Campus León

División de Ciencias e Ingenierías
Departamento de Física

Cosmology: Project 1.

Gabriel Missael Barco.

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Abstract

This project simulates the evolution of a homogeneous, isotropic, and spatially-flat universe using the standard co smological model Λ-CDM and the cosmological parameters given by the Planck experiment [1]. We use the Runge-Kutta method of order 4 to solve for the scale factor a and at the same time obtain the evolution of the different matter-energy species densities. With that information, we analyze the development of those variables as a function of redshift, time, and scale factors (in the case of densities). Finally, using the same information, we obtain different distance measures for such a universe and compare the predicted distance modulus with observational supernova data from [2].

1 Introduction

In many science areas, we can create different experiments under specific circumstances and then vary some of its variables to observe and measure the effects that that variation has on the properties and behavior of the system. That way, several theories can be tested against the experiment and iterated over to improve our approaches.

Other areas, such as biology or astronomy, don't have freedom of creation: you can only observe the existing elements, such as the present biodiversity of species or the different starts we can observe in the sky. However, there are several elements, and it is still possible to test the theory with those "natural" experiments. They still have a wide range of characteristics and properties, which we can treat as variables.

However, in cosmology, something remarkable happens: you don't have freedom of creation, and in addition, there is only one Universe, which is the object of study of cosmology. We only have one natural element of analysis. Because of this, it's essential to observe all the properties of our Universe very carefully, whit as high precision as possible. Also, given a specific model or theory, we can simulate the Universe's evolution to the present day and compare our observations to that simulation.

Nowadays, several essential observations have led our understanding of our Universe to the present state (with the ΛCMB model being the current best model), such as:

- The galaxies that are further away from us are going out faster (Hubble law).

- The observations of the cosmic microwave background suggest the Big Bang as the correct theory for the birth of the Universe.
- The general relativity theory's tests show that it accurately describes space-time behavior.
- Our location in the Universe is not unique.

This last observation has led to the cosmological principle, which is the most important base of our current understanding of our Universe. It states that there are no privileged positions or directions in our Universe. In other words, our Universe is homogeneous and isotropic. It's important to note that this principle is true at large scales, but not necessarily at every scale. As stated before, it's essential to simulate the evolution of the Universe given a particular model. Our current best model is the Lambda-CMB cosmological model, so we must emulate the development of such a model to compare it to our observations. We can find in the literature that this model accurately predicts most of our cosmological observations.

In general, the Λ-CMB model correctly predicts the characteristics of our Universe at a large scale, probably because one of its bases works only at that scale (the cosmological principle). However, it fails to predict the existence of galaxies, stars, planets, humans, and everything smaller than a Parsec. That may seem like a problem, but in cosmology, we are interested in the global evolution, structure, and properties of our Universe as a whole, and not in very local phenomena. We don't say that Newtonian physics doesn't work in predicting a free-fall experiment because it doesn't describe the quantum processes happening in the object. It's simply out of the scope of the theory and the events we aim to predict and explain.

Finally, we must say that we can make different observations to compare the model with them. For example, we could measure:

- The baryonic acoustic oscillations (BAO) radio.
- The CMB temperature (and its very existence).
- The distribution of galaxies (matter) in the Universe.
- The curvature of space-time (overall and in presence of matter).
- The Hubble constant value.
- Matter-energy species densities.
- The position of different standard candles, such as specific types of supernovas.

In this project, we will simulate the evolution of a spatially flat Universe using the Lambda-CMB model and the matter-energy species reported in [1], and we will compare the predicted distance modulus of this model with the supernova data provided in [2].

2 Methods

2.1 Runge-Kutta method, order four.

The Runge-Kutta methods are a family of methods to solve differential equations that have the high-order local truncation error and eliminate the need to compute and evaluate the derivatives that other methods have [3]. The Runge-Kutta methods solve differential equations of the form:

$$\frac{dy}{dt} = f(t, y) \quad ; \quad a \leq t \leq b \quad ; \quad y(a) = \alpha \tag{1}$$

where we observe that we need to specify an initial condition. For this project, we will use the **Runge-Kutta method of order four**, which has a local truncation error of $O(h^4)$ [3], where h is the step size of the numerical procedure. This is a iterative method, where the solution for the next step depends on the solution on the previous step and the derivative $f(t, y)$, namely:

$$\begin{aligned} w_0 &= \alpha \\ k_1 &= hf(t_i, w_i) \\ k_2 &= hf(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1) \\ k_3 &= hf(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2) \\ k_4 &= hf(t_{i+1}, w_i + k_3) \end{aligned}$$

And the solution in the next step:

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{2}$$

for each $i = 0, 1, 2, ..., (b - a)//h$. Finally, we should observe that we can change this method to go backwards, starting from $y(b) = \beta$, just specifying a negative step size h.

2.2 Numerical integration.

For this project, we will use a simple approximation for numerical integration. Namely:

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} \Delta x f(x_i) \tag{3}$$

where

$$\Delta x = \frac{b - a}{n} \qquad x_i = a + \Delta x \cdot i$$

This simply comes from the limit of a Riemann sum to an integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} \Delta x f(x_i) \tag{4}$$

This could be considered the most basic form of a **numerical quadrature** method for integration, but without interpolating to Lagrange polynomials [3]. In general, this approximation works as long as the step size h (here Δx) is sufficiently small, and the function $f(x)$ sufficiently smooth. Otherwise, other methods with higher precision and stability should be used.

2.3 Matter-energy species evolution.

In a spatially flat universe ($k = 0$), which we will consider trough this project, there is a **critical density** for a given Hubble parameter [4]:

$$\varepsilon_c(t) = \frac{3c^2}{8\pi G} H(t)^2 \tag{5}$$

In this project, we will consider three types of matter-energy species: matter, radiation and dark energy (Λ). If we know the present density of a given specie $\varepsilon(t_0) = \varepsilon_0$, and we consider $a(t_0) = 1$, we are able to calculate the density at any other moment as a function of the expansion factor a. The exact dependence on a depends on the type of matter-energy, namely [4]:

$$\varepsilon_r = \frac{\varepsilon_{r,0}}{a^4} \quad ; \quad \varepsilon_m = \frac{\varepsilon_{m,0}}{a^3} \quad ; \quad \varepsilon_\Lambda = \varepsilon_{\Lambda,0} \tag{6}$$

We can express the densities as a fraction of the critical density, that is:

$$\Omega = \frac{\varepsilon}{\varepsilon_c} \tag{7}$$

For convenience, we will express the Ωs in therms of Ω_0, a, H_0 and H:

$$\Omega_r = \frac{\varepsilon_r}{\varepsilon_c} = \frac{\varepsilon_{r,0}}{a^4 \varepsilon_c} = \frac{\Omega_{r,0}\varepsilon_{c,0}}{a^4 \varepsilon_c} = \frac{\Omega_{r,0}}{a^4}\frac{H_0^2}{H^2} \implies \Omega_r = \frac{\Omega_{r,0}}{a^4}\frac{H_0^2}{H^2} \tag{8}$$

Analogously for matter and Λ:

$$\Omega_m = \frac{\Omega_{m,0}}{a^3}\frac{H_0^2}{H^2} \quad ; \quad \Omega_m = \frac{\Omega_{m,0}}{a^3}\frac{H_0^2}{H^2} \tag{9}$$

Finally, it's important to note that:

$$\Omega_0 = \Omega_r + \Omega_m + \Omega_\Lambda = 1 \tag{10}$$

Because we have a spatially flat universe ($k = 0$, $\varepsilon = \varepsilon_c$).

2.4 Friedmann Equation.

The Friedmann equation, which describes the expansion of the universe as a function of the curvature and the matter-energy species densities in our universe, is given by:

$$H^2 = (\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2}\frac{1}{a^2}$$

where $\varepsilon(t)$ represents the total matter-energy density. Separating the $\varepsilon(t)$ in it's components and considering $k = 0$ gives us:

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3c^2}[\varepsilon_r + \varepsilon_\Lambda + \varepsilon_m] \tag{11}$$

2.5 Differential equation for $\dot{a}$ in terms of Ωs.

An analogous expression for the Friedmann equation in therms of the Ωs [4]:

$$\frac{H^2}{H_0^2} = \frac{\dot{a}^2}{a^2 H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} \tag{12}$$

We have $\Omega_0 = 1$, therefore:

$$\dot{a} = H_0 \sqrt{\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + a^2 \Omega_{\Lambda,0}} \tag{13}$$

2.6 Distances.

There are different measure to obtain the distance to a certain object. In this project, we will compute how those distances vary given the redshift z.

2.6.1 Proper distance

The proper distance is given by [4]:

$$d_p(t_0) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} \tag{14}$$

Let's see that:

$$H = \frac{\dot{a}}{a} \implies \frac{da}{dt} = aH \implies dt = \frac{da}{aH} \tag{15}$$

Therefore, we can re-express 14 as:

$$d_p = c \int_a^1 \frac{1}{a} \frac{da}{aH} = c \int_a^1 \frac{da}{a\dot{a}} \tag{16}$$

2.6.2 Luminosity distance.

In the case of a spatially flat universe ($k = 0$), we can express the luminosity distance d_L as [4]:

$$d_L = d_p(1+z) \tag{17}$$

2.6.3 Angular-diameter distance.

Finally, and again considering $k = 0$, the angular-diameter distance:

$$d_A = \frac{d_L}{(1+z)^2} \tag{18}$$

2.7 Distance modulus.

If the luminosity distance is given in Mpc units, then the distance modulus is defined as:

$$\mu = 5 \log_{10} \frac{d_L}{1Mpc} + 25 \tag{19}$$

2.8 Redshift z as a function of the scale factor a.

Given that we set the present scale factor to $a(t_0) = 1$, and with the cosmological model we are considering in this project, we can obtain the corresponding redshift z to a given scale factor a with [4]:

$$z = \frac{1}{a} - 1 \tag{20}$$

3 Results and Discussion

All the code is available in the GitHub respository `https://github.com/GabrielMissael/cosmologia_p1/blob/master/proyecto.ipynb`. Furthermore, a PDF version of the Python Notebook is available at the end of the document for reference.

First, we need to solve the expansion factor a as a function of time. For that, we will use the Runge-Kutta method of order four given by the iterative equation 2:

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

We need to express the derivative of the expansion factor $\dot{a}$ as:

$$\frac{dy}{dt} = f(t, y) \quad ; \quad a \leq t \leq b \quad ; \quad y(a) = \alpha$$

For this, we will use equation 13:

$$\frac{da}{dt} = H_0 \sqrt{\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + a^2 \Omega_{\Lambda,0}}$$

We treat the different Ωs as parameters of our method in order to being able to simulate different universes. Finally, we are interested in two intervals, with the same initial condition:

$$a(t_0) = 1 \quad ; \quad 0 < t < t_0 \quad and \quad t_0 < t < t_f(a(t_f) = 20)$$

For that, we will use the method two times, with positive and negative step-size h for going back and advancing in time.

As an experiment, we simulate the three single-component universes, presented in Figure 1. We observe that the Λ case presents an accelerated expansion, as expected. For the matter and radiation cases, the behavior as a decelerated expansion.

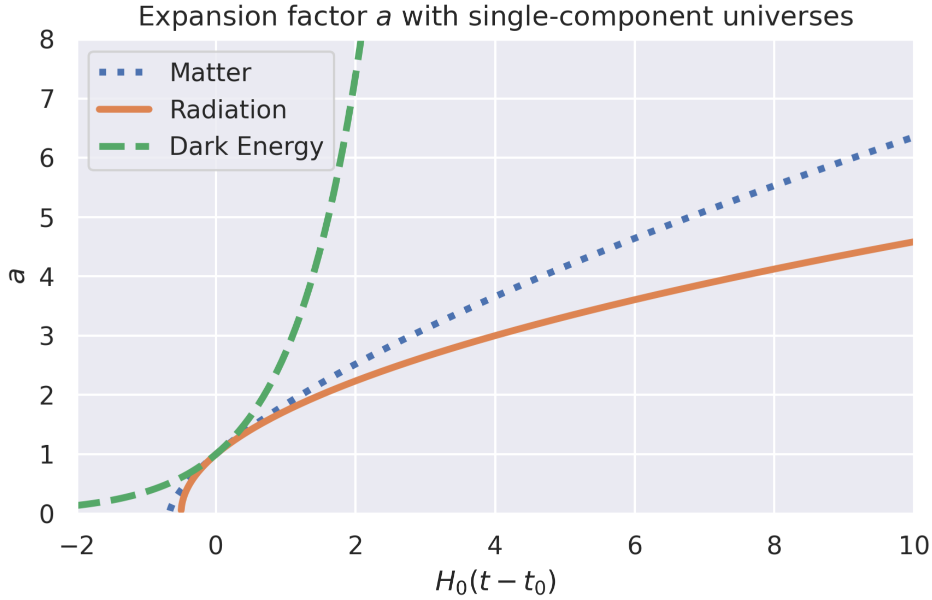

Figure 1: Expansion factor a for different single-component universes as a function of time (Hubble time units $H_0(t-t_0)$)

If we perform the same simulation but now using the cosmological parameters given by the Planck experiment [1], we obtain the scale factor evolution presented in Figure 2, with time in Hubble time units. In this case, the expansion seems to be decelerated at the very beginning, but then it start to grow in an accelerated way, indicating the probable dominance of dark matter that we will analyze further.

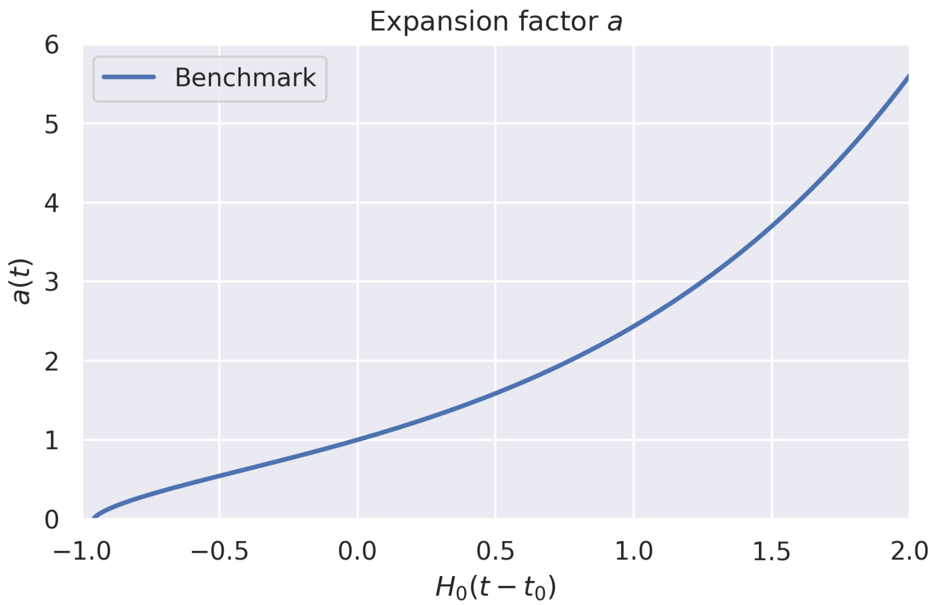

Figure 2: Expansion factor a for universe benchmark model (matter, radiation, and dark energy, with current values from [1]).

We changed the time axis to a log-scale in Figure 3 for comparison with the plot presented in [4]. Also, we indicate the moment of matter-Λ equality, and we observe that the accelerated expansion stars with the dark energy dominance.

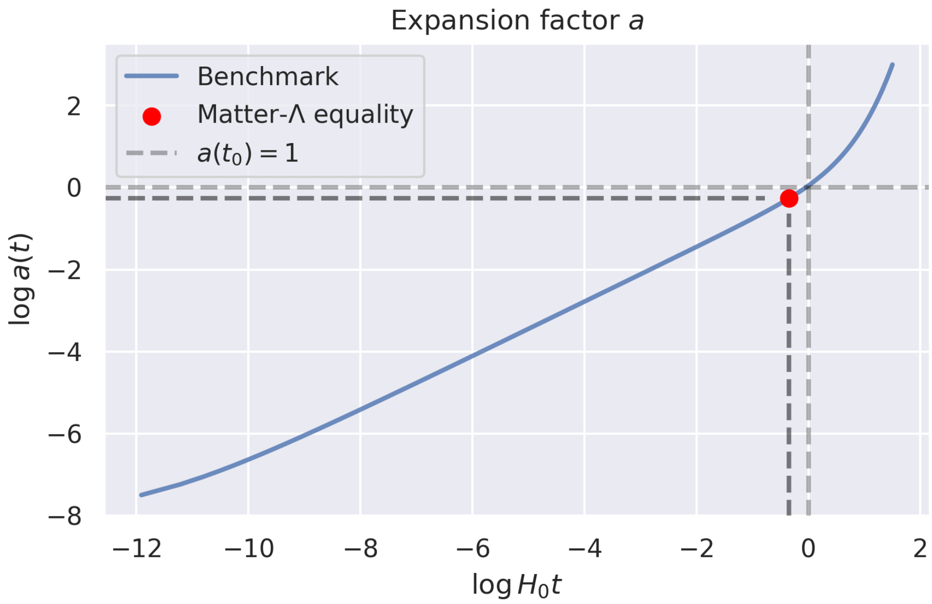

Figure 3: Expansion factor a for universe benchmark model as a function of time (log-scale in both axes). We observe the matter-Λ density equality in the plot.

Inside our Runge-Kutta method, we stored the Ω densities in each iteration, because we calculated them and used to solve for a. Because of this, it's straight-forward to plot the densities as a function of it's corresponding a, as presented in Figure 4.

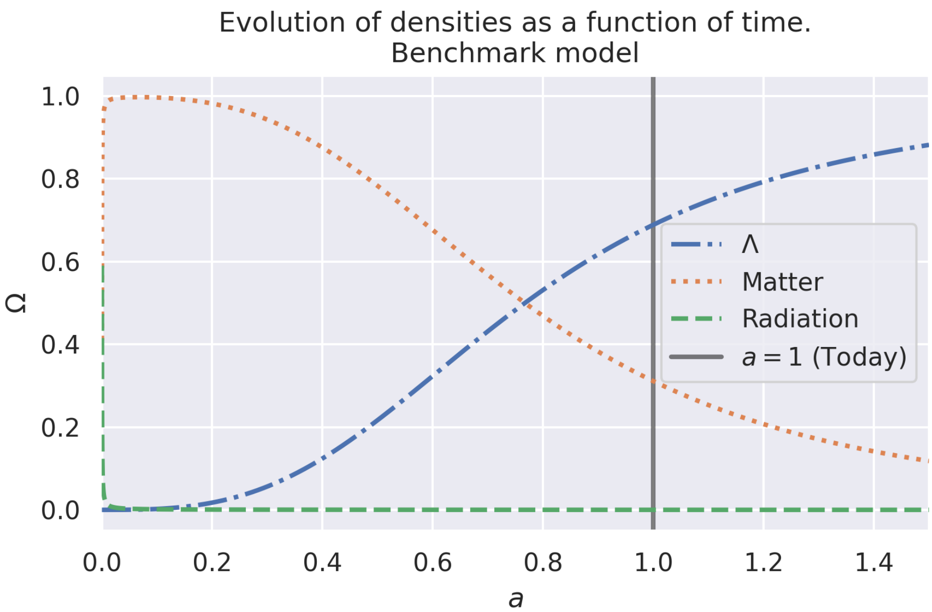

Figure 4: Evolution of matter-energy species as a function of the scale factor a in the benchmark model.

However, we can observe that the matter and radiation densities values vary considerably near the $a = 0$. In order to observe that behavior, we changed the a axis to a log-scale in Figure 5. In that plot, we can observe two important moments: the matter-radiation equality, which happens in a very early stage of the universe evolution, and the matter-Λ equality, which happened recently in the evolution of the universe. Finally, we observe that, in the future, the dark energy will become the complete dominant matter-energy specie in our Universe.

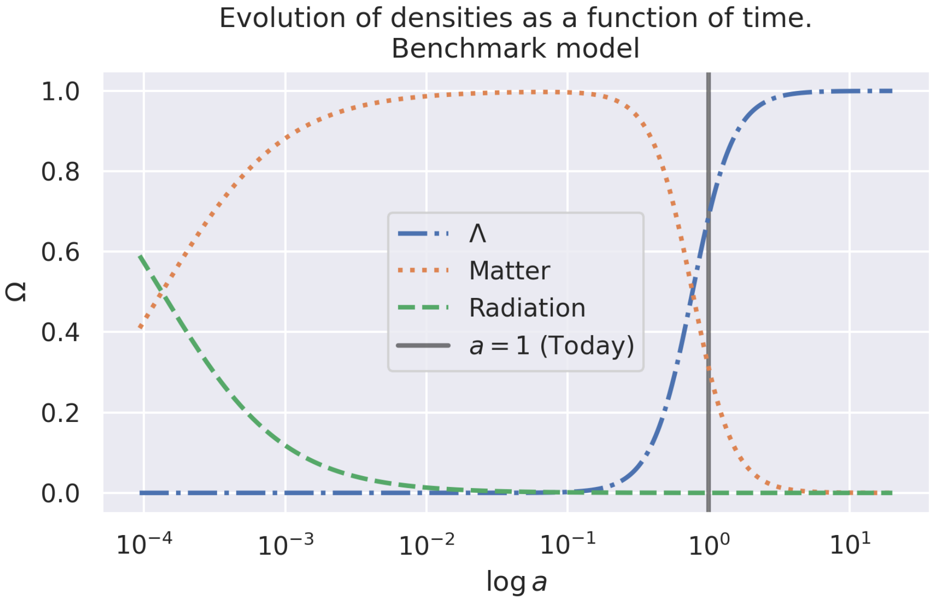

Figure 5: Evolution of matter-energy species as a function of the scale factor a in the benchmark model with log-scale in the a axis for resolution at low a values. Note: The x-axis label is incorrect, it should be just a, not $log(a)$, because we just changed the scale, we didn't obtained the logarithm of a.

In the next Figures, we plot the densities as a function of redshift and time.

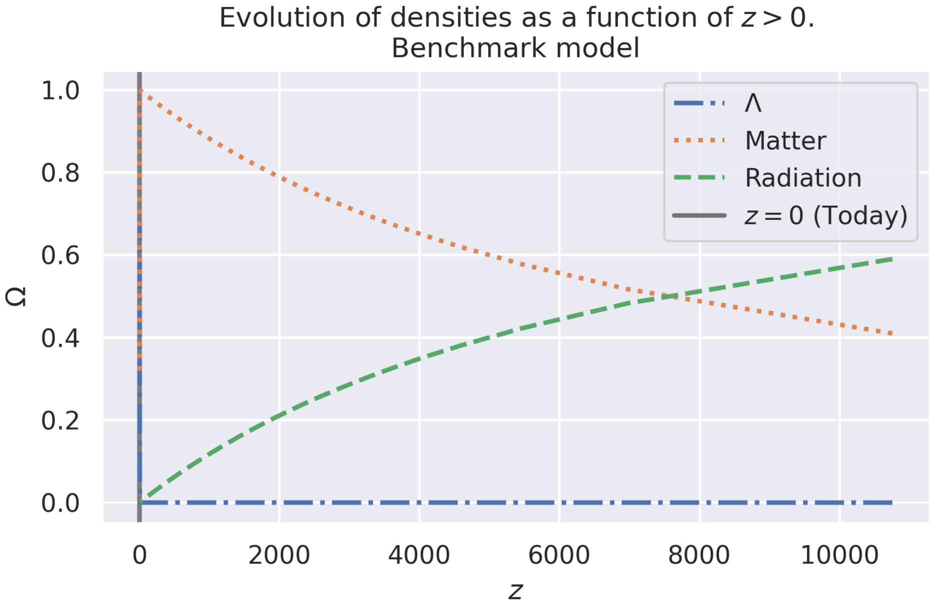

Figure 6: Evolution of matter-energy species as a function of the redshift z.

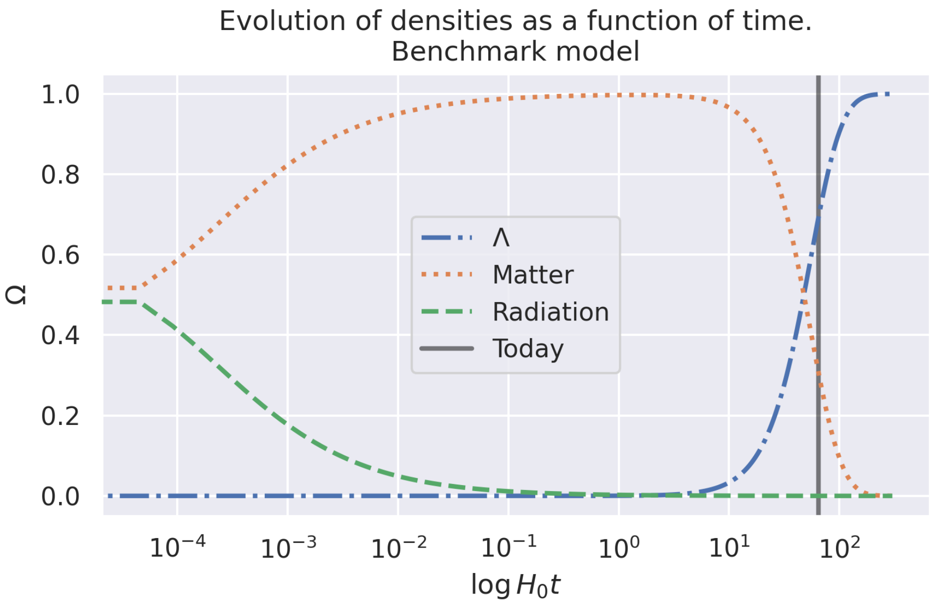

Figure 7: Evolution of matter-energy species as a function of time t (Hubble time units H_0t) with log-scale for resolution.

Now, we are interested in obtaining the proper distance, the luminosity distance and the angular-diameter distance. For that, we will use the naive approximation for numerical integration presented in equation 3:

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} \Delta x f(x_i)$$

for integrating the proper distance in form of the equation 16:

$$d_p = c\int_a^1 \frac{1}{a}\frac{da}{aH} = c\int_a^1 \frac{da}{a\dot{a}}$$

From our Runge-Kutta routine, we also stored the derivative $\dot{a}$, so obtaining the proper distance for each redshift z is also straight-forward. In Figure 8, we observe the obtained proper distance of our model as a function of the redshift z. We obtained the redshift z using the scale factor we obtained and the equation 20:

$$z = \frac{1}{a} - 1$$

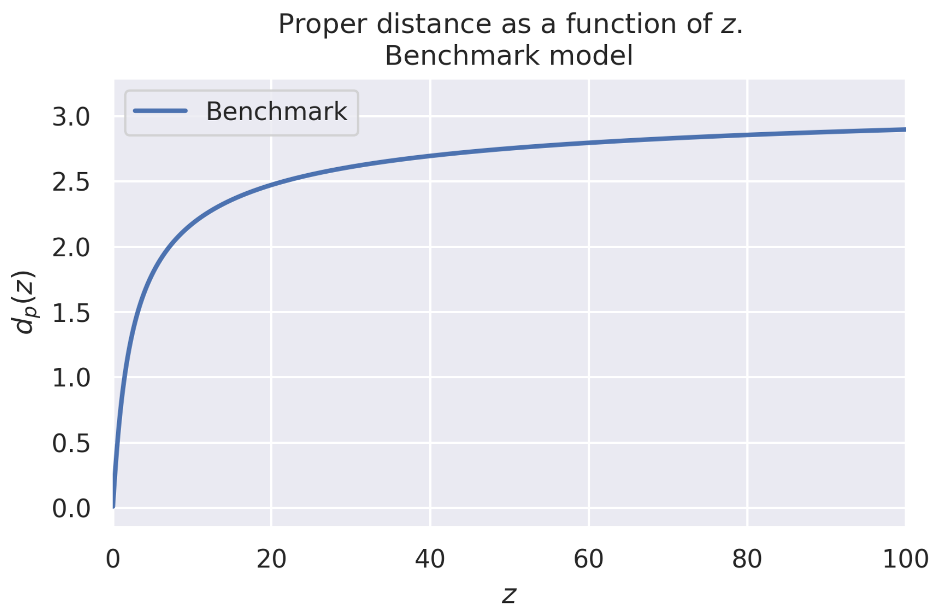

Figure 8: Proper distance as a function of the redshift z in the benchmark model. Distance in Hubble distance units.

Again, for comparison with the plots available at the Ryden' book [4], the same plot but with log-scale is presented in Figure 9.

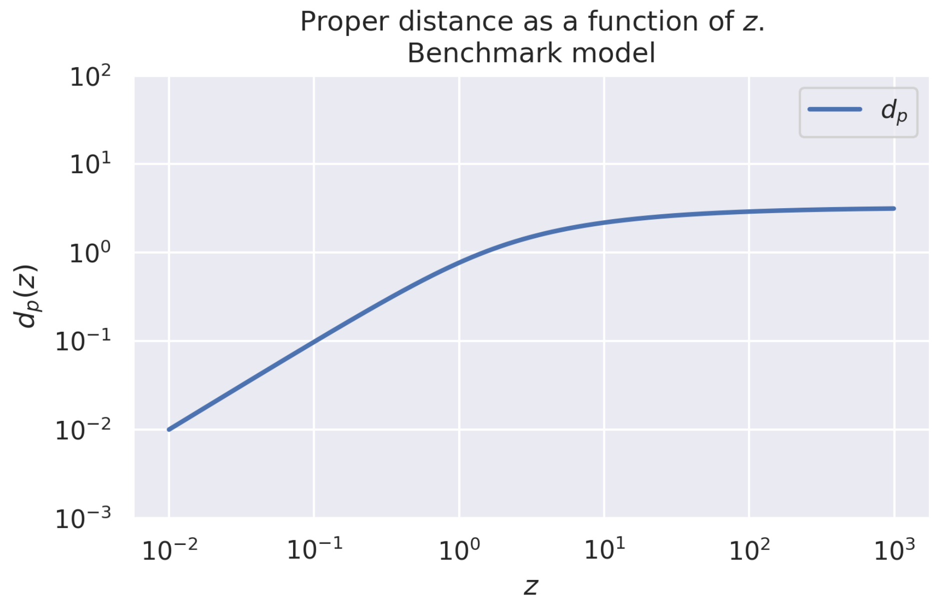

Figure 9: Proper distance as a function of the redshift z in the benchmark model with log-scale for comparison. Distance in Hubble distance units.

Now, using equation 17 and the proper distance:

$$d_L = d_p(1 + z)$$

We obtain the luminosity distance as a function of the redshift, as presented in Figure 10.

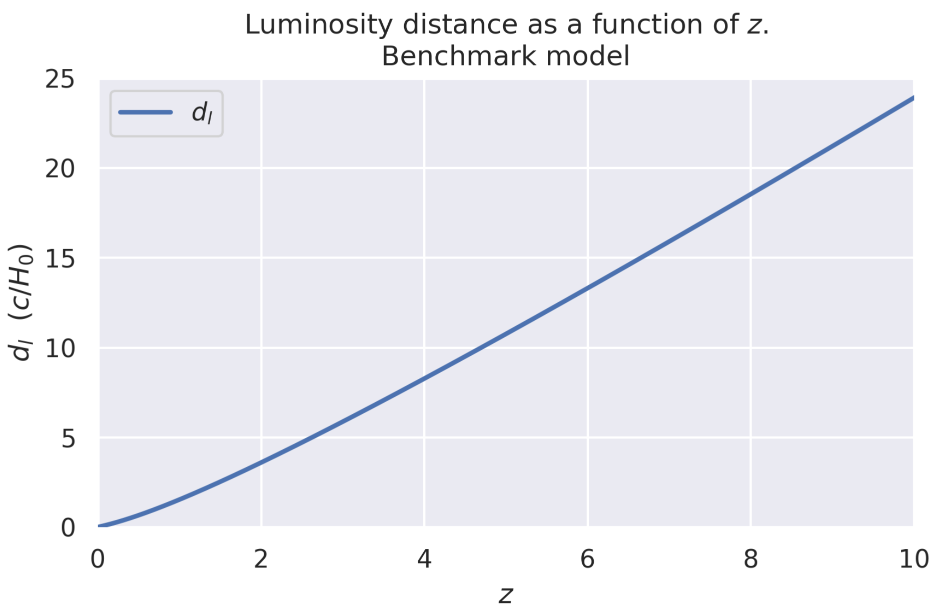

Figure 10: Luminosity as a function of the redshift z in the benchmark model. Distance in Hubble distance units.

Analogously, using equation 18 and the luminosity distance:

$$d_A = \frac{d_L}{(1+z)^2}$$

We obtain the angular-diameter distance as a function of z (Figure 11).

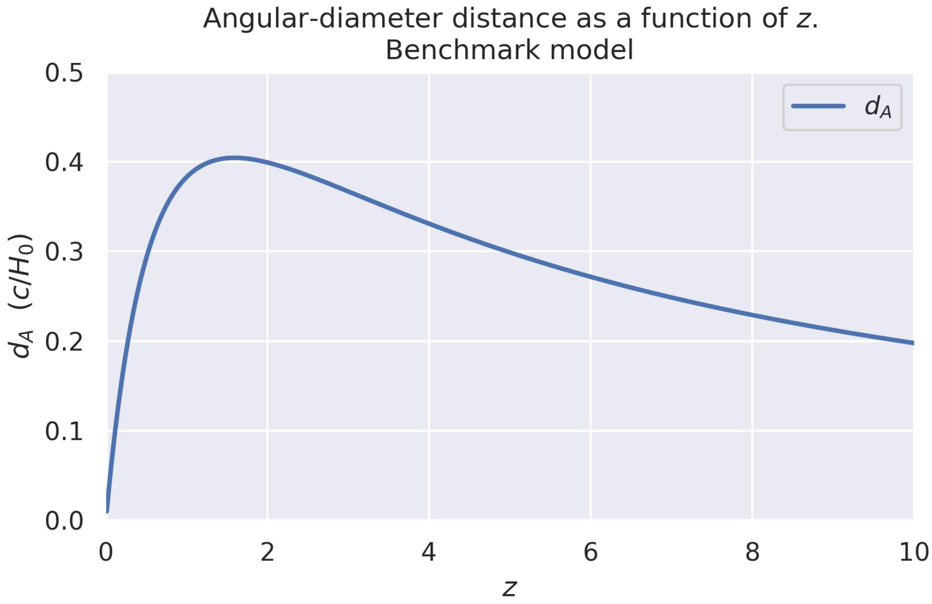

Figure 11: Angular-diameter distance as a function of the redshift z in the benchmark model. Distance in Hubble distance units.

Finally, for comparison, the three different distance measures are presented in Figure 12, as a function of z.

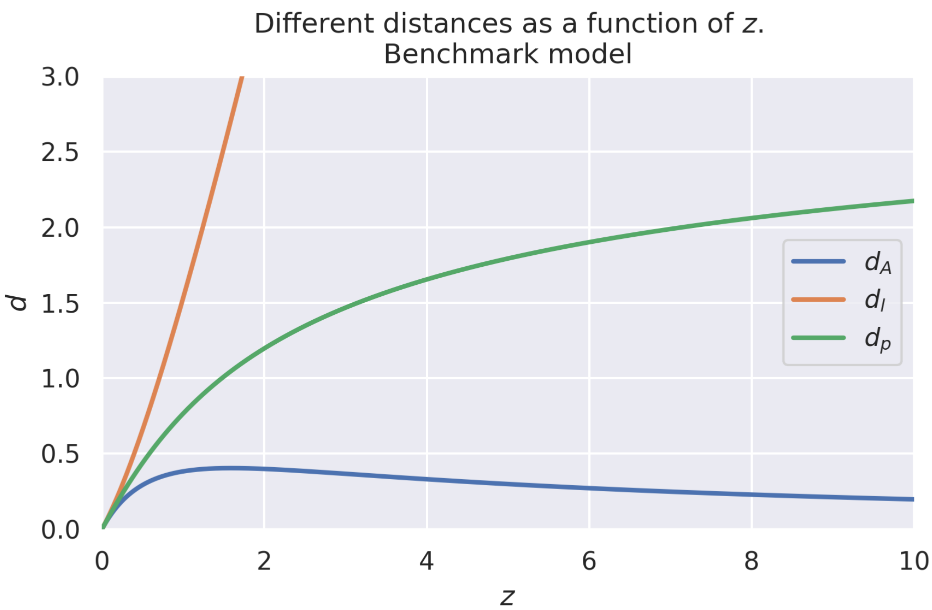

Figure 12: Comparison between the different distance measures as a function of the redshift z. Distance in Hubble distance units.

Finally, using the equation 19:

$$\mu = 5 \log_{10} \frac{d_L}{1Mpc} + 25$$

and the luminosity distance d_L we obtained from our simulation, we can get the distance modulus. In Figure 13 and Figure 14 we observe the result of this distance modulus compared with supernova data obtained from [2]. We should observe that the simulation results fits successfully to the data.

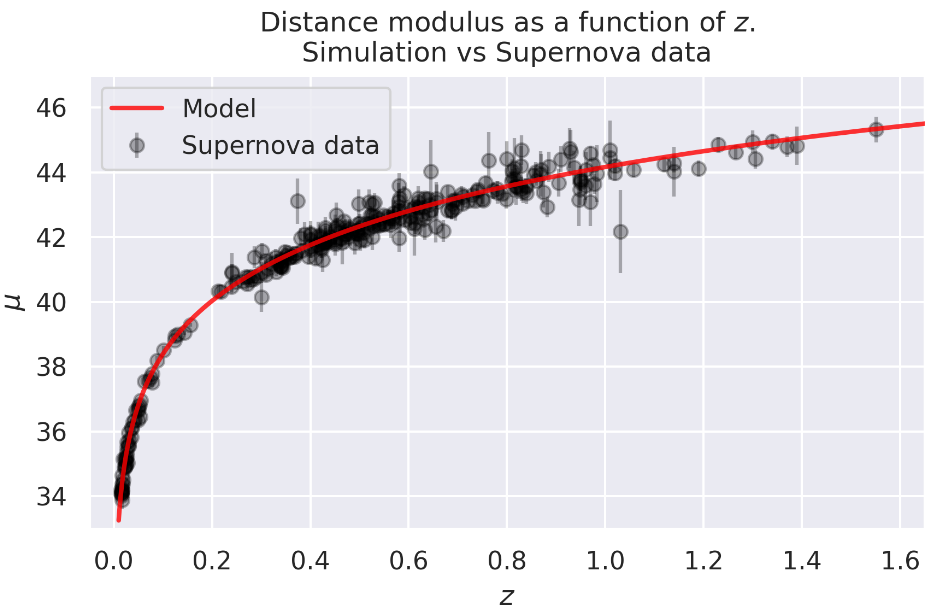

Figure 13: Distance modulus as a function of z.

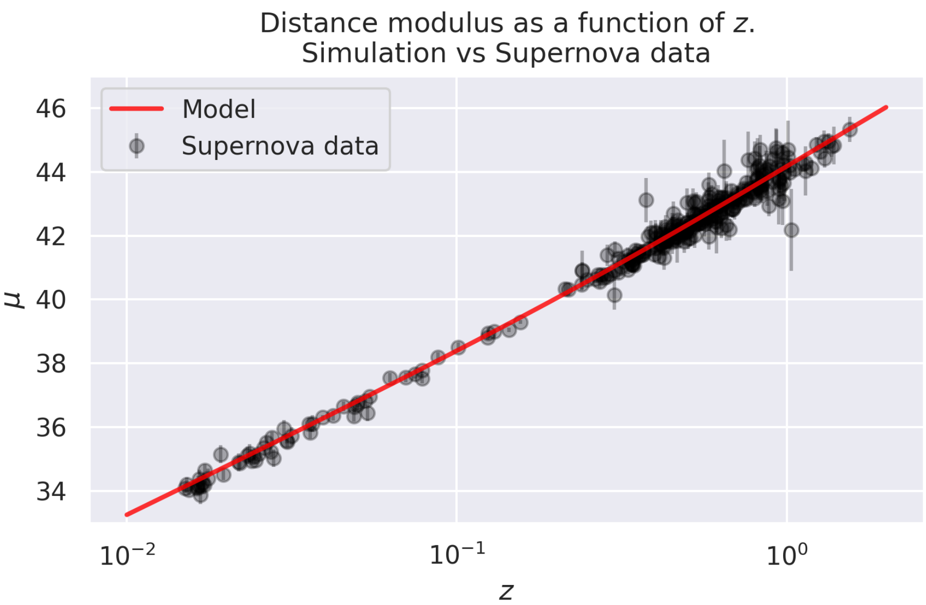

Figure 14: Distance modulus as a function of z with log-scale.

4 Conclusions

In this project, we simulated the evolution of a homogeneous, isotropic, and spatially-flat universe using the standard cosmological model Λ-CDM and the cosmological parameters given by the Planck experiment. We obtained different essential properties, such as the evolution of the energy-matter species (radiation, matter, and dark energy), the three distance measures (proper, luminosity, and angular-diameter distances), and the expansion factor.

The plots obtained in this project successfully reproduce the analogous plots presented in [], as proof of a correct simulation of the model. More importantly, the model successfully describes one observational variable given by supernova data as standard candles: the distance modulus. We conclude that this Universe model is successfully numerically solved using the Runge-Kutta method of order four and a naive approximation for integration (for the proper distance). Furthermore, this model successfully describes the observational data used in this project.

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