Options Pricing via Continuous-Time Markov Chain (CTMC) Approximation

Marie-Claude Vachon (UQAM)

June 2021

Problem

The objective is to find an approximation for

$$\mathbb{E}[e^{-rT} g(S_T)] \text{ (European)},$$

or,

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[e^{-r\tau} g(S_\tau)] \text{ (American)},$$

where

- $\{S_t\}_{t \geqslant 0}$: Stock process (diffusion process),
- $g(\cdot)$: Payoff function - continuous bounded function,
- T: Maturity, and
- r: risk-free rate.

1. Problem
2. Continuous-Time Markov Chain
3. One-Dimensional Diffusion Process
   - Grid Construction
   - Generator Construction
   - Convergence
   - Options Pricing
   - Numerical Example
4. Two-Dimensional Process (briefly)

Continuous-Time Markov Chain I

- Stochastic process: $X = \{X_t\}_{t \geqslant 0}$
- Countable state space: $\mathcal{S}_X$
- Markov property:

$$\mathbb{P}\left(X_{t_n} = j | X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_{n-1}} = i_{n-1}\right) = \mathbb{P}\left(X_{t_n} = j | X_{t_{n-1}} = i_{n-1}\right),$$

for all $j, i_1, \ldots, i_{n-1} \in \mathcal{S}_X$ and $t_1 < t_2 < \ldots < t_n$.

Continuous-Time Markov Chain II

- Homogeneous:

$$p_{ij}(s,t) = \mathbb{P}\left(X_t = j | X_s = i\right) = \mathbb{P}\left(X_{t-s} = j | X_0 = i\right) = p_{ij}(0, t-s) = p_{ij}(t-s)$$

- Transition matrix: $\mathbf{P}_t = [p_{ij}(t)]$

- On small time interval $h > 0$:

$$p_{ij}(h) \simeq q_{ij} h, \text{ if } i \neq j, \quad p_{ii}(h) \simeq 1 + q_{ii} h$$

with $q_{ij} \geqslant 0$ for $i \neq j$, $q_{ii} \leqslant 0$ and $\sum_j q_{ij} = 0$ for all i (since $\sum_j p_{ij}(h) = 1$).

- Generator: $\mathbf{Q} = [q_{ij}]$

- $\mathbf{P}_t = \exp(\mathbf{Q}t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbf{Q}^n$ (under some technical conditions)

One-Dimensional Diffusion Process

$$\mathrm{d}S_t = \mu(S_t)\,\mathrm{d}t + \sigma(S_t)\,\mathrm{d}W_t, \quad 0 \leqslant t \leqslant T, \tag{1}$$

where W is a standard Brownian motion, $\mu(\cdot)$ and $\sigma(\cdot)$ are defined such that

- (1) has a unique solution (strong or weak solution)
- Unique-in-law's weak solution is sufficient for our discussions.

CTMC Approximation

The approximation $\{S_t^N\}_{0 \leqslant t \leqslant T}$ is done in two steps:

- Construct a finite state space (grid) $\mathcal{S}^N = \{s_1, s_2, \ldots, s_N\}$, $N \in \mathbb{N}$, and
- Construct a generator $\mathbf{Q}^N = [q_{ij}]_{N \times N}$.

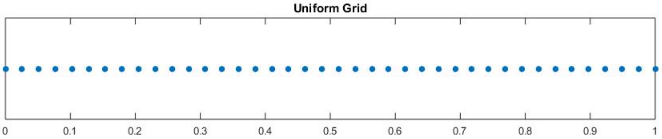
Grid Construction I

- Choose s_1 and s_N

- **Uniform Grid**:

$$s_i = s_1 + (i-1)h, \quad \text{for} \quad i = 2, \ldots, N-1,$$

where $h = (s_N - s_1)/(N-1)$.



Grid Construction II

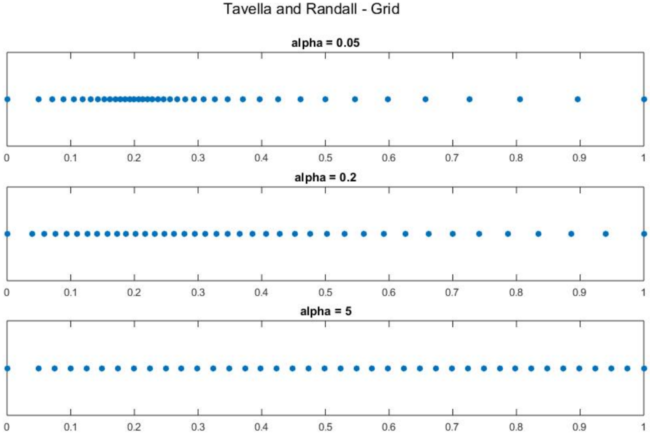
- **Non-Uniform Grid**: (Tavella and Randall (2000))

$$s_i = S_0 + \alpha \sinh\left(c_2 \frac{i}{N} + c_1 \left[1 - \frac{i}{N}\right]\right), \quad i = 2, \ldots, N-1,$$

where $c_1 = \sinh^{-1}\left(\frac{s_1 - s_0}{\alpha}\right), \quad c_2 = \sinh^{-1}\left(\frac{s_N - s_0}{\alpha}\right).$

- α: Non- uniformity parameter

Grid Construction III



Grid Construction IV

Remark 1

- *If S_0 is NOT in the grid $\Rightarrow$ insert it.*

  Suppose $s_{j_0} < S_0 < s_{j_0+1}$ for some $s_{j_0}, s_{j_0+1} \in \mathcal{S}^N$, set $s'_j = s_j + S_0 - s_{j_0}$, $j \geqslant 2$, so $s'_{j_0} = S_0$.

- *Non-uniform grids can converge faster (Lo and Skindilias (2014)).*

- *Grid designs and convergence behaviour (Zhang and Li (2019)).*

Generator Construction I

Local consistency condition:

$$
\begin{aligned}
\mathbb{E}_t\left[S_{t+h}^N - S_t^N\right] &= \mathbb{E}_t\left[S_{t+h} - S_t\right] &&\simeq \mu(S_t)h \\
\mathbb{E}_t\left[\left(S_{t+h}^N - S_t^N\right)^2\right] &= \mathbb{E}_t\left[\left(S_{t+h} - S_t\right)^2\right] &&\simeq \sigma^2(S_t)h,
\end{aligned}
$$

for an infinitesimal period of time $h > 0$.

Generator Construction II

For $i = 2, 3, \ldots, N-1$,

$$\begin{aligned}
\mathbb{E}\left[S^N_{t+h} - S^N_t | S^N_t = s_i\right] &= p_{i,i-1}(h)(s_{i-1} - s_i) + p_{ii}(h)(s_i - s_i) + p_{i,i+1}(h)(s_{i+1} - s_i) \\
&\simeq -hq_{i,i-1}\delta_{i-1} + hq_{i,i+1}\delta_i.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[(S^N_{t+h} - S^N_t)^2 | S^N_t = s_i\right] &= p_{i,i-1}(h)(s_{i-1} - s_i)^2 + p_{ii}(h)(s_i - s_i)^2 + p_{i,i+1}(h)(s_{i+1} - s_i)^2 \\
&\simeq -hq_{i,i-1}\delta^2_{i-1} + hq_{i,i+1}\delta^2_i.
\end{aligned}$$

with $\delta_i = s_{i+1} - s_i$.

Generator Construction III

- We obtain the following system of equations:

$$\begin{aligned} -hq_{i,i-1}\delta_{i-1} + hq_{i,i+1}\delta_i &= \mu(s_i)h \\ hq_{i,i-1}\delta_{i-1}^2 + hq_{i,i+1}\delta_i^2 &= \sigma^2(s_i)h. \end{aligned}$$

- Using $\sum_j q_{ij} = 0$, $q_{ij} \geqslant 0$ and $q_{ii} \leqslant 0$, we have

$$q_{ij} = \begin{cases} \frac{\sigma^2(s_i) - \delta_i \mu(s_i)}{\delta_{i-1}(\delta_{i-1}+\delta_i)} & \text{if } j = i-1 \\ -q_{i,i-1} - q_{i,i+1} & \text{if } j = i \\ \frac{\sigma^2(s_i) + \delta_{i-1}\mu(s_i)}{\delta_i(\delta_{i-1}+\delta_i)} & \text{if } j = i+1 \\ 0 & \text{if } j \neq i,\ i-1,\ i+1, \end{cases} \tag{2}$$

for $i = 2, 3, \ldots, N-1$.

Generator Construction IV

- At end points $(i = 1, N)$:
  - $q_{12} = |\mu(s_1)|/\delta_1$ and $q_{11} = -q_{12}$,
  - $q_{N,N-1} = |\mu(s_N)|/\delta_{N-1}$ and $q_{N,N} = -q_{N,N-1}$.
  - Some authors use : $q_{1j} = q_{N,j} = 0$, $j = 1, 2, \dots, N$

Generator Construction V

- **Additional conditions**:
  - $q_{i,i-1} \geqslant 0 \Rightarrow \frac{\sigma^2(s_i)-\delta_i\mu(s_i)}{\delta_{i-1}(\delta_{i-1}+\delta_i)} \geqslant 0 \Rightarrow \delta_i \leqslant \frac{\sigma^2(s_i)}{\mu(s_i)}$, if $\mu(s_i) > 0$. $\delta_i = s_{i+1} - s_i$.
  - $q_{i,i+1} \geqslant 0 \Rightarrow \frac{\sigma^2(s_i)+\delta_{i-1}\mu(s_i)}{\delta_i(\delta_{i-1}+\delta_i)} \geqslant 0 \Rightarrow \delta_{i-1} \leqslant -\frac{\sigma^2(s_i)}{\mu(s_i)}$, if $\mu(s_i) < 0$
- **Sufficient condition**: $\max_{2\leqslant i\leqslant N-1} \delta_i \leqslant \min_{2\leqslant i\leqslant N-1} \frac{\sigma^2(s_i)}{|\mu(s_i)|}$.
- - If the sufficient condition is NOT satisfied $\Rightarrow$ check additional conditions.
  - If additional conditions are NOT satisfied $\Rightarrow$ increase N.

Convergence

- $S^N \Rightarrow S$ as $N \to \infty$, where "$\Rightarrow$" denotes the convergence in distribution (or weak-convergence).
  - Proof: Mijatović and Pistorius (2013) for details.
  - Idea: Distance between the generators of S^N and S tends to 0 as $N \to \infty$ for a sufficiently large class of functions.
  - Semi-group theory, Ethier and Kurtz (2005), Theorem 4.2.11.
  - For $t \geqslant 0$, $\mathbb{E}[f(S_t^N)] \to \mathbb{E}[f(S_t)]$ for every bounded continuous real function f, Billingsley (1979), Theorem 25.8.

European Option Pricing I

Given $S_0 = s_i$,

- $\mathbb{E}[e^{-rT} g(S_T)] \approx \mathbb{E}[e^{-rT} g(S_T^N)]$
- $\mathbf{P}_T = \exp\{T\mathbf{Q}^N\}$, $\mathbf{Q}^N = [q_{ij}]_{N\times N}$ defined in (2).
- $$\mathbb{E}[e^{-rT} g(S_T^N)] = e^{-rT} \sum_{j=1}^{N} p_{ij}(T) g(s_j) = e^{-rT} \mathbf{e}_i \exp\{T\mathbf{Q}^N\}\mathbf{G}.$$
    - $\mathbf{e}_i$ row vector of size $1 \times N$ having a value of 1 on the i-th entry and 0 elsewhere,
    - $\mathbf{G} = [g_k]_{N\times 1}$, column vector of size $N \times 1$ whose k-entry $g_k = g(s_k)$.

American Option Pricing I

Given $S_0 = s_i$,

- $\Delta_M = T/M$, $M > 0$
- $\mathcal{H}_M(0,T) = \{t_0, t_1, \dots, t_M\}$, $t_k = k\Delta_M$, $k = 0, 1, \dots M$.
- $\mathcal{T}_{\Delta_M(t,T)}$, set of stopping times taking values in $\mathcal{H}_M(t,T)$.
- Bermudan approximation:

$$\sup_{\tau \in \mathcal{T}_{\Delta_M(0,T)}} \mathbb{E}[e^{-r\tau} g(S_\tau^N)] \approx \sup_{\tau \in \mathcal{T}_{\Delta_M(0,T)}} \mathbb{E}[e^{-r\tau} g(S_\tau)] \xrightarrow[M \to \infty]{} \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[e^{-r\tau} g(S_\tau)]$$

.

American Option Pricing II

- Bermudan option admits the following representation (dynamic programming principle):

$$\begin{cases} B_M^N & = g(S_T^N) \\ B_k^N & = \max\left(g(S_{t_k}^N), e^{-r\Delta_M}\mathbb{E}[B_{k+1}^N|\mathcal{F}_{t_k}]\right), \qquad 0 \leqslant k \leqslant M-1, \end{cases},$$

- equivalent to:

$$\begin{cases} \mathbf{B}_M^N & = \mathbf{G}, \\ \mathbf{B}_k^N & = \max\{\mathbf{G}, e^{-r\Delta_M}\exp\{\Delta_M \mathbf{Q}^N\}\mathbf{B}_{k+1}^N\}, \quad 0 \leqslant k \leqslant M-1, \end{cases}$$

$\mathbf{G} = [g_k]_{N\times 1}$, column vector of size $N \times 1$ whose k-entry $g_k = g(s_k)$, $\mathbf{B}_k^N$, column vector of size $N \times 1$, $k = 0, 1, \dots, M$

- $\sup_{\tau\in\mathcal{T}_{\Delta_M(0,T)}} \mathbb{E}[e^{-r\tau}g(S_\tau^N)] = \mathbf{e}_i\mathbf{B}_0^N$.

Numerical Example I

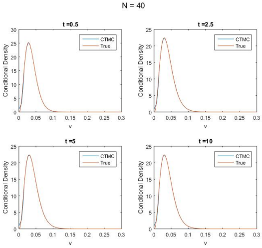
- Square-Root diffusion process:

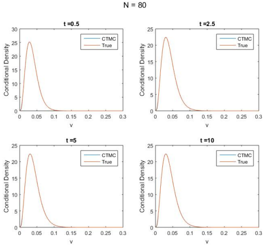
$$\mathrm{d}V_t \quad = \kappa(\theta - V_t)\,\mathrm{d}t + \sigma\sqrt{V_t}\,\mathrm{d}W_t$$

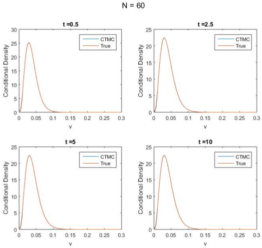
  $\theta, \kappa, \sigma > 0$, and W Brownian motion.

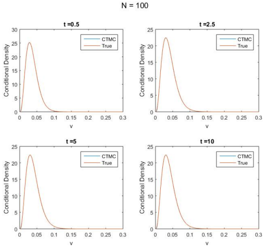
- $\theta = 0.04$, $\kappa = 2$, $\sigma = 0.2$, and $V_0 = 0.03$.

Example: Tavella and Randall grid, $\alpha \approx 0.45$, $v_1 = V_0/100$, $v_N = 6V_0$.

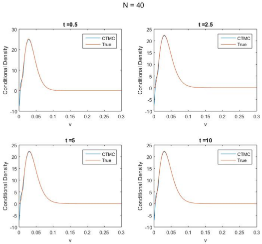


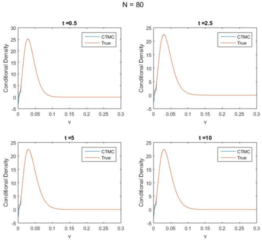


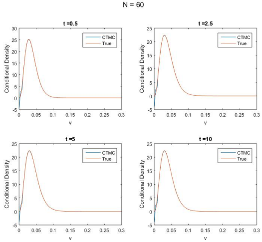


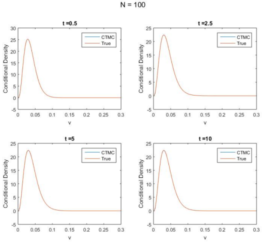


Example: Uniform grid, $v_1 = V_0/100$, $v_N = 6V_0$.









Two-Dimensional Process

Extension to two-dimensional processes works similarly (e.g., Stochastic Volatility Models):

- CTMC approximation of the variance process.
- Plug the variance CTMC approximation in the stock diffusion $\Rightarrow$ Regime-switching diffusion.
- CTMC approximation of the regime-switching diffusion $\Rightarrow$ Regime-switching CTMC.
- Regime-switching CTMC map onto a one-dimensional CTMC on an enlarged state space.
- Back to the one-dimensional CTMC case. Pricing works as explained previously.
- See Cui, Kirkby and Nguyen (2018) for details.

Numerical Tips

- Define $\mathbf{Q}^N$ as a sparse matrix.
- Matrix exponential may be calculated using function *expm* in Matlab or R.
- If the calculation of the matrix exponential is too heavy (usually for 2D-Process), use Expokit of Sidje (1998) downloadable at `https://www.maths.uq.edu.au/expokit`.

Conclusion

- CTMC approximation can be applied to different types of diffusion processes (one-dimensional or two-dimensional)
- European option: Closed-form matrix expression.
- American option: Bermudan approximation, works similarly as in a tree (easy to implement).
- See Cui, Kirkby and Nguyen (2019)

References I

Billingsley, P.
Probability and measure.
John Wiley & Sons, 1979.

Cui, Z., Kirkby, J. L. and Nguyen, D.
A General Valuation Framework for SABR and Stochastic Local Volatility Models.
SIAM Journal on Financial Mathematics, 9(2):520–563, 2018.

Cui, Z., Kirkby, J. L. and Nguyen, D.
Continuous-Time Markov Chain and Regime Switching Approximations with Applications to Options Pricing.
In Modeling, Stochastic Control, Optimization, and Applications, Springer, Cham, 115–146, 2019.

References II

Lo, C. C. and Skindilias, K.

An Improved Markov Chain Approximation Methodology: Derivative Pricing and Model Calibration.

International Journal of Theoretical and Applied Finance, 17(7):1450047, 2014.

Mijatović, A., and Pistorius, M.

Continuously Monitored Barrier Options under Markov Processes.

Mathematical Finance, 23(1):1–38, 2013.

Sidge, R. B.

Expokit: A Software Package for Computing Matrix Exponentials.

ACM Transactions on Mathematical Software, 24(1):130–156, 1998.

References III

Tavella, D. and Randall, C.

Pricing Financial Instruments: The Finite Difference Method.

John Willey & Sons, 2000.

Zhang, G. and Li, L.

Analysis of Markov Chain Approximation for Option Pricing and Hedging: Grid Design and Convergence Behavior.

Operations Research, 67(2):407–427, 2019.