_{\mathrm{A}}(M)$ to be contained in a full satisfaction class for a countable recursively saturated $M \vDash PA$, namely such a satisfaction class exists just in case that $\{\varphi\}$ is M-consistent. For example, if $a \in M \setminus \mathbb{N}$, then the 'obviously false' $^*\mathscr{L}_{\mathrm{A}}(M)$-sentence

$$\exists x(\overbrace{(\cdots(x \neq x \vee x \neq x) \vee x \neq x) \vee \cdots) \vee x \neq x}^{a})$$

is M-consistent (since, in proofs from the standard sentence $\exists x((R(x) \vee x \neq x \vee \cdots \vee x \neq x)$, $R(x)$ could be interpreted as 'true' for all x), so the nonstandard sentence above can be made 'true' in some satisfaction class. (Note, however, that the non-standard sentence above is false in all *inductive* nonstandard satisfaction classes (exercise).)

There are also versions of Theorem 15.6 for languages $\mathscr{L}$ extending $\mathscr{L}_{\mathrm{A}}$. For example, if $\mathscr{L}$ is a recursive extension of $\mathscr{L}_{\mathrm{A}}$ then there is a corresponding notion of an M-consistent set of (possible nonstandard) $\mathscr{L}$-sentences. It turns out that a *standard* set of $\mathscr{L}$-sentences T is M-consistent if and only if $T + \mathrm{Th}(M, a)_{a \in M}$ is consistent, and any recursive set of M-consistent $\mathscr{L}$-sentences T can be extended to a full satisfaction class S, provided M is countable and recursively saturated. We can then *define* interpretations of the extra symbols in $\mathscr{L}$ according to this satisfaction class, and obtain an expansion of our model M to the language $\mathscr{L}$, the expansion satisfying T. This argument can be made to give a proof that all countable recursively saturated models of PA are *resplendent*, an important notion that will be defined properly and studied in detail in the next section.

Exercises for Section 15.1

15.1 A very useful kind of partial satisfaction class is a Σ_α-satisfaction class.
(a) Working in PA, define a formula $\psi(x, y)$ expressing 'y is the Gödel-number of a Σ_x formula.' Prove that your formula is $\Delta_1(PA)$ and has some of the expected properties.
(b) Define a *partial nonstandard Σ_α-satisfaction class* S for a model $M \vDash PA$ to be a set $S \subseteq M^2$ such that

$$\forall a, \varphi[S(\varphi, a) \leftrightarrow \psi(\alpha, \varphi) \wedge \bigwedge\!\!\!\bigwedge_{i=1}^{7} T_i(\varphi, a)],$$

where α is a nonstandard element of M. Using the $\mathrm{Sat}_{\Sigma_n}(\varphi, a)$ relations, prove that for all $M \vDash PA$ there is $N > M$ and a partial nonstandard *inductive* Σ_α-satisfaction class $S \subseteq N^2$ for N, and, moreover, any such N is recursively saturated.

15.2* Let $\mathscr{L}$ extend $\mathscr{L}_A$ by at most finitely many relation, function and constant symbols, and suppose M is a nonstandard $\mathscr{L}$-structure whose $\mathscr{L}_A$-reduct $M \upharpoonright \mathscr{L}_A$ is a nonstandard model of PA.
(a) By adding new clauses to $T_1, \ldots, T_7$ define notions of full and partial $\mathscr{L}$-satisfaction classes for M, i.e. satisfaction classes that tell the truth value of all $\mathscr{L}$-formulas.
(*Hint:* If $\mathscr{L}$ has an extra function symbol f, you might like to consider clauses such as

$$\exists t_0, \ldots, t_k \left\{ \begin{array}{l} \varphi = (t_0 = f(t_1, \ldots, t_k) \wedge \exists b_1, \ldots, b_k, r \\ \left[\bigwedge\!\!\!\bigwedge_{j=0}^{k} r > t_j \wedge \bigwedge\!\!\!\bigwedge_{j=1}^{k} S((\mathsf{v}_r = t_j), a[b_j/r]) \right. \\ \left. \wedge S((t_0 = \mathsf{v}_r), a[f(b_1, \ldots, b_k)/r]) \right] \end{array} \right\}.$$

In this clause, the formula $r > t_j$ is simply a convenient formula that implies that 'the variable v_r does not occur in the term t_j'.)
(b) Prove analogues of Theorems 15.5 and 15.6 for $\mathscr{L}$-satisfaction classes, and also of Propositions 15.1 and 15.4 for partial nonstandard inductive $\mathscr{L}$-satisfaction classes under the additional assumption that M satisfies induction for all $\mathscr{L}$-formulas.
(c) Given that $\mathscr{L}$ is a recursive extension of $\mathscr{L}_A$ (not necessarily finite), show that the same results hold except that 'S is a full $\mathscr{L}$-satisfaction class' is then expressed by a set of $\mathscr{L} \cup \{S\}$ sentences, rather than a single sentence.
15.3 Let $M \vDash PA$ and let $a \in M$ be nonstandard. Let K be the structure for the language $R_+, R_\cdot, <, 0, 1$ (where $R_+, R_\cdot$ are new ternary relation symbols) defined by:

the domain of K is $\{x \in M \mid M \vDash x \leqslant a\}$;
for all $x, y, z \in K$, $K \vDash R_+(x, y, z) \Leftrightarrow M \vDash x + y = z$;
for all $x, y, z \in K$, $K \vDash R_\cdot(x, y, z) \Leftrightarrow M \vDash x \cdot y = z$;
$0, 1$ are interpreted in K by the same elements of M interpreting these symbols.

Show that K is recursively saturated.
(*Hint:* Use $\mathrm{Sat}_{\Delta_0}(x, y)$ in M.)
15.4 (For those with some knowledge of axiomatic set theory.) A model $M \vDash ZF$ is said to be ω-nonstandard iff there are elements $a_0, a_1, \ldots \in M$ such that $M \vDash (a_{i+1} \in a_i \wedge a_i \in \omega)$ for each $i \in \mathbb{N}$.
(a) Show that $ZF + \mathrm{Con}(ZF) \vdash \exists \omega$-nonstandard models of ZF.
(b) Suppose $a, \bar{R}, \bar{f}, \bar{c} \in M \vDash$ '$(a, \bar{R}, \bar{f}, \bar{c})$ is a structure for the first-order language $\mathscr{L} = \{\bar{R}, \bar{f}, \bar{c}\}$' where M is an ω-nonstandard model of ZF (and the sentence in quotation marks is interpreted in some standard way as a sentence in the first-order language of set theory). Define an $\mathscr{L}$-structure $^*(a, \bar{R}, \bar{f}, \bar{c})$ with domain $^*a = \{x \in M \mid M \vDash x \in a\}$ and relations, functions, constants inherited from $\bar{R}, \bar{f}$, and $\bar{c}$ in M. Show that $^*(a, \bar{R}, \bar{f}, \bar{c}) \vDash \sigma$ for every standard sentence σ such that $M \vDash$ '$(a, \bar{R}, \bar{f}, \bar{c}) \vDash \sigma$' and that $^*(a, \bar{R}, \bar{f}, \bar{c})$ is recursively saturated.
15.5 Let $PA(S)$ denote the $\mathscr{L}_S$-theory with axioms of PA, the axiom 'S is a full satisfaction class', and induction axioms for all formulas of $\mathscr{L}_S$.
(a) Complete the sketch proof just before Theorem 15.6 and show that $\mathrm{PA(S)} \vdash \mathrm{Con}(PA)$.

(b) By formalizing Lemma 14.16 in $PA(S)$ (using induction on certain $\mathscr{L}_S$-formulas) show that $PA(S)$ proves both the Paris–Harrington principle (PH) and the Kanamori–McAloon principle (KM) of Section 14.3.

15.6* (Constuction of ω_1-like recursively saturated models.)

(a) Let $M_0 \prec M_1 \prec M_2 \prec \cdots \prec M_i \prec \cdots (i < \lambda)$ be an elementary chain of recursively saturated models for a given language $\mathscr{L}$, where λ is an arbitrary ordinal. Show that their union $\bigcup_{i<\lambda} M_i$ is also recursively saturated.

(b) Let $M \vDash PA$ be countable and recursively saturated. Modify the proof of Friedman's theorem (Theorem 12.4) to show that there is a proper initial segment $I \prec M$ with $I \cong M$.

(c) Using (a) and (b) construct an elementary chain of countable models $M_0 \prec M_1 \prec \cdots \prec M_i \prec \cdots (i < \omega_1)$ such that $M_i \vDash PA$ is recursively saturated for each i, and $M_i \subseteq_e M_{i+1}$ is a proper end-extension. Show that the union is an ω_1-like recursively saturated model of PA.

15.7+ Let $M \vDash PA$ be countable and recursively saturated, and let $S \subseteq {}^*\mathscr{L}_A(M)$ be a finite M-consistent set of sentences. Show that there is an ${}^*\mathscr{L}_A(M)$-sentence φ such that both $S \cup \{\varphi\}$ and $S \cup \{\neg\varphi\}$ are M-consistent. Deduce that if $M \vDash PA$ is countable, and nonstandard, and has at least one full satisfaction class, then it has $2^{\aleph_0}$ of them. them.

15.8+ (*M-logic*) Define $\Gamma \vdash_M \varphi$ for $\{\varphi\} \cup \Gamma \subseteq {}^*\mathscr{L}_A(M)$ as for $\Gamma \vdash_{FA} \varphi$ except add the following new rule of inference (called the *M-rule*):

$$\text{if } \Gamma \vdash_M \varphi(\text{clterm}(a)) \text{ for all } a \in M \text{ deduce that } \Gamma \vdash_M \forall x \varphi(x).$$

(a) Show that $\vdash_M \varphi(\text{clterm}(\bar{a}))$ for any standard $\mathscr{L}_A$-formula $\varphi(\bar{x})$, and any $\bar{a} \in M$ such that $M \vDash \varphi(\bar{a})$.

(b) By re-examining the proof of Theorem 15.6 show that, if M is recursively saturated then for all $\varphi \in {}^*\mathscr{L}_A(M)$

$$\text{Th}(M, a)_{a \in M} \vdash_{FA} \varphi \Leftrightarrow \vdash_M \varphi$$

and hence M-logic is consistent.

(c) (Smith 1987) Modify the proof of Theorem 15.5 to prove that, if M is nonstandard and not recursively saturated, then M-logic is inconsistent (i.e. $\vdash_M 0 = 1$). Hence, for $M \vDash PA$, M is recursively saturated iff M is nonstandard and M-logic is consistent.

15.2 RESPLENDENCY

We now turn to the more 'classical' aspects of the theory of recursively saturated models. Many of the results here can be expressed and proved without reference to PA or to satisfaction classes and, since they have numerous applications in other parts of model theory too, the presentation here is as self-contained as possible. Thus, in the main, the only prerequisites for this section are the basic model-theoretic and recursion-theoretic background in Chapter 0, the definitions and remarks at the beginning of

Section 11.2, and Proposition 11.4. There are, however, three results in this section that are of general interest but require a little more arithmetic. These are: the theorem on chronic resplendency (Theorem 15.8); Kleene's theorem (Theorem 15.11) (both of which require some ideas about satisfaction classes from Section 15.1); and the classification theorem for countable recursively saturated models of rich theories (Theorem 15.22) which requires some of the results on Scott sets from Section 13.1. This notwithstanding, a reader whose main interest is in these 'classical' aspects of recursive saturation is recommended to start here and follow up the other details as and when necessary.

As indicated in the title, the key notion here is that of *resplendency*, which we now define.

Definition. Let $\mathscr{L}$ be a first-order language. A Σ_1^1 *formula over* $\mathscr{L}$ is a formula of the form $\exists X_1, \ldots, X_n \varphi(X_1, \ldots, X_n, \bar{x})$ where $X_1, \ldots, X_n$ are new relation or function symbols and $\varphi(X_1, \ldots, X_n, \bar{x})$ is a formula of the expanded language $\mathscr{L} \cup \{X_1, \ldots, X_n\}$. If $\varphi(X_1, \ldots, X_n)$ has no free first-order variables $\bar{x}$, then $\exists X_1, \ldots, X_n \varphi(X_1, \ldots, X_n)$ is a Σ_1^1 *sentence*. An $\mathscr{L}$-structure M *satisfies* the Σ_1^1 sentence $\Phi =$

$$\exists X_1, \ldots, X_n \varphi(X_1, \ldots, X_n)$$

iff there is an expansion $(M, X_1, \ldots, X_n)$ of M to $\mathscr{L} \cup \{X_1, \ldots, X_n\}$ such that

$$(M, X_1, \ldots, X_n) \models \varphi(X_1, \ldots, X_n).$$

If T is a first-order $\mathscr{L}$-theory then $T + \Phi$ is *consistent* iff it has a model, i.e., if the $\mathscr{L} \cup \{X_1, \ldots, X_n\}$-theory $T + \varphi(X_1, \ldots, X_n)$ has a model.

Definition. An $\mathscr{L}$-structure M is *resplendent* iff for all tuples $\bar{a} \in M$ of finite length and for all Σ_1^1 sentences $\Psi(\bar{a})$ in $\mathscr{L} \cup \{a\}$

$$\begin{aligned}&\text{if } \mathrm{Th}(M, \bar{a}) + \Psi(\bar{a}) \text{ is consistent}\\ &\text{then } (M, \bar{a}) \models \Psi(\bar{a}).\end{aligned}$$

Thus resplendent models satisfy as many Σ_1^1 formulas as possible. The existence of countable resplendent models follows from the following theorem due to Barwise and Schlipf, and (independently) Ressayre.

Theorem 15.7. Suppose M is a countable recursively saturated model for a recusive language $\mathscr{L}$, $\bar{a} \in M$ is a tuple of elements of finite length from M, the language $\mathscr{L}'$ is a recursive extension* of $\mathscr{L} \cup \{\bar{a}\}$, and T is a recursively

* See Exercise 11.9. In this section, as in the last, we shall identify formulas in a recursive language with their Gödel-numbers.

axiomatized $\mathscr{L}'$-theory. Then, if $\mathrm{Th}(M, \bar{a}) + T$ is consistent, there is an expansion of $(M, \bar{a})$ to $\mathscr{L}'$ satisfying T.

Proof. Let $\mathscr{L}(M)$, $\mathscr{L}'(M)$ denote the languages $\mathscr{L}$, $\mathscr{L}'$ expanded by adding constant symbols for each $b \in M$. (As usual we denote the constant symbol representing b by b itself.) We shall construct a *complete* $\mathscr{L}'(M)$ theory T' extending T such that

(1) If $T' \vdash \sigma$ where σ is an $\mathscr{L}(M)$-sentence, then $M \vDash \sigma$ (where M is considered as an $\mathscr{L}(M)$-structure in the obvious way), and

(2) if $T' \vdash \exists x \psi(x)$, where $\psi(x)$ is an $\mathscr{L}'(M)$ formula with only x free, then $T' \vdash \psi(b)$ for some $b \in M$.

A simple induction on $\mathscr{L}'(M)$-formulas (similar to that in the proof of Theorem 13.4) then shows that the expansion of M to $\mathscr{L}'(M)$ defined by

$$M \vDash R(b_1, \ldots, b_k) \Leftrightarrow T' \vdash R(b_1, \ldots, b_k)$$
$$M \vDash F(b_1, \ldots, b_l) = b_{l+1} \Leftrightarrow T' \vdash F(b_1, \ldots, b_l) = b_{l+1},$$

and

$$M \vDash b = c \Leftrightarrow T' \vdash b = c$$

or every new relation symbol R, function symbol F, and constant symbol c of $\mathscr{L}'$, and every $\bar{b} \in M$, has the property that

$$M \vDash \sigma \Leftrightarrow T' \vdash \sigma \tag{*}$$

for every sentence σ of $\mathscr{L}'(M)$. It follows that this expansion satisfies T. (Notice that (*) holds for $\mathscr{L}(M)$ sentences σ by (1) and the completeness of T'. (*) holds for atomic $\mathscr{L}'(M)$ sentences by definition. The induction step for (*) when σ is $\sigma_1 \wedge \sigma_2$, $\sigma_1 \vee \sigma_2$ or $\neg \sigma_1$ is easy, using the completeness of T'. The induction step when σ is $\exists x \sigma_1(x)$ is the important one: in this case $T' \vdash \exists x \sigma_1(x)$ iff $T' \vdash \sigma_1(b)$ for some $b \in M$ (by (2)) iff $M \vDash \sigma_1(b)$ for some $b \in M$ (by (*) for σ_1) iff $M \vDash \exists x \sigma_1(x)$.)

T' will be constructed as $T + \{\tau_i \mid i \in \mathbb{N}\}$ where the τ_i are $\mathscr{L}'(M)$-sentences, the τ_is being constructed inductively. Our inductive hypothesis on $\tau_0, \tau_1, \ldots, \tau_{n-1}$ is that

(3) Whenever $T + \{\tau_0, \tau_1, \ldots, \tau_{n-1}\} \vdash \sigma$ where σ is an $\mathscr{L}(M)$-sentence, then $M \vDash \sigma$.

When $n = 0$, (3) is satisfied automatically, since if $T \vdash \sigma$ we may assume that the only constants from M in σ are in the tuple $\bar{a}$. (As these are the only such contants in T, all other such constants can be replaced by new variables and 'quantified out' with a universal quantifier.) But then, if $M \nvDash \sigma$, $T + \mathrm{Th}(M, \bar{a})$ would be inconsistent, contradicting our assumption.

Now enumerate all $\mathscr{L}(M)$-formulas with only one free-variable, as $\varphi_0(x), \varphi_1(x), \ldots, \varphi_i(x), \ldots$ $(i \in \mathbb{N})$. (This is where we use the assumption that M is countable.) Suppose $\tau_0, \ldots, \tau_{n-1}$ have been found and satisfy (3). We shall let τ_n be *either* $\forall x \neg \varphi_n(x)$ *or* $\varphi_n(b)$ for some $b \in M$. Recursive saturation will be used to show that at least one of these choices can be made so that $\tau_0, \ldots, \tau_{n-1}, \tau_n$ also satisfies (3). To clarify our notation, let $c_1, \ldots, c_k$ denote the constants from M that appear in $T + \tau_0 + \cdots + \tau_{n-1}$, and let $\bar{d}$ denote any further constants from M that apear in φ_n. We shall write τ_i as $\tau_i(\bar{c})$ and φ_n as $\varphi_n(x, \bar{c}, \bar{d})$ to keep this in mind. Now suppose that no choice of τ_n as $\forall x \neg \varphi_n(x, \bar{c}, \bar{d})$ or $\varphi_n(b, \bar{c}, \bar{d})$ can be made to satisfy (3). Then there are $\mathscr{L}(M)$-sentences $\sigma(\bar{c}, \bar{d})$, $\eta_i(\bar{c}, \bar{d})$ (for $1 \leqslant i \leqslant k$), and $\zeta_b(\bar{c}, \bar{d}, b)$ (for each $b \in M \setminus \{\bar{c}\}$) such that

$$\left.\begin{array}{l} T + \tau_0(\bar{c}) + \cdots + \tau_{n-1}(\bar{c}) + \forall x \neg \varphi_n(x, \bar{c}, \bar{d}) \vdash \sigma(\bar{c}, \bar{d}) \\ T + \tau_0(\bar{c}) + \cdots + \tau_{n-1}(\bar{c}) + \varphi_n(c_i, \bar{c}, \bar{d}) \vdash \eta_i(\bar{c}, \bar{d}) \\ T + \tau_0(\bar{c}) + \cdots + \tau_{n-1}(\bar{c}) + \varphi_n(b, \bar{c}, \bar{d}) \vdash \zeta_b(\bar{c}, \bar{d}, b) \end{array}\right\} \tag{4}$$

but $M \nvDash \sigma(\bar{c}, \bar{d})$, $M \nvDash \eta_i(\bar{c}, \bar{d})$, and $M \nvDash \zeta_b(\bar{c}, \bar{d}, b)$ for all $1 \leqslant i \leqslant k$ and all $b \in M \setminus \{\bar{c}\}$. Since $b, \bar{d}$ do not appear in $T + \tau_0 + \cdots + \tau_{n-1}$, from (4) we have

$$T + \tau_0(\bar{c}) + \cdots + \tau_{n-1}(\bar{c}) \vdash \forall \bar{y} (\neg \sigma(\bar{c}, \bar{y}) \rightarrow \exists x \varphi_n(x, \bar{c}, \bar{y}))$$

and

$$T + \tau_0(\bar{c}) + \cdots + \tau_{n-1}(\bar{c}) \vdash \forall x, \bar{y} (\varphi_n(x, \bar{c}, \bar{y}) \wedge \textstyle\bigwedge_i \neg \eta_i(\bar{c}, \bar{y}) \rightarrow \bigwedge_i x \neq c_i).$$

Thus if $\theta(x, \bar{c}, \bar{y})$ is an $\mathscr{L} \cup \{\bar{c}\}$ formula and

$$T + \tau_0(\bar{c}) + \cdots + \tau_{n-1}(\bar{c}) \vdash \forall x, \bar{y} (\varphi_n(x, \bar{c}, \bar{y}) \wedge \textstyle\bigwedge_i x \neq c_i \rightarrow \theta(x, \bar{c}, \bar{y}))$$

then

$$T + \tau_0(\bar{c}) + \cdots + \tau_{n-1}(\bar{c}) \vdash \forall \bar{y} \exists x (\neg \sigma(\bar{c}, \bar{y}) \wedge \textstyle\bigwedge_i \neg \eta_i(\bar{c}, \bar{y}) \rightarrow \theta(x, \bar{c}, y)).$$

So, by our induction hypothesis (3), the formula on the right-hand side here is true in M (since it is an $\mathscr{L} \cup \{\bar{c}\}$-formula). Thus putting $\bar{y} = \bar{d}$ we obtain

$$M \vDash \exists x \theta(x, \bar{c}, \bar{d}).$$

This shows that the set $p(x)$ consisting of all $\mathscr{L} \cup \{\bar{c}, \bar{d}\}$ formulas of the form

$$\overbrace{\theta(x, \bar{c}, \bar{d}) \wedge \cdots \wedge \theta(x, \bar{c}, \bar{d})}^{m}$$

such that there is a proof from $T + \tau_0 + \cdots + \tau_{n-1}(\bar{c})$ of the sentence

$$\forall x, \bar{y}(\varphi_n(x, \bar{c}, \bar{y}) \wedge \bigwedge_i (x \neq c_i) \rightarrow \theta(x, \bar{c}, \bar{y}))$$

in $\leqslant m$ steps is finitely satisfied in M, i.e., is a type. It is also recursive, since $T + \tau_0(\bar{c}) + \cdots + \tau_{n-1}(\bar{c})$ is a recursive theory in the language $\mathscr{L}' \cup \{\bar{c}\}$, hence by recursive saturation, $p(x)$ is realized by some $b \in M$, which is clearly not equal to any of the c_i. But then by (4)

$$T + \tau_0(\bar{c}) + \cdots + \tau_{n-1}(\bar{c}) \vdash \forall \bar{y}(\varphi_n(x, \bar{c}, \bar{y}) \rightarrow \zeta_b(\bar{c}, \bar{y}, x))$$

so $\zeta_b(\bar{c}, \bar{d}, x) \wedge \cdots \wedge \zeta_b(\bar{c}, \bar{d}, x)$ is in $p(x)$ for some suitable number of conjuncts. Thus $M \models \zeta_b(\bar{c}, \bar{d}, b)$, contradicting our asusmption on ζ_b. Thus φ_n can be found satisfying (3).

It is obvious that, once all the τ_ns have been constructed, $T' = T \cup \{\tau_i \mid i \in \mathbb{N}\}$ is complete and has both properties (1) and (2). Thus the proof is finished. □

REMARKS. Theorem 15.7 shows that countable recursively saturated structures are resplendent. It seems to show rather more, since infinitely many new relations and functions may be added, provided that the new language is a recursive extension of the old language. In fact we will see soon that this apparently strong notion of resplendency in the conclusions to Theorem 15.7 is, for finite languages $\mathscr{L}$ at least, actually equivalent to the original notion. The reader should also note the reason why *recursive* types appear in the proof of Theorem 15.7: it is because of the recursive nature of the notion of *proof* in a recursively axiomatized theory. (This idea has already been exploited elsewhere in this book, in particular in Corollary 3.10 and Theorem 13.3.)

Theorem 15.7 is not true in general for uncountable models M. For example, let $M = \bigcup_{i<\omega_1} M_i$ be the recursively saturated ω_1-like model of PA constructed in Exercise 15.7, where each $M_i \subseteq M_{i+1}$ is an end-extension, $M_\lambda = \bigcup_{i<\lambda} M_i$, for limit ordinals $\lambda \leqslant \omega_1$, and each M_i is countale and recursively saturated. Then M is not resplendent, since for example the Σ_1^1 sentence $\exists f$ 'f is an embedding of M onto a proper initial segment of M' is consistent with $\mathrm{Th}(M)$, by Friedman's theorem and the downward Löwenheim–Skolem theorem, yet is not true in M since M is ω_1-like.

Theorem 15.7 is extremely powerful. There are many interesting model-theoretic properties that are expressed by recursive set of sentences in languages $\mathscr{L}'$ extending the language $\mathscr{L}$ of the given model. (The reader might like to consider 'there exists a partial inductive satisfaction class' and 'there exists a proper elementary submodel isomophic to the original model' as examples. More examples will be considered soon.) In fact, using the idea of satisfaction classes from Section 15.1, Theorem 15.7 can be made stronger still. We say an $\mathscr{L}$-structure M is *chronically resplendent* iff for all $\bar{a}\in M$ and all Σ_1^1 sentences $\exists \bar{x}\varphi(\bar{X}, \bar{a})$ over the language $\mathscr{L}\cup\{\bar{a}\}$ that is consistent with $\mathrm{Th}(M, \bar{a})$, there is an expansion $(M, \bar{X})$ of M such that $(M, \bar{X})\models\varphi(\bar{x}, \bar{a})$ and $(M, \bar{X})$ is resplendent. The next theorem shows that all countable recursively saturated $\mathscr{L}$-structures are chronically resplendent.

THEOREM 15.8. Let M be a countable recursively saturated structure for the recursive language $\mathscr{L}$, $\bar{a}\in M$, $\mathscr{L}'$ a recusive extension of $\mathscr{L}\cup\{\bar{a}\}$, and let T be a recursively axiomatized $\mathscr{L}'$-theory consistent with $\mathrm{Th}(M, \bar{a})$. Then there is an expansion of $(M, \bar{a})$ to $\mathscr{L}'$ satisfying T that is recursively saturated (and hence also resplendent) as an $\mathscr{L}'$-structure.

Proof. Let $\mathscr{L}''$ be the language $\mathscr{L}'\cup\{+, \cdot, 0, 1, c, <, S\}$, where S is a binary relation symbol and c is a new constant symbol, and consider the $\mathscr{L}''$-theory T^* consisting of axioms from T together with axioms expressing:

(a) $(M, +, \cdot, 0, 1, <)$ is a model of PA;

(b) $c>1+1+\cdots+1$ (n 1s) for each $n\in\mathbb{N}$;

(c) S is a full satisfaction class for M, for the language $\mathscr{L}'\cup\{+, \cdot, 0, 1, <\}$.

(See Exercise 15.2 for hints on how to express (c). Note also that if $\mathscr{L}'$ has finitely many non-logical symbols, (c) is a single $\mathscr{L}''$-sentence. Otherwise (c) will be a recursive axiom scheme of sentences such as

$$\forall\varphi, b, \bar{r}[\varphi=R(\mathsf{v}_{r_1}, \ldots, \mathsf{v}_{r_k})\rightarrow(S(\varphi, b)\leftrightarrow R([b]_{r_1}, \ldots, [b]_{r_k}))]$$
$$\forall\varphi, \psi, \theta, b, r[\varphi=(\psi\wedge\theta)\rightarrow(S(\varphi, b)\leftrightarrow S(\psi, b)\wedge S(\theta, b))]$$

for all symbols R, $\wedge, \ldots$ of $\mathscr{L}'\cup\{+, \cdot, 0, 1, c, <\}$.) By compactness, T^* is consistent with $\mathrm{Th}(M, \bar{a})$: this is proved as follows. M has an expansion $M_T\models T$, the domain of M_T is countable, so can be identified with $\mathbb{N}$ via some bijection, and hence $+, \cdot, 0, 1, <$ can be interpreted in M_T in the obvious way given the identification of the domains of M_T and $\mathbb{N}$. This interpretation clearly satisfies PA, and we may expand the model further by adding $S\subseteq M_T^2$ to $(M_T, +, \cdot, 0, 1, <)$, where S is the obvious standard satisfaction class

$$S=\{(\varphi, [b_0, \ldots, b_n])\mid(M_T, +, \cdot, 0, 1, <)\models\varphi(b_0, \ldots, b_n)\}.$$

Interpreting c in M_T by arbitrarily large integers n, we see that any finite set of axioms from T^* can be made to be true in this expansion, thus $\mathrm{Th}(M, \bar{a})+T^*$ is consistent.

Thus by Theorem 15.7 there is an expansion M_T^* of M satisfying T^*, and by Exercise 15.2 (the analogue of Lachlan's theorem, Theorem 15.5, for $\mathscr{L}''$) the reduct of M_T^* to $\mathscr{L}'$ is recursively saturated. □

We shall now show how to construct resplendent models of arbitrary cardinalities. We first need a well-known result from 'classical' model theory which has a particularly easy proof using Theorem 15.7.

COROLLARY 15.9 (Robinson's joint consistency test). Let $\mathscr{L}$ be a first-order language, let $\mathscr{L}_1$ and $\mathscr{L}_2$ be first-order languages extending $\mathscr{L}$ such that $\mathscr{L}_1 \cap \mathscr{L}_2 = \mathscr{L}$, i.e., the only symbols in both $\mathscr{L}_1$ and $\mathscr{L}_2$ are those of $\mathscr{L}$. Let T be a complete $\mathscr{L}$-theory, and let S_1, S_2 be $\mathscr{L}_1$- and $\mathscr{L}_2$-theories (respectively) extending T. Then $S_1 \cup S_2$ is consistent iff both S_1 and S_2 are consistent.

Proof. We first use compactness to reduce to the case when $\mathscr{L}, \mathscr{L}_1, \mathscr{L}_2$ have only finitely many nonlogical symbols. Suppose $S_1 \cup S_2$ is inconsistent. Then there are finite subsets $R_1 \subseteq S_1$ and $R_2 \subseteq S_2$ such that $R_1 \cup R_2$ is inconsistent. Let $\mathscr{L}_1'$ be the first-order language formed from the nonlogical symbols in R_1, $\mathscr{L}_2'$ that formed from R_2, and $\mathscr{L}'$ that formed from those symbols in both $\mathscr{L}_1'$ and $\mathscr{L}_2'$. Let $T' = \{\sigma \in \mathscr{L}' \mid T \vdash \sigma\}$, $S_1' = T' \cup R_1$ and $S_2' = T' \cup R_1$. Then T' is a complete $\mathscr{L}'$-theory. We must show that either S_1' is inconsistent or S_2' is.

Suppose T' is consistent, and (by Proposition 11.4 and Theorem 15.7) let $M \models T'$ be a countable resplendent model of T'. Then $T' = \mathrm{Th}(M)$, since T' is complete, so by the definition of resplendency, if both S_1' and S_2' are consistent then M has expansions $(M, \bar{R}_1, \bar{F}_1, \bar{C}_1) \models S_1'$ and $(M, \bar{R}_2, \bar{F}_2, \bar{C}_2) \models S_2'$. But since the languages $\mathscr{L}_1'$ and $\mathscr{L}_2'$ are disjoint except for the common sublanguage $\mathscr{L}'$ of M,

$$(M, \bar{R}_1, \bar{R}_2, \bar{F}_1, \bar{F}_2, \bar{C}_1, \bar{C}_2) \models S_1' \cup S_2',$$

so $S_1' \cup S_2'$ is consistent, a contradiction. □

THEOREM 15.10. Let M be an $\mathscr{L}$-structure, where $\mathscr{L}$ is a first-order language of cardinality $\leqslant \mathrm{card}(M)$. Then there is an elementary extension $N \succ M$ with $\mathrm{card}(N) = \mathrm{card}(M)$ and N is resplendent.

Proof. We construct N as the reduct to $\mathscr{L}$ of $\bigcup_{i \in \mathbb{N}} M_i$, where each M_i is an $\mathscr{L}_i$-structure, M_0 being M itself, $\mathscr{L}_0 = \mathscr{L}$, each $\mathscr{L}_{i+1}$ being an expansion of $\mathscr{L}_i$, and $M_i \prec M_{i+1} \upharpoonright \mathscr{L}_i$ for each i. To construct M_{i+1} from M_i we assume inductively that $\mathrm{card}(M_i) = \mathrm{card}(M)$ and enumerate all Σ_1^1 formulas over $\mathscr{L}$ with parameters $\bar{a}$ from M_i as $\{\Phi_j(\bar{a}) \mid j < \mathrm{card}(M)\}$. (By the assumption on card $(\mathscr{L}) \leqslant \mathrm{card}(M)$ there are at most $\mathrm{card}(M)$ such formulas.) M_{i+1} will be the union of a chain $\{M_{i,j} \mid j < \mathrm{card}(M)\}$ where $M_{i,j}$ is a structure for some language $\mathscr{L}_{i,j} \supseteq \mathscr{L}_i$, $\mathscr{L}_{i,j+1} \supseteq \mathscr{L}_{i,j}$, and $M_{i,j} \prec M_{i,j+1} \upharpoonright \mathscr{L}_{i,j}$ for each $j < \text{card}(M)$, $\mathscr{L}_{i,\lambda} = \bigcup_{j<\lambda} \mathscr{L}_{i,j}$ and $M_{i,\lambda} = \bigcup_{j<\lambda} M_{i,j}$ for limit ordinals $\lambda \leqslant \text{card}(M)$. We will ensure that each $M_{i,j}$ has cardinality $\text{card}(M)$, so M_{i+1} will also have this cardinality.

$M_{i,0}$ is taken equal to M_i, and $\mathscr{L}_{i,0} = \mathscr{L}_i$. Given $M_{i,j}$ let $\bar{Y}$ denote the new relation and function symbols in $\mathscr{L}_{i,j}$ but not in $\mathscr{L}$, and consider the jth Σ_1^1 formula $\Phi_j(\bar{a}) = \exists \bar{X} \varphi(\bar{X}, \bar{a})$ in our enumeration. Without loss of generality we may assume X, Y are disjoint. If $\Phi_j(\bar{a})$ is consistent with the theory of

$$M_i \upharpoonright \mathscr{L} \cup \{\bar{a}\}$$

(which is $\text{Th}(M_{i,j} \upharpoonright \mathscr{L} \cup \{\bar{a}\})$, since $\bar{a} \in M_i$ and $M_i \prec M_{i,j} \upharpoonright \mathscr{L}_i$) then

$$\varphi(\bar{X}, \bar{a}) + \text{Th}(M_{i,j}, c)_{c \in M_{i,j}}$$

is consistent by Robinson's joint consistency test. In this case we set $\mathscr{L}_{i,j+1} = \mathscr{L}_{i,j} \cup \{\bar{X}\}$ and

$$M_{i,j+1} \vDash \varphi(\bar{X}, \bar{a}) + \text{Th}(M_{i,j}, c)_{c \in M_{i,j}}$$

of cardinality $\text{card}(M)$, by the completeness theorem, and as usual we may assume $M_{i,j} \prec M_{i,j+1} \upharpoonright \mathscr{L}_{i,j}$. Otherwise we set $\mathscr{L}_{i,j+1} = \mathscr{L}_{i,j}$ and $M_{i,j+1} = M_{i,j}$.

It is now only necessary to check that N, the reduct of $\bigcup_i M_i = \bigcup_{i,j} M_{i,j}$ to $\mathscr{L}$, is resplendent. But if $\exists \bar{X} \varphi(\bar{X}, \bar{a})$ is a Σ_1^1 formula consistent with $\text{Th}(N, \bar{a})$. then $\bar{a} \in M_i$ for some i, $\exists \bar{X} \varphi(\bar{X}, \bar{a}) = \Phi_j(\bar{a})$ some j, and, since $M_i \upharpoonright \mathscr{L} \prec N$,

$$\text{Th}(N, \bar{a}) = \text{Th}(M_i, \bar{a}) \upharpoonright \mathscr{L} \cup \{\bar{a}\}.$$

Therefore $M_{i,j+1}$ has relations and functions $\bar{X}$ satisfying $\varphi(\bar{X}, \hat{a})$, and so $\bigcup_{i,j} M_{i,j} \vDash \varphi(\bar{X}, \bar{a})$ since this union is an elementary extension of $M_{i,j+1}$ for the language $\mathscr{L} \cup \{\bar{X}\}$ by the lemma on elementary chains. □

The next result uses the idea of a satisfaction class from Section 15.1 to increase our stock of Σ_1^1 formulas considerably.

THEOREM 15.11 (Kleene 1952). Let $\mathscr{L}$ be a first-order language with finitely many relations, functions and constant symbols, and let $\{\theta_i(\bar{x}) \mid i \in \mathbb{N}\}$ be a recursive set of $\mathscr{L}$-formulas involving finitely many free-variables $\bar{x}$. Then there is a Σ_1^1 formula $\Theta(\bar{x})$ over $\mathscr{L}$ such that, in any infinite $\mathscr{L}$-structure M,

$$M \vDash \forall \bar{x} (\Theta(\bar{x}) \leftrightarrow \bigwedge\!\!\bigwedge_{i=0}^{\infty} \theta_i(\bar{x})).$$

Proof. We first reduce to the case when $\mathscr{L}$ only contains finitely many *relation* symbols. To get rid of the constants $\bar{c}$ in L, replace them by new variables $\bar{y}$ obtaining a recursive set $\{\theta_i'(\bar{x},\bar{y}) \mid i \in \mathbb{N}\}$ of formulas. Then, if we can find $\Theta(\bar{x},\bar{y})$ equivalent to $\bigwedge\!\!\!\bigwedge_{i=0}^{\infty}\theta_i'(\bar{x},\bar{y})$, $\Theta(\bar{x},\bar{c})$ is equivalent to $\bigwedge\!\!\!\bigwedge_{i=0}^{\infty}\theta_i(\bar{x})$. To get rid of the function symbols $\bar{f}$ in $\mathscr{L}$, replace each n-ary function symbol f by an $(n+1)$-ary relation symbol $R_f(x_1,\ldots,x_n,x_{n+1})$ representing '$f(x_1,\ldots,x_n)=x_{n+1}$'. Then, for each formula $\theta_i(\bar{x})$, replace any atomic subformula $R(\ldots,f(t_1,\ldots,t_n),\ldots)$ by

$$\exists y(R_f(t_1,\ldots,t_n,y)\wedge R(\ldots,y,\ldots)),$$

where R is any relation symbol, f is a function symbol of $\mathscr{L}$, $t_1,\ldots,t_n$ are $\mathscr{L}$-terms, and y is a new variable not occurring elsewhere in θ_i. By repeating this operation, all function symbols f of $\mathscr{L}$ are eliminated in favour of the new relation symbols and, moreover, this operation is recursive, so the set of formulas $\{\theta_i'(\bar{x}) \mid i \in \mathbb{N}\}$ obtained in this way is a recursive set of formulas in the new language. If $\Theta(\bar{x})$ is Σ_1^1 and equivalent to $\bigwedge\!\!\!\bigwedge_{i=0}^{\infty}\theta_i'(\bar{x})$ in this new language, then

$$\exists R_{f_1},\ldots,R_{f_n}[\bigwedge\!\!\!\bigwedge_{j=1}^{n}\forall y_1,\ldots,y_{k_j},y_{k_j+1}(R_{f_j}(y_1,\ldots,y_{k_j},y_{k_j+1}) \leftrightarrow f(y_1,\ldots,y_{k_j})=y_{k_j+1})\wedge\Theta(\bar{x})]$$

is a Σ_1^1 formula over $\mathscr{L}$ equivalent to $\bigwedge\!\!\!\bigwedge_{i=0}^{\infty}\theta_i(\bar{x})$, where $f_1,\ldots,f_n$ are the function symbols of $\mathscr{L}$, f_j having arity k_j.

Now suppose that $\mathscr{L}$ contains as its non-logical symbols only relation symbols $R_1,\ldots,R_n$, where each R_i is an l_i-ary relation symbol, and suppose that we have some Gödel-numbering $\ulcorner\cdot\urcorner$ for formulas of $\mathscr{L}$. Since $\{\theta_i \mid i \in \mathbb{N}\}$ is a recursive set of formulas of $\mathscr{L}$, by Corollary 3.7 there is a formula $\tau(v)$ of $\mathscr{L}_A$ that represents $\{\ulcorner\theta_i(\mathrm{v}_0,\ldots\mathrm{v}_k)\urcorner \mid i \in \mathbb{N}\}$. Now consider the Σ_1^1 formula $\exists +,\cdot,<,0,1,M,S,f,g\,\psi(x_0,\ldots,x_k)$, where $+,\cdot,f$ are binary function symbols, g is a ternary function symbol, $<,S$ are binary relation symbols, M is a unary relation symbol, 0, 1 are constant symbols, and $\psi(\bar{x})$ is the following formula:

$$
\begin{aligned}
&M(0)\wedge M(1)\wedge\forall x,y(M(x)\wedge M(y)\to M(x+y)\wedge M(x\cdot y))\\
&\quad\wedge\text{`}(M,+,\cdot,<,0,1)\text{ satisfies }PA^{-}\text{'}\\
&\quad\wedge\forall u,v,z,w(M(u)\wedge M(v)\wedge u\neq v\to\\
&\qquad\qquad f(g(w,z,v),v)=z\wedge f(g(w,z,v),u)=f(w,u))\\
&\quad\wedge\text{`}S\text{ is a satisfaction class for }\mathscr{L}\text{'}\\
&\quad\wedge\forall u(M(u)\wedge\text{`}(M,+,\cdot,<,0,1)\vDash\tau(u)\text{'}\to\\
&\qquad\quad S(u,g(\cdots g(g(g(0,x_0,0),0),x_1,1),x_2,2),\ldots),x_k,k))).
\end{aligned}
$$

$\psi(\bar{x})$ says that M (when considered as a subset of the model in hand) forms a model of PA^- with $+, \cdot, <, 0, 1$, and that $f(v, y)$ and $g(w, z, y)$ behave like $[v]_y$ and $w[z/y]$ respectively (see Section 9.1) over the model in hand, provided $y \in M$. 'S is a full satisfaction class for $\mathscr{L}$' is taken to be an abbreviation for

$$\forall u, a(S(u, a) \leftrightarrow M(u)$$
$$\wedge[\exists i, j \in M(u = (\mathsf{v}_i = \mathsf{v}_j) \wedge f(a, i) = f(a, j))$$
$$\vee \textstyle\bigvee\!\!\bigvee_{j=1}^{n} \exists i_1, \ldots, i_{l_j} \in M(u = R_j(\mathsf{v}_{i_1}, \ldots, \mathsf{v}_{i_{l_j}}) \wedge R_j(f(a, i_i), \ldots, f(a, i_{l_j})))$$
$$\vee \exists u_1 \in M(u = \neg u_1 \wedge \neg S(u_1, a))$$
$$\vee \exists u_1, u_2 \in M(u = (u_1 \wedge u_2) \wedge S(u_1, a) \wedge S(u_2, a))$$
$$\vee \exists u_1, u_2 \in M(u = (u_1 \vee u_2) \wedge (S(u_1, a) \vee S(u_2, a)))$$
$$\vee \exists u_1, i \in M(u = \exists \mathsf{v}_i u_1 \wedge \exists b S(u_1, g(a, b, i)))$$
$$\vee \exists u_1, i \in M(u = \forall \mathsf{v}_i u_1 \wedge \forall b S(u_1, g(a, b, i)))])$$

where, for example '$u = (\mathsf{v}_i = \mathsf{v}_j)$' is an $\mathscr{L}_A$-formula interpreted in $(M, +, \cdot, <, 0, 1)$ that represents the set

$$\{(x, y, z) \mid x = \ulcorner(\mathsf{v}_y = \mathsf{v}_z)\urcorner\} \subseteq \mathbb{N}^3$$

in PA^-, according to the definition in Section 3.2, and all the other syntactic operations are treated similarly. The final conjunct of $\psi(\bar{x})$ states that 'S thinks each $\theta_i(x_0, \ldots, x_k)$ is true',

$$g(\cdots g(g(0, x_0, 0), x_1, 1), \ldots), x_k, k)$$

being a convenient code for the sequence $x_0, x_1, \ldots, x_k$ and '2', '3', ..., 'k' are interpreted as $(\cdots(1+1)+\cdots+1)$ as usual, for the appropriate number of 1s.

We must show that this Σ_1^1 formula is equivalent to $\bigwedge\!\!\bigwedge_{i=0}^{\infty} \theta_i(\bar{x})$ in all infinite $\mathscr{L}$-structures. For one direction, if we have an $\mathscr{L}$-structure N with an expansion satisfying $\psi(\bar{x})$, then (by representability) it is easy to check that, for all $y_0, \ldots, y_r \in N$,

$$\forall a S(\theta(\mathsf{v}_0, \ldots, \mathsf{v}_r), g(\cdots(g(g(a, y_0, 0), y_1, 1), \ldots), y_r, r))$$

holds in this expansion if and only if $N \vDash \theta(y_0, \ldots, y_r)$. Then by the last clause of $\psi(\bar{x})$ we have $N \vDash \theta_i(x_0, \ldots, x_k)$ for all $i \in \mathbb{N}$. Conversely, if N is an infinite $\mathscr{L}$-structure satisfying $\bigwedge\!\!\bigwedge_{i=0} \theta_i(\bar{x})$, let $M \subseteq N$ be a countable subset of N, and identify M with $\mathbb{N}$. Thus M inherits $\mathscr{L}_A$-structure from $\mathbb{N}$. (These functions $+, \cdot$, and relation $<$ on M can be extended to the whole of N in any way you like). Let $h: N^{<\omega} \to N$ be any bijection from the set $N^{<\omega}$ of sequences from N of finite length onto the set N, and define $f(a, i)$ = the ith term of the sequence $h^{-1}(a)$, for each $a \in N$ and $i \in \mathbb{N}$, and define

$$g(a, b, i) = h(c_0, c_1, \ldots, c_{i-1}, b, c_{i+1}, \ldots, c_n)$$

where $c_0, c_1, \ldots, c_i, \ldots, c_n$ is the sequence $h^{-1}(a)$. (This sequence is padded out with zeros as in Section 9.1 if $i > n$.) f, g can be extended to functions on N in any way whatsoever. Finally, let S be the relation on N^2 given by

$$(u, a) \in S \Leftrightarrow \text{`}u = \theta(\mathrm{v}_0, \ldots, \mathrm{v}_r) \text{ for some } \mathscr{L}\text{-formula } \theta$$
$$\text{and } N \models \theta(f(a, 0), \ldots, f(a, r))\text{'}.$$

It is then obvious that this expansion of N satisfies $\psi(\bar{x})$. □

Our first application of Kleene's theorem is the promised converse to Proposition 15.4.

COROLLARY 15.12. Let $M \models PA$ be resplendent. Then M has a partial inductive nonstandard satisfaction class. In particular, if M is countable then M is recursively saturated iff such a satisfaction class exists for M.

Proof. 'There exists a partial inductive satisfaction class' is expressed by

$$\exists S, a\{\textstyle\bigwedge\!\!\!\bigwedge_{n\in\mathbb{N}} a > n \wedge \forall \varphi, a(S(\varphi, a) \leftrightarrow \mathrm{form}(\varphi) \wedge \varphi < a$$
$$\wedge \textstyle\bigwedge\!\!\!\bigwedge_{i=1}^{7} T_i(\varphi, a)) \wedge \bigwedge\!\!\!\bigwedge_{\psi \in \mathscr{L}_S} I_x\psi\}.$$

Both of the infinite conjunctions are recursive, so by Theorem 15.11 equivalent to Σ_1^1 formulas, hence this whole formula is Σ_1^1. It is consistent with $\mathrm{Th}(M)$ by Proposition 15.1, hence is satisfied by M, by the definition of resplendency.

The 'in particular ...' part now follows from Theorem 15.7 and Proposition 15.4. □

Corollary 15.12 and Proposition 15.4 show that any resplendent model of PA is recursively saturated. This is true in general.

COROLLARY 15.13. Let M be a resplendent $\mathscr{L}$-structure, where $\mathscr{L}$ contains at most finitely many relation, function and constant symbols. Then M is recursively saturated.

Proof. If M is finite this is trivial.

If M is infinite, $\bar{a} \in M$, and $p(\bar{x}) = \{\theta_i(\bar{x}, \bar{a}) \mid i \in \mathbb{N}\}$ is a recursive type over M then, since $\mathscr{L}$ has finitely many nonlogical symbols, by Kleene's theorem

$$\exists \bar{x} \bigwedge\!\!\!\bigwedge_{i=0}^{\infty} \theta_i(\bar{x}, \bar{a}) \tag{*}$$

is Σ_1^1 over M, and since $p(\bar{x})$ is finitely satisfied, (*) is consistent with $\mathrm{Th}(M, \bar{a})$, hence is true in M, i.e., $p(\bar{x})$ is realized in M. □

Note that countability of M is not required for Corollary 15.13, unlike Theorem 15.7.

We conclude this chapter by applying these results to some properties of countable recursively saturated models and of resplendent models. In particular we shall prove a classification theorem for the countable recursively saturated models of a wide class of theories, including PA.

DEFINITION. Let M be an $\mathscr{L}$-structure. Recall that

$$tp_M(\bar{a}) = \{\theta(\bar{x}) \mid M \models \theta(\bar{a}),\ \theta \text{ an } \mathscr{L}\text{-formula}\}$$

for any tuple $\bar{a} \in M$ of finite length. We define

$$S(M) = \{tp_M(a_1, \ldots, a_n) \mid n \in \mathbb{N} \text{ and } a_1, \ldots, a_n \in M\}.$$

LEMMA 15.14. Let M, N be recursively saturated $\mathscr{L}$-structures, where $\mathscr{L}$ is a recursive language. Suppose $\mathrm{Th}(M) = \mathrm{Th}(N)$ and $S(M) = S(N)$. Then if $\bar{a} \in M, \bar{b} \in N$ are tuples of the same finite length, with $tp_M(\bar{a}) = tp_N(\bar{b})$ and $c \in M$ is arbitrary, then there exists $d \in N$ such that $tp_M(\bar{a}, c) = tp_N(\bar{b}, d)$.

Proof. Let $\bar{e}, f \in N$ satisfy $tp_N(\bar{e}, f) = tp_M(\bar{a}, c)$, using $S(M) = S(N)$. d is chosen to satisfy the recursive type

$$p(x) = \{\theta(\bar{e}, f) \to \theta(\bar{b}, x) \mid \theta \text{ an } \mathscr{L}\text{-formula}\}.$$

This is clearly recursive, and is a type since $tp_N(\bar{e}, f) = tp_M(\bar{a}, c)$ and $tp_M(\bar{a}) = tp_N(\bar{b})$, so if $N \models \bigwedge\!\!\!\bigwedge_{i=1}^{n} \theta_i(\bar{e}, f)$ then $M \models \bigwedge\!\!\!\bigwedge_{i=1}^{n} \theta_i(\bar{a}, c)$ so $M \models \exists x \bigwedge\!\!\!\bigwedge_{i=1}^{n} \theta_i(\bar{a}, x)$; hence $N \models \exists x \bigwedge\!\!\!\bigwedge_{i=1}^{n} \theta_i(\bar{b}, x)$. □

COROLLARY 15.15. Let M, N be countable, recursively saturated structures for a recursive language $\mathscr{L}$. Then $M \cong N$ iff $\mathrm{Th}(M) = \mathrm{Th}(N)$ and $S(M) = S(N)$.

Proof. Enumerate $M = \{a_i \mid i \in \mathbb{N}\}$, $N = \{b_i \mid i \in \mathbb{N}\}$, and, assuming $\mathrm{Th}(M) = \mathrm{Th}(N)$ and $S(M) = S(N)$, construct an isomorphism $h: M \to N$ by 'back-and-forth', using Lemma 15.14. That is, define $h: u_i \mapsto v_i$ for each $i \in \mathbb{N}$ where $u_0, u_1, \ldots \in M$ and $v_0, v_1, \ldots \in M$ are constructed so that

$$tp_M(v_0, v_1, \ldots, v_n) = tp_N(u_0, u_1, \ldots, u_n) \qquad (*)$$

for each $n \in \mathbb{N}$. If $n = 2k$, we set $v_n = a_k$ and find $u_n \in N$ satisfying (*) using the lemma. If $n = 2k+1$ is odd we set $u_n = b_k$ and find $v_n \in M$ satisfying (*) by the lemma again. This way h will be an isomorphism.

The converse, $M \cong N \Rightarrow \mathrm{Th}(M) = \mathrm{Th}(N) \& S(M) = S(N)$ is trivial. □

DEFINITION. Let M be an $\mathscr{L}$-structure. M is *ω-homogeneous* iff for all tuples $\bar{a}, \bar{b} \in M$ of finite length such that $tp_M(\bar{a}) = tp_M(\bar{b})$ and for all $c \in M$ there is $d \in M$ such that $tp_M(\bar{a}, c) = tp_M(\bar{b}, d)$. M is *homogeneous* (some authors use: *strongly ω-homogeneous*) iff for all $\bar{a}, \bar{b} \in M$ of finite length, $tp_M(\bar{a}) = tp_M(\bar{b})$ implies that there is an automorphism h of M sending the tuple $\bar{a}$ to $\bar{b}$.

Obviously homogeneous implies ω-homogeneous, and (by a back-and-forth argument) if M is countable and ω-homogeneous then it is homogeneous. Lemma 15.14 then implies

COROLLARY 15.16. Any recursively saturated model M for a recursive language $\mathscr{L}$ is ω-homogeneous. If $\mathscr{L}$ has finitely many non-logical symbols and M is resplendent, then M is homogeneous.

Proof. For the second part, if $\mathscr{L}$ is finite, note that

'there exists an automorphism h sending $\bar{a}$ to $\bar{b}$'

is Σ_1^1, and if $tp_M(\bar{a}) = tp_M(\bar{b})$ this sentence is consistent with $\mathrm{Th}(M, \bar{a}, \bar{b})$ by the existence of countable recursively saturated (and hence homogeneous) models of $\mathrm{Th}(M, \bar{a}, \bar{b})$. □

Next, we shall show that resplendent models have non-trivial automorphisms. The main lemma needed to show the consistency of this statement is

LEMMA 15.17. Let M be an infinite $\mathscr{L}$-structure. Then there exists $N > M$ and $a, b \in N$ such that $a \neq b$ and $tp_N(a) = tp_N(b)$.

Proof. We must show that

$$\mathrm{Th}(M, c)_{c\in M} + \{\varphi(a) \leftrightarrow \varphi(b) \mid \varphi(x) \text{ an } \mathscr{L}\text{-formula}\} + a \neq b$$

is consistent in the language $\mathscr{L} \cup M \cup \{a, b\}$, where elements of $M \cup \{a, b\}$ are regarded as new constants. This follows by compactness, since given $\varphi_1, \ldots, \varphi_n$ and writing $(\neg)^\varepsilon\varphi(x)$ for $\varphi(x)$ if $\varepsilon = 0$ or $\neg\varphi(x)$ if $\varepsilon = 1$, there are at most 2^n subsets

$$S_{\varepsilon_1,\ldots,\varepsilon_n} = \{x \in M \mid M \vDash (\neg)^{\varepsilon_1}\varphi_1(x) \wedge \cdots \wedge (\neg)^{\varepsilon_n}\varphi_n(x)\}$$

where $\varepsilon_i \in \{0, 1\}$ for each i. As M is infinite, one such subset is infinite, so there are $a \neq b$ in this set, which clearly satisfy $\varphi_i(a) \leftrightarrow \varphi_i(b)$ for $i = 1, \ldots, n$. □

COROLLARY 15.18. Let M be an infinite $\mathscr{L}$-structure, where
(a) M is countable, recursively saturated and $\mathscr{L}$ is a recursive language, or
(b) M is resplendent and $\mathscr{L}$ is finite.
Then for all $a_1, \ldots a_n \in M$ of finite length n, there is an automorphism h of M sending some b to some $c \neq b$, but fixing each a_i.

Proof. Consider the type

$$p(x, y) = \{x \neq y\} \cup \{\varphi(x, \bar{a}) \leftrightarrow \varphi(y, \bar{a}) \mid \varphi \text{ an } \mathscr{L}\text{-formula}\}.$$

This is clearly recursive and is a type over M, by Lemma 15.17, so is realized by some $b, c \in M$. Since $tp_M(\bar{a}, b) = tp_M(\bar{a}, c)$, by homogeneity of M there is an automorphism h sending $\bar{a}, b$ to $\bar{a}, c$. □

It actually follows from Corollary 15.18 that if M is a *countable* recursively saturated $\mathscr{L}$-structure, then M has $2^{\aleph_0}$ automorphisms. This is proved by constructing $\bar{a}_\sigma, \bar{b}_\sigma \in M$ for all finite sequences σ of 0s and 1s (where $\bar{a}_\sigma$ and $\bar{b}_\sigma$ are all triples) such that whenever $\tau: \mathbb{N} \to \{0, 1\}$, then the function defined by $\bar{a}_{\sigma\restriction i} \mapsto \bar{b}_{\sigma\restriction i}$ for each $i \in \mathbb{N}$ is an automorphism. To do this, suppose

$$\bar{a} = (\bar{a}_\varnothing, \ldots, \bar{a}_{\sigma\restriction i}, \ldots, \bar{a}_\sigma) \quad \text{and} \quad \bar{b} = (\bar{b}_\varnothing, \ldots, \bar{b}_{\sigma\restriction i}, \ldots, \bar{b}_\sigma)$$

have been constructed so that $tp_M(\bar{a}) = tp_M(\bar{b})$ and $M = c_0, c_1, \ldots, c_i, \ldots$ is some fixed enumeration. Set $s = r = c_n$, where n is the length of σ, and, using ω-homogeneity twice, find $r', s' \in M$ such that $tp_M(\bar{a}, r, s') = tp_M(\bar{b}, r', s)$. Let h be an automorphism of M sending $\bar{a}, r, s'$ to $\bar{b}, r', s$, and, using Corollary 15.18, let $u \neq v \in M$ and an automorphism k of M fixing a, b, r, s, r', s' and sending u to v. We then put $a_{\sigma 0} = a_{\sigma 1} = (r, s', u)$, $b_{\sigma 0} =$

$(r', s, h(u))$ and $b_{\sigma 1} = (r', s, h(k(u)))$. It is then easy to check that

$$tp_M(\bar{a}, \bar{a}_{\sigma 0}) = tp_M(\bar{b}, \bar{b}_{\sigma 0}) = tp_M(\bar{b}, \bar{b}_{\sigma 1}),$$

and that the map $\bar{a}_{\tau \restriction i} \mapsto \bar{b}_{\tau \restriction i} (i \in \mathbb{N})$ is an automorphism of M for any given $\tau: \mathbb{N} \to \{0, 1\}$. Yet since $v = k(u) \neq u$, $h(k(u)) \neq h(u)$, so two different functions $\mathbb{N} \to \{0, 1\}$ give rise to two different automorphisms in this way. In fact this argument shows that if M is countable and for any finite set $A \subseteq M$ there is a nontrivial automorphism $h \in \mathrm{Aut}(M)$ such that $h(a) = a$ for all $a \in A$, then M has $2^{\aleph_0}$ automorphisms. (It is a nice exercise to prove a converse to this, that if M is countable and has fewer than $2^{\aleph_0}$ automorphisms then there is a finite set $A \subseteq M$ such that no non-trivial automorphism h of M fixes every $a \in A$. Deduce that if M has $> \aleph_0$ automorphisms then it has exactly $2^{\aleph_0}$ automorphisms.)

We now show how to define a notion of *standard system* for a large class of recursively saturated models. The following definition is from Jensen and Ehrenfeucht (1976).

DEFINITION. A theory T in a recursive language $\mathscr{L}$ is *rich* iff there is a recursive sequence $\{\varphi_n(v) \mid n \in \mathbb{N}\}$ of $\mathscr{L}$-formulas such that, for all disjoint finite sets $X, Y \subseteq \mathbb{N}$,

$$T \vdash \exists v (\textstyle\bigwedge\!\!\!\bigwedge_{n \in X} \varphi_n(v) \wedge \bigwedge\!\!\!\bigwedge_{m \in Y} \neg \varphi_m(v)).$$

The idea of this definition is that models M of rich theories can code arbitrary finite sets $A \subseteq \{0, 1, \ldots, n\}$ by some element $a \in M$ so that $M \vDash \varphi_k(a) \Leftrightarrow k \in A$ for all $k \leqslant n$. We shall see in a moment that recursively saturated models can code much more than this. As examples of rich theories, taking $\varphi_n(v) =$ '$(v)_n \neq 0$' (see p. 63), we see that PA is rich. Also, by putting $\varphi_n(v)$ equal to

$$\exists x (v = \overbrace{x + x + x + \cdots + x}^{p_n})$$

where p_n is the nth prime, $\mathrm{Th}(\mathbb{N}, +, 0, 1)$ is rich. There are many non-rich theories too, however: see Exercise 15.14 for two examples.

DEFINITION. If T is a rich theory in a recursive language $\mathscr{L}$ and $M \vDash T$ is recursively saturated, we define the *standard system* of M, $SSy(M)$, to be

$$SSy(M) = \{A \subseteq \mathbb{N} \mid A = \{n \in \mathbb{N} \mid M \vDash \varphi_n(a)\} \text{ for some } a \in M\},$$

the set of all $A \subseteq \mathbb{N}$ *coded* in M.

The following lemma shows that the definition of 'standard system' does not depend on the choice of the sequence φ_n. It is much more important than it may seem at first sight.

LEMMA 15.19. Let $M \models T$ be recursively saturated, where T is rich and both sequences $\{\varphi_n(v) \mid n \in \mathbb{N}\}$ and $\{\psi_n(v) \mid n \in \mathbb{N}\}$ satisfy the hypotheses in the definition of 'T is rich'. Then for all $a \in M$ there is $b \in M$ such that

$$\{n \in \mathbb{N} \mid \varphi_n(a)\} = \{n \in \mathbb{N} \mid M \models \psi_n(b)\}.$$

Proof. Given $a \in M$ consider

$$p(x) = \{\psi_n(x) \leftrightarrow \varphi_n(a) \mid n \in \mathbb{N}\}.$$

This is recursive since both $\{\psi_n \mid n \in \mathbb{N}\}$ and $\{\varphi_n \mid n \in \mathbb{N}\}$ are, and is finitely satisfied by the hypotheses on $\{\psi_n \mid n \in \mathbb{N}\}$. Thus, by recursive saturation, $p(x)$ is satisfied by some $b \in M$ with $\{n \mid M \models \psi_n(b)\} = \{n \mid M \models \varphi_n(a)\}$. $\square$

It follows from this lemma that, in the case when $M \models PA$ is recursively saturated, the notion $SSy(M)$ just defined agrees with the definition given in Chapter 11, and in particular $SSy(M)$ is a Scott set. This is, in fact, true in general.

THEOREM 15.20. Let M be a recursively saturated model of a rich theory T. Then $SSy(M)$ is a Scott set.

Proof. This can be proved directly, in a way similar to the proof of Theorem 13.2, or alternatively, we can use Theorem 13.2 and the other results of this section.

Suppose, for example, that $A = \{n \in \mathbb{N} \mid M \models \varphi_n(a)\}$ is a tree and we wish to find an infinite path of A in $SSy(M)$. By the downward Löwenheim–Skolem theorem there is a countable model $L_0 \prec M$ containing a. By the proof of Proposition 11.4, making sure each recursive type that is realized is realized *inside* M, L_0 has a countable elementary extension $L \prec M$ that is recursively saturated. By Theorem 15.8 there is a recursively saturated expansion $L' = (L, +, \cdot, <, 0, 1)$ of L such that the arithmetic part $L' \restriction \mathcal{L}_A$ is a model of PA. L' has associated with it two notions of standard system, namely that of its reduct $L' \restriction \mathcal{L}_A$, considered as a model of PA, and that of L considered as a model of T. By Lemma these notions are the same, and by Theorem 13.2 this standard system is a Scott set. Thus there is $b \in L$ such that $B = \{n \in \mathbb{N} \mid L \models \varphi_n(b)\}$ is an infinite path of A. But as $L \prec M$, $B = \{n \in \mathbb{N} \mid M \models \varphi_n(b)\}$, and so $SSy(M)$ is closed under König's lemma. The other closure properties are proved in exactly the same way. □

The next lemma should seem very familiar after Lemmas 12.1 and 12.2.

LEMMA 15.21. Let M be a recursively saturated model of a rich theory T. Then

(a) for all $\bar{a} \in M$ (of finite length) $tp_M(\bar{a}) \in SSy(M)$, and
(b) if $p(\bar{x})$ is a type over M and

$$\{\varphi(\bar{v}, \bar{w}) \mid \varphi(\bar{x}, \bar{a}) \in p(\bar{x})\} \in SSy(M),$$

then $p(\bar{x})$ is realized in M.

Proof. (a) Consider the type

$$q(x) = \{\theta(\bar{a}) \leftrightarrow \varphi_{\theta(\bar{v})}(x) \mid \theta(\bar{v}) \text{ an } \mathscr{L}\text{-formula}\}.$$

This is a type over M by the assumption on the sequence $\{\varphi_n(v) \mid n \in \mathbb{N}\}$ in the definition of 'rich', and it is recursive since $\mathscr{L}$ is recursive and so is $\{\varphi_n \mid n \in \mathbb{N}\}$. Clearly any b realizing $q(x)$ in M has $tp_M(\bar{a}) = \{\theta(\bar{v}) \mid M \models \varphi_{u(\bar{v})}(\bar{b})\}$.

(b) Consider the type

$$r(\bar{x}) = \{\varphi_{\theta(\bar{v}, \bar{w})}(b) \to \theta(\bar{x}, \bar{a}) \mid \theta(\bar{v}, \bar{w}) \text{ an } \mathscr{L}\text{-formula}\},$$

where $b \in M$ is such that

$$\{\theta(\bar{v}, \bar{w}) \mid \theta(\bar{x}, \bar{a}) \in p(\bar{x})\} = \{n \in \mathbb{N} \mid M \models \varphi_n(b)\}.$$

Then $r(\bar{x})$ is finitely satisfied since $p(\bar{x})$ is, and $r(\bar{x})$ is recursive. Hence $r(\bar{x})$ is realized by some $\bar{c} \in M$. Clearly any such $\bar{c}$ then realizes the type $p(\bar{x})$. □

COROLLARY 15.22. Let M be a countable recursively saturated structure for a recursive language $\mathscr{L}$. Then M is $\mathscr{X}$-saturated for some Scott set $\mathscr{X}$.

Proof. By Theorem 15.8 let $M' = (M, +, \cdot, <, 0, 1)$ be a recursively saturated expansion of M, so that the 'arithmetic part' $M' \upharpoonright \mathscr{L}_A$ is a model of PA.

Let $\mathscr{X} = SSy(M')$. Then $\mathscr{X}$ is a Scott set, and, for every complete type $p(\bar{x}) = \{\theta_i(\bar{x}, \bar{a}) \mid i \in \mathbb{N}\}$ over M in the language $\mathscr{L}$, and for all $\bar{b} \in M$,

$\bar{b}$ realizes $p(\bar{x})$ in M

$\Leftrightarrow \bar{b}$ realizes $p(\bar{x})$ in M'

$\Leftrightarrow tp_{M'}(\bar{b}, \bar{a}) \in \mathscr{X}$, and

$$p(\bar{x}) = \left\{ \theta(\bar{x}, \bar{a}) \middle| \begin{array}{l} \theta \text{ is an } \mathscr{L}\text{-formula and} \\ \theta(\bar{x}, \bar{a}) \in tp_{M'}(\bar{b}, \bar{a}) \end{array} \right\}, \qquad \text{(by Lemma 15.21)},$$

$\Leftrightarrow \{\theta_i(\bar{x}, \bar{y}) \mid i \in \mathbb{N}\} \in \mathscr{X}$, since this set is recursive in the oracle $tp_{M'}(\bar{b}, \bar{a}) \in \mathscr{X}$, as the language of M' is a recusive extension of $\mathscr{L}$. □

Finally, we have the theorem classifying recursively saturated models of rich theories:

THEOREM 15.2 3 (Classification of countable recursively saturated models of rich theories). Let T be a rich theory in a recursive language $\mathscr{L}$. Then:
(a) for all countable recursively saturated models $M, N \vDash T$, $M \cong N$ if and only if $\mathrm{Th}(M) = \mathrm{Th}(N)$ and $SSy(M) = SSy(N)$;
(b) for all complete extensions T^* of T and all countable $\mathscr{X} \subseteq \mathscr{P}(\mathbb{N})$ there is a countable recursively saturated $M \vDash T^*$ with $SSy(M) = \mathscr{X}$ if and only if $T^* \in \mathscr{X}$ and $\mathscr{X}$ is a Scott set.

Proof. (a) If $M, N \vDash T$ are recursively saturated and countable then $SSy(M) = SSy(N) \Leftrightarrow S(M) = S(N)$, by Lemma 15.21, so $M \cong N \Leftrightarrow SSy(M) = SSy(N)$ and $\mathrm{Th}(M) = \mathrm{Th}(N)$, by Corollary 15.15.

(b) If $M \vDash T$ is recursively saturated, then

$$p(x) = \{\varphi_\sigma(x) \leftrightarrow \sigma \mid \sigma \text{ an } \mathscr{L}\text{-sentence}\}$$

is a recursive type realized by some $a \in M$ coding $\mathrm{Th}(M)$, so $\mathrm{Th}(M) \in SSy(M)$. Conversely, if $\mathscr{X}$ is a countable Scott set and $T^* \in X$ is a complete extension of T, then by Theorem 13.6 there is a countable $\mathscr{X}$-saturated model $M \vDash T^*$ and by Lemma 13.5 and the argument directly following it, M is recursively saturated with $SSy(M) = \mathscr{X}$. □

Exercises for Section 15.2

15.9 Let M be an infinite countable recursively saturated model for a recursive language $\mathscr{L}$. Show that there is a proper elementary substructure $N \prec M$ with $M \cong N$.

15.10 Let $\mathscr{L}$ be a recursive extension of $\mathscr{L}_A$ and let T be a recursively axiomatized $\mathscr{L}$-theory. Show that there is a well-behaved indicator $Y(x, y)$ in PA for initial segments I that have an expansion to the language $\mathscr{L}$ satisfying T.

Deduce that, for all $\mathscr{L}_A$-theories S and all $M \vDash PA$ the classes of initial segments

$$\{I \subseteq_e M \mid I \vDash S\}$$

and

$$\{I \subseteq_e M \mid I \models S \text{ and } I \text{ is recursively saturated}\}$$

are symbiotic, thus improving Corollary 14.10. (See Exercise 14.6 for the definition of 'symbiotic'.)
(*Hint:* For the last part, use a satisfaction class.)
15.11 Use Exercise 15.8 to show that, for all languages $\mathscr{L}$ with finitely many function, relation, and constant symbols, there is a Σ_1^1 sentence Ψ over $\mathscr{L}$ such that, for all infinite $\mathscr{L}$-structures M

$$M \models \Psi \text{ iff } M \text{ is recursively saturated.}$$

15.12 Let M be a homogeneous countably infinite $\mathscr{L}$-structure, where $\mathscr{L}$ is a recursive language, and suppose that for all complete types $p(\bar{x}) = \{\theta_i(\bar{x}) \mid i \in \mathbb{N}\}$ over M, *where $p(\bar{x})$ does not contain parameters from M*, $p(\bar{x})$ is realized in M iff $p(\bar{x}) \in \mathscr{X}$, for some fixed Scott set $\mathscr{X}$. Show that M is recursively saturated.
15.13 (For those who attempted Exercise 15.4). Let M be a countable $\mathscr{X}$-saturated structure for the language $\mathscr{L}$ with finitely many nonlogical symbols, where $\mathscr{X}$ is a countable Scott set. Working in $ZFC + \mathrm{Con}(ZF)$, show that there is a model $V \models ZFC$ that is countable, ω-nonstandard, and has standard system $\mathscr{X}$ such that $M \cong {}^*(a, \bar{R}, \bar{f}, \bar{c})$ for some $a, \bar{R}, \bar{f}, \bar{c} \in V$.
15.14 Let T be a consistent theory in a recursive language $\mathscr{L}$. Show that T has $2^{\aleph_0}$ non-isomorphic countable models. Deduce that neither DLO nor T_{DIS} (see Section 6.2) is rich.
15.15 Let T be a consistent rich theory in a recursively saturated language $\mathscr{L}$, and suppose that the formulas $\varphi_n(x)$ in the definition of 'rich' can be all taken to be of the form $\exists \bar{y} \psi_n(x, \bar{y})$ with ψ_n quantifier-free. Show that T has no recursive recursively saturated model M.
(*Hint:* Any recursively saturated $M \models T$ codes a complete extension of PA; however, no complete extension of PA is r.e., for if it were, then it would be recursive. This exercise is due to Smoryński (1984).)

16

Suggestions for further reading

> No reasonable little Child expects
> A Grown-up Man to make a rhyme on X.
>
> MORAL
> These verses teach a clever child to find
> Excuse for doing all that he's inclined.
>
> Hilaire Belloc. *A moral alphabet*

Arithmetic is a large subject and touches on many different branches of mathematics and logic. The last fifteen chapters have been devoted to what could be described as the 'core' of the subject of models of arithmetic, and there are many directions that the reader could now pursue; I would like to indicate some suggestions here. Inevitably these suggestions reflect my own interests, and some areas of study will be unjustly under-represented. I have concentrated mainly on those papers with a specifically model-theoretic interest: this has made my task more tractable, but it has meant that proof-theoretic aspects of arithmetic and also reverse mathematics (to name just two further areas with a large literature) have been largely omitted.

As a reasonably gentle introduction to the literature on models of arithmetic (especially on initial segments and independence results) I can recommend the two volumes (Berline *et al.* 1981; McAloon 1980). Some of the material in these volumes has already been covered in Chapters 1–15 above, but there is other interesting information there too. In a more purely model-theoretic direction I can strongly recommend Wilfrid Hodges' book (Hodges 1985) to put some of the constructions I have been using in this book in their proper model-theoretic setting. Sections 6.1, 6.2 and 7.1 of Hodges (1985) are especially relevant to arithmetic.

NON-STANDARD METHODS

It was Abraham Robinson more than anyone else who advocated the use of nonstandard methods in Mathematics. (For example, see Robinson (1961).) A particularly beautiful example of the use of nonstandard integers in number theory appears in Robinson and Roquette (1975). For nonstandard aspects of Hilbert's irreducibility theorem see Roquette (1975). See also (Macintyre 1980b).

'ALGEBRAIC' FRAGMENTS OF ARITHMETIC

Another lively area where number theory and algebra meet models of arithmetic is in the study of those 'algebraic' fragments of PA. These can be loosely described as being those fragments that fail to satisfy Tennenbaum's theorem (Theorem 11.7) in some way; that is, they have recursive nonstandard models. Of particular interest in the $\mathscr{L}_A$-theory IOpen of PA^- together with axioms $I\theta$ of induction for *quantifier-free* $\mathscr{L}_A$-formulas θ. Shepherdson (1964, 1965) proved that if $M \vDash PA^-$ and $\mathscr{RC}(M)$ is the real closure of the ordered ring corresponding to M, then $M \vDash$ IOpen if $\forall r \in \mathscr{RC}(M)$ with $r > 0$ such that $0 \leqslant r - s < 1$, and he used this characterization to construct recursive nonstandard models of IOpen. This topic has been expanded on in a sequence of excellent papers, notably Wilkie (1978), van den Dries (1980), and Macintrye and Marker (1989). The key open problem in this area is: is the set of diophantine equations $p(\bar{x}) = q(\bar{x})$ solvable in *some* model of IOpen recursive? (Here $p(\bar{x})$ and $q(\bar{x})$ are polynomials with coefficients from $\mathbb{N}$. Compare this question with the solution of Hilbert's 10th problem which says that the set of all $p(\bar{x}) = q(\bar{x})$ solvable in $\mathbb{N}$ is *not* recursive.)

THE LIMITS OF TENNENBAUM'S THEOREM

It is obviously of great interest to investigate the limits of the Tennenbaum phenomenon further, and this study is intimately related to the MRDP theorem (Result 7.8). McAloon (1982) shows that nonstandard models of $I\Delta_0$ cannot be recursive, and, using the MRDP theorem, Wilmers (1985) extended this result to the theory IE_1 of PA^- together with induction on formulas of the form $\exists \bar{x} < t(\bar{y})(r(\bar{x}, \bar{y}) = s(\bar{x}, \bar{y}))$ $(r, s, t$ $\mathscr{L}_A$-terms). McAloon (1982) also showed that a nonstandard model of $I\Delta_0$ has a nonstandard initial segment satisfying PA, and this was extended to IE_1 by Paris (1984). Kaye (1990; 1989) contain further information concerning the MRDP theorem for related theories, and this is used to give refinements of Wilmers' and Paris' results. In another line of research, Jensen and Ehrenfeucht (1976) studied the reducts $M \restriction +$ and $M \restriction \cdot$ of a nonstandard model $M \vDash PA$. The finishing touches on this work, that $M \restriction +$ and $M \restriction \cdot$ are both recursively saturated, were given by Lessan (1978) and (independently) Lipshitz and Nadel (1978) for $M \restriction +$, and by Cegielski (1980; 1981) using the Presburger–Skolem–Mal'cev *et al.* elimination of quantifiers for $\mathrm{Th}(\mathbb{N}, +)$ and $\mathrm{Th}(\mathbb{N}, \cdot)$. See Cegielski *et al.* (1982) for details on these issues. By Exercise 15.15 these results may be regarded as a strong version of Tennenbaum's theorem for PA. More recently they have been extended to nonstandard models $M \models I\Delta_0 + \exp$ (Cegielski *et al.* 1982) and (for addition) to nonstandard models of IE_1 (Wilmers 1985).

TURING DEGREES, SCOTT SETS AND MODELS OF ARITHMETIC

Also related to the Tennenbaum phenomena and to recursive saturation is some recent work on the (recursion-theoretic) complexity of models of arithmetic and Turing degrees. The key papers on this line to date are Knight and Nadel (1982a; 1982b), Marker (1982), Macintyre and Marker (1984) and Knight *et al.* (1984).

There are still many open problems arising from this work, of which I would like to mention three that concern Scott sets.

PROBLEM 1. Given a Scott set $\mathscr{X}$ (not necessarily countable) and a theory $T \in \mathscr{X}$ extending PA, when does T have a model M with $SSy(M) = \mathscr{X}$?

If $\mathscr{X}$ has cardinality $\aleph_1$, it is reasonably easy to modify the proof we gave in Section 13.1 to construct a model M with $SSy(M) = \mathscr{X}$. What happens for higher cardinalities?

PROBLEM 2. If M is a recursively saturated model (for a recursive language $\mathscr{L}$), is M $\mathscr{X}$-saturated for some Scott set $\mathscr{X}$?

The answer is 'yes' if Th(M) is rich, or if M is countable, by Theorem 15.20, Lemma 15.21 and Corollary 15.22. What happens in the other cases?

PROBLEM 3. If T is a theory and $\mathscr{X}$ is a Scott set (but not necessarily with $T \in \mathscr{X}$), when does T have a model M with $SSy(M) = \mathscr{X}$?

Notice that, if $T \neq \mathrm{Th}(\mathbb{N})$ is a complete extension of PA, then $\mathscr{X} = SSy(K_T)$ is a Scott set but $T \notin SSy(K_T)$. For some information on Problem 3 see Jensen and Ehrenfeucht (1976).

END EXTENSIONS AND INITIAL SEGMENTS

Following Friedman's results (1973) and Gaifman's results (1972 and 1976) there has been a large number of papers on these themes, often linking all the ideas together. Gaifman's (1976) paper is still well worth reading, and contains many results that go beyond those presented in Chapter 8. In particular he constructs *minimal* elementary end-extensions of a given model $M \models PA$, i.e. end-extensions $K > M$ such that for no L with $K < L < M$ is $K \neq L \neq M$. Wilkie's results (1975, 1977) extending Rabin's theorem (Rabin 1962) are particularly nice: Rabin proved that every nonstandard model M of Th($\mathbb{N}$) has a diophantine equation $p(\bar{x}) = 0$ over M that is unsolvable in M but solvable in some extension $M \subseteq N \models \mathrm{Th}(\mathbb{N})$. Gaifman asked if '$M \subseteq N$' here can be taken to be an end-extension, and Wilkie solved this affirmatively, using techniques form the proof of Friedman's theorem.

Also, and more directly inspired by the type of argument employed in Gaifman (1976), are an important sequence of papers concerning refinements and variations on these arguments: Knight (1973; 1975); Schmerl (1973); Phillips (1974); Kaufmann (1977); Schmerl (1981); Kaufmann and Schmerl (1984). I should also add that the notion of definable type has since found its way into modern abstract model theory, and it is now a key notion of *stability theory*, see e.g. Pillay (1983).

Another obvious direction in which this work can be taken, is in extending it to the fragments of PA, $I\Sigma_n$, $B\Sigma_n$, etc. Paris and Kirby (1978) contains constructions of Σ_n-elementary initial segments of models of $B\Sigma_n$ for $n \geqslant 2$. Kirby (1984) contains furher information on these types of constructions, and Wilkie and Paris (1989) and Adamowicz (to appear) consider the apparently very difficult problem of constructing end-extensions $K \models I\Delta_0$ of a given countable model $M \models B\Sigma_1$. (It is still unknown if *all* countable models $M \models B\Sigma_1$ have such an end-extension K.) Dimitracopoulos and Paris (1988) consider versions of Friedman's theorem for fragments of PA. The essential difficulty in these questions is in finding the right notion of *saturation* here. For example, it is easy to check that every countable recursively saturated model $M \models B\Sigma_1$ has a proper end-extension $K \models I\Delta_0$, so the first-order theory '$B\Sigma_1$' is the correct one for the end-extension problem. In fact it suffices here to take M short Π_1-recursively saturated. But *exact* necessary and sufficient saturation conditions on M are difficult to obtain. There are similar problems in proving embedding theorems for models of fragments: given two countable models M, $K \models I\Sigma_n$, where $n \in \mathbb{N}$, under what conditions is there an embedding $M \xrightarrow{\cong} I \subseteq_e K$ with $I <_{\Sigma_n} K$ and $I \neq K$? Again, one must have $K \models \Sigma_{n+1}^+ - \mathrm{Th}(M)$ (see Exercise 7.2) and $M \models B\Sigma_{n+1}$, and in fact this suffices if K is Σ_{n+1}^+-recursively saturated and M is short Π_{n+1}-recursively saturated, but is this amount of saturation necessary?

COMBINATORIAL PROPERTIES OF CUTS

Initially inspired by several fascinating combinatorial properties of large cardinals in ZFC-set theory, Kirby and Paris investigated several such properties of cuts $I \subseteq_e M \models PA$. They called their cuts *semiregular*, *regular*, *strong*, *Ramsey*, *extendible*, etc. The original motivation was to classify these cuts in terms of their indicators and deduce independence results. See Wilkie (1986c) for a very clear description of this programme of research. Of course the work did eventually lead to independence results (Paris 1978b; Paris and Harrington 1977), but it produced mathematics that is interesting in its own right, and often very useful too in the study of models of PA.

The starting point for all this is in the paper by Kirby and Paris (1977). These ideas were taken up and elaborated later in McAloon (1978), Paris (1978b; 1980; 1981), Kirby (1982), and Mills and Paris (1984). One interesting problem still left open from this work is

PROBLEM 4. Characterize those properties $P(M,I)$ of an initial segment $I \subseteq_e M$ that have a well-behaved indicator $Y(x, y)$.

Here, we are asking for a sort of 'preservation theorem' (in the sense of Chang and Keisler (1973)).

COMBINATORICS AND ARITHMETIC

Obviously, this is another vast area. The volume (Simpson 1987) is a good introductory reference, and Buchholz and Wainer's article therein contains a nice self-contained introduction to the proof-theoretic arguments that are commonly used. Ketonen and Solovay's (1981) paper concentrates mainly on the combinatorial issues raised by the Paris–Harrington incompleteness phenomena, and is still essential reading. The papers by Paris (1980) and Mills (1980) give a detailed account of how the incompleteness phenomena arise, with emphasis on the various model-theoretic aspects, and give information for the fragments $I\Sigma_n$ too. For further information on the approach used in Section 14.3 above, see Kanamori and McAloon (1987).

There have also been many interesting independence results for theories stronger that PA (usually fragments of second-order arithmetic). Friedman *et al.* (1982) contains one such. Another interesting and related line is that in Macintyre (1980) (and also, independently, Morgenstern (1982)), where Macintyre introduces the 'Ramsey quantifier' $\mathrm{Q}^2 x_1, x_2\, \theta(x_1, x_2)$ for 'there exists an unbounded set H such that for all $\{x_1, x_2\} \in [H]^2\ \theta(x_1, x_2)$'. The resulting system proves the Paris–Harrington statement, and has an interesting model theory, but Schmerl and Simpson (1982) relate it to a certain fragment of second-order arithmetic for which independence results are known.

FRAGMENTS OF *PA*

Essential background for the model theory of the fragments $I\Delta_0$, $B\Sigma_1$, $I\Sigma_1, \ldots,$ of PA is knowledge of how exponentiation $x^y = z$ can be expressed in $I\Delta_0$ by means of a Δ_0 formula $\eta(x, y, z)$, and the properties of the theory $I\Delta_0 + \exp =$

$$I\Delta_0 + \forall x, y \exists z \eta(x, y, z).$$

The existence of such a formula η is essentially due to Bennett (1962), and expositions relevant to $I\Delta_0$ appear in Gaifman and Dimitracopoulos (1982), Nelson (1986), and Wilkie (1987).

The fragments $I\Sigma_n$, $B\Sigma_n$, etc. crop up throughout the study of models of PA, but they are of particular interest in relation to finite-combinatorial statements in arithmetic and classes of recursive functions. The general types of problem are as follows.

PROBLEM-TYPE 1. Given a natural class $\mathscr{S}$ of recursive functions, find a theory T such that the functions of $\mathscr{S}$ are exactly the provably recursive functions of T.

PROBLEM-TYPE 2. Given a fragment T of PA, find a recursion-theoretic characterization of the provably recursive functions of T.

In Problem-type 1 it is natural (at least at first) to consider the classes $\mathscr{S} = \mathscr{E}^\alpha$, defined to be the least class containing 0, 1, the successor function, and F_β for each $\beta < \alpha$ (where F_β is as in Exercise 14.13) and closed under composition and the following recursion scheme:

if $b(\bar{x}, y)$, $g(\bar{x})$ and $h(\bar{x}, y, z)$ are in $\mathscr{E}^\alpha$, then so is $f(\bar{y}, y)$ defined by

$$f(\bar{x}, 0) = g(\bar{x})$$

and

$$f(\bar{x}, y+1) = \min(h(\bar{x}, y, f(\bar{x}, y)), b(\bar{x}, y)).$$

(Exercise: verify that $\mathscr{E}^\omega = \mathscr{PR}$.)

An answer to Problem-type 1 is the more interesting the stronger the theory T is. There is a surprise here though: most fragments T of PA one might consider turn out to be either equal to one of the $I\Sigma_n$s or to have the same provably recursive functions as one of these theories. (An example is the theory $B\Sigma_{n+1}$, which has the same provably recursive functions as $I\Sigma_n$ by Corollary 10.9. Hájek and Paris (1986) give another interesting hierarchy of fragments which have, it turns out, the same provably recursive functions as the $I\Sigma_n$.) Put another way, this says that many of the fragments of PA have *conservation results* connecting them, such as that between $I\Sigma_n$ and $B\Sigma_{n+1}$. The paper (Kaye *et al.* 1988) contains other examples of these phenomena, and also exhibits theories T characterizing the classes

$$\mathscr{E}^{\omega_n^{\omega^k} \cdot m}$$

(where $\omega_0^a = a$ and $\omega_{n+1}^a = \omega^{\omega_n^a}$) for various $n, k, m \in \mathbb{N}$. Some open problems of type 2 are suggested there too.

SATISFACTION CLASSES

Historically, the material presented in Section 15.1 is from pioneering papers by Robinson (1963), Krajewski (1976), Kotlarski *et al.* (1981), and Lachlan (1981). This material has been taken further, notably by Kotlarski, Krajewski, Kossak, Murawski, J. H. Schmerl, and S. T. Smith. Of particular interest is Smith's extension of the Lachlan and Kotlarski–Krajewski–Lachlan results to obtain characterizations of recursive saturation and resplendency (Smith 1987). It was known from Wilmers' work that recursive saturation has a Σ_1^1 characterization and resplendency is Δ_2^1, however exact Σ_1^1 and Δ_2^1 sentences characterizing these notions were not known until Smith's work.

Recursive saturation and satisfaction classes are related to *expandability* of models $M \vDash PA$ to fragments of second-order arithmetic. Barwise and Schlipf (1975) proved that a countable $M \vDash PA$ is expandable to the fragment $\Delta_1^1 - CA$ of second-order arithmetic if and only if M is recursively saturated. For other background information and a different approach to recursive saturation via *admissible sets*, see Barwise (1975).

A more recent and very interesting line of research concerns questions of expandability of a given countable recursively saturated $M \vDash PA$ to (M, S), where S is a full satisfaction class and (M, S) satisfies extra induction axioms. The key paper here is by Kotlarski and Ratajczyk (to appear), where these authors describe exactly when such M have a full *inductive* satisfaction class. Their characterization involves an iterated ω-rule, suitably formalized in PA. Two other relevant papers in this area are Kotlarski (1986; 1989).

RECURSIVE SATURATION AND RESPLENDENCY

There is a growing literature on the automorphism group Aut(M) (considered as a permutation group acting on the domain of M) where M is a countably infinite $\aleph_0$-categorical structure. (See, for example, Macpherson (1986), Hodges (1989), and Cameron (1990).)

Recursive saturation can be seen as a natural generalization of this notion, and has many powerful features which allow us to show that, for countably infinite recursively saturated M, $\mathrm{Aut}(M)$ is very rich. (See Kaye (to appear, b), for example.) One very striking result here is due to J. H. Schmerl, who proved that any countable recursively saturated $M \models PA$ has a set $I \subseteq M$ of order-indiscernibles that *generate* M, i.e. $M = K(M; I)$. In particular we may take I to have the order-type of a dense linear order, so $\mathrm{Aut}(\mathbb{Q}, <)$ embeds in $\mathrm{Aut}(M)$ in a fairly natural way. (See Schmerl 1985; 1989). The 1989 paper extends this result further, but is still reasonably self-contained.) Indeed, any countable recursively saturated M (not just of PA) has an expansion (M, β) where $\beta: M^2 \to M$ and a set of order-indiscernibles $I \subseteq M$ of order-type $\mathbb{Q}$ such that M is generated by I via the β function.

For further information on automorphisms of a countable recursively saturated model $M \models PA$ see also Kotlarski (1983b; 1984a) and Kossak and Kotlarski (1988).

Jim Schmerl has asked whether every resplendent model M for a finite language $\mathscr{L}$ is chronically resplendent. This question is still open. Two other interesting open problems here are

Problem 5 (Barwise and Schlipf). Does there exist a consistent theory T in a countable recursive language $\mathscr{L}$ such that every countably infinite model $M \models T$ is recusively saturated but T is not $\aleph_0$-categorical?

Problem 6. What permuation groups $G \leqslant \mathrm{Sym}(\Omega)$ (the symmetric group acting on a countably infinite set Ω) arise as $G = \mathrm{Aut}(M)$ for a recursively saturated model M with domain Ω?

The answer to Problem 6 is known for $\aleph_0$-categorical M, by Svenonius's formulation of the Ryll-Nardzewski–Svenonius–Engeler characterization of $\aleph_0$-theories, and by Frassé's construction of a canonical structure with domain Ω and a given closed $G \leqslant \mathrm{Sym}(\Omega)$ as its automorphism group. See one of the references by Hodges, Macpherson, or Cameron cited above.

FRAGMENTS OF ARITHMETIC AND COMPLEXITY THEORY

As early ago as 1971, R. Parikh observed that weak fragments such as $I\Delta_0$ are intimately related to 'feasible computatons'. With the recent growth of interest in computational complexity theory, and the $P = NP$ problem, it is not surprising that there has been a corresponding growth in the number of papers written on weak fragments. (Here, *weak* fragments will be fragments of PA that cannot prove the axiom *exp*, i.e. $\forall x, y \exists z \eta(x, y, z)$, where η is Bennett's Δ_0 definition of '$x^y = z$'. This is true of $I\Delta_0$ by Parikh's theorem (see Exercise 6.5).

One point of contact between weak fragments and complexity was observed by Wilkie and Paris (1987), who observed that if $\bar{a} \in M \models I\Delta_0 + \exp + \theta(\bar{a})$, $\theta(\bar{a})$ a Σ_1 formula, then M contains a proof $p \in M$ of $\theta(\mathrm{clterm}(\bar{a}))$ from the axioms of PA^-. (This is just a version of our Corollary 2.9 formalized in $I\Delta_0 + \exp$). If M satisfies $I\Delta_0 + \Omega_1$, where the axiom Ω_1 is '$\forall x x^{[\log_2 x]}$ exists' (and this axiom is given a precise meaning via Bennett's definition of exponentiation), then M will contain a proof from PA^- of every true formula of the form $\theta(\mathrm{clterm}(\bar{a}))$, where $\theta(\bar{y})$ is an NP predicate (and the canonical term function has to be chosen rather more carefully). Thus, in $I\Delta_0 + \Omega_1$, NP predicates play the role usually enjoyed by r.e. predicates in stronger theories. Another point of contact with complexity theory was provided by Buss (1986), developing ideas of Cook (1975) and Paris (1978*a*). Buss gave a theory S_2^1 (which is, essentially, a subtheory of $I\Delta_0 + \Omega_1$) such that the 'provably recursive' functions in S_2^1 are exactly the polynomial-time computable functions, P. (I write 'provably recursive' in inverted commas, since as for Wilkie and Paris's $I\Delta_0 + \Omega_1$, NP formulas take the place of Σ_1 formulas, and to define a 'provably recursive' function f in S_2^1 it is necessary to give a definition of the graph of f in some explicit NP form. However, if $f(\bar{x}) = y$ is in this form and $S_2^1 \vdash \forall x \exists ! y f(\bar{x}) = y$, then it turns out that f is actually in P.) It follows from Buss's results, for example, that if S_2^1 proves the MRDP theorem (and it is still unknown if *any* 'weak' fragment proves this theorem) then $P = NP$. A further connection with complexity theory is that Σ_1^1 predicates over finite structures of the form

input: a (natural) code c for a finite relational structure $(M, \bar{R})$
output: 'yes' or 'no' depending on whether $(M, \bar{R}) \models \Theta$

where Θ is a fixed Σ_1^1 sentence over the first-order language $\mathscr{L} = \{\bar{R}\}$ are, essentially, just the NP predicates, whereas the same predicates for *first-order* sentences, are just the polynomial-time predicates. Questions about Σ_1^1 and first-order definability are studied in detail by Ajtai (1983; 1989).

As introductory references to these areas, Wilkie and Paris' (1987), Buss' (1986), and Ajtai's (1983, 1989) can be recommended. Paris and Dimitracopoulos (1982), Wilkie (1987), McAloon (1986), and Wilkie and Paris (1981) are also good introductions to some of the other aspects of these connections between complexity theory and models of arithmetic. As follow-ups to these, the reader may like to consult the following work concerning (amongst other things) cuts, interpretations, and consistency statements in Pudlák (1985; 1986), Wilkie (1986a), Nelson (1986), Krajíček (1987), and also work by various authors on the structure of the sets definable by Δ_0 formulas in Wilkie (1978), Woods (1986), Paris and Wilkie (1985; 1987). Concerning Wilkie's problem whether $I\Delta_0$ proves the existence of unboundedly many primes (see p. 70) and Macintyre's problem whether $I\Delta_0$ proves the pigeonhole principle for Δ_0 formulas (see Exercise 5.12), see Paris *et al.* (1988).

I am fully aware that, in the last three paragraphs, proof-theoretic results in this area are particularly poorly represented. In particular I have omitted much important work by Pudlák, Krajíček and Takeuti. However the beautiful results in Krajíček *et al.* (to appear) are well worth reading.

I close by mentioning that most of the important open problems concerning weak fragments of arithmetic are about the provability of certain Π_1 sentences in fragments of arithmetic. (See also the remarks at the end of Section 14.3.) Shelah (1984) succeeded in exhibiting a Π_1 sentence that is independent of PA, and his methods may turn out to be useful in this area too.

Bibliography

[This bibliography, like Chapter 16, is not claimed to be comprehensive. It is, rather, a personal selection of some of the many papers and books on models of arithmetic and other topics touched upon in this book. The bibliographies in Hájek and Pudlák (1989) and in Smoryński (1984) are more comprehensive, and have been invaluable in preparing the list below. The reader could also consult the excellent Ω-bibliography (*Perspectives in mathematical logic*, Springer-Verlag, 1987) for further information.]

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Index

[An index of notation precedes the main index. This lists the defined symbols, in the order of appearance, along with page references to their definitions only. (Sometimes there are several page references: this is either because the symbol(s) in question are defined in several different contexts, or because an earlier definition is superceded by a later, more general, one.) In the main index, I have included entries for all of the key notions in the book, such as: nonstandard model; recursive saturation. The corresponding references give the definitions of these notions, but sometimes no more. For further information on these rather general topics, see other (more specific) entries in the index, the contents page, or the main text of the book.]