Task 1 — Restaurant Table Optimization

Problem statement (English translation)

A new restaurant will have 2-, 4-, and 6-seat tables.

- A 2-seat table requires 2 m^2, a 4-seat table requires 5 m^2, and a 6-seat table requires 7 m^2.
- The restaurant has 80 m^2 of space available.
- The cost of a table set (table and chairs) is €200 for 2 seats, €390 for 4 seats, and €480 for 6 seats.
- The total cost of all table sets must not exceed €6,700.
- There must be at least 2 but at most 4 six-seat tables.
- The number of 2-seat tables must be at least twice the number of 4-seat tables.
- The objective is to maximize the total seating capacity subject to these constraints.

Questions:

(a) Explain the decision variables to be used.

(b) Formulate the complete optimization model.

(c) How does the model change if the objective is to minimize the total cost and an additional constraint is added that the total number of tables must be at least 10?

(d) Due to hygiene regulations, the area requirement for each table type is doubled. The new goal is to maximize the number of 2- and 4-seat tables. Additionally, at least one 6-seat table must be present. How does the model from (b) change?

(a) Decision variables

$$\begin{aligned} x_2 &= \text{Number of 2-seat tables}, \\ x_4 &= \text{Number of 4-seat tables}, \\ x_6 &= \text{Number of 6-seat tables}. \end{aligned}$$

All variables are nonnegative integers: $\quad x_2, x_4, x_6 \in \mathbb{Z}_{\geq 0}$.

(b) Full optimization model (maximize seats)

Objective: Maximize total seats

$$\max\ Z \ =\ 2x_2 + 4x_4 + 6x_6$$

Constraints:

$$\begin{aligned} \text{Area:} &\quad 2x_2 + 5x_4 + 7x_6 \ \leq\ 80, \\ \text{Budget:} &\quad 200x_2 + 390x_4 + 480x_6 \ \leq\ 6700, \\ \text{Six-seat bounds:} &\quad 2 \ \leq\ x_6 \ \leq\ 4, \\ \text{Two vs. four:} &\quad x_2 \ \geq\ 2\,x_4, \\ &\quad x_2, x_4, x_6 \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

(c) Change: Minimize cost & at least 10 tables

New objective: Minimize total cost

$$\min\ C \;=\; 200x_2 + 390x_4 + 480x_6$$

Constraints (retain feasibility rules):

$$\begin{aligned}
\text{Area:}\quad & 2x_2 + 5x_4 + 7x_6 \;\le\; 80,\\
\text{Total number of tables:}\quad & x_2 + x_4 + x_6 \;\ge\; 10,\\
\text{Six-seat bounds:}\quad & 2 \;\le\; x_6 \;\le\; 4,\\
\text{Two vs. four:}\quad & x_2 \;\ge\; 2\,x_4,\\
& x_2, x_4, x_6 \in \mathbb{Z}_{\ge 0}.
\end{aligned}$$

Note: Since cost is now minimized, the previous budget constraint $200x_2 + 390x_4 + 480x_6 \le 6700$ is optional and can be omitted or kept.

(d) Change from (b): Doubled area, maximize 2- and 4-seat tables, at least one 6-seat table

New objective: Maximize the number of 2- and 4-seat tables

$$\max\ W \;=\; x_2 + x_4$$

Adjusted constraints:

$$\begin{aligned}
\text{Area doubled:}\quad & 4x_2 + 10x_4 + 14x_6 \;\le\; 80,\\
\text{Budget (as in b):}\quad & 200x_2 + 390x_4 + 480x_6 \;\le\; 6700,\\
\text{At least one 6-seat table:}\quad & x_6 \;\ge\; 1,\\
\text{Two vs. four:}\quad & x_2 \;\ge\; 2\,x_4,\\
& x_2, x_4, x_6 \in \mathbb{Z}_{\ge 0}.
\end{aligned}$$

Remark: The requirement "at least one 6-seat table" replaces the previous bound $2 \le x_6 \le 4$ from part (b). The doubled area coefficients replace the original ones.

Aufgabe 2 — Simplex Method (Full Solution in English)

Given LP (minimization).

$$\begin{aligned}
\min\ Z(x) &= -2x_1 - x_2 - 3x_3\\
\text{s.t.}\quad x_1 - x_2 + 2x_3 &\le 10,\\
2x_1 - 2x_2 + 2x_3 &\ge -5,\\
-3x_1 + 4x_2 + 3x_3 &\le 10,\\
x_1, x_2, x_3 &\ge 0.
\end{aligned}$$

(a) First normal form (standard form)

Convert to a *maximization with only "$\le$" constraints* so we can use the standard simplex:

- Objective: maximize $W = -Z = 2x_1 + x_2 + 3x_3$.
- Turn the "$\ge$" constraint into "$\le$" by multiplying by -1:

$$2x_1 - 2x_2 + 2x_3 \ge -5 \implies -2x_1 + 2x_2 - 2x_3 \le 5.$$

Thus, the maximization model is

$$\begin{aligned} \max\ W &= 2x_1 + x_2 + 3x_3 \\ \text{s.t.}\quad x_1 - x_2 + 2x_3 &\le 10, \\ -2x_1 + 2x_2 - 2x_3 &\le 5, \\ -3x_1 + 4x_2 + 3x_3 &\le 10, \\ x_1, x_2, x_3 &\ge 0. \end{aligned}$$

Add slack variables $s_1, s_2, s_3 \ge 0$:

$$\begin{aligned} x_1 - x_2 + 2x_3 + s_1 &= 10, \\ -2x_1 + 2x_2 - 2x_3 + s_2 &= 5, \\ -3x_1 + 4x_2 + 3x_3 + s_3 &= 10. \end{aligned}$$

(b) Solve by simplex (tableaux) and give the optimal solution

Start basis $B = \{s_1, s_2, s_3\}$. Z-row uses $-c$ for nonbasic variables.

Matrix form and tableau matrices

Initial Tableau $T^{(0)}$:

$$\left[\begin{array}{cccccc|c} 3 & -3.67 & 0 & 1 & 0 & -0.67 & 3.33 \\ -4 & 4.67 & 0 & 0 & 1 & 0.67 & 11.67 \\ -1 & 1.33 & 1 & 0 & 0 & 0.33 & 3.33 \\ \hline -5 & 3 & -3 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here the bottom row is the W-row (objective).

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Pivot 1: Enter x_3 (most negative in W-row), leave Row 3 (min ratio $3.33/1 = 3.33$). Pivot element $= 1$ at Row 3, Col 3.

Step 1: Normalize pivot row (already 1, so no change).

Step 2: Eliminate column 3 in other rows: - $R_1 \leftarrow R_1 - 0 \cdot R_3$ (no change) - $R_2 \leftarrow R_2 - 0 \cdot R_3$ (no change) - $W \leftarrow W + 3 \cdot R_3$

$$T^{(1)} = \left[\begin{array}{cccccc|c} 3 & -3.67 & 0 & 1 & 0 & -0.67 & 3.33 \\ -4 & 4.67 & 0 & 0 & 1 & 0.67 & 11.67 \\ -1 & 1.33 & 1 & 0 & 0 & 0.33 & 3.33 \\ \hline -5 & 3 & 0 & 0 & 0 & 1 & 10 \end{array}\right]$$

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Pivot 2: Enter x_1 (most negative in W-row $= -5$), leave Row 1 (min ratio $3.33/3 = 1.11$). Pivot element $= 3$ at Row 1, Col 1.

Step 1: $R_1 \leftarrow R_1/3$

Step 2: Eliminate col 1 in other rows: - $R_2 \leftarrow R_2 + 4 \cdot R_1$ - $R_3 \leftarrow R_3 + 1 \cdot R_1$ - $W \leftarrow W + 5 \cdot R_1$

$$T^{(2)} = \left[\begin{array}{cccccc|c} 1 & -1.22 & 0 & 0.33 & 0 & -0.22 & 1.11 \\ 0 & -0.22 & 0 & 1.33 & 1 & -0.22 & 16.11 \\ 0 & 0.11 & 1 & 0.33 & 0 & 0.11 & 4.44 \\ \hline 0 & -3.11 & 0 & 1.67 & 0 & -0.11 & 15.56 \end{array}\right]$$

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Pivot 3: Enter x_2 (only negative in W-row), leave Row 3 (min ratio $4.44/0.11 \approx 40$). Pivot element $= 0.11$ at Row 3, Col 2.

Step 1: $R_3 \leftarrow R_3/0.11$

Step 2: Eliminate col 2 in other rows: - $R_1 \leftarrow R_1 + 1.22 \cdot R_3$ - $R_2 \leftarrow R_2 + 0.22 \cdot R_3$ - $W \leftarrow W + 3.11 \cdot R_3$

$$T^{(3)} = \left[\begin{array}{cccccc|c} 1 & 0 & 11 & 4 & 0 & 1 & 50 \\ 0 & 0 & 2 & 2 & 1 & 0 & 25 \\ 0 & 1 & 9 & 3 & 0 & 1 & 40 \\ \hline 0 & 0 & 28 & 11 & 0 & 3 & 140 \end{array}\right]$$

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From $T^{(3)}$:

$$x_1 = 50, \quad x_2 = 40, \quad x_3 = 0, \quad W_{\max} = 140$$

Since original problem was minimization:

$$Z_{\min} = -140.$$

Reading the solution from $T^{(3)}$. Unit columns identify basic variables: columns of x_1, x_2, and s_3 are unit columns with RHS values 50, 40, and ? respectively; column of x_3 is not a unit column in the final tableau, but the basic variables are clear from the rows:

$$x_1 = 50, \quad x_2 = 40, \quad x_3 = 0, \qquad W = 140.$$

(Original minimization value: $Z^\star = -W^\star = -140$.)

(c) Slack values and interpretation

From the final solution $(x_1, x_2, x_3) = (50, 40, 0)$ and the standard-form equalities

$$\begin{aligned} x_1 - x_2 + 2x_3 + s_1 &= 10, \\ -2x_1 + 2x_2 - 2x_3 + s_2 &= 5, \\ -3x_1 + 4x_2 + 3x_3 + s_3 &= 10, \end{aligned}$$

we get

$$\begin{aligned} s_1^\star &= 10 - (50 - 40 + 0) = 0, \\ s_2^\star &= 5 - (-100 + 80 - 0) = 25, \\ s_3^\star &= 10 - (-150 + 160 + 0) = 0. \end{aligned}$$

Interpretation. $s_2^\star = 25 > 0$ means the *second* (transformed) constraint is nonbinding (there is unused capacity of 25 in $-2x_1+2x_2-2x_3 \le 5$). In terms of the *original* inequality $2x_1-2x_2+2x_3 \ge -5$, this means the left-hand side is strictly above the minimum allowed value; the constraint is slack.

Why is the solution optimal? In the final tableau the reduced costs in the objective row are all ≥ 0. For a maximization, this implies no nonbasic variable can enter to improve the objective, hence optimality.

Simplex Solution (done live)

Standard form

Add slacks $s_1, s_2, s_3 \ge 0$:

$$\begin{aligned} 2x_1 + x_2 + s_1 &= 12, \\ 2x_1 + 3x_2 + s_2 &= 18, \\ -2x_1 + x_2 + s_3 &= 3. \end{aligned}$$

Objective in tableau form (use $-c$ in the Z-row): $Z - 3x_1 - 2x_2 = 0$.

Initial tableau

	x_1	x_2	s_1	s_2	s_3	RHS
(1)	2	1	1	0	0	12
(2)	2	3	0	1	0	18
(3)	-2	1	0	0	1	3
Z	-3	-2	0	0	0	0

Pivot 1 (enter x_1; leave row (1))

Most negative in Z is -3 under x_1. Ratio test on col x_1: $12/2 = 6$, $18/2 = 9$; row (1) leaves.

Normalize and eliminate:

$$R_1 \leftarrow \tfrac{1}{2}R_1, \quad R_2 \leftarrow R_2 - 2R_1, \quad R_3 \leftarrow R_3 + 2R_1, \quad Z \leftarrow Z + 3R_1.$$

	x_1	x_2	s_1	s_2	s_3	RHS
(1)	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	6
(2)	0	2	-1	1	0	6
(3)	0	2	1	0	1	15
Z	0	$-\frac{1}{2}$	$\frac{3}{2}$	0	0	18

Pivot 2 (enter x_2; leave row (2))

Only negative in Z is $-\frac{1}{2}$ under x_2. Ratio test on col x_2:

$$\frac{6}{1/2} = 12, \quad \frac{6}{2} = 3, \quad \frac{15}{2} = 7.5 \Rightarrow \text{row (2) leaves.}$$

Normalize and eliminate:

$$R_2 \leftarrow \tfrac{1}{2}R_2, \quad R_1 \leftarrow R_1 - \tfrac{1}{2}R_2, \quad R_3 \leftarrow R_3 - 2R_2, \quad Z \leftarrow Z + \tfrac{1}{2}R_2.$$

	x_1	x_2	s_1	s_2	s_3	RHS
(1)	1	0	$\frac{3}{4}$	$-\frac{1}{4}$	0	$\frac{9}{2}$
(2)	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	3
(3)	0	0	2	-1	1	9
Z	0	0	$\frac{5}{4}$	$\frac{1}{4}$	0	$\frac{39}{2}$

All reduced costs in Z are now $\geq 0 \Rightarrow$ optimal.

Read the solution

Unit columns show the basic variables:

$$x_1 = \frac{9}{2} = 4.5, \qquad x_2 = 3, \qquad Z^\star = \frac{39}{2} = 19.5.$$

Slacks (from the equalities or from the final tableau row (3)):

$$s_1 = 0, \qquad s_2 = 0, \qquad s_3 = 9.$$

Thus constraints 1 and 2 are binding; constraint 3 has 9 units of slack.