MASTER'S THESIS

A Deep Learning Approach to Optimal Investment With Transaction Costs

Author:
Gabriel NOBIS

Supervisor:
Prof. Dr. Cristoph BELAK

Co-Supervisor:
Dr. Wojciech SAMEK

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Abstract

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by Gabriel NOBIS

Exploiting the shadow price approach of Czichowsky et al. [11] for continuous price processes, we develop a numerical method based on machine learning to approximate the optimal trading strategy in a continuous time financial model with transaction costs. To formulate the transaction cost problem as a learning problem, we generalize the PAC-learning framework of Mohri et al. [29] from a data generating random variable to a data generating process. We state sufficient conditions for the proposed method to satisfy our notion of a weak PAC-learning algorithm in a financial model driven by a recurrent Markov process with bounded coefficient functions. In addition, this work comprises an implementation of the proposed method in PyTorch that can be accessed at `https://github.com/GabrielNobis/Thesis`.

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Contents

Notational Conventions

$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space
$[0, T]$	Time horizon with terminal time $T > 0$
λ, γ	Transaction cost parameters $\lambda, \gamma \in (0, 1)$ with $\lambda = \frac{2\gamma}{1+\gamma}$
$W = (W_t)_{t \in [0,T]}$	Brownian motion
$(\mathcal{F}_t)_{t \in [0,T]}$	Filtration
$(\mathcal{F}_t^W)_{t \in [0,T]}$	Filtration generated by Brownian motion W
$S = (S_t)_{t \in [0,T]}$	Continuous stochastic price process
μ, σ	Coefficient functions of the price process
$\tilde{S} = (\tilde{S}_t)_{t \in [0,T]}$	Shadow price process
g	Smooth pasting function
$[1, \bar{s}]$	Domain of a smooth pasting function with $\bar{s} > \frac{1}{1-\lambda}$
$\tilde{\mu}, \tilde{\sigma}$	Coefficient functions of the shadow price process
$\bar{\mu}, \bar{\sigma}$	Rewritten form of functions $\tilde{\mu}$ and $\tilde{\sigma}$
$P = (P_t)_{t \in [0,T]}$	Continuous stochastic price process $P = \frac{S}{(1+\gamma)}$
$X = (X_t)_{t \in [0,T]}$	Continuous stochastic process $X = \frac{1}{\gamma}\left(\frac{\tilde{S}}{P} - 1\right)$
$\mathcal{C}$	Concept class of a learning problem
$\mathcal{H}$	Hypothesis set of a learning problem
$R(\Phi)$	Neural network Φ with realization R
$\bar{\mu}_{R_1(\Phi_1)}, \bar{\sigma}_{R_2(\Phi_2)}$	Neural network approximation of $\bar{\mu}$ and $\bar{\sigma}$
$\mathbb{R}_+$	Set of strictly positive real numbers
L_h	Lipschitz constant of the function h
C_h, c_h	function h is bounded from above by C_h and bounded from below by c_h
$X_1 = X_2$	Equality holds $\mathbb{P} - a.s.$ for the random variables X_1 and X_2

Chapter 1

Introduction

Under the assumption that the prices of shares are driven by an underlying Brownian motion, Black and Scholes [5] 1973 invented the Black-Scholes model based on the theory of stochastic processes in continuous time. The model consists of a geometric Brownian motion

$$P_t = P_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right), \quad P_0 = p_0 > 0$$

modelling the price evolution of a risky asset accompanied by a deterministic process

$$B_t = \exp(rt)$$

called the Bank account with interest rate $r \geq 0$. For a trader with logarithmic preferences, the optimal way of trading in the absence of transaction costs is to keep at any time the constant fraction

$$\pi_M = \frac{\mu}{\sigma^2}$$

of her total wealth invested in the risky share [26]. This optimal fraction π_M is called the *Merton fraction*. In the presence of transaction costs, a trader following the Merton fraction is instantaneously ruined, since the variation of the underlying price process implies that trading at any point in time is necessary to keep the risky fraction constant and results in infinite losses during any trading period. The incorporation of transaction costs ensures therefore that an optimal way of trading does not require to trade at any point in time.

To characterize an optimal trading strategy in the presence of proportional transaction costs, the hereinafter presented shadow price approach [21], [15],[9],[10],[11] exploits the well known theory of optimal investment without transaction costs. Based on the shadow price approach and based on neural network approximation of continuous functions [17], we develop a numerical method to approximate the optimal risky fraction and the optimal trading strategy in a continuous time financial model with transaction costs. In contrast to Tourin and Zariphopoulou [34], Herzog et al. [18] and Belak and Sass [4] the proposed method does not rely on numerical schemes to solve a PDE and may therefore be extended to solve the transaction cost problem with an underlying higher-dimensional price process. In addition, we design the method to be applicable in a more general financial model driven by a recurrent Markov process with bounded coefficient functions.

This thesis consists of three parts: In the first part, we summarize the shadow price approach in the general setting of strictly positive and continuous price processes and in the second part we translate the transaction cost problem into a learning problem within a generalized PAC-learning framework and specify the numerical method. We finish with numerous experiments, deploying our developed method in the Black-Scholes model with transaction costs. A more detailed outline is presented here in the following.

In Chapter 2 we present the shadow price approach in the general setting of strictly positive and continuous price processes and show in Corollary 2.5 that in the setting of [11], the assumption of an underlying Brownian filtration is sufficient for the shadow price process to be an Itô process. In preparation of the theoretical justification of the proposed method, we define the set $\boldsymbol{\mathcal{S}}_{LS}$ of price processes consisting of recurrent Markov processes satisfying Assumption 2 and show in Theorem 2.10 that a geometric Brownian motion with $\sigma^2 > \mu > 0$ is an element of $\boldsymbol{\mathcal{S}}_{LS}$.

Chapter 3 begins with the formulation of the learning problem in the setting of the PAC-learning framework of Mohri et al. [29] and continues with the generalization of this framework from a data generating random variable to a data generating stochastic process in order to formulate the transaction cost problem as a learning problem. The proposed numerical method is described in Section 3.3.1 to Section 3.3.4 accompanied by theoretical justifications of each step, stating the sufficient conditions for the method to converge. Doing so, we restrict the set $\boldsymbol{\mathcal{S}}_{LS}$ to the set $\boldsymbol{\mathcal{S}}_{LS,b}$ of recurrent Markov processes with bounded coefficient functions. Finally, in Theorem 3.23 we show that the proposed learning algorithm of Section 3.3.5 satisfies under the derived assumptions our notion of a weak PAC-learning algorithm.

In Chapter 4 we apply the proposed method in the Black-Scholes model with transaction cost, testing four different parametrizations of the shadow price process and four different approximation schemes of the underlying process X defined in (3.15). We find that the method *Oracle* shows the best performance, where the name comes from an unexpected approximation scheme for X. We conclude our experiments by a comparison of two neural network architectures, changing as well the input from Y as defined in Lemma 2.7 to (Y, m).

Finally, we discuss in Chapter 5 short comings of the proposed method and outline future research directions.

Chapter 2

The Shadow Price Approach for Continuous Processes

We begin with the formulation of the optimal investment problem under transaction costs for a market driven by a strictly positive and continuous stochastic process. For the definitions and concepts of Stochastic Analysis used in this work we refer the reader to Cohen and J.Elliott [7]. Unless stated otherwise we closely follow Czichowsky et al. [11] in the subsequent Section 2.1 and Section 2.2.

2.1 The Transaction Cost Problem

Let $\boldsymbol{\mathcal{S}}$ be the set of continuous, $(0,\infty)$-valued stochastic processes defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to the augmented natural filtration[1] $(\mathcal{F})_{t\in[0,T]}$. We consider a market over the fixed time horizon $T > 0$ with proportional transaction costs according to the transaction cost parameter $\lambda \in (0,1)$, where shares of the risky asset $S \in \boldsymbol{\mathcal{S}}$ are bought at the ask price S and sold at the bid price $(1-\lambda)S$. A trading strategy of infinite variation in a market with transaction costs leads to instantaneous ruin. For that reason we only consider trading strategies of finite variation.

Definition 1 (Self-financing trading strategy)**.** *A self-financing trading strategy is an $\mathbb{R}^2$-valued, càdlàg process $\varphi = (\varphi^0, \varphi^1)$ of finite variation satisfying*

$$\int_s^t d\varphi_u^0 \leq -\int_s^t S_u d\varphi_u^{1,\uparrow} + \int_s^t (1-\lambda) S_u d\varphi_u^{1,\downarrow}, \quad \text{for all} \quad 0 \leq s < t \leq T, \tag{2.1}$$

where we denote by $\varphi^1 = \varphi^1_{0-} + \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ its Jordan-Hahn decomposition into two non-decreasing processes $\varphi^{1,\uparrow}$ and $\varphi^{1,\downarrow}$ with corresponding signed measure $d\varphi^1 = d\varphi^{1,\uparrow} - d\varphi^{1,\downarrow}$.

A self-financing trading strategy does not allow to add money to the initial amount of investment during the trading period. Furthermore, equality in (2.1) ensures no withdraw of money throughout the whole trading period, but this is not excluded in general in this setting. For more details see Appendix C.1. The definition of a process indicating the wealth of the trader needs to distinguish between the executed purchases and the executed sells.

Definition 2 (Liquidation value)**.** *For an $\mathbb{R}^2$-valued process $\varphi = (\varphi_t^0, \varphi_t^1)_{t\in[0,T]}$, define the liquidation value $V_t^{liq}(\varphi)$ at time $0 \leq t \leq T$ by*

$$V_t^{liq}(\varphi) := \varphi_t^0 + (\varphi_t^1)^+ (1-\lambda) S_t - (\varphi_t^1)^- S_t.$$

[1] See Appendix C.3 for a definition.

The process φ is called admissible if

$$V_t^{liq}(\varphi) \geq 0.$$

For $x > 0$, denote by $\mathcal{A}(x)$ the set of admissible and self-financing strategies $\varphi = (\varphi^0, \varphi^1)$ starting with initial endowment $\varphi_{0-} = (\varphi_{0-}^0, \varphi_{0-}^1) = (x, 0)$. We call

$$\mathcal{C}(x) := \left\{ V_T^{liq}(\varphi) \mid \varphi \in \mathcal{A}(x) \right\}$$

the set of admissible terminal wealth.

The notation φ_{0-} is used since $\varphi \in \mathcal{A}(x)$ is càdlàg and might have a jump at time zero. For that reason denote by

$$\varphi_{0-} = (\varphi_{0-}^0, \varphi_{0-}^1) := \lim_{t \nearrow 0} (\varphi_t^0, \varphi_t^1)$$

the left side and by

$$\varphi_0 = (\varphi_0^0, \varphi_0^1) = \lim_{t \searrow 0} (\varphi_t^0, \varphi_t^1)$$

the limit from the right side. Given an initial endowment $(\varphi_{0-}^0, \varphi_{0-}^1) = (x, 0)$ the self-financing assumption (2.1) ensures

$$x \geq \varphi_0^0 + S_0 \left(\varphi_0^1 \right)^+ - (1 - \lambda) S_0 \left(\varphi_0^1 \right)^- \tag{2.2}$$

and hence that at most $x > 0$ is used for the initial investment [11]. The inequality (2.2) is derived in Appendix C.2. For a given trading strategy, the traders utility of the corresponding liquidation value is measured by a utility function U. To model the law of diminishing marginal utility, the function used to measure the traders preference must satisfy the following conditions [26].

Definition 3 (Utility function)**.** *A strictly concave and continuously differentiable function $U : (0, \infty) \to \mathbb{R}$ with*

$$\lim_{x \downarrow 0} U'(x) = +\infty \quad \textit{and} \quad \lim_{x \to \infty} U'(x) = 0$$

is called utility function.

A utility function is in particular strictly monotonic increasing and hence every additional unit of wealth leads to a higher utility. Since U' is strictly monotonic decreasing the increase of utility diminishes for every additional unit. The concavity of the utility functions forces the trader to act risk-averse. By Jensen's-inequality a deterministic pay-out has strictly greater utility than a risky investment with the same expected utility [26]. Given a utility function U and an initial endowment $x > 0$ the transaction cost problem is to find $g^\star \in \mathcal{C}(x)$ such that

$$\sup_{g \in \mathcal{C}(x)} \mathbb{E}[U(g)] \tag{2.3}$$

is attained. This corresponds to the traders decision on a self-financing and admissible trading strategy to maximize her expected terminal utility.

2.2 On the Existence of a Shadow Market

The shadow price approach links the problem of optimal investment under transaction costs described in (2.3) to the optimal trading strategy in a shadow market. In a shadow market, transaction costs are absent again and hence the theory of optimal investment in a frictionless market outlined in Appendix B may be applied. The underlying price process of the shadow market - the shadow price process - is defined such that the optimal trading strategy coincides with the optimal trading strategy in the market with frictions. In this section, we state Theorem 3.2. from [11] on the existence and the composition of a shadow price process. For the proof and a more detailed description of the concepts we refer to [11]. The main result of this section is that the shadow price is an Itô-diffusion, if the underlying price process is adapted to a Brownian filtration. A similar result holds true even for a price process driven by fractional Brownian motion [10]. The theory presented here is taken from [11] and [10]. For the remainder of this section fix an initial endowment $x > 0$ and the transaction cost parameter $0 < \lambda < 1$. We begin with the central definition of a shadow price process.

Definition 4 (Shadow price process). *A semimartingale $\tilde{S} = (\tilde{S}_t)_{0 \le t \le T}$ is called a shadow price process associated to $S \in \boldsymbol{\mathcal{S}}$ if*

(i) $\tilde{S}$ takes its values in the bid-ask interval $[(1-\lambda)S, S]$.

(ii) The optimizer to the corresponding frictionless utility maximization problem

$$\mathbb{E}\left[U(\tilde{g})\right] \to \max!, \quad \tilde{g} \in \tilde{\mathcal{C}}(x), \tag{2.4}$$

exists and coincides with the solution $\hat{g} \in \mathcal{C}(x)$ for the optimization problem (2.3) under transaction costs. In (2.4) the set $\tilde{\mathcal{C}}(x)$ consists of all non-negative random variables, which are attainable by starting with initial endowment x and then trading the stock process $\tilde{S}$ in a frictionless admissible way following the definitions in Appendix B.1.

(iii) The optimal trading strategy $\tilde{\varphi}^1$ is equal to the left-continuous version of the finite variation process φ^1 of the unique optimizer $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \le t \le T}$ of the optimization problem (2.3).

The shadow price process $\tilde{S}$ defines a frictionless financial model $(B, \tilde{S})$ to which we refer as the shadow market. By the above definition the value function $\tilde{u}$ in the shadow market and the value function u in the market with transaction costs coincide. To be more precise

$$\tilde{u}(x) := \sup_{\tilde{g} \in \tilde{\mathcal{C}}(x)} E[U(\tilde{g})] = \sup_{g \in \mathcal{C}(x)} E[U(g)] =: u(x)$$

holds true and the set of admissible terminal wealth $\mathcal{C}(x)$ is a subset of the admissible terminal wealth $\tilde{\mathcal{C}}(x)$ in the shadow market [11]. Towards the existence of a shadow price we first need a sufficient condition such that the supremum in (2.3) is attained.

Definition 5 (λ-consistent price system). *A λ-consistent price system for $S \in \boldsymbol{\mathcal{S}}$ is a two dimensional strictly positive process $Z = (Z_t^0, Z_t^1)_{0 \le t \le T}$ with $Z_0^0 = 1$, that consists of a martingale Z^0 and a local martingale Z^1 under $\mathbb{P}$ such that*

$$\tilde{S}_t := \frac{Z_t^1}{Z_t^0} \in [(1-\lambda)S_t, S_t], \quad a.s. \quad for\ 0 \le t \le T.$$

The process S satisfies the condition (CPS^λ) of admitting a λ-consistent price system, if a λ-consistent price system exists. We say that S satisfies locally the condition (CPS^λ), if there exists a strictly positive process Z and a sequence $(\tau)_{n\in\mathbb{N}}$ of $[0,T]\cup\{\infty\}$-valued stopping times, increasing to infinity, such that each stopped process Z^{τ_n} defines a λ-consistent price system for the stopped process S^{τ_n}. Define the set of processes

$$\boldsymbol{\mathcal{S}}_{\mu,\lambda} := \{S \in \boldsymbol{\mathcal{S}} \mid S \text{ satisfies } (CPS^\mu) \text{ locally for all } 0 < \mu < \lambda\}\,.$$

Remark 2.1. *Corollary 2 of Bayraktar et al. [2] states that a process $S = \exp(f(X))$ admits a λ-consistent price system if X is a continuous process with conditional full support and $f : \mathbb{R} \to \mathbb{R}$ is monotone, continuous and surjective. Following Example 4.1 of Guasoni et al. [16] every continuous Markov process S satisfies the conditional support condition. Hence, $\exp(S) \in \boldsymbol{\mathcal{S}}_{\mu,\lambda}$ for all continuous Markov processes S.*

The following Theorem states that $S \in \boldsymbol{\mathcal{S}}_{\mu,\lambda}$ is sufficient for the supremum in (2.3) to be a maximum. For more details see Theorem 2.4 of Czichowsky et al. [11] and Lemma A.1 of Czichowsky and Schachermayer [9].

Theorem 2.2. *For a price process $S \in \boldsymbol{\mathcal{S}}_{\mu,\lambda}$ the set $\mathcal{C}(x) \subset L^0_+(\Omega,\mathcal{F},\mathbb{P})$ is convex, closed and bounded with respect to the topology of convergence in measure.*

Definition 6 $(NUPBR)$**.** *Let $\tilde{\mathcal{C}}(x)$ be the set of admissible wealth for initial endowment $x > 0$ defined in Appendix B.1. A semimartingale S is said to satisfy the condition of "no unbounded profit with bounded riks" $(NUPBR)$, if the set $\tilde{\mathcal{C}}(1)$ is bounded in L^0. We define the set*

$$\boldsymbol{\mathcal{S}}_{\boldsymbol{NUPBR}} := \{S \in \boldsymbol{\mathcal{S}}_{\mu,\lambda} \mid S \text{ satisfies } (NUPBR)\}. \tag{2.5}$$

To state the existence of a shadow price process for processes in $\boldsymbol{\mathcal{S}}_{\boldsymbol{NUPBR}}$ we limit the utility functions to functions of reasonable asymptotic elasticity.

Definition 7 (Reasonable asymptotic elasticity)**.** *A utility function U satisfies the condition of reasonable asymptotic elasticity [27] if*

$$AE(U) := \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1. \tag{2.6}$$

Remark 2.3. *The logarithmic utility function $U(x) = \ln = \log_e$ is of reasonable asymptotic elasticity since*

$$AE(\ln) = \lim_{x\to\infty} \frac{1}{\ln(x)} = 0 < 1.$$

The following Theorem states that every $S \in \boldsymbol{\mathcal{S}}_{\boldsymbol{NUPBR}}$ induces a shadow price and that this shadow price is a composition of two local martingales. The existence of a shadow price for cádlág price processes is proved in Proposition 3.7 of Czichowsky and Schachermayer [9] and the decomposition of the shadow price into two local martingales is the Main Theorem in Czichowsky et al. [11].

Theorem 2.4. *Fix $S \in \boldsymbol{\mathcal{S}}_{\boldsymbol{NUPBR}}$ and let U be a utility function of reasonable asymptotic elasticity with finite value function $u(x)$. It exists a continuous local martingale $(\hat{Z}^0_t)_{t\in[0,T]}$ and a continuous local martingale $(\hat{Z}^1_t)_{t\in[0,T]}$ such that the process defined by*

$$\tilde{S} := \frac{\hat{Z}^1}{\hat{Z}^0}$$

is a shadow price process.

Similar to the proof of Lemma 5.1 in Czichowsky and Schachermayer [10], the martingale representation theorem [7] implies that the shadow price process is an Itô process, if the underlying filtration is Brownian.

Corollary 2.5. *Assume that $S \in \boldsymbol{\mathcal{S}}_{\boldsymbol{NUPBR}}$ is adapted to a Brownian filtration*

$$\mathcal{F}_t^W := \sigma(\{W_s \,|\, s \leq t\}).$$

Then, the shadow price $\tilde{S}$ is an Itô-process with dynamics

$$d\tilde{S}_t = \frac{1}{2}\tilde{S}_t\left(\tilde{\mu}_t - \frac{\tilde{\sigma}_t^2}{2}\right)dt + \tilde{S}_t\tilde{\sigma}_t dW_t, \tag{2.7}$$

for some predictable processes $\hat{\sigma}$ and $\hat{\mu}$.

Proof. We apply the arguments in the proof of Lemma 5.1 in Czichowsky and Schachermayer [10] to show that the shadow price is an Itô process. By the martingale representation theorem [7] the local martingales $\hat{Z}^1$ and $\hat{Z}^0$ can be decomposed in

$$\hat{Z}_t^0 = \hat{Z}_0^0 - \int_0^t \bar{a}_s dW_s$$

and

$$\hat{Z}_t^1 = \hat{Z}_0^1 - \int_0^t \bar{b}_s dW_s$$

with

$$\bar{a}, \bar{b} \in L(W) := \left\{\theta \, predictable \,\Big|\, \int_0^t \theta_s^2 dt < \infty \, \mathbb{P} \, a.s., \, t \in [0,T]\right\}.$$

In particular both processes are continuous. Set $\hat{a} := \frac{\bar{a}}{\hat{Z}^0}$ and $\hat{b} := \frac{\bar{b}}{\hat{Z}^1}$. Since $\hat{Z}^0$ and $\hat{Z}^1$ are continuous and strictly positive , deduce by Itôs formular

$$\begin{aligned} d(\ln(\hat{Z}_t^0)) &= -\frac{1}{\hat{Z}_t}d\hat{Z}_t - \frac{1}{2}\frac{1}{\hat{Z}_t^2}d\langle\hat{Z},\hat{Z}\rangle_t = -\frac{\bar{a}_t}{\hat{Z}_t}dW_t - \frac{1}{2}\frac{\bar{a}_t^2}{\hat{Z}_t^2}dt \\ &= -\hat{a}_t dW_t - \frac{1}{2}\hat{a}_t^2 dt \end{aligned}$$

and similar

$$d(\ln(\hat{Z}_t^1)) = -\hat{b}dW_t - \frac{1}{2}\hat{b}^2 dt.$$

Hence

$$\hat{Z}_t^0 = \hat{Z}_0^0 \exp\left(-\int_0^t \hat{a}_s dW_s - \frac{1}{2}\int_0^t \hat{a}_s^2 ds\right)$$

and

$$\hat{Z}_t^1 = \hat{Z}_0^1 \exp\left(-\int_0^t \hat{b}_s dW_s - \frac{1}{2}\int_0^t \hat{b}_s^2 ds\right).$$

This yields for $\hat{X} = \ln(\hat{S})$

$$\begin{aligned} \hat{X} &= \ln(\hat{Z}_t^1) - \ln(\hat{Z}_t^0) = \hat{X}_0 + \int_0^t \hat{a}_s - \hat{b}_s dW_s + \frac{1}{2}\int_0^t \hat{a}_s^2 - \hat{b}_s^2 ds \\ &= \hat{X}_0 + \int_0^t \hat{\sigma}_s dW_s + \int_0^t \left(\hat{\mu}_s - \frac{\hat{\sigma}_s^2}{2}\right) ds, \end{aligned}$$

with $\hat{\sigma}_t = \hat{a}_t - \hat{b}_t$ and $\hat{\mu}_t = (\hat{a}_t^2 - \hat{a}_t\hat{b}_t + \hat{b}_t^2)$. Finally, the dynamics of the shadow price $\hat{S} = \exp(\hat{X})$ are given by

$$\hat{S}_t = \frac{\hat{Z}_0^1}{\hat{Z}_0^0} + \int_0^t \hat{S}_u \hat{\sigma}_u dW_u + \frac{1}{2}\int_0^t \hat{S}_u \left(\hat{\mu}_u - \frac{\hat{\sigma}_u^2}{2}\right) du.$$

□

To ensure that the shadow price process is an Itô process we restrict the set of price processes to the set

$$\boldsymbol{\mathcal{S}}_W := \left\{ S \in \boldsymbol{\mathcal{S}}_{NUPBR} \mid S \text{ is adapted to a Brownian filtration } (\mathcal{F}_t^W)_{t\in[0,T]} \right\}.$$

2.3 Optimal Trading in the Shadow Market

For a price process $S \in \boldsymbol{\mathcal{S}}_W$, Corollary 2.5 ensures the existence of a shadow price $\tilde{S}$ with predictable and hence $(\mathcal{F}_t)_{t\in[0,T]}$ progressive measurable coefficient processes $(\tilde{\mu}_t)_{t\in[0,T]}$ and $(\tilde{\sigma}_t)_{t\in[0,T]}$. Assuming in addition that these processes are uniformly bounded in (t,ω) enables us to apply the theory of optimal investment outlined in Appendix B.2 to the shadow market $(B,\tilde{S})$. The log-optimal risky fraction in this shadow market is given by

$$\pi_t = \frac{\tilde{\mu}_t}{\tilde{\sigma}_t^2} \tag{2.8}$$

leading to an optimal expected terminal wealth of

$$\tilde{u}(x) = \sup_{\tilde{V}_T \in \tilde{\mathcal{C}}(x)} \mathbb{E}\left[\ln(\tilde{V}_T)\right] = \mathbb{E}\left[\ln(\tilde{V}_T^{\varphi})\right] = \frac{1}{2}\mathbb{E}\left[\int_0^T \theta_s^2 ds\right] + \ln\left(\frac{x}{T+1}\right). \tag{2.9}$$

The value function in the shadow market coincides with the value function in the market with transaction costs and hence

$$\frac{1}{2}\mathbb{E}\left[\int_0^T \theta_s^2 ds\right] + \ln\left(\frac{x}{T+1}\right) = \tilde{u}(x) \overset{(2.2)}{=} u(x) = \sup_{V_T \in \mathcal{C}(x)} \mathbb{E}\left[\ln(V_T)\right] \tag{2.10}$$

is as well the optimal expected terminal wealth in the financial model (B,S) with transaction costs.

2.4 The Shadow Market for Recurrent Markov Processes

By the observations of the above section, we need to estimate the coefficient processes

$$\hat{\mu}_t \approx \tilde{\mu}_t \quad \textit{and} \quad \hat{\sigma}_t \approx \tilde{\sigma}_t \tag{2.11}$$

in order to approximate the optimal trading strategy in the shadow market. So far, we only know that these processes are predictable if the price process is an element of $\boldsymbol{\mathcal{S}}_W$. Since the shadow price process is induced by the price process, it is natural that the assumptions on the coefficients of the price process affect the properties of $\tilde{\mu}$ and $\tilde{\sigma}$. We aim for deterministic coefficient functions $\tilde{\mu}, \tilde{\sigma}$ and a stochastic process Y such that

$$\tilde{\mu}_t = \tilde{\mu}(Y_t) \quad \textit{and} \quad \tilde{\sigma}_t = \tilde{\sigma}(Y_t) \quad \textit{for all } t \in [0,T].$$

If we are additionally able to estimate a realization of the path $t \mapsto Y_t$, knowing $\tilde{\mu}$ and $\tilde{\sigma}$ enables us to estimate the optimal risky fraction (2.8) in the shadow market and hence the optimal risky fraction the financial model (B, S) with transaction costs. Towards that goal let μ and σ of the price process

$$S_t = s_0 \exp\left(\int_0^t \mu(S_u) - \frac{1}{2}\sigma^2(S_u)du + \int_0^t \sigma(S_u)dW_u \right), \quad s_0 > 0 \tag{2.12}$$

satisfy the growth bound

$$|\mu(s)|^2 + |\sigma(s)|^2 \leq K(1+s^2), \tag{2.13}$$

and the Lipschitz conditions

$$|\mu(s) - \mu(s')|^2 \leq K|s-s'|^2 \quad \textit{and} \quad |\sigma(s) - \sigma(s')|^2 \leq K|s-s'|^2. \tag{2.14}$$

for some $K > 0$ [7]. In addition, assuming that

$$\sigma(s) > 0 \quad \textit{for all } s \in (0, \infty) \tag{2.15}$$

holds true. Define the set of price processes

$$\boldsymbol{\mathcal{S}}^{MP} := \{S \in \boldsymbol{\mathcal{S}}_W \mid S \textit{ satisfies } (2.12), (2.13), (2.14) \textit{ and } (2.15)\}.$$

With these assumptions, a stochastic processes $S \in \boldsymbol{\mathcal{S}}^{MP}$ is a continuous strong Markov process and the unique strong solution to the stochastic differential equation [7]

$$dS_t = S_t\mu(S_t)dt + S_t\sigma(S_t)dW_t, \quad S_0 = s_0 > 0. \tag{2.16}$$

Let $(\mathbb{P}_{s0})_{s0>0}$ be the family of probability measures associated to the strong Markov process S satisfying $\mathbb{P}_{s0}(S_0 = s_0) = 1$. For a fixed starting point $s_0 > 0$ we write $\mathbb{P} = \mathbb{P}_{s0}$. According to Lemma 16.1.4 in Cohen and J.Elliott [7] it exists a constant $\tilde{C}_S$ depending on T and K such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} S_T^2\right] \leq \tilde{C}_S\left(s_0^2 + T\mu^2(0) + T\sigma^2(0)\right) =: C_S.$$

Hence, by Fubinis Theorem [7]

$$\mathbb{E}\left[\int_u^v S_t^2 dt\right] = \int_u^v \mathbb{E}\left[S_t^2\right] dt \leq C_S(u-v), \quad 0 \leq u \leq v \leq T. \tag{2.17}$$

The strong Markov process S is called *recurrent* if for all $a > 0$ the stopping time

$$T_a = \inf\{t \geq 0 \mid S_t = a\}$$

is $\mathbb{P}_{s0} - a.s.$ finite for all $s_0 > 0$. We define the set of recurrent price processes by

$$\boldsymbol{\mathcal{S}}_r^{MP} := \{S \in \boldsymbol{\mathcal{S}}^{MP} \mid S \textit{ is recurrent}\}.$$

To examine if a process is recurrent, a sufficient and necessary condition for the recurrence of a process is established in Durrett [13] and presented here. The condition relies on the natural scale function

$$p(x) = \int_0^x \exp\left(-2\int_0^y \frac{\mu(z)}{\sigma^2(z)}dz\right)dy$$

associated to $S \in \boldsymbol{\mathcal{S}}^{MP}$. Following Corollary 3.4 of Durrett [13] with

$$p(\infty) := \lim_{x\to\infty} p(x) \quad and \quad p(-\infty) := \lim_{x\to-\infty} p(x)$$

a process $S \in \boldsymbol{\mathcal{S}}^{MP}$ is recurrent if and only if

$$p(\infty) = \infty \quad and \quad p(-\infty) = -\infty.$$

For a price process $S \in \boldsymbol{\mathcal{S}}_r^{MP}$ we construct a process m that scales the process S into some interval $[1, \bar{s}]$ [15].

Definition 8. *For a real number $\bar{s} > 1$ set $\varrho_0 = 0$ and define recursively the sequence of stopping times $\varrho = (\varrho_n)_{n\in\mathbb{N}_0}$ and $\tau = (\tau_n)_{n\in\mathbb{N}_0}$ by the relative running minimum process*

$$m_t = \inf_{\varrho_{k-1}\le u\le t} S_u, \quad \varrho_{k-1} \le t \le \tau_k \quad with \quad \tau_k = \inf\left\{t \ge \varrho_{k-1} \mid \frac{S_t}{m_t} \ge \bar{s}\right\}$$

and the relative running maximum process

$$M_t = \sup_{\tau_k\le u\le t} S_u, \quad \tau_k \le t \le \varrho_k \quad with \quad \varrho_k = \inf\left\{t \ge \tau_k \mid \frac{S_t}{M_t} \le \frac{1}{\bar{s}}\right\}.$$

In order to have a.s. continuous processes m and M are set to

$$M_t := \bar{s}m_t \quad for\ t \in \bigcup_{n=0}^{\infty} [[\varrho_n, \tau_{n+1}]]$$

and

$$m_t := \frac{M_t}{\bar{s}} \quad for\ t \in \bigcup_{n=1}^{\infty} [[\tau_n, \varrho_n]]$$

In the following lemma we state some properties of the sequences $(\varrho_n)_{n\in\mathbb{N}_0}$ and $(\tau_n)_{n\in\mathbb{N}}$. A proof is provided in Appendix A.1.

Lemma 2.6. *Define $(\varrho_n)_{n\in\mathbb{N}_0}$ and $(\tau_n)_{n\in\mathbb{N}}$ according to the definition above.*

(i) $(\varrho_n)_{n\in\mathbb{N}_0}$ and $(\tau_n)_{n\in\mathbb{N}}$ are sequences of a.s. finite $(\mathcal{F}_t)_{t\in[0,T]}$ stopping times

(ii) $(\varrho_n)_{n\in\mathbb{N}_0}$ and $(\tau_n)_{n\in\mathbb{N}}$ increase a.s. to infinity

(iii) For all $T > 0$ we find

$$(0, T] = \bigcup_{k\in\mathbb{N}} \left(]]\varrho_{k-1}, \tau_k]] \cup]]\tau_k, \varrho_k]]\right) \cap [0, T], \quad a.s. \tag{2.18}$$

We proceed with properties of the process $\frac{S}{m}$. For a proof see Appendix A.2.

Lemma 2.7. *Let $Y = (Y_t)_{t\in[0,T]}$ with $Y_t := \frac{S_t}{m_t}$.*

(i) Y takes values a.s. in $[1, \bar{s}]$

(ii) m is continuous

(iii) m is of finite variation

(iv) There exists constants $C_{dt} > 0$ and $C_m > 0$ such that

$$\mathbb{E}\left[\int_{t_n}^{t_{n+1}} \left|\mu(S_u)\frac{S_u}{m_u}\right| du\right] \leq \bar{s}C_{dt}(t_{n+1} - t_n) \tag{2.19}$$

and

$$\mathbb{E}\left[\int_{t_n}^{t_{n+1}} \left|\frac{S_u}{m_u}\right| |d\ln(m_u)|\right] \leq 2\bar{s}C_m\sqrt{(t_{n+1} - t_n)} \tag{2.20}$$

for all $0 \leq t_n < t_{n+1} \leq T$.

(v) Y follows the dynamics

$$d(Y) = \mu(S_t)\frac{S_t}{m_t}dt + \frac{S_t}{m_t}d\ln(m_t) + \sigma(S_t)\frac{S_t}{m_t}dW_t.$$

In particular, Y is a continuous semimartingale with decomposition

$$Y = A + N$$

for a process of bounded variation A and a martingale N.

For the proof of the second inequality in (2.20) we use that $S_t = s_0 \exp(I_t)$ for some Itô process I. For that reason, we write S this way. However, assuming (2.12) is not a restriction for $S > 0$, since the exponential function is strictly positive and injective. To derive deterministic functions as described in (2.11) we need to find a function that maps the process Y in an interval such that $\tilde{S}$ takes only values in the bid and ask interval.

Definition 9. *Let $\bar{s}$ be a real number with $\bar{s} > 1$. The C^2 function*

$$g : [1, \bar{s}] \to [1, (1 - \lambda)\bar{s}]$$

is called a smooth pasting function if the smooth pasting conditions

$$g(1) = g'(1) = 1 \tag{2.21}$$

$$g(\bar{s}) = (1 - \lambda)\bar{s} \quad \text{and} \quad g'(\bar{s}) = 1 - \lambda \tag{2.22}$$

are satisfied and

$$g'(s) > 0, \quad 1 \leq s \leq \bar{s}$$

holds true.

For the subsequently proposed method to estimate the coefficient processes of the shadow price we need further assumptions in addition to $S \in \boldsymbol{\mathcal{S}}_r^{MP}$. The following assumption on the existence of a smooth pasting function is sufficient for our method to work, if the price process is a geometric Brownian motion. To prove the existence of a smooth pasting function for a price process following a geometric Brownian motion is the subject of [15].

Assumption 1. *For the process $S \in \boldsymbol{\mathcal{S}}_r^{MP}$ exists a smooth pasting function g such that the stochastic process*

$$\tilde{S}_t = m_t g\left(\frac{S_t}{m_t}\right) \tag{2.23}$$

is a shadow price process. Moreover $|g''(s)| \le L_{g'}$ *for all* $s \in [1, \bar{s}]$ *and a constant* $L_{g'} > 0$.

By construction of the set $\boldsymbol{\mathcal{S}}_r^{MP}$, Theorem 2.4 yields the existence of a shadow price process for all $S \in \boldsymbol{\mathcal{S}}_r^{MP}$. Thus, the above assumption is not the existence of a shadow price but only the existence of a smooth pasting function and the decomposition (2.23). Corollary 2.5 already implies that the shadow price is an Itô process, but under Assumption 1 explicit dynamics are derived in the subsequent Lemma as a generalization of Proposition 2.1 in Gerhold et al. [15]. The proof is similar to the proof in [15] and provided in Appendix A.3.

Lemma 2.8. *Assume that Assumption 1 holds true for some* $S \in \boldsymbol{\mathcal{S}}_r^{MP}$. *The shadow price process* $\tilde{S}$ *is an Itô process satisfying the stochastic differential equation*

$$d\tilde{S}_t = g'\left(\frac{S_t}{m_t}\right) dS_t + \frac{1}{2m_t} g''\left(\frac{S_t}{m_t}\right) d\langle S, S\rangle. \tag{2.24}$$

Corollary 2.9. *Under Assumption 1, the shadow price from Lemma 2.8 induced by* $S \in \boldsymbol{\mathcal{S}}_r^{MP}$ *satisfies*

$$d\tilde{S}_t = S_t \tilde{\mu}\left(\frac{S_t}{m_t}, S_t\right) dt + S_t \tilde{\sigma}\left(\frac{S_t}{m_t}, S_t\right) dW_t \tag{2.25}$$

for deterministic functions $\tilde{\mu}, \tilde{\sigma} : [1, \bar{s}] \times (0, \infty) \to \mathbb{R}$ *with*

$$\tilde{\mu}(y, s) := g'(s)\mu(y) + \frac{1}{2} y g''(y)\sigma^2(s) \quad \text{and} \quad \tilde{\sigma}(y, s) := g'(y)\sigma(s).$$

Proof. Using (2.24) and (2.16) we calculate for $S \in \boldsymbol{\mathcal{S}}_r^{MP}$ by Itôs rule

$$\begin{aligned}
d\tilde{S}_t &= g'\left(\frac{S_t}{m_t}\right) dS_t + \frac{1}{2m_t} g''\left(\frac{S_t}{m_t}\right) d\langle S, S\rangle \\
&= S_t \left[g'\left(\frac{S_t}{m_t}\right) \mu(S_t) dt + \frac{S_t}{2m_t} g''\left(\frac{S_t}{m_t}\right) \sigma^2(S_t) dt + g'\left(\frac{S_t}{m_t}\right) \sigma(S_t) dW_t \right] \\
&= S_t \tilde{\mu}\left(\frac{S_t}{m_t}, S_t\right) dt + S_t \tilde{\sigma}\left(\frac{S_t}{m_t}, S_t\right) dW_t.
\end{aligned}$$

□

Assumption 2. *Assume that* $S \in \boldsymbol{\mathcal{S}}_r^{MP}$ *induces a shadow price* $\tilde{S}$ *following the dynamics*

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \frac{S_t}{\tilde{S}_t} \tilde{\mu}(Y_t) dt + \frac{S_t}{\tilde{S}_t} \tilde{\sigma}(Y_t) dW_t \tag{2.26}$$

for a stochastic process $Y = (Y_t)_{t \in [0,T]}$ *with* $Y \in \boldsymbol{\mathcal{S}}$, *a compact set* $K \subset \mathbb{R}$ *and deterministic functions* $\tilde{\mu}, \tilde{\sigma}$ *with*

$$\tilde{\mu}_t = \tilde{\mu}(Y_t) \quad \text{and} \quad \tilde{\sigma}_t = \tilde{\sigma}(Y_t) \quad \text{with} \quad \tilde{\mu}, \tilde{\sigma} : K \to \mathbb{R} \tag{2.27}$$

satsifying the Lipschitz conditions (2.14) and $\tilde{\sigma} > 0$. *Assume further that the function*

$$f(s) := \frac{\tilde{\mu}^2(s)}{\tilde{\sigma}^2(s)} \tag{2.28}$$

has a bounded derivative

$$|f'(s)| \le L_f.$$

Assumption 1 and (2.25) already yield (2.26) and (2.27) for the process $Y = (\frac{S}{m}, S)$. However, in Assumption 2 we assume that Y is one-dimensional and defined on a compact domain. As a consequence there exist constants $C_{\tilde{\mu}}, c_{\tilde{\sigma}}, C_{\tilde{\sigma}} > 0$ such that

$$|\tilde{\mu}(y)| \leq C_{\tilde{\mu}} \quad \textit{and} \quad C_{\tilde{\sigma}} > \tilde{\sigma}(y) > c_{\tilde{\sigma}} \quad \textit{for all } y \in K. \tag{2.29}$$

Under Assumption 1 the coefficient functions $\tilde{\mu}$ and $\tilde{\sigma}$ do not need to be differentiable since g is only in C^2 and not in C^3. Imposing Assumption 2 the function f is differentiable and Lipschitz-continuous with Lipschitz constant $L_f > 0$. We denote by

$$\boldsymbol{\mathcal{S}}_{LS} := \left\{ S \in \boldsymbol{\mathcal{S}}_r^{MP} \mid S \textit{ satisfies Assumption } 2 \right\}.$$

the set of processes satisfying Assumption 2. With some additional assumptions on the drift and the volatility, geometric Brownian motion is an element of $\boldsymbol{\mathcal{S}}_{LS}$.

Theorem 2.10. *A geometric Brownian motion S with $\sigma^2 > \mu > 0$ satisfies Assumption 1 and Assumption 2. Moreover $S \in \boldsymbol{\mathcal{S}}_r^{MP}$ and hence $S \in \boldsymbol{\mathcal{S}}_{LS}$.*

Proof. First note that $S = \exp(X)$ for the continuous Markov process X with

$$X_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

Hence, $S \in \boldsymbol{\mathcal{S}}_{\mu,\lambda}$ by Remark 2.1. The financial model (B, S) satisfies the condition $(NFLVR)$ defined in Delbaen and Schachermayer [12]. Indeed, S is a continuous semimartingale and $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 - \theta W_T\right)$$

and

$$\mathbb{Q}(A) := \mathbb{E}\left[\mathbb{1}_A \frac{d\mathbb{Q}}{d\mathbb{P}}\right] \quad \textit{for all } A \in \mathcal{F}_T$$

is an equivalent martingale measure to $\mathbb{P}$ [26]. The condition $(NFLVR)$ is a stronger assumption than $(NUPBR)$ by Proposition 3.2 in Karatzas and Kardaras [22] and hence $S \in \boldsymbol{\mathcal{S}}_{\boldsymbol{NUPBR}}$ ensures the existence of a shadow price $\tilde{S}$ by Theorem 2.4. Since S is adapted to $\{\mathcal{F}_t^W\}$ we also find $S \in \boldsymbol{\mathcal{S}}_{\boldsymbol{W}}$. To show $S \in \boldsymbol{\mathcal{S}}_r^{MP}$, note that S satisfies the growth bound and the Lipschitz condition with $K := \mu^2 + \sigma^2$ and $\sigma > 0$. Towards the recurrence of S set $r := \frac{2\mu}{\sigma^2}$ and calculate for all $y > 0$ and all $x \in \mathbb{R}$

$$\begin{aligned}
p(x) &= \int_0^x \exp\left(\int_0^y -\frac{2\mu(z)}{\sigma^2(z)} dz\right) dy = \int_0^x \exp\left(-\frac{2\mu}{\sigma^2} \lim_{a \searrow 0} \int_a^y \frac{1}{z} dz\right) dy \\
&= \lim_{a \searrow 0} \int_0^x \exp\left(-\frac{2\mu}{\sigma^2} \int_a^y \frac{1}{z} dz\right) dy = \lim_{a \searrow 0} \int_0^x \exp\left(\ln\left(\frac{y^{-r}}{a^{-r}}\right)\right) dy \\
&= \lim_{a \searrow 0} \int_0^x \left(\frac{y}{a}\right)^{-r} dy = \lim_{a \searrow 0} \frac{a^{-r}}{1-r} \left(\frac{x}{a}\right)^{1-r} = \lim_{a \searrow 0} \frac{1}{a(1-r)} x^{1-r}.
\end{aligned}$$

Taking the limit leads to

$$p(\infty) = \infty \quad \textit{and} \quad p(-\infty) = -\infty.$$

Therefore S is recurrent and $S \in \boldsymbol{\mathcal{S}}_r^{MP}$. The subject of Gerhold et al. [15] is to show the existence of a smooth pasting function g implying a shadow price process $\tilde{S}$. The existence of the smooth pasting functions follows for $\frac{\mu}{\sigma^2} < 1$ by Lemma 4.3 in [15] and Proposition 1.2 in [15] states that $\tilde{S} = mg\left(\frac{S}{m}\right)$ is a shadow price process. For Assumption 1 to hold, it is left to show that g'' is bounded. Equation (5.1) in [15] states

$$\tilde{\mu}(s) = \frac{\sigma^2 g'(s)^2 s}{g(s)\,(c + g(s))}, \quad \tilde{\sigma}(s) = \frac{\sigma g'(s) s}{g(s)} \tag{2.30}$$

with $c > 0$ by lemma 4.2 in [15] and

$$g''(s) = \frac{2g'(s)^2}{c + g(s)} - \frac{2\theta g'(s)}{s} \tag{2.31}$$

by lemma 4.3 in [15]. The smooth pasting conditions (2.21) and (2.22) imply that g' is bounded. Equation (2.31) implies therefore that g'' is bounded as well. Hence, S satisfies Assumption 1 and (2.27) follows by (2.25) with $X = \frac{S}{m}$, since $\mu(P_t) \equiv \mu$ and $\sigma(P_t) \equiv \sigma$. Note that the smooth pasting function g takes only values in $[1, (1-\lambda)\bar{s}]$ by definition. Calculating the derivative of f using (2.30) and (2.31) shows the existence of a constant $L_f > 0$ such that

$$|f'(s)| \leq L_f \quad \textit{for all } s \in [1, \bar{s}]. \tag{2.32}$$

For an explicit derivation of the constant L_f see Appendix A.4. This shows that $S \in \boldsymbol{\mathcal{S}}_r^{MP}$ satisfies Assumption 2 □

For processes $S \in \boldsymbol{\mathcal{S}}_{LS}$ a method to numerically approximate (2.11) is proposed in the subsequent Chapter 3. The theoretical justification of the method is derived for a subset of price processes in $\boldsymbol{\mathcal{S}}_{LS}$.

Chapter 3

A PAC-Learning Algorithm for the Transaction Cost Problem

In *Machine learning*, computational methods are developed to infer the state of unseen instances by the deployment of experiences gained from observations. The observations of a *supervised learning task* consist of labeled instances whereas the observations of an *unsupervised learning task* are composed only of instances. In case of one-dimensional and real valued states, the task is called *regression*. Therefore, the approximation of the coefficient functions in (2.27) is an unsupervised regression task. By the absence of labels it is difficult to quantify the performance of methods for unsupervised tasks and to guarantee a certain performance of the method. The PAC-learning framework in Mohri et al. [29] establishes learning guarantees for a supervised learning task, that ensure for a sufficient number of observations a high probability to achieve an approximate solution. In the subsequent Section 3.1, Section 3.2 and Section 3.3 we present the PAC-learning framework and adapt its definitions to formulate our unsupervised regression problem. We closely follow the definitions and the notation of Mohri et al. [29].

3.1 The Learning Problem

Let $X : \Omega \to \mathcal{X}$ be the *data generating random variable* on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in the Borel measure space $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$. We call $\mathcal{X} \subset \left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ the space of instances and denote by

$$\mathcal{D} := \mathbb{P} \circ X^{-1}$$

the data generating unknown[1] *target distribution*. Define

$$\mathcal{C}_0 := \{c : \mathcal{X} \to \mathcal{Y} \mid c \textit{ is measurable}\}$$

where $\mathcal{Y} \subset \left(\mathbb{R}^k, \mathcal{B}\left(\mathbb{R}^k\right)\right)$ is the *space of target values*. A *concept class* $\mathcal{C}$ is a subset $\mathcal{C} \subset \mathcal{C}_0$ of *concepts* we may wish to learn. In order to approximate a fixed but unknown *target concept* c, the learner selects a *hypothesis set*

$$\mathcal{H} \subset \mathcal{C}_0$$

and a *learning algorithm* $\mathcal{A} = (\mathcal{A}_M)_{M \in \mathbb{N}}$ consisting of maps

$$\mathcal{A}_M : \mathcal{X}^M \to \mathcal{H}$$

[1] $\mathcal{D}$ is unknown if the underlying probability measure $\mathbb{P}$ is unknown.

for an *unsupervised learning task* and

$$\mathcal{A}_M : \mathcal{X}^M \times \mathcal{Y}^M \to \mathcal{H}$$

for a *supervised learning task*. Given a *sample* of observations $x_1, ..., x_M \in \mathcal{X}$

$$\mathcal{S} := \{x_1, ..., x_M\}$$

for an *unsupervised learning task* and

$$\mathcal{S} := \{(x_1, c(x_1)), ..., (x_M, c(x_M))\}$$

for a *supervised learning task*, the learning algorithm assigns a hypothesis $h \in \mathcal{H}$ to the observed sample. The output $h_{\mathcal{S}} := \mathcal{A}(S)$ of a learning algorithm is called the *learned hypothesis*. The standing assumption is the

$(i.i.d)$**-Learning Assumption** (I). *The observations $x_1, ..., x_M$ are realisations of independent copies $X_1, ..., X_M : \Omega \to \mathcal{X}$ of the data generating random variable such that*

$$X_i \sim \mathcal{D} \quad \textit{for all}\, 1 \leq i \leq M.$$

A measurable *loss function*

$$\mathcal{L} : \mathbb{R}^k \times \mathbb{R}^k \to [0, \infty]$$

quantifies for a given input $x \in \mathcal{X}$ how well the prediction $h_{\mathcal{S}}(x) = \hat{y} \in \mathcal{Y}$ matches the target $c(x) \in \mathcal{Y}$. The expected loss of a hypothesis is called the *generalization error*.

Definition 10 (Generalization error I)**.** *Let c be a target concept and X the data generating random variable. For $h \in \mathcal{H}$ we call*

$$R(h) := \mathbb{E}\left[\mathcal{L}(h(X), c(X))\right]$$

the generalization error wrt. $\mathcal{L}$.

Remark 3.1. *In order to learn a binary image classifier given by the target concept*

$$c : \mathbb{R}^d \to \{0, 1\}$$

the learner determines a hypothesis set $\mathcal{H} \subset \{h : \mathbb{R}^d \to \{0, 1\} \,|\, h \text{ is measurable}\}$ and the loss function

$$\mathcal{L} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (y, y') \mapsto \mathbb{1}_{\{y \neq y'\}}$$

with generlization error

$$R(h) = \mathbb{E}\left[\mathbb{1}_{\{h(X) \neq c(X)\}}\right] = \mathbb{P}\left(h(X) \neq c(X)\right)$$

that is the probability of a false prediction.

Having fixed a hypothesis set $\mathcal{H}$, the learner aims for the hypothesis $h^\star \in \mathcal{H}$ such that

$$R(h^\star) = \inf_{h \in \mathcal{H}} R(h).$$

The approximation quality of $h^\star$ depends on the approximation ability of the hypothesis set $\mathcal{H}$, measured by the *approximation error*[2]

$$\mathcal{E}(\mathcal{H}) := \left|\hat{R}^\star - R(c)\right| \quad \textit{with } \hat{R}^\star := \inf_{h\in\mathcal{H}} R(h).$$

The error decomposition

$$|R(h) - R(c)| = \underbrace{R(h) - \hat{R}^\star}_{\textbf{estimation error}} + \underbrace{\hat{R}^\star - R(c)}_{\textbf{approximation error}} \tag{3.1}$$

illustrates the different error sources [29]. A simple hypothesis set might lead to a small estimation error with the drawback of a large approximation error. On the other hand, a complex hypothesis set might provide a small approximation error but induces a large estimation error. The problem of balancing these two errors is known as the *model selection problem* [29]. The *learning problem* consists of both, the model selection problem and the design of a learning algorithm:

Definition 11. *(Learning problem I) The learning problem for a target concept $c \in \mathcal{C}$ is to*

(1) find a hypothesis set $\mathcal{H}$ with a small approximation error $\mathcal{E}(\mathcal{H})$.

(2) define a learning algorithm such that $R(h_\mathcal{S}) \to \hat{R}^\star$ for $M \to \infty$.

Having determined a suitable hypothesis set, a remaining difficulty is that c and $\mathcal{D}$ are unknown so that it is not possible to determine the generalization error analytically. However, the generalization error can be approximated by use of the $(i.i.d)$-Learning Assumption and the strong law of large numbers.

Definition 12 (Empirical error I)**.** *Let c be a target concept and X the data generating random variable. For a given sample $\mathcal{S}$, a given hypothesis $h \in \mathcal{H}$ and a loss function $\mathcal{L}$, the empirical error is defined by*

$$\hat{R}_\mathcal{S}(h) := \frac{1}{M}\sum_{m=1}^{M} \mathcal{L}(h(x_m), c(x_m))).$$

Remark 3.2. *First, note that, since $\mathcal{S}$ is random, the empirical error is in contrast to the generalization error a random quantity. Moreover, under the $(i.i.d)$-Learning Assumption the expectation of the empirical error equals the generalization error by*

$$\mathbb{E}\left[\hat{R}_\mathcal{S}(h)\right] = \frac{1}{M}\sum_{m=1}^{M} \mathbb{E}[\mathcal{L}\left(h(X_m), (c(X_m)\right)] = R(h).$$

By the the strong law of large numbers

$$\hat{R}_\mathcal{S}(h) \to R(h) \ a.s. \quad \textit{for } M \to \infty \tag{3.2}$$

under the assumption of a finite generalization error [25], [29].

The recipe to solve a learning problem under the $(i.i.d)$-Learning Assumption is therefore:

(1) find a hypothesis set that is densely contained in the concept class.

[2]The Bayes error $R^\star$ [29] is replaced by $R(c)$, since we assume deterministic labels. Nevertheless, it appears as we do not impose $\mathcal{L}(x,x) = 0$ resulting in $R(c) \geq 0$ instead of $R(c) = 0$.

(2) define a learning algorithm that minimizes the empirical error.

The *Probably Approximately Correct* (PAC)-learning framework [29] gives a rigorous definition of a well-posed learning problem for a fixed concept class. In this framework, a well-posed learning problem has to provide convergence in probability of the estimation error to zero with at least a polynomial rate of convergence.

Definition 13. *(PAC-learnable I) A concept class $\mathcal{C}$ is said to be PAC-learnable if there exists a learning algorithm $\mathcal{A}$ and a polynomial p such that for every $\epsilon > 0$ and any $\delta > 0$, for every target distribution $\mathcal{D}$ and all target concepts $c \in \mathcal{C}$, we have*

$$\mathbb{P}\left(R(h_{\mathcal{S}}) - \hat{R}^{\star} > \epsilon\right) < \delta$$

for any sample $\mathcal{S}$ with at least

$$M \geq p\left(\frac{1}{\epsilon}, \frac{1}{\delta}\right)$$

observations. When such an algorithm $\mathcal{A}$ exists, it is called a PAC-learning algorithm for $\mathcal{C}$.

Remark 3.3. *Since $\mathcal{D}$ is in general unknown, a PAC-learnable concept class $\mathcal{C}$ has to satisfy convergence in probability*

$$\mathbb{P} - \lim_{M\to\infty} R(h_{\mathcal{S}}) = \hat{R}^{\star} \tag{3.3}$$

independent of the target distribution $\mathcal{D}$. In addition, for a PAC-learnable concept class it exists a polynomial $p : (0,1)^2 \to \mathbb{N}$ such that for every $\epsilon > 0$ and any $\delta > 0$ with probability of at least $1 - \delta$ we find

$$0 \leq R(h_{\mathcal{S}}) - \hat{R}^{\star} \leq \epsilon \quad \textit{if } M \geq p\left(\frac{1}{\epsilon}, \frac{1}{\delta}\right).$$

The above equation contains only the estimation error and not the approximation error, so that the approximation quality of the learned hypothesis strongly depends on the selected hypothesis set. For example, we may construct a PAC-learning algorithm for every concept class $\mathcal{C}$ by

$$\mathcal{H} = \{h\} \subset \mathcal{C}_0 \quad \textit{and} \quad \mathcal{A}(\mathcal{S}) = h_{\mathcal{S}} := h, \tag{3.4}$$

where h is any measurable function. However, selecting a hypothesis set that is densely contained in the concept class and assuming

$$R(h) \geq R(c) \quad \textit{for all } h \in \mathcal{H},$$

the generalization error of the learned hypothesis of a PAC-learning algorithm converges in probability to the generalization error of the target concept with at least a polynomial rate of convergence wrt. to the number of observations.

In order to approximate the optimal risky fraction (2.8) and the optimal expected terminal wealth (2.9) in the shadow market, we propose a learning algorithm that approximates the target concept $\frac{\tilde{\mu}}{\tilde{\sigma}}$ implied by a price process $S \in \boldsymbol{\mathcal{S}}_{LS}$. To formulate a corresponding learning problem and a PAC-learning definition we generalize the learning problem and the PAC-learnable framework from an underlying data generating random variable to an underlying data generating process. Since we might know the distribution of the starting value and the distribution of the increments of the data generating process we weaken the assumption that (3.3) has to hold for *every* target distribution.

3.2 A Generalized PAC-Learning Framework

Let $\boldsymbol{\mathcal{S}}_0$ be the set of continuous stochastic processes defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to the augmented natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$ taking values in the space of instances $\mathcal{X}$. Define a set $\mathcal{X}(\boldsymbol{\mathcal{S}}) \subset \boldsymbol{\mathcal{S}}_0$ of *data generating processes*. For every $X \in \mathcal{X}(\boldsymbol{\mathcal{S}})$ we fix $\mathcal{C}(X) \subset \mathcal{C}_0$ and denote by

$$\mathcal{C} = \bigcup_{X\in\mathcal{X}(\boldsymbol{\mathcal{S}})} \mathcal{C}(X)$$

the concept class. For a fixed data generating process $X \in \mathcal{X}(\boldsymbol{\mathcal{S}})$ and a fixed target concept $c \in \mathcal{C}(X)$ the learner wishes to approximate the path

$$t \mapsto c(X_t) \tag{3.5}$$

of the *target process* $c(X) = (c(X_t))_{t\in[0,T]}$. In order to approximate a fixed but unknown target process, the learner selects a hypothesis set

$$\mathcal{H} \subset \mathcal{C}_0,$$

a time grid $0 = t_0 < t_1 < ... < t_N = T$ of $[0, T]$ and a *learning algorithm* $\mathcal{A} = (\mathcal{A}_{N,M})_{(M,N)\in\mathbb{N}^2}$ consisting of maps

$$\mathcal{A}_{M,N} : \mathcal{X}^{M,N} \to \mathcal{H}$$

for an unsupervised learning task and

$$\mathcal{A}_{M,N} : (\mathcal{X} \times \mathcal{Y})^{M,N} \to \mathcal{H}$$

for a supervised learning task. Denote by

$$Z_N = (Z_{t_n})_{0\le n\le N}$$

a discretized path of a stochastic process $Z = (Z_t)_{t\in[0,T]}$. The learner might not observe a discretized path X_N of the true data generating process X but only an *observable path* $\bar{X}_N$ that is an approximation of X_N. Given *sample paths* of observations $\bar{x}_N^1, ..., \bar{x}_N^M \in \mathcal{X}^N$

$$\mathcal{SP} = \left\{\bar{x}_N^1, ..., \bar{x}_N^M\right\}$$

for an unsupervised learning problem and

$$\mathcal{SP} = \left\{\left(\bar{x}_N^1, c\left(\bar{x}_N^1\right)\right), ..., \left(\bar{x}_N^M, c\left(\bar{x}_N^M\right)\right)\right\}$$

for a supervised learning problem , the learning algorithm assigns a hypothesis $h \in \mathcal{H}$ to the observed sample paths. It is not reasonable to assume that $\bar{x}_{t_n}^i$ and $\bar{x}_{t_{n+1}}^i$ are independent realisations, since they are subsequent values of $\bar{X}_N^i$. However, we impose that the given sample paths are independent realisations satisfying the

$(i.i.d)$**-Learning Assumption** (II). *The observations $\bar{x}_N^1, ..., \bar{x}_N^M$ are realisations of independent copies $\bar{X}_N^1, ..., \bar{X}_N^M$ of the observable process $\bar{X}_N$ such that*

$$\bar{X}_{t_n}^i \sim \mathbb{P} \circ \bar{X}_{t_n}^{-1} \quad \textit{for all } 1 \le i \le M \textit{ and } 0 \le n \le N.$$

In order to determine $h \in \mathcal{H}$ such that

$$h(X_t) \approx c(X_t) \quad \textit{for all } t \in [0,T],$$

the learner fixes a loss function $\mathcal{L}$ and aims to minimize the *integrated loss*

$$I_T\left(h(X), c(X)\right) := \int_0^T \mathcal{L}\left(h(X_t), c(X_t)\right) dt \tag{3.6}$$

over the time period $[0, T]$. The integral in (3.6) is defined as a ω-pathwise Stieltjes integral wrt. the process $(t)_{t\in[0,T]}$ of finite variation [7]. We define the expected integrated loss for a hypothesis as the *generalization error*.

Definition 14 (Generalization error II). *Let $c(X)$ be the target process. For $h \in \mathcal{H}$ we call*

$$R(h) := \mathbb{E}\left[I_T\left(h(X), c(X)\right)\right] = \mathbb{E}\left[\int_0^T \mathcal{L}\left(h(X_t), c(X_t)\right) dt\right]$$

the generalization error wrt. $\mathcal{L}$.

Remark 3.4. *To recover the setting of binary classification or regression, let X be the data generating random variable and set $\mathcal{X}(\boldsymbol{\mathcal{S}}) = \{S\}$ with $S_t \equiv X$ for all $t \in [0,1]$. For regression choose $\mathcal{Y} = \mathbb{R}$ and fix the loss function*

$$\mathcal{L} : \mathbb{R} \times \mathbb{R} \to [0, \infty], (y, y') \mapsto |y - y'|^p$$

resulting in the generalization error

$$R(h) = \mathbb{E}\left[\int_0^1 |h(X) - c(X)|^p \, dt\right] = \mathbb{E}\left[|h(X) - c(X)|^p\right]$$

that is the expected ℓ_p loss.

Similar to the setting of a data generating random variable, we define for a fixed target concept c the approximation error

$$\mathcal{E}(\mathcal{H}) := \left|\hat{R}^\star - R(c)\right| \quad \textit{with} \quad \hat{R}^\star := \inf_{h\in\mathcal{H}(S)} R(h)$$

and for a given hypothesis $h \in \mathcal{H}$ the estimation error

$$\mathcal{E}(h) := R(h) - \hat{R}^\star \geq 0.$$

Towards an error decomposition we need to take into account two additional error sources. First, we define the error arising from the discretization.

Definition 15 (Discretization error). *Let $c(X)$ be the target process, $\mathcal{L}$ a loss function and $0 = t_0 < t_1 < ... < t_N = T$ a time grid of $[0, T]$. The discrete generalization error is defined by*

$$R_N(h) := \frac{T}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[\mathcal{L}\left(h(X_{t_n}), c(X_{t_n})\right)\right]$$

and the discretization error is defined by

$$\mathcal{E}_N(h) := \left|R_N(h) - R(h)\right|.$$

Remark 3.5. *Note that for continuous h and continuous c the integrated loss is for a deterministic and continuous process X a Riemann integral with*

$$\lim_{N\to\infty} R_N(h) = \lim_{N\to\infty} \frac{T}{N} \sum_{n=0}^{N-1} \mathcal{L}\left(h(X_{t_n}), c(X_{t_n})\right) = \int_0^T \mathcal{L}\left(h(X_t), c(X_t)\right) dt.$$

The second source of error is that the learner does not observe true realisations of X_N but only realisations of the observable process $\bar{X}_N$.

Definition 16 (Error of observation)**.** *Define*

$$\bar{R}_N(h) := \frac{T}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[\mathcal{L}\left(h(\bar{X}_{t_n}), c(\bar{X}_{t_n})\right)\right]$$

and denote by

$$\mathcal{E}_O(h) := |\bar{R}_N(h) - R_N(h)|$$

the error of observation.

For a given hypothesis $h \in \mathcal{H}$ we decompose the different sources of error into

$$|\bar{R}_N(h) - R(c)| = \underbrace{\mathcal{E}_O(h)}_{\textbf{observation error}} + \underbrace{\mathcal{E}_N(h)}_{\textbf{discretization error}} + \underbrace{\mathcal{E}(h)}_{\textbf{estimation error}} + \underbrace{\mathcal{E}(\mathcal{H})}_{\textbf{approximation error}} .$$

Remark 3.6. *In the setting of binary classification or regression introduced in Remark 3.4 the above error decomposition reduces to*

$$|\bar{R}_N(h) - R(c)| = R(h) = \underbrace{\mathcal{E}(h)}_{\textbf{estimation error}} + \underbrace{\hat{R}^\star}_{\textbf{approximation error}} ,$$

since

$$\begin{aligned}\mathcal{E}_N(h) &= \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[\mathcal{L}\left(h(X), c(X)\right) - \int_0^1 \mathcal{L}\left(h(X), c(X)\right) dt\right]\right| \\ &= |\mathbb{E}\left[\mathcal{L}\left(h(X), c(X)\right) - \mathcal{L}\left(h(X), c(X)\right)\right]| \\ &= 0\end{aligned}$$

as well as

$$\begin{aligned}\mathcal{E}_O(h) &= \left| \frac{1}{N} \sum_{n=0}^{N-1} \left(\mathbb{E}\left[\mathcal{L}\left(h(\bar{X}), c(\bar{X})\right) - \mathcal{L}\left(h(X), c(X)\right)\right]\right)\right| \\ &= |\mathbb{E}\left[\mathcal{L}\left(h(X), c(X)\right) - \mathcal{L}\left(h(X), c(X)\right)\right]| \\ &= 0\end{aligned}$$

and $R(c) = 0$ implies $\mathcal{E}(\mathcal{H}) = \hat{R}^\star$.

In the formulation of a learning problem we need to accommodate the minimization of the observation error and the minimization of the discretization error:

Definition 17. *(Learning problem II) The learning problem for a target process $c(X)$ is to*

(1) *find a hypothesis set $\mathcal{H}$ with a small approximation error $\mathcal{E}(\mathcal{H})$.*

(2) *choose a discretization scheme with a small discretization error $\mathcal{E}_N(h)$ for all $h \in \mathcal{H}$.*

(3) *determine a process $\bar{X}_N$ that approximates a discrete path of the data generating process X with a small observation error $\mathcal{E}_O(h)$ for all $h \in \mathcal{H}$.*

(4) *define a learning algorithm such that $R(h_{\mathcal{SP}}) \to \hat{R}^\star$ for $M \to \infty$.*

In the setting of a data generating random variable, the learner has access to the empirical error that converges *a.s.* towards the generalization error. For a data generating process, the learner has only access to the empirical error of observation that we call the *empirical error* as well. This is justified, since by Remark 3.6 it coincides with the empirical error defined in the setting of a data generating random variable.

Definition 18 (Empirical loss II)**.** *Let $\bar{X}_N$ be the observable process to the target process $c(X)$. For a given sample path $\mathcal{SP}$, a given hypothesis $h \in \mathcal{H}$ and a loss function $\mathcal{L}$ define the empirical error by*

$$\hat{R}^N_{\mathcal{SP}}(h) := \frac{T}{NM} \sum_{m=1}^{M} \sum_{n=0}^{N-1} \mathcal{L}\left(h(\bar{X}^m_{t_n}), c(\bar{X}^m_{t_n})\right). \tag{3.7}$$

Remark 3.7. *The empirical error is a random quantity. Under the $(i.i.d)$-Learning II the expectation of the empirical error equals the error of observation*

$$\mathbb{E}\left[\hat{R}^N_{\mathcal{SP}}(h)\right] = \hat{R}_N(h).$$

By the strong law of large numbers

$$\hat{R}^N_{\mathcal{SP}}(h) \to \hat{R}_N(h) \text{ a.s.} \quad \text{for } M \to \infty.$$

under the assumption of a finite error of observation [25].

In addition to the convergence in probability

$$\mathbb{P} - \lim_{M \to \infty} \bar{R}_N(h_{\mathcal{S}}) = R_N(h)$$

of the error of observation towards the error of discretization, a generalized PAC-learning framework should ensure

$$\lim_{N \to \infty} R_N(h) = \hat{R}^\star \quad a.s$$

with at least a polynomial rate of convergence of N and M.

Definition 19. *(PAC-learnable II) A concept class $\mathcal{C}$ is said to be weak PAC-learnable if for every data generating process $X \in \mathcal{X}(\boldsymbol{\mathcal{S}})$ with initial distribution $\mathcal{D}_0$ it exists a learning algorithm $\mathcal{A}$, a observable process $\bar{X}_N$ and polynomials p_1, p_2 such that for every $\epsilon > 0$ and any $\delta > 0$, for all concepts $c \in \mathcal{C}(X)$*

$$\mathbb{P}\left(\left|\hat{R}_N(h_{\mathcal{SP}}) - \hat{R}^\star\right| > \epsilon\right) < \delta$$

for any sample path $\mathcal{SP}$ with

$$N \geq p_1\left(\frac{1}{\epsilon}, \frac{1}{\delta}\right)$$

discretization stepts and

$$M \geq p_2\left(\frac{1}{\epsilon}, \frac{1}{\delta}\right)$$

observations. When such an algorithm $\mathcal{A}$ exists, it is called a weak PAC-learning algorithm for $\mathcal{C}$. The concept class $\mathcal{C}$ is said to be PAC-learnable if the learning algorithm and the polynomials do not depend on the initial distribution $\mathcal{D}_0$ of X_0. In this case the learning algorithm is called a PAC-learning algorithm.

To recover the PAC-learning framework for a data generating random variable X, set $\mathcal{X}(\boldsymbol{\mathcal{S}}) = \{S\}$ with $S_t \equiv \bar{X} = X$ for all $t \in [0,1]$ such that $\hat{R}_N(h) = R(h)$. A weak PAC-learning algorithm $\mathcal{A}$ allows to adjust the learning algorithm wrt. some assumptions on the target distribution $\mathcal{D} = \mathcal{D}_0$ whereas a PAC-learning algorithm is independent of the target distribution, which is a strong assumption, since the hypothesis has to be learned for *every* distribution $\mathcal{D}$. This is suitable if we observe truly unknown distributed data but not expedient to learn the transformed path of a stochastic process, with a known distribution of the starring value. For example, if we know the starting value x_0 of the data generating process we have $\mathcal{D}_0 = \delta_{s_0}$ for the initial distribution. In the setting of a data generating random variable this models a scenario where the target concept c is unknown but the target distribution is known. Hence, towards learning a concept class $\mathcal{C}$ from a set of data generating processes with known initial values, the learner aims to design a weak PAC-learning algorithm.

3.3 The Transaction Cost Problem as a Learning Problem

We aim to design a weak PAC-learning algorithm $\mathcal{A}$ that outputs an approximation of the target concept $\frac{\tilde{\mu}}{\tilde{\sigma}}$ as defined in (2.28). To be more precise, we aim for a PAC-learning algorithm $\mathcal{A}(\mathcal{SP})\frac{h_1}{h_2}$ such that

$$h_1 \approx \tilde{\mu} \quad \textit{and} \quad h_2 \approx \tilde{\sigma}.$$

Starting with the choice of a hypothesis set, describe the four tasks of this learning problem in the following Sections.

3.3.1 The Hypothesis Set of Neural Networks

The concept we aim to learn is a continuous function on a compact domain. In this section we explain why we use a subclass of neural networks as the hypothesis set for our learning problem. Let us begin with the definition of a neural network from [31]. This definition distinguishes between a neural network as a sequence of weights and the function implemented by the neural network.

Definition 20. *A neural network (NN) Φ with input dimension $d \in \mathbb{N}$ and $L \in \mathbb{N}$ layers is a sequence of matrix-vector tuples*

$$\Phi = ((A_1, b_1), (A_2, b_2), ..., (A_L, b_L)) \tag{3.8}$$

where $N_0 = d$ and $N_1, ..., N_L \in \mathbb{N}$ and where each $A_l \in \mathbb{R}^{N_l \times N_{l-1}}$ and $b_l \in \mathbb{R}^{N_l}$. The integer N_l is called the width of $l - th$ layer for $1 \leq l \leq L - 1$ and N_L is called the output dimension. The integer L is called the depth of Φ and the first $L - 1$ matrix-vector tuples in (3.8) are named the hidden layers. For a NN Φ we define the associated ϱ-realization over $K \subset \mathbb{R}^d$ as a map $R_\varrho(\Phi) : K \to \mathbb{R}^{L_L}$ such that

$$R_\varrho(\Phi)(x) := x_{k+1}$$

with the output x_{k+1} resulting from the scheme

$$x_0 := x, \quad x_i := \varrho(A_i x_{i-1} + b_i), \quad x_{k+1} := A_{k+1} x_k + b_{k+1},$$

where the activation function ϱ is applied componentwise. A NN with at least two hidden layers is called a deep neural network (DNN).

The entries of the matrices and vectors in (3.8) are *the parameters* of the NN Φ. We denote by

$$M_p(\Phi) = \sum_{l=1}^{L} (N_l N_{l-1} + N_l) \,. \tag{3.9}$$

the *total number of parameters*. A fixed parameter configuration $\Theta \in \mathbb{R}^{M_p(\Phi)}$ leading to R_ϱ^Θ is called the *weight vector* and the entries of the weight vector are called the *weights* of the realization R_ϱ^Θ.[3] The *universal approximation theorem* [19], [8], [20] states that the hypothesis set of NNs with only one hidden layer and non-polynomial ϱ-realization is already dense in the space of real-valued continuous functions on a compact domain $K \subset \mathbb{R}^d$. Therefore, for an arbitrary continuous real-valued function f on K and for every $\epsilon > 0$ exists Φ with only one hidden layer and a realization R_ϱ such that

$$\sup_{x \in K} |f(x) - R^K(\Phi)(x)| < \epsilon. \tag{3.10}$$

This gives a justification to approximate continuous functions by a NN with one hidden layer, however, it does not specify the necessary number of parameters in the hidden layer in order to satisfy (3.10). A tremendous amount of research on the so-called expressivity of NN, subjcet to the number of layers and the number of weights, has been carried out [33], [36], [14], [28] for NNs with *rectified linear unit* activation functions

$$\varrho_0(x) := \max\{0, x\}.$$

In the following we denote by $\mathcal{N}$ the set of NNs with ϱ_0-realization. The identity

$$x = \varrho_0(x) - \varrho_0(-x) \quad \textit{for all } x \in \mathbb{R}$$

ensures for every $\Phi \in \mathcal{N}$ with $L \geq 2$ layers, and widths

$$N_1 = 2d, ..., N_{L-1} = 2d, N_L = d$$

the existence of a realization R such that

$$R(\Phi) = \mathrm{Id}_{\mathbb{R}^d}$$

is the identity [31]. Hence, for a ϱ_0-realization, the expressivity of a DNN is at least as good as the expressivity of a NN with only one hidden layer. The deployment of DNN have shown great success in a number of tasks, raising the conjecture that an increase of the depth is more efficient than an increase of the width of a NN with only one hidden layer and rigorous results substantiate this conjecture. Thus, when selecting a hypothesis set of NNs for a specific task in practice, the question "How many parameters are sufficient in the hidden layer?" is additionally expanded to "How many layers are sufficient?". Using the rigorous results of Hanin and Sellke

[3] In contrast, Petersen and Voigtlaender [31] define *the size* $M(\Phi)$ of Φ as the number of non-zero parameters. For a non-degenerate NN the total number of parameters $M_p(\Phi)$ is bounded by $M(\Phi)^2 + M(\Phi)$ [14].

[17], one my formulate the rule of thumb "Do not use too many parameters in each layer and scale the depth with the input dimension, with the diameter of the input domain and with the level of continuity of the target concept." To be more precise, denote by

$$diam(K) := \max \{\|x-y\|_2 \,|\, x,y \in K\}$$

the diameter of the compact set $K \subset \mathbb{R}^d$ and by

$$\omega_f(\epsilon) := \sup_{\|x-y\|_2 \leq \epsilon} \{\|f(x)-f(y)\|_2\}$$

the modulus of continuity for a fixed continuous function f with domain K [17]. For a L_f-Lipschitz function f deduce

$$\omega_f(\epsilon) = L_f \epsilon \quad \textit{and} \quad \omega_f(\epsilon)^{-1} = \frac{\epsilon}{L_f} \qquad \textit{for every } \epsilon > 0.$$

The following Theorem summarizes the main results of Hanin and Sellke [17] applied to Lipschitz continuous functions.

Theorem 3.8. *For every compact set $K \subset \mathbb{R}^d$, any L_f-Lipschitz continuous function $f : K \to \mathbb{R}^{N_L}$ with output dimension $N_L \in \mathbb{N}$ and each $\epsilon > 0$ exists a constant $C_K > 0$ and a NN $\Phi \in \mathcal{N}$ of widths*

$$w := \max\{N_1, ..., N_L\} \leq d + N_L$$

and depth

$$L \leq \left(\frac{L_f C_K diam(K)}{\epsilon}\right)^{d+1}$$

with realization R such that

$$\sup_{x \in K} \|f(x) - R(\Phi)(x)\|_2 \leq \epsilon. \tag{3.11}$$

The total number of parameters is therefore bounded by

$$M_p(\Phi) = \sum_{l=1}^{L} (N_l N_{l-1} + N_l) \leq \left(\frac{L_f C_K diam(K)}{\epsilon}\right)^{d+1} \left[(d+N_L)^2 + d + N_L\right].$$

and depends for fixed input dimension d polynomial on the error of the approximation (3.11).

3.3.2 A Hypothesis Set for the Transaction Cost Problem

From now on we assume for $P \in \boldsymbol{\mathcal{S}}_{LS}$ a bit price of $(1-\gamma)P$ and an ask price of $(1+\gamma)P$, reflecting a scenario where proportional transaction costs are present during the purchase and the sale. To translate this scenario in the notation we used in Chapter 2 we follow Gerhold et al. [15] and set

$$S := (1+\gamma)P \quad \textit{and} \quad \lambda := \frac{2\gamma}{1+\gamma}. \tag{3.12}$$

In order to approximate $\tilde{\mu}$ and $\tilde{\sigma}$ defined in (2.27) by neural network realizations $R_1(\Phi_1)$ and $R_2(\Phi_2)$ we need to ensure that the approximation $\bar{S} \approx \tilde{S}$ takes only values in the random bid and ask interval $[(1-\lambda)S, S]$. For a fixed pair of neural

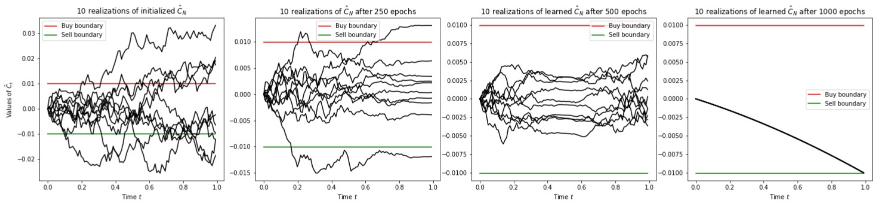

FIGURE 3.1: On the LHS in the first plot we see ten realizations of the initialization of C. After 250 optimization steps, we observe that the volatility already vanishes in the last steps. the realizations are squeezed into $[\ln(1-\gamma), \ln(1+\gamma)]$

networks $\Phi_1, \Phi_2 \in \mathcal{N}$ we find for most price processes a realization such that

$$\bar{S}_{t_n} \notin [(1-\lambda)S_{t_n}, S_{t_n}] \quad \textit{for some } 1 \leq n \leq N \tag{3.13}$$

with high probability [4]. In a first approach we used the parametrization

$$\tilde{S} = P \exp\left(C\right),$$

from the proof of the existence of a shadow price process in J.Kallsen and J.Muhle-Karbe [21] with some unknown process $C \subset [\ln(1-\lambda), \ln(1+\lambda)]$ following

$$dC_t = \tilde{\alpha}(t, C_t)dt + \tilde{\beta}(t, C_t)dW_t.$$

To circumvent (3.13) we added two additional terms to the empirical loss (3.7) of our learning problem: First a term to adhere the Fichera condition [6] close to the boundary points and second a penalty term to penalize realizations where the process $\bar{S}_N$ leaves the interval. A recurring problem was

$$R_2(\Phi_2)(t_n, C_{t_n}) \to 0 \tag{3.14}$$

during the minimization of the empirical loss, resulting in a deterministic process C. A careful scaling of the different error terms did not resolve the problem. The observation might be a consequence of the deployed Euler-Maruyama scheme, [5] since within this scheme every realization $R(\Phi_2)$ except for $R(\Phi_2) \equiv 0$ provides in every time step a strict positive probability that $\bar{S}_N$ leaves the interval, resulting eventually in a penalization of every realization except for $R(\Phi_2) \equiv 0$. See Figure 3.1 for an illustration of the problem.

A more convenient approach is to ensure a priori that $\bar{S}_N$ does not leave the interval. For a fixed price process $S \in \boldsymbol{\mathcal{S}}_{LS}$ and the induced shadow price process $\tilde{S}$ define the process

$$X := \frac{1}{\gamma}\left(\frac{\tilde{S}}{P} - 1\right) \subset [-1, 1] \tag{3.15}$$

such that

$$\tilde{S} = P + \gamma P X.$$

[4] Seting the weights of Φ_1 and Φ_2 to zero resulting in $\bar{S} \equiv s_0$ and hence (3.13) if $(1-\lambda)S_{t_n} > s_0$ or $S_{t_n} < s_0$ for some $n \leq N$. We also find non trivial realizations: For arbitrary weights of Φ_2 choose the weights of the last layer of Φ_1 sufficient small or sufficient large such that $\bar{S}_T > S_T$ or $\bar{S}_T < S_T$ with high probability.

[5] See Definition 21.

By *Assumption* 2

$$d\tilde{S}_t = S_t\tilde{\mu}\left(Y_t\right)dt + S_t\tilde{\sigma}\left(Y_t\right)dW_t \quad \textit{with} \quad Y_t = \frac{S_t}{m_t} \tag{3.16}$$

holds true. Towards the dynamics of X calculate

$$d\left(\frac{1}{P_t}\right) = \left\{-\frac{1}{P_t}\mu(P_t) + \frac{1}{P_t}\sigma(P_t)^2\right\}dt - \frac{1}{P_t}\sigma(P_t)dW_t$$

and using (3.16) and (3.12)

$$dX_t = \frac{1}{\gamma}\tilde{a}\left(Y_t, P_t, \tilde{S}_t\right)dt + \frac{1}{\gamma}\tilde{b}\left(Y_t, P_t, \tilde{S}_t\right)dW_t \tag{3.17}$$

with

$$\tilde{a} : [1, \bar{s}] \times \mathbb{R}_+^2 \to \mathbb{R}, \quad (y, p, s) \mapsto \tilde{a}_1\left(y, p\right) + \tilde{a}_2\left(p, s\right),$$

where

$$\tilde{a}_1\left(y, p\right) := (1+\gamma)\left[\tilde{\mu}\left(y\right) - \tilde{\sigma}\left(y\right)\sigma(p)\right], \tilde{a}_2\left(p, s\right) := \frac{s}{p}\left[\sigma(p)^2 - \mu(p)\right] \tag{3.18}$$

and

$$\tilde{b} : [1, \bar{s}] \times \mathbb{R}_+^2 \to \mathbb{R}, \quad (y, p, s) \mapsto \tilde{b}_1\left(y\right) + \tilde{b}_2\left(p, s\right)$$

where

$$\tilde{b}_1\left(y\right) := (1+\gamma)\tilde{\sigma}\left(y\right), \quad \tilde{b}_2\left(p, s\right) := \frac{s}{p}\sigma(p). \tag{3.19}$$

For the calculations see Appendix A.5. Inline with (2.27) in Assumption 2, we assume that the domain of $\tilde{a}_1$ is one-dimensional and compact and in addition that the coefficient functions of the price process are bounded.

Assumption 3. *Assume that for $S \in \mathcal{S}_{LS}$ the coefficient function σ is bounded from above by $C_\sigma > 0$ and μ is bounded by $C_\mu > 0$ such that $|\mu(p)| \leq C_\mu$ for all $p \in \mathbb{R}$. Assume further that it exists a $L_{\tilde{a}_1}$-Lipschitz continuous function $\tilde{a}_1 : [1, \bar{s}] \to \mathbb{R}$ such that*

$$\tilde{a}_1\left(Y_t\right) = (1+\gamma)\left[\tilde{\mu}\left(Y_t\right) - \tilde{\sigma}\left(Y_t\right)\sigma(P_t)\right] \quad a.s.$$

Denote by $C_{\tilde{a}_1} > 0$ the maximum of the function $|\tilde{a}_1|$.

Remark 3.9. *Note that a geometric Brownian motion S satisfies Assumption 3 since*

$$\sigma(S_t) \equiv \sigma =: C_\sigma \quad \mu(S_t) \equiv \mu =: C_\mu$$

and

$$\tilde{a}_1(Y_t) = (1+\gamma)\left[\tilde{\mu}(Y_t) - \tilde{\sigma}(Y_t)\sigma\right]$$

is $L_{\tilde{\mu},\tilde{\sigma}}$-Lipschitz continuous with

$$L_{\tilde{\mu},\tilde{\sigma}} := (1+\gamma)(L_{\tilde{\mu}} + \sigma L_{\tilde{\sigma}}).$$

We assume from now on that the price process is an element of the set

$$\mathcal{S}_{LS,b} := \{S \in \mathcal{S}_{LS} \mid S \text{ satisfies Assumption } 3\}.$$

By *Remark* 3.9 a geometric Brownian motion with $\sigma^2 > \mu > 0$ is an element of $\boldsymbol{\mathcal{S}}_{LS,b}$. Towards a hypothesis set consisting of maps $h = \frac{h_1}{h2}$ so that the process

$$d\bar{X}_t = h_1(Y_t, P_t, \tilde{S}_t)dt + h_2(Y_t, P_t, \tilde{S}_t)dW_t, \quad \bar{X}_0 = 0 \tag{3.20}$$

takes only values in $[-1, 1]$, we define

$$g_\alpha : \mathbb{R}^2 \to \mathbb{R}, \quad (p, x) \mapsto p + \gamma p \alpha(x)$$

with a C^2 function $\alpha : \mathbb{R} \to [-1, 1]$. The stochastic process

$$\bar{S}^\alpha := g_\alpha(P, \bar{X})$$

satisfies

$$\bar{S}^\alpha_t \in [(1-\gamma)P_t, (1+\gamma)P_t] \quad \textit{for all } t \in [0, T]$$

and any $\bar{X} \in \boldsymbol{\mathcal{S}}$ by definition of g_α and α. Note that $\bar{X} = X$ implies

$$\bar{S}^\alpha = g_\alpha(P, \alpha^{-1}(\bar{X})) = \tilde{S} \quad \textit{on } \{\tilde{S} \neq S\} \cup \{\tilde{S} \neq (1-\lambda)S\}$$

if α is bijective and hence invertible on $(-1, 1)$. The idea of the proposed learning algorithm is to approximate the coefficient functions $\tilde{a}_1$ and $\tilde{b}_1$ defined in (3.18) and (3.19), while α ensures that the approximation $\bar{S}^\alpha \approx \tilde{S}$ of the shadow price process never leaves the bid and ask interval. Towards the dynamics of $\bar{S}^\alpha$ we calculate the partial derivatives of g_α

$$\frac{\partial g_\alpha}{\partial p}(p, x) = 1 + \gamma\alpha(x), \quad \frac{\partial^2 g_\alpha}{\partial^2 p}(p, x) = 0, \quad \frac{\partial^2 g_\alpha}{\partial x \partial p}(p, x) = \gamma\alpha(x)'$$

and

$$\frac{\partial g_\alpha}{\partial x}(p, x) = \gamma p \alpha(x)', \quad \frac{\partial^2 g_\alpha}{\partial p \partial x}(p, x) = \gamma\alpha(x)', \quad \frac{\partial^2 g_\alpha}{\partial^2 x}(p, x) = \gamma p \alpha(x)''.$$

to derive

$$d\bar{S}^\alpha_t = \bar{S}^\alpha_t \hat{\mu}\left(Y_t, P_t, \tilde{S}_t, X_t\right) dt + \bar{S}^\alpha_t \hat{\sigma}\left(Y_t, P_t, \tilde{S}_t, X_t\right) dW_t \tag{3.21}$$

with $\hat{\mu} := \hat{\mu}_1 + \hat{\mu}_2$ where

$$\hat{\mu}_1 : [1, \bar{s}] \times \mathbb{R}^2_+ \times \mathbb{R} \to \mathbb{R}, (y, p, x, s) \mapsto \mu(p) + \frac{\alpha'(x)}{1 + \gamma\alpha(x)}\tilde{a}(y, p, s)$$

$$\hat{\mu}_2 : [1, \bar{s}] \times \mathbb{R}^2_+ \times \mathbb{R} \to \mathbb{R}, (y, p, x, s) \mapsto \frac{\alpha'(x)}{1 + \gamma\alpha(x)}\sigma(p)\tilde{b}(y, p, s) + \frac{1}{2\gamma}\frac{\alpha''(x)}{1 + \gamma\alpha(x)}\tilde{b}^2(y, p, s)$$

and

$$\hat{\sigma} : [1, \bar{s}] \times \mathbb{R}^2_+ \times \mathbb{R} \to \mathbb{R}, \quad (y, p, x, s) \mapsto \sigma(p) + \frac{\alpha'(x)}{1 + \gamma\alpha(x)}\tilde{b}(y, p, s).$$

For the calculations see Appendix A.5. Following (*B.4*) the optimal risky fraction in the financial model $(B, \bar{S}^\alpha)$ is

$$\bar{\pi}^\star_t = \frac{\hat{\mu}\left(Y, P, X, \tilde{S}\right)}{\hat{\sigma}\left(Y, P, X, \tilde{S}\right)^2}$$

and $\bar{\theta}$ as defined in (*B*.3) is given by

$$\bar{\theta} = \frac{\hat{\mu}\left(Y, P, X, \tilde{S}\right)}{\hat{\sigma}\left(Y, P, X, \tilde{S}\right)}. \tag{3.22}$$

Let $\alpha : \mathbb{R} \to [-1, 1]$ be the *hardtanh* [6] with

$$x \mapsto \begin{cases} 1, & \textit{if } x \geq 1 \\ 1, & \textit{if } x \leq -1 \\ x & \textit{otherwise.} \end{cases}$$

where we define for the non-differentiable points

$$\alpha'(-1) := 1, \alpha'(1) := 1 \quad \textit{and} \quad \alpha''(-1) := 0, \alpha''(1) := 0.$$

For this choice of α we find

$$\bar{S}^{\alpha} = g_{\alpha}(P, \bar{X}) = \tilde{S}.$$

if $\bar{X} = X$ and may rewrite (3.22) to

$$\hat{\theta} = \frac{\hat{\mu}\left(Y, P, X, g_{\alpha}(P, X)\right)}{\hat{\sigma}\left(Y, P, X, g_{\alpha}(P, X)\right)} =: \frac{\bar{\mu}\left(Y, P, X\right)}{\bar{\sigma}\left(Y, P, X\right)} \tag{3.23}$$

Moreover, by (3.16) and (3.16)

$$\bar{\mu}(Y, P, X) = \frac{S}{\tilde{S}}\tilde{\mu}(Y) \quad \textit{and} \quad \bar{\sigma}(Y, P, X) = \frac{S}{\tilde{S}}\tilde{\sigma}(Y) \tag{3.24}$$

so that

$$\hat{\theta} = \frac{\frac{S}{\tilde{S}}\tilde{\mu}(Y)}{\frac{S}{\tilde{S}}\tilde{\sigma}(Y)} = \frac{\tilde{\mu}(Y)}{\tilde{\sigma}(Y)} \tag{3.25}$$

and we define the target concept of our learning problem by

$$c : [1, \bar{s}] \times (0, \infty) \times \mathbb{R} \to (0, \infty), \quad (y, p, x) \mapsto \frac{\bar{\mu}(y, p, x)}{\bar{\sigma}(y, p, x)}$$

to find by (3.25) for f from (2.28) in Assumption 2

$$f(Y) = c^2(Y, P, X). \tag{3.26}$$

In accordance of the generalized PAC-learning framework defined in **??**, let

$$\mathcal{X} = \left([1, \bar{s}] \times (0, \infty) \times \mathbb{R}, \mathcal{B}\left([1, \bar{s}] \times (0, \infty) \times \mathbb{R}\right)\right)$$

be the set of instances and $\mathcal{Y} := (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ the space of target values. We define the set of data generating processes by

$$\mathcal{X}(\boldsymbol{\mathcal{S}}) := \left\{(Y, P, X) \,\middle|\, (1+\gamma)P = S \in \boldsymbol{\mathcal{S}}_{LS,b}, \textit{ m from Definition 8 and X as in (3.15)}\right\}$$

and

$$\mathcal{C}(Z) := \{c : [1, \bar{s}] \times (0, \infty) \times \mathbb{R} \to \mathbb{R} \textit{ with } \hat{\mu} \textit{ and } \hat{\sigma} \textit{ as defined in (3.23) induced by } Z \in \mathcal{X}(\boldsymbol{\mathcal{S}})\}$$

[6] In accordance with the name of the PyTorch implementation `https://pytorch.org/docs/stable/generated/torch.nn.Hardtanh.html`.

to fix the concept class

$$\mathcal{C} := \bigcup_{Z \in \mathcal{X}(\boldsymbol{\mathcal{S}})} \mathcal{C}(Z).$$

Note that the concept c induced by $Z \in \mathcal{X}(\boldsymbol{\mathcal{S}})$ is already induced by S since it consists of the coefficient functions μ and σ of S and of the coefficient functions $\tilde{\mu}$ and $\tilde{\sigma}$ of the shadow price process induces by S from Assumption 2. We choose $c = \frac{\hat{\mu}}{\hat{\sigma}}$ instead of $c = \frac{\tilde{\sigma}}{\tilde{\mu}}$ to have the same domain for the target concepts and the hypotheses. By Assumption 2 and (3.24) we find $c_{\tilde{\sigma}} > 0$ such that

$$\bar{\sigma}\,(y,p,x) = \frac{(1+\gamma)p}{g_\alpha(p,x)}\tilde{\sigma}(y) \geq \tilde{\sigma}(y) \geq c_{\tilde{\sigma}}, \tag{3.27}$$

since $\frac{(1+\gamma)p}{g_\alpha(p,x)} \geq 1$.

Lemma 3.10. *Let $\bar{\mu}$ and $\bar{\sigma}$ be the maps defined in (3.23) induced by some $S \in \boldsymbol{\mathcal{S}}_{LS,b}$ bounded by*

$$|\bar{\mu}(y,p,x)| \leq C_{\bar{\mu}} \quad \textit{and} \quad \bar{\sigma}(y,p,x) \geq c_{\bar{\sigma}} \quad \textit{for all } (y,p,x) \in [1,\bar{s}] \times (0,\infty) \times \mathbb{R}_+.$$

Let $\bar{f}$ be the map with

$$\bar{f} : [1,\bar{s}] \times (0,\infty) \times \mathbb{R} \to (0,\infty), \quad (y,p,x) \mapsto \frac{\bar{\mu}^2\,(y,p,x)}{\bar{\sigma}^2\,(y,p,x)}.$$

Define for fixed $(y,x) \in [1,\bar{s}] \times \mathbb{R}$

$$\bar{f}_{(y,x)} : \mathbb{R} \to (0,\infty), p \mapsto \bar{f}(y,p,x)$$

and for fixed $(y,p) \in [1,\bar{s}] \times \mathbb{R}$

$$\bar{f}_{(y,p)} : \mathbb{R} \to (0,\infty), x \mapsto \bar{f}(y,p,x).$$

The functions $\bar{f}_{(y,x)}$ and $\bar{f}_{(y,p)}$ are Lipschitz continuous.

Proof. First note that

$$\alpha'(x) = 1 \quad \textit{and} \quad \alpha''(x) = 0 \quad \textit{for all } x \in [-1,1]$$

implies

$$\frac{\partial^2 g_\alpha}{\partial^2 x}(p,x) = 0 \quad \textit{for all } (p,x) \in \mathbb{R} \times [-1,1].$$

Since μ, σ and $\hat{a}_1$ are bounded by Assumption 3 we find

$$\begin{aligned} |\bar{\mu}(p,x,y)| &\leq |\mu(p)| + \frac{C_{\hat{a}_1}}{1-\gamma} + \sigma^2(p) + |\mu(p)| + \frac{(1+\gamma)\tilde{\sigma}(y)\sigma(p)}{1-\gamma} + \sigma(p)^2 \\ &\leq 2C_\mu + \frac{C_{\hat{a}_1}}{1-\gamma} + 2C_\sigma^2 + \frac{(1+\gamma)}{1-\gamma}C_{\tilde{\sigma}}C_\sigma =: C_{\bar{\mu}} \end{aligned} \tag{3.28}$$

and

$$\begin{aligned}\bar{\sigma}(p,x,y) &= \sigma(p) + \frac{1}{1+\gamma\alpha(x)}\left[(1+\gamma)\tilde{\sigma}(y) - (1+\gamma\alpha(x))\sigma(p)\right] \\ &= \sigma(p) + \frac{1+\gamma}{1+\gamma\alpha(x)}\tilde{\sigma}(y) - \sigma(p) \\ &\geq \frac{1+\gamma}{1-\gamma}c_{\tilde{\sigma}} =: c_{\bar{\sigma}} \end{aligned} \tag{3.29}$$

Fix $(y,x) \in [1,\bar{s}] \times \mathbb{R}$ and note that $p \mapsto \bar{\mu}(y,p,x)$ is Lipschitz, since for $p,p' \in \mathbb{R}$

$$\begin{aligned}|\bar{\mu}(y,p,x) - \bar{\mu}(y,p',x)| \leq\ & L_\mu|p-p'| + 2C_\sigma L_\sigma|p-p'| + L_\mu|p'-p| \\ &+ \frac{1+\gamma}{1-\gamma}C_{\tilde{\sigma}}L_\sigma|p-p'| + 2C_\sigma L_\sigma|p-p'| + L_\mu|p'-p|\end{aligned}$$

and $p \mapsto \frac{1}{\bar{\sigma}(y,p,x)}$ is constant. Hence $p \mapsto \frac{\bar{\mu}}{\bar{\sigma}}(y,p,x)$ is Lipschitz continuous. By (3.28) and (3.29) both maps are bounded and $p \mapsto \frac{\bar{\mu}^2}{\bar{\sigma}^2}(y,p,x) = \bar{f}_{(y,x)}(p)$ is Lipschitz continuous as the product of bounded and Lipschitz continuous functions. Let us now fix $(y,p) \in [1,\bar{s}] \times \mathbb{R}$ and note that $x \mapsto \bar{\mu}(y,p,x)$ is Lipschitz continuous since,

$$\begin{aligned}|\bar{\mu}(y,p,x) - \bar{\mu}(y,p,x')| &\leq [\tilde{a}_1(y) + \sigma(p)\tilde{\sigma}(y)(1+\gamma)]\left|\frac{1}{1+\gamma\alpha(x)} - \frac{1}{1+\gamma\alpha(x')}\right| \\ &\leq [\tilde{a}_1(y) + \sigma(p)\tilde{\sigma}(y)(1+\gamma)]\left|1+\gamma\alpha(x') - 1 - \gamma\alpha(x)\right| \\ &\leq [\tilde{a}_1(y) + \sigma(p)\tilde{\sigma}(y)(1+\gamma)]\,\gamma L_\alpha\left|x'-x\right| \\ &\leq [C_{\tilde{a}_1} + C_\sigma C_{\tilde{\sigma}}(1+\gamma)]\,\gamma L_\alpha\left|x'-x\right|.\end{aligned}$$

and $x \mapsto \frac{1}{\bar{\sigma}(y,p,x)}$ is Lipschitz continuous by

$$\begin{aligned}\left|\frac{1}{\bar{\sigma}(y,p,x)} - \frac{1}{\bar{\sigma}(y,p,x')}\right| &= \frac{1}{(1+\gamma)\tilde{\sigma}(y)}\left[1+\gamma\alpha(x) - 1 - \gamma\alpha(x')\right] \\ &\leq \frac{1}{(1+\gamma)\tilde{\sigma}(y)}\gamma L_\alpha\left[x-x'\right] \\ &\leq \frac{1}{(1+\gamma)c_{\tilde{\sigma}}}\gamma L_\alpha\left[x-x'\right]\end{aligned}$$

Hence $x \mapsto \frac{\hat{\mu}^2(y,p,x)}{\hat{\sigma}^2(y,p,x)} = \hat{f}_2(x)$ is Lipschitz continuous as the product of bounded and Lipschitz continuos functions. □

For $\Phi_1, \Phi_2 \in \mathcal{N}$ and some ϱ_0-realizations R_1, R_2 denote by $\bar{\mu}_{R_1(\Phi_1)}$ and $\bar{\sigma}_{R_2(\Phi_2)}$ the maps defined by $\bar{\mu}$ and $\bar{\sigma}$ but with

$$\tilde{a}_1(t,Y_t) := R_1(\Phi_1)(t,Y_t)$$

and

$$\tilde{b}_1(t,Y_t) := R_2(\Phi_2)(t,Y_t)$$

instead of (3.18) and (3.19). We finally define the hypothesis set of our learning problem

$$\mathcal{H}_\alpha := \left\{ \frac{\bar{\mu}_{R_1(\Phi_1)}}{\bar{\sigma}_{R_2(\Phi_2)}} : [0,T] \times [1,\bar{s}] \times (0,\infty) \times \mathbb{R} \to \mathbb{R} \mid R_1, R_2 \textit{ realizations of } \Phi_1, \Phi_2 \in \mathcal{N} \right\}.$$

In the following theorem we show that the above hypothesis approximates the target concept arbitrarily well.

Theorem 3.11. *Let $S \in \boldsymbol{\mathcal{S}}_{LS,b}$ satisfy Assumption 3. Set $c_{min} := \{1, c_{\tilde{\sigma}}\}$ for $c_{\tilde{\sigma}}$ defined in (2.29). For every $\epsilon \in (0, c_{min})$ it exists $\Phi_1, \Phi_2 \in \mathcal{N}$ and realizations R_1, R_2 on the compact domain $K := [0,T] \times [1,\bar{s}]$ with*

$$\max\{M_p(\Phi_1), M_p(\Phi_2)\} \leq 12 C^3_{\sigma,\tilde{\mu},\tilde{\sigma}} \epsilon^{-3} \tag{3.30}$$

such that for all $(p,x) \in (0,\infty) \times \mathbb{R}$

$$\sup_{(t,y)\in[0,T]\times K} \left| \frac{\bar{\mu}^2\,(y,p,x)}{\bar{\sigma}^2\,(y,p,x)} - \frac{\bar{\mu}^2_{R_1(\Phi_1)}\,(t,y,p,x)}{\bar{\sigma}^2_{R_1(\Phi_2)}\,(t,y,p,x)} \right| \leq \epsilon \tag{3.31}$$

for some constant $C_{\sigma,\tilde{\mu},\tilde{\sigma}} > 0$.

Proof. Let $\tilde{a}_1$ and $\tilde{b}_1$ be the maps defined in (3.18) and (3.19). Set

$$f_1 : K \to \mathbb{R}, (t,y) \mapsto \tilde{a}_1(y) \quad \textit{and} \quad f_2 : K \to \mathbb{R}, (t,y) \mapsto \tilde{b}_1(y).$$

Towards the application of Theorem 3.8 calculate

$$d + N_L = 3 \quad \textit{and} \quad \textit{diam}(K) = \sqrt{T^2 + \bar{s}^2}.$$

Note that f_1 is Lipschitz continuous by Assumption 3 and f_2 is Lipschitz continuous since $\tilde{\sigma}$ is Lipschitz continuous by Assumption 2. Denote by

$$L_{\tilde{a}_1,\tilde{b}_1} := \max\{L_{\tilde{a}_1}, L_{\tilde{b}_1}\}$$

the maximum over the associated Lipschitz constants. Set further

$$C_{\sigma,\tilde{\mu}} := \max\{C_\sigma, C_{\tilde{\mu}}\}.$$

According to Theorem 3.8, for every $\epsilon \in (0, c_{min})$ we find $\Phi_1, \Phi_2 \in \mathcal{N}$ with realizations R_1, R_2 and bounded number of parameters

$$\max\{M(\Phi_1), M(\Phi_2)\} \leq 12 \left(\frac{4(C_{\sigma,\tilde{\mu}}+1)^2 L_{\tilde{\mu},\tilde{\sigma}} C_K \sqrt{T^2+\bar{s}^2}}{(1-\gamma)c^2_{min}} \right)^3 \left(\frac{1}{\epsilon}\right)^3 \tag{3.32}$$

such that

$$\sup_{(t,s)\in K} |\tilde{a}_1(y) - R_1(\Phi_1)(t,y)| \leq \frac{\epsilon}{8} \frac{(1-\gamma)c^2_{min}}{(C_{\sigma,\tilde{\mu}}+1)^2} \tag{3.33}$$

and

$$\sup_{(t,s)\in K} \left|\tilde{b}_1(y) - R_2(\Phi_1)(t,y)\right| \leq \frac{\epsilon}{8} \frac{(1-\gamma)c^2_{min}}{(C_{\tilde{\mu}}+1)^2}. \tag{3.34}$$

Note that

$$\alpha'(x) = 1 \quad \textit{and} \quad \alpha''(x) = 0 \quad \textit{for all } x \in [-1,1]$$

implies

$$\frac{\partial^2 g_\alpha}{\partial^2 x}(p,x) = 0 \quad \textit{for all } (p,x) \in \mathbb{R} \times [-1,1].$$

For fixed $(t,y,p,x) \in [0,T] \times K \times (0,\infty)\mathbb{R}$ observe

$$\begin{aligned}\left|\bar{\mu}\,(y,p,x) - \bar{\mu}_{R_1(\Phi_1)}\,(t,y,p,x)\right| &\leq \frac{1}{1-\gamma}\left[\frac{\epsilon}{8}\frac{(1-\gamma)c_{min}^2}{(C_{\sigma,\bar{\mu}}+1)^2} + C_\sigma \frac{\epsilon}{8}\frac{(1-\gamma)c_{min}^2}{(C_{\sigma,\bar{\mu}}+1)^2}\right] \\ &\leq \frac{\epsilon}{4}\frac{c_{min}^2}{(C_{\sigma,\bar{\mu}}+1)} \end{aligned} \tag{3.35}$$

and

$$\left|\bar{\sigma}\,(y,p,x) - \bar{\sigma}_{R_2(\Phi_2)}\,(t,y,p,x)\right| \leq \frac{1}{1-\gamma}\frac{\epsilon}{8}\frac{(1-\gamma)c_{min}^2}{(C_{\sigma,\bar{\mu}}+1)^2} = \frac{\epsilon}{8}\frac{c_{min}^2}{(C_{\sigma,\bar{\mu}}+1)^2}. \tag{3.36}$$

Set

$$I_1(t,y,p,x) := \left|\frac{\bar{\mu}^2(y,p,x)}{\bar{\sigma}^2(y,p,x)} - \frac{\bar{\mu}^2_{R_1(\Phi_1)}(t,y,p,x)}{\bar{\sigma}^2(y,p,x)}\right|.$$

By (3.35)

$$|\bar{\mu}_{R_1(\Phi_1)}\,(t,y,p,x)\,| \leq C_{\bar{\mu}} + \frac{\epsilon}{4}\frac{(1-\lambda)c_{min}^2}{(C_{\sigma,\bar{\mu}}+1)} \leq C_{\bar{\mu}}+1 \tag{3.37}$$

and hence

$$\begin{aligned} I_1(t,y,p,x) &\overset{(3.27),(3.35)}{\leq} \frac{2(C_{\bar{\mu}}+1)}{c_{min}^2}\left|\bar{\mu}(y,p,x) - \bar{\mu}_{R_1(\Phi_1)}(t,y,p,x)\right| \\ &\overset{(3.35)}{\leq} \frac{2(C_{\bar{\mu}}+1)}{c_{min}^2}\frac{\epsilon}{4}\frac{c_{min}^2}{(C_{\sigma,\bar{\mu}}+1)} \leq \frac{\epsilon}{2}. \end{aligned} \tag{3.38}$$

Define

$$I_2(t,y,p,x)_2 := \left|\frac{\bar{\mu}^2_{R_1(\Phi_1)}(t,y,p,x)}{\bar{\sigma}^2(y,p,x)} - \frac{\bar{\mu}^2_{R_1(\Phi_1)}(t,y,p,x)}{\bar{\sigma}^2_{R_2(\Phi_2)}(t,y,p,x)}\right|.$$

If $\bar{\sigma}_{R_2(\Phi_2)}(t,y,p,x) - \bar{\sigma}(y,p,x) < 0$ we have

$$\bar{\sigma}_{R_2(\Phi_2)}(t,y,p,x) > \bar{\sigma}(y,p,x) \geq c_{\bar{\sigma}} \geq c_{min}$$

Otherwise assuming

$$\bar{\sigma}_{R_2(\Phi_2)}(t,y,p,x) < c_{min} - \frac{\epsilon}{2}$$

contradicts to (3.36) by

$$c_{min} - \frac{\epsilon}{2} < c_{min} - \frac{2(C_{\bar{\mu}}+1)}{c_{min}^2}\frac{\epsilon}{4}\frac{c_{min}^2}{(C_{\sigma,\bar{\mu}}+1)^2}.$$

Hence

$$|\bar{\sigma}_{R_2(\Phi_2)}(t,y,p,x)| = \bar{\sigma}_{R_2(\Phi_2)}(t,y,p,x) \geq c_{min} - \frac{\epsilon}{2} \geq c_{min} - \frac{c_{min}}{2} = \frac{c_{min}}{2}. \tag{3.39}$$

Using (3.37),(3.39) and (3.36) we observe

$$
\begin{aligned}
I_2(t,y,p,x) &\leq \left[\frac{(C_{\bar{\mu}}+1)^2}{\bar{\sigma}^2_{R_2(\Phi_2)}(t,y,p,x)} + \frac{(C_{\bar{\mu}}+1)^2}{\bar{\sigma}^2(y,p,x)}\right] \left|\bar{\sigma}(y,p,x) - \bar{\sigma}_{R_2(\Phi_2)}(t,y,p,x)\right| \\
&\leq \left[\frac{(C_{\bar{\mu}}+1)^2}{c^2_{min}} + \frac{(C_{\bar{\mu}}+1)^2}{c^2_{min}}\right] \left|\bar{\sigma}(y,p,x) - \bar{\sigma}_{R_2(\Phi_2)}(t,y,p,x)\right| \\
&\leq \frac{2(C_{\bar{\mu}}+1)^2}{c^2_{min}} \frac{\epsilon}{8} \frac{c^2_{min}}{(C_{\sigma,\bar{\mu}}+1)^2} \leq \frac{\epsilon}{4}
\end{aligned} \tag{3.40}
$$

Hence for Φ_1 and Φ_2 and some realizations R_1 and R_2 we have for all $(p,x) \in (0,\infty) \times \mathbb{R}$

$$
\begin{aligned}
\sup_{(t,y)\in[0,T]\times K} \left|\frac{\bar{\mu}^2(y,p,x)}{\bar{\sigma}^2(y,p,x)} - \frac{\bar{\mu}^2_{R_1(\Phi_1)}(t,y,p,x)}{\bar{\sigma}^2_{R_2(\Phi_2)}(t,y,p,x)}\right| &\leq I_1(t,y,p,x) + I_2(t,y,p,x) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{4} \leq \epsilon
\end{aligned}
$$

and by (3.32)

$$
\max\{M_p(\Phi_1), M_p(\Phi_2)\} \leq 12 \left(\underbrace{\frac{4(C_{\sigma,\bar{\mu}}+1)^2 L_{\bar{\mu},\bar{\sigma}} C_K \sqrt{T^2+\bar{s}^2}}{(1-\gamma)c^2_{min}}}_{=:C_{\sigma,\bar{\mu},\bar{\sigma}}}\right)^3 \left(\frac{1}{\epsilon}\right)^3.
$$

□

For the loss function of our learning problem set

$$
\mathcal{L} : \mathbb{R} \times \mathbb{R} \to [0,\infty], (y,y') \mapsto y^2
$$

with associated generalization error for $Z \in \mathcal{X}(\boldsymbol{\mathcal{S}})$ and $h \in \mathcal{H}_\alpha$

$$
R(h) = \mathbb{E}\left[\int_0^T \mathcal{L}(h(t,Z),c(Z))dt\right] = \mathbb{E}\left[\int_0^T h^2(t,Z_t)dt\right]
$$

First note that the loss function does not depend on the target concept since we face an unsupervised learning problem. The following Lemma is implied by equation (2.9) in Czichowsky et al. [11] and shows that the hypothesis set $\mathcal{H}_\alpha$ approximates the generalization error of the target concept from above.

Lemma 3.12. *For fixed $Z \in \mathcal{X}(\boldsymbol{\mathcal{S}})$ with associated target concept*

$$
c := \frac{\bar{\mu}}{\bar{\sigma}}
$$

we find for all $h \in \mathcal{H}_\alpha$

$$
R(h) \geq R(c) = \mathbb{E}\left[\int_0^T c^2(Z_t)dt\right]. \tag{3.41}
$$

Proof. Any $h = \frac{h_1}{h_2} \in \mathcal{H}_\alpha$ implies by construction a financial model $(B, \bar{S}^\alpha)$ with

$$
(1-\lambda)S \leq \bar{S}^\alpha \leq S.
$$

The associated value function in this frictionless model is by (B.5) for initial endowment $x > 0$

$$\tilde{u}(x) = \sup_{\tilde{V}_T \in \tilde{C}(x)} \mathbb{E}[\ln(\tilde{V}_T)] = \frac{1}{2}\mathbb{E}\left[\int_0^T h^2(t, Z_t)dt\right] + \ln\left(\frac{x}{T+1}\right).$$

Let φ be the optimal trading strategy in (B, S) with transaction costs from Definition 4 of the shadow price process. In $(B, \bar{S}^\alpha)$ the trader sells at $\bar{S}^\alpha \geq (1-\lambda)S$ and buys at $\bar{S}^\alpha \leq S$. Hence following φ in $(B, \bar{S}^\alpha)$ without transaction costs results in

$$\tilde{u}(x) \geq \mathbb{E}\left[\ln\left(\varphi_T^0 + \varphi_T^1 \bar{S}_T^\alpha\right)\right] \geq V_T^{liq}(\varphi)$$

where we used in the first inequality that $\tilde{u}$ is optimal in $(B, \bar{S}^\alpha)$. Moreover by (2.10) and the construction of the target concept c

$$V_T^{liq}(\varphi) = \frac{1}{2}\mathbb{E}\left[\int_0^T c^2(Z_t)dt\right] + \ln\left(\frac{x}{T+1}\right)$$

and hence

$$R(h) = \mathbb{E}\left[\int_0^T h^2(t, Z_t)dt\right] \geq \mathbb{E}\left[\int_0^T c^2(Y_t)dt\right] = R(c).$$

□

By Lemma 3.12 and Theorem 3.11 we conclude that the hypothesis set $\mathcal{H}_\alpha$ approximates the target concept arbitrary well.

Corollary 3.13. *For fixed $Z \in \mathcal{X}(\boldsymbol{\mathcal{S}})$ with associated target concept*

$$c := \frac{\bar{\mu}}{\bar{\sigma}}$$

we have

$$\hat{R}^\star = \inf_{h \in \mathcal{H}_\alpha} R(h) = \mathbb{E}\left[\int_0^T c^2(Z_t)dt\right] = R(c). \tag{3.42}$$

and hence

$$\mathcal{E}\left(\mathcal{H}_\alpha\right) = 0.$$

Proof. Since $Y_t \in [1, \bar{s}]$ *a.s.*, it exists for every $n \in \mathbb{N}$ by Theorem 3.11 a hypothesis $h_n \in \mathcal{H}_\alpha$ such that

$$\left|c^2(Z_t) - h_n^2(t, Z_t)\right| \leq \frac{1}{n} \quad a.s.$$

Therefore

$$|R(c) - R(h_n)| \leq \mathbb{E}\left[\int_0^T \left|c^2\left(Z_t\right) - h_n^2\left(t, Z_t\right)\right| dt\right] \leq \frac{T}{n}$$

and we find a sequence $(h_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ such that

$$\lim_{n \to \infty} R(h_n) = R(c).$$

By Lemma 3.12 we conclude

$$\inf_{h \in \mathcal{H}_\alpha} R(h) = R(c).$$

□

By the above Corollary we have for every $h \in \mathcal{H}$

$$\left|R(h) - \hat{R}^\star\right| = |R(h) - R(c)|\,.$$

For that reason we replace the hypothesis in the discretization error and in the observation error by the target concept and decompose the sources of error for $h \in \mathcal{H}$ into

$$\left|\hat{R}_N(h) - R(c)\right| = \underbrace{\left|\hat{R}_N(h) - \hat{R}_N(c)\right|}_{\textbf{estimation error}} + \underbrace{\mathcal{E}_O(c)}_{\textbf{observation error}} + \underbrace{\mathcal{E}_N(c)}_{\textbf{discretization error}}\,.$$

We begin with the derivation of an upper bound for the discretization error.

3.3.3 Analysis of the Discretization Error

To bound the discretization error

$$\mathcal{E}_N(c) := |R_N(c) - R(c)|$$

with

$$R_N(c) := \frac{T}{N}\sum_{n=0}^{N-1}\mathbb{E}\left[c^2(Z_{t_n})\right]$$

observe by (3.26)

$$R_N(c) = \frac{T}{N}\sum_{n=0}^{N-1}\mathbb{E}\left[f(Y_{t_n})\right].$$

This observations enables uss to use f instead of c^2 for the derivation of a bound for $\mathcal{E}_N(c)$. We first calculate a bound four the expected difference of Y_t and Y_{t_n}.

Lemma 3.14. *For $Y = \frac{S}{m}$ with $S \in \boldsymbol{\mathcal{S}}_{LS}$ and all $t \in [t_n, t_{n+1}] \subset [0, T]$ is exists a constant $c_1 > 0$ such that for all numbers $N \in \mathbb{N}$ we find*

$$\mathbb{E}\left[|Y_t - Y_{t_n}|\right] \leq \frac{\bar{s}c_1}{\sqrt{N}} \tag{3.43}$$

Proof. By Itô

$$\begin{aligned} d\left(\frac{S_t}{m_t}\right) &= \frac{1}{m_t}dS_t + S_t d\left(\frac{1}{m_t}\right) + d\left\langle S, \frac{1}{m_t}\right\rangle_t \\ &= \mu(S_t)\frac{S_t}{m_t}dt + \sigma(S_t)\frac{S_t}{m_t}dW_t + S_t d\left(\frac{1}{m_t}\right) + d\left\langle S, \frac{1}{m_t}\right\rangle_t \\ &= \mu(S_t)\frac{S_t}{m_t}dt + \sigma(S_t)\frac{S_t}{m_t}dW_t + \frac{S_t}{m_t}d\ln(m_t) \end{aligned}$$

and hence for $t \in [t_n, t_{n+1}]$

$$\left|\frac{S_t}{m_t} - \frac{S_{t_n}}{m_{t_n}}\right| \leq \left|\int_{t_n}^t \mu(S_t)\frac{S_u}{m_u}du\right| + \left|\int_{t_n}^t \sigma(S_t)\frac{S_u}{m_u}dW_u\right| + \left|\int_{t_n}^t \frac{S_u}{m_u}d\ln(m_u)\right| = I_1 + I_2 + I_3.$$

Using the Lipschitz continuity of μ and (2.17) leads to

$$\begin{aligned} I_1 &\leq \bar{s}\mathbb{E}\left[\int_{t_n}^{t} |\mu(S_u)|\,du\right] \leq \bar{s}\mathbb{E}\left[\int_{t_n}^{t} |\mu(S_u)-\mu(0)|\,du + |\mu(0)|(t-t_n)\right] \\ &= \bar{s}\mathbb{E}\left[K\int_{t_n}^{t} S_u du + |\mu(0)|(t-t_n)\right] \leq \bar{s}K\mathbb{E}\left[\int_{t_n}^{t} S_u^2 du\right] + |\mu(0)|(t-t_n) \\ &\leq \bar{s}(KC_S + |\mu(0)|)(t-t_n). \end{aligned} \tag{3.44}$$

Towards a bound for I_2 note that

$$M_u := \int_{t_n}^{t_n+u} \sigma(S_r)\left(\frac{S_r}{m_r}\right) dW_r$$

is a continuous local martingale with $M_0 = 0$. By the Burkholder-Davis-Gundy inequality (BDG) we find a constant $C_{BDG} > 0$ such that for all $t > 0$

$$\mathbb{E}\left[\sup_{t_n \leq u \leq t} |M_u|\right] \leq \bar{s}C_{BDG}\mathbb{E}\left[\left(\int_{t_n}^{t} \sigma^2(S_r)dr\right)^{\frac{1}{2}}\right].$$

Hence using Jensens-inequality and the growth bound of σ leads to

$$I_2 \leq \bar{s}C_{BDG}\left(\mathbb{E}\left[\int_{t_n}^{t} \sigma^2(S_r)dr\right]\right)^{\frac{1}{2}} \leq \bar{s}C_{BDG}K\left(\mathbb{E}\left[\int_{t_n}^{t} (1+S_r^2)dr\right]\right)^{\frac{1}{2}}.$$

Therefore by (2.17)

$$I_2 \leq \bar{s}C_{BDG}K\left((t-t_n) + C_S(t-t_n)\right)^{\frac{1}{2}} \leq \bar{s}C_{BDG}K(1+C_S)\sqrt{t-t_n}. \tag{3.45}$$

To derive a bound for I_3 we define

$$\mathcal{I}(\omega) := \left\{[\varrho_{k-1}(\omega),\sigma_k(\omega)] \cap [t_n,t] \,\middle|\, k \in \mathbb{N},\ [\varrho_{k-1}(\omega),\sigma_k(\omega)] \cap [t_n,t] \neq \emptyset\right\}$$

$$\mathcal{J}(\omega) := \left\{[\sigma_k(\omega),\varrho_k(\omega)] \cap [t_n,t] \,\middle|\, k \in \mathbb{N},\ [\sigma_k(\omega),\varrho_k(\omega)] \cap [t_n,t] \neq \emptyset\right\}$$

and

$$\begin{aligned} \underline{X}_u &:= \inf_{\varrho \leq v \leq u}\left\{\left(\int_0^v \mu(S_r) - \frac{1}{2}\sigma^2(S_r)\right)dr + \int_0^v \sigma(S_r)dW_r\right\}, \\ \overline{X}_u &:= \sup_{\sigma \leq v \leq u}\left\{\left(\int_0^v \mu(S_r) - \frac{1}{2}\sigma^2(S_r)\right)dr + \int_0^v \sigma(S_r)dW_r\right\} \end{aligned}$$

to calculate

$$
\begin{aligned}
I_3 &\leq \mathbb{E}\left[\left|\sum_{[\varrho,\sigma]\in\mathcal{I}} \int_{\varrho}^{\sigma} \frac{S_u}{m_u} d\ln\left(\inf_{\varrho\leq v\leq u} S_v\right)\right|\right] \\
&+ \mathbb{E}\left[\left|\sum_{[\sigma,\varrho]\in\mathcal{J}} \int_{\sigma}^{\varrho} \frac{\bar{s}S_u}{m_u} d\ln\left(\frac{1}{\bar{s}}\sup_{\sigma\leq v\leq u} S_v\right)\right|\right] \\
&= \mathbb{E}\left[\left|\sum_{[\varrho,\sigma]\in\mathcal{I}} \int_{\varrho}^{\sigma} \frac{S_u}{m_u} d\underline{X}_u\right|\right] + \mathbb{E}\left[\left|\sum_{[\sigma,\varrho]\in\mathcal{J}} \int_{\sigma}^{\varrho} \frac{S_u}{m_u} d\overline{X}_u\right|\right] \\
&\leq \bar{s}\mathbb{E}\left[\sum_{[\varrho,\sigma]\in\mathcal{I}} \int_{\varrho}^{\sigma} |d\underline{X}_u|\right] + \bar{s}\mathbb{E}\left[\sum_{[\sigma,\varrho]\in\mathcal{J}} \int_{\sigma}^{\varrho} |d\overline{X}_u|\right] \\
&\leq \bar{s}\mathbb{E}\left[\int_{t_n}^{t} |d\underline{X}_u|\right] + \bar{s}\mathbb{E}\left[\int_{t_n}^{t} |d\overline{X}_u|\right] \\
&= \bar{s}\mathbb{E}\left[|\underline{X}_t - \underline{X}_{t_n}|\right] + \bar{s}\mathbb{E}\left[|\overline{X}_t - \overline{X}_{t_n}|\right] && (3.46) \\
&= \bar{s}\mathbb{E}\left[\underline{X}_{t_n} - \underline{X}_t\right] + \bar{s}\mathbb{E}\left[\overline{X}_t - \overline{X}_{t_n}\right] && (3.47)
\end{aligned}
$$

Note first

$$
\underline{X}_{t_n} - \underline{X}_t = \sup_{0\leq u\leq t-t_n} \int_{t_n}^{t_n+u} \left(\mu(S_r) - \frac{1}{2}\sigma^2(S_r)\right) dr + \int_{t_n}^{t_n+u} \sigma(S_r)W_r \quad (3.48)
$$

and hence by the Lipschitz continuity of μ

$$
\underline{X}_{t_n} - \underline{X}_t \leq K\int_{t_n}^{t_{n+1}} S_r^2 dr + |\mu(0)|(t-t_n) + \sup_{0\leq u\leq t-t_n} \int_{t_n}^{t_n+u} \sigma(S_r)W_r
$$

Using (2.17), the BDG inequality and Jensens inequality leads to

$$
\mathbb{E}[\underline{X}_{t_n} - \underline{X}_t] \leq (KC_s + |\mu(0))|(t-t_n) + \tilde{C}_{BDG}\left(\mathbb{E}\left[\left(\int_{t_n}^{t_{n+1}} \sigma^2(S_r)dr\right)\right]\right)^{\frac{1}{2}}.
$$

Since

$$
\mathbb{E}\left[\int_{t_n}^{t_{n+1}} \sigma^2(S_r)dr\right] \leq K\mathbb{E}\left[\int_{t_n}^{t_{n+1}} (1+S_r^2)dr\right] \leq K(1+C_s)(t-t_n)
$$

by the growth bound of σ we arrive at

$$
\begin{aligned}
\mathbb{E}[\underline{X}_{t_n} - \underline{X}_t] &\leq (KC_s + |\mu(0)|(t-t_n) + \tilde{C}_{BDG}\sqrt{K(1+C_s)(t-t_n)} \\
&\leq \left(KC_s + |\mu(0)| + \tilde{C}_{BDG}\sqrt{K(1+C_s)}\right)\sqrt{(t-t_n)}
\end{aligned}
$$

and similar using (4)

$$
\mathbb{E}\left[\underline{X}_{t_n} - \underline{X}_t\right] \leq \left(KC_s + |\mu(0)| + \tilde{C}_{BDG}\sqrt{K(1+C_s)}\right)\sqrt{(t-t_n)}.
$$

Therefore by (*A*.9)

$$I_3 \leq 2\bar{s}\left(KC_s + |\mu(0)| + \tilde{C}_{BDG}\sqrt{K(1+C_s)}\right)\sqrt{(t-t_n)}. \tag{3.49}$$

With (*A*.8),(3.45) and (3.49) we find in total

$$I_1 + I_2 + I_3 \leq \bar{s}\left\{3KC_S + |3\mu(0)| + C_{BDG}\sqrt{K(1+C_S)}\left[\sqrt{K(1+C_S)}+2\right]\right\}\frac{1}{\sqrt{N}}.$$

□

As a consequence of (3.43) we find that the discretization error is of order $N^{-\frac{1}{2}}$. To proof this bound, we deploy the Lipschitz continuity of f that holds by Assumption 2 for every $S \in \boldsymbol{\mathcal{S}}_{LS}$.

Lemma 3.15. *For $Y = \frac{S}{m}$ with $S \in \boldsymbol{\mathcal{S}}_{LS}$ and all $N \in \mathbb{N}$ we have*

$$\mathcal{E}_N(c) = \left|\frac{T}{N}\sum_{n=0}^{N-1}\mathbb{E}\left[f(Y_{t_n})\right] - \mathbb{E}\left[\int_0^T f(Y_t)dt\right]\right| \leq \frac{L_f T \bar{s} c_1}{\sqrt{N}} \tag{3.50}$$

for the constant $c_1 > 0$ derived in Lemma 3.14 and in particular

$$\lim_{N\to\infty}\mathcal{E}_N(c) = \lim_{N\to\infty}|R_N(c) - R(c)| = 0.$$

for the discretization error.

Proof. By linearity of the expected value and of the integral using the triangle inequality

$$\begin{aligned}\mathcal{E}_N(c) = |R_N(c) - R(c)| &= \left|\sum_{n=0}^{N-1}\mathbb{E}\left[\int_{t_n}^{t_{n+1}} f(Y_t) - f(Y_{t_n})dt\right]\right| \\ &\leq \sum_{n=0}^{N-1}\mathbb{E}\left[\int_{t_n}^{t_{n+1}} |f(Y_t) - f(Y_{t_n})|dt\right].\end{aligned}$$

Since f is Lipschitz continuous we find with Fubini's Theorem [7]

$$\mathcal{E}_N(c) \leq L_f\sum_{n=0}^{N-1}\mathbb{E}\left[\int_{t_n}^{t_{n+1}}|Y_t - Y_{t_n}|dt\right] = L_f\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\mathbb{E}\left[|Y_t - Y_{t_n}|\right]dt$$

Hence by (3.43)

$$\mathcal{E}_N(c) \leq L_f\sum_{n=0}^{N-1}\frac{\bar{s}c_1}{\sqrt{N}}(t_{n+1}-t_n) = \frac{L_f T\bar{s}c_1}{\sqrt{N}}.$$

□

So far we showed

$$\left|\hat{R}_N(h) - R(c)\right| = \left|\hat{R}_N(h) - \hat{R}_N(c)\right| + \mathcal{E}_O(c) + c_1\bar{s}L_f T N^{-\frac{1}{2}}$$

turning next to an upper bound for the error of observation.

3.3.4 Approximation of the Data Generating Process

To approximate the data generating process $Y \in \mathcal{X}(\boldsymbol{\mathcal{S}})$ we need to approximate the Markov-processes $S, P \in \boldsymbol{\mathcal{S}}_{LS,b}$, the hidden process $X \in \boldsymbol{\mathcal{S}}$ and the relative running minimum process m of S. For the simulation of the Markov-process S we use the Euler-Maruyama scheme [1], [32]:

Definition 21 (Euler approximation)**.** *Let $S \in \boldsymbol{\mathcal{S}}^{MP}$ be a Markov-process satisfying*

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t, \quad S_0 = s_0.$$

Fix an equidistant time grid $0 = t_0 < t_1 < ... < t_N = T$ on $[0, T]$, set $\Delta t := \frac{T}{N}$ and

$$\Delta W_n := W_{t_{n+1}} - W_{t_n} \sim \mathcal{N}(0, \Delta t) \quad \text{for } 0 \leq n \leq N.$$

Define the recursive Euler-Maruyama scheme

$$\bar{S}_{t_{n+1}} := \bar{S}_{t_n} + \mu\left(\bar{S}_{t_n}\right)\Delta t + \sigma\left(\bar{S}_{t_n}\right)\Delta W_n, \quad \bar{S}_0 = s_0.$$

We refer to the discrete process $\bar{S}_N = (\bar{S}_{t_{n+1}})_{0 \leq 1 \leq N}$ as the Euler approximation of S.

Remark 3.16. *Note that the Euler approximation is a discrete time Markov chain taking values in $\mathbb{R}$ with normal distributed transition probabilites [32]*

$$p_n(x, \cdot) = \mathcal{N}\left(\Delta t\mu(x), \Delta t^2\sigma^2(x)\right).$$

Note further that this transition probability depends on the current state $x \in \mathbb{R}$ of the Markov chain resulting for every n in uncountable many transition probabilites depending on $\mu(x)$ and $\sigma(x)$. Hence, the estimation of the unknown $\tilde{\mu}$ and $\tilde{\sigma}$ associated to the discretized shadow price process $\tilde{S}_N$ is the estimation of uncountable many normal distributed transition probabilites of a time discrete Markov chain.

An Euler approximation provides an error of order $N^{-\frac{1}{2}}$ [1].

Theorem 3.17. *Let $S \in \boldsymbol{\mathcal{S}}^{MP}$ be a Markov-process and $\bar{S}_N = (\bar{S}_{t_n})_{0 \leq 1 \leq N}$ the Euler approximation of S. It exists a constant C_E such that*

$$\mathbb{E}\left[\sup_{1 \leq m \leq N}\left|(S_{t_n} - \bar{S}_{t_n})^2\right|\right] \leq \frac{C_E T}{N} \tag{3.51}$$

and hence by Hölders inequality

$$\mathbb{E}\left[\sup_{1 \leq m \leq N}\left|S_{t_n} - \bar{S}_{t_n}\right|\right] \leq \frac{C_E T}{\sqrt{N}}.$$

Lemma 3.18. *Fix $Z = (Y, P, X) \in \mathcal{X}(\boldsymbol{\mathcal{S}})$ with associated target concept*

$$c : \frac{\bar{\mu}}{\bar{\sigma}} \quad \text{and} \quad \hat{f} := c^2.$$

For the Euler approximation $\bar{P}_N$ of P and the Euler approximation $\bar{X}_N$ of X

$$\mathbb{E}\left[\left|\bar{f}(y, P_{t_n}, X_{t_n}) - \bar{f}(y, \bar{P}_{t_n}, \bar{X}_{t_n})\right|\right] \leq \frac{C_{E,\bar{f}} T}{\sqrt{N}}.$$

for a constant $C_{E,\bar{f}} > 0$ that does not depend on $y \in [1.\bar{s}]$.

Proof. By Lemma 3.10 and its proof we know that $p \mapsto \bar{f}(y,p,x)$ and $x \mapsto \bar{f}(y,p,x)$ are Lipschitz continuous with Lipschitz constants $L_{\bar{f}_1}$ and $L_{\bar{f}_2}$ that do not depend on (y,x) and (y,p). Hence, Theorem 3.17 yields by the triangle inequality

$$I_1 := \mathbb{E}\left[|\bar{f}(y,P_{t_n},X_{t_n}) - \bar{f}(y,\bar{P}_{t_n},X_{t_n})|\right] \leq L_{\bar{f}_1}\mathbb{E}\left[|P_{t_n} - \bar{P}_{t_n}|\right] \leq \frac{L_{\bar{f}_1} C_{E_1} T}{\sqrt{N}}$$

and

$$I_2 := \mathbb{E}\left[|\bar{f}(y,\bar{P}_{t_n},X_{t_n}) - \bar{f}(y,\bar{P}_{t_n},\bar{X}_{t_n})|\right] \leq L_{\bar{f}_2}\mathbb{E}\left[|X_{t_n} - \bar{X}_{t_n}|\right] \leq \frac{L_{\bar{f}_2} C_{E_2} T}{\sqrt{N}}.$$

In total again by the triangle inequality

$$\mathbb{E}\left[|\bar{f}(y,P_{t_n},X_{t_n}) - \bar{f}(y,\bar{P}_{t_n},\bar{X}_{t_n})|\right] \leq I_1 + I_2 \leq \underbrace{(L_{\bar{f}_1} C_{E_1} + L_{\bar{f}_2} C_{E_2})}_{:=C_{E,\bar{f}}} \frac{T}{\sqrt{N}}.$$

□

The approximation of m provides two difficulties:

1) $\bar{s}$ is unknown.

2) the stopping times $(\tau_k)_{k\in\mathbb{N}}$ and $(\varrho_k)_{k\in\mathbb{N}}$ are unknown.

Note that by definition of g we find $\bar{s} > \frac{1}{1-\lambda}$. For that reason we add $\varrho_0\left(-\bar{s} + \frac{1}{1-\lambda}\right)$ to the empirical loss as defined in (3.58). To address the second difficulty we first simulate the minimum and the maximum of S on $[t_n, t_{n+1}]$ as described in Appendix D, using the approach of Becker [3] to simulate the final, the minimal and the maximal value of Brownian motion. With these realizations at hand, we estimate the values of $(m_{t_n})_{1\leq n\leq N}$ by Algorithm 1.
In the following we assume the existence of a constant $C_{\bar{m}} > 0$

$$\mathbb{E}\left[\left(\frac{1}{m_{t_n}} - \frac{1}{\bar{m}_{t_n}}\right)^2\right] \leq \frac{C_{\bar{m}}}{N^{\frac{\alpha}{2}}} \tag{3.52}$$

and set

$$C_2(\bar{m}) := \sqrt{\max_{1\leq n\leq N} \mathbb{E}[\bar{m}_{t_n}^{-2}]}.$$

Under this assumption we derive an upper bound on the expected error of observation on a single time step.

Lemma 3.19. *For $Y = \frac{S}{m}$ with $S \in \boldsymbol{\mathcal{S}}_{LS}$ let $\bar{S}_N$ be the Euler approximation of S and $\bar{m}$ an approximation of m satisfying (3.52). It exists a constant $c_2 > 0$ such that*

$$\mathbb{E}\left[\left|f\left(\frac{S_{t_n}}{m_{t_n}}\right) - f\left(\frac{\bar{S}_{t_n}}{\bar{m}_{t_n}}\right)\right|\right] \leq \frac{c_2 L_f}{N^{\frac{\alpha}{2}}} \quad \textit{for all } N \in \mathbb{N}$$

Proof. By the Lipschitz-continuity of f

$$\mathbb{E}\left[\left|f\left(\frac{S_{t_n}}{m_{t_n}}\right) - f\left(\frac{\bar{S}_{t_n}}{\bar{m}_{t_n}}\right)\right|\right] \leq L_f\mathbb{E}\left[\left|\frac{S_{t_n}}{m_{t_n}} - \frac{\bar{S}_{t_n}}{\bar{m}_{t_n}}\right|\right]$$

Algorithm 1 Approximation of m from Definition 8

Input: A realization of the triplet $(\bar{S}_N, A_N, B_N)$ on the equidistant time grid $0 = t_0 < t_1 < ... < t_N$ of $[0, T]$ with $N \in \mathbb{N}$ time steps as defined in (D.2) and an estimation $\hat{s} > \frac{1}{1-\lambda}$ for $\bar{s}$.
Output: An approximation $\hat{m}_N$ of the discrete process $m_N = (m_{t_n})_{0 \le n \le N}$.

Set $\hat{m}_{t_0} \leftarrow \bar{S}_{t_0}$; $currentstate(1) \leftarrow minimum$ and $priorstate(1) \leftarrow minimum$
for $i = 1, ..., N$ **do**
    if $currentstate(i)$ **is** $mininum$ **then**
        $priorstate(i+1) \leftarrow mininum$ and $\hat{m}_{t_i} \leftarrow A_{t_i}$
        if $priorstate(i)$ **is** $mininum$ **then**
            $\hat{m}_{t_i} \leftarrow \arg\min \left\{\hat{m}_{t_i}, \hat{m}_{t_{i-1}}\right\}$
        $currentstate(i+1) \leftarrow minimum$
        if $\frac{\bar{S}_{t_n}}{m_{t_i}} \ge \hat{s}$ **then**
            $currentstate(i+1) \leftarrow maximum$
    else
        $priorstate(i+1) \leftarrow maximum$ and $\hat{m}_{t_i} \leftarrow \frac{B_{t_i}}{\hat{s}}$
        if $priorstate$ **is** $maximum$ **then**
            $\hat{m}_{t_i} \leftarrow \arg\max \left\{\hat{m}_{t_i}, \hat{m}_{t_{i-1}}\right\}$
        $currentstate(i+1) \leftarrow maximum$
        if $\frac{\bar{S}_{t_n}}{\hat{s} m_{t_i}} \le \frac{1}{\hat{s}}$ **then**
            $currentstate(i+1) \leftarrow minimum$

and with the triangle inequality

$$\mathbb{E}\left[\left|\frac{S_{t_n}}{m_{t_n}} - \frac{\bar{S}_{t_n}}{\bar{m}_{t_n}}\right|\right] = \underbrace{\mathbb{E}\left[\left|\frac{S_{t_n}}{m_{t_n}} - \frac{S_{t_n}}{\bar{m}_{t_n}}\right|\right]}_{=:I_1} + \underbrace{\mathbb{E}\left[\left|\frac{S_{t_n}}{\bar{m}_{t_n}} - \frac{\bar{S}_{t_n}}{\bar{m}_{t_n}}\right|\right]}_{=:I_2}.$$

With Höders inequality

$$I_1 \le \sqrt{\mathbb{E}\left[S_{t_n}^2\right] \mathbb{E}\left[\left(\frac{1}{m_{t_n}} - \frac{1}{\bar{m}_{t_n}}\right)^2\right]} \overset{(2.4)\,(3.52)}{\le} \sqrt{C_S} C_{\bar{m}} \frac{1}{N^{\frac{\alpha}{2}}}$$

and

$$I_2 \le \sqrt{\mathbb{E}\left[\bar{m}_{t_n}^{-2}\right] \mathbb{E}\left[\left(S_{t_n} - \bar{S}_{t_n}\right)^2\right]} \overset{(3.51)}{\le} C_2(\bar{m}) \sqrt{C_E T} \frac{1}{\sqrt{N}}.$$

In summary

$$\begin{aligned} \mathbb{E}\left[\left| f\left(\frac{S_{t_n}}{m_{t_n}}\right) - f\left(\frac{\bar{S}_{t_n}}{\bar{m}_{t_n}}\right)\right|\right] &\le L_f(I_1 + I_2) \le L_f\left(\sqrt{C_S} C_{\bar{m}} \frac{1}{N^{\frac{\alpha}{2}}} + C_2(\bar{m})\sqrt{C_E T}\frac{1}{\sqrt{N}}\right) \\ &\le L_f \left(\underbrace{\sqrt{C_S} C_{\bar{m}} + C_2(\bar{m})\sqrt{C_E T}}_{=:c_2}\right) \frac{1}{N^{\frac{\alpha}{2}}} \end{aligned}$$

shows the claim. □

Corollary 3.20. *Fix $Z = (Y, P, X) \in \mathcal{X}(\boldsymbol{\mathcal{S}})$ with associated target concept*

$$c : \frac{\bar{\mu}}{\bar{\sigma}} \quad \text{and} \quad \hat{f} := c^2.$$

Let $\bar{P}_N$ be an Euler approximation of P, $\bar{X}_N$ and Euler approximation of X, $\bar{S}_N$ the Euler approximation of S and $\bar{m}$ an approximation of m satisfying (3.52). Define the observable process

$$\bar{Z}_{t_n} := \left(\frac{\bar{S}_{t_n}}{\bar{m}_{t_n}}, \bar{P}_{t_n}, \bar{X}_{t_n} \right).$$

For all $N \in \mathbb{N}$ we find

$$\mathcal{E}_O(c) = \left| \frac{T}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[c^2(\bar{Z}_{t_n})\right] - \frac{T}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[f(Y_{t_n})\right] \right| \leq \frac{c_3 T}{N^{\frac{\alpha}{2}}} \tag{3.53}$$

for some constant $c_3 > 0$ and in particular

$$\lim_{N \to \infty} \mathcal{E}_O(c) = |\hat{R}_N(c) - R_N(c)|| = 0.$$

Proof. Define

$$I_{t_n} := \mathbb{E}\left[\left| \bar{f}\left(\bar{Y}_{t_n}, P_{t_n}, X_{t_n}\right) - \bar{f}\left(\bar{Y}_{t_n}, \bar{P}_{t_n}, \bar{X}_{t_n}\right) \right| \right]$$

and note that (3.26) holds for all $y \in [1, \bar{s}]$ such that

$$\mathbb{E}\left[f(\bar{Y}_{t_n})\right] = \mathbb{E}\left[c^2(\bar{Y}_{t_n}, P_{t_n}, X_{t_n})\right] = \mathbb{E}\left[\bar{f}(\bar{Y}_{t_n}, P_{t_n}, X_{t_n})\right]$$

and hence Lemma 3.18, Lemma 3.19, and the Lipschitz continuity of f yield

$$\begin{aligned} \mathbb{E}\left[\left| \bar{f}(\bar{Y}_{t_n}, \bar{P}_{t_n}, \bar{X}_{t_n}) - f(Y_{t_n}) \right| \right] &\leq I_{t_n} + \mathbb{E}\left[\left| f(\bar{Y}_{t_n}) - f(Y_{t_n}) \right| \right] \\ &\leq I_{t_n} + \frac{c_2 L_f}{N^{\frac{\alpha}{2}}} \\ &\leq \frac{C_{E,\bar{f}} T}{\sqrt{N}} + \frac{c_2 L_f}{N^{\frac{\alpha}{2}}} \\ &\leq \underbrace{\left(C_{E,\bar{f}} T + c_2 L_f \right)}_{=: c_3} \frac{1}{N^{\frac{\alpha}{2}}} \end{aligned}$$

By the triangle inequality

$$\begin{aligned} \left| \frac{T}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[c^2(\bar{Z}_{t_n})\right] - \frac{T}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[f(Y_{t_n})\right] \right| &\leq \frac{T}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[\left| \bar{f}(\bar{Y}_{t_n}, \bar{P}_{t_n}, \bar{X}_{t_n}) - f(Y_{t_n}) \right| \right] \\ &\leq \frac{c_3 T}{N^{\frac{\alpha}{2}}}. \end{aligned}$$

□

Under assumption (3.52) we find for the target concept associated to $S \in \boldsymbol{\mathcal{S}}_{LS,b}$ the error decomposition

$$\left| \hat{R}_N(h) - R(c) \right| \leq \left| \hat{R}_N(h) - \frac{T}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[\bar{f}(\bar{Z}_{t_n})\right] \right| + c_3 T N^{-\frac{\alpha}{2}} + c_1 \bar{s} L_f T N^{-\frac{1}{2}}. \tag{3.54}$$

Since $\bar{Y}_{t_n} \in [1, \bar{s}]$ by construction , Theorem 3.11 yields the existence of a sequence of hypothesis $(h_N)_{N\in\mathbb{N}} \subset \mathcal{H}$ such that

$$\left|h_N^2(t_n, \bar{Z}_{t_n}) - \bar{f}(\bar{Z}_{t_n})\right| \leq \frac{1}{N} \quad a.s. \quad \textit{for all } 0 \leq n \leq N \tag{3.55}$$

and hence by (3.54)

$$\lim_{N\to\infty} \left|\hat{R}_N(h_N) - R(c)\right| = 0$$

with convergence rate of order $N^{-\frac{\alpha}{2}}$. Based on this result, we design in the following section a weak PAC-learning algorithm $\mathcal{A}$.

3.3.5 The Learning Algorithm

Fix two neural networks $\Phi_1, \Phi_2 \in \mathcal{N}$ with a sufficient number of layers and a sufficient depth such that Theorem 3.11 implies

$$\max_{0 \le n \le N, 0 \le m \le M} \left| h^2(t_n, \bar{Z}_{t_n}) - \hat{f}(\bar{Z}_{t_n}) \right| \le \frac{1}{N} \tag{3.56}$$

for some realizations R_1 and R_2. Denote by $h_\Theta \in \mathcal{H}_\alpha$ the hypothesis with

$$h_1 = R_{\Theta_1}(\Phi_1) \quad \textit{and} \quad h_2 = R_{\Theta_2}(\Phi_2)$$

associated to the weight vector $\Theta = (\Theta_1, \Theta_2) \in \mathbb{R}^{M_p(\Phi_1)} \times \mathbb{R}^{M_p(\Phi_2)}$. Towards an approximation of the unknown process

$$X = \frac{1}{\gamma}\left(\frac{\tilde{S}}{P} - 1\right) \subset [-1, 1]$$

define for R_{Θ_1} and R_{Θ_2} the process $\bar{X}_N^\Theta$ by

$$\begin{aligned} \bar{X}_{t_{n+1}}^\Theta = \bar{X}_{t_n}^\Theta &+ \left\{ R_{\Theta_1}(\Phi_1)(t_n, \bar{Y}_{t_n}) + \frac{(1 + \gamma\alpha(\bar{X}_{t_n}^\Theta))}{\gamma} \left[\sigma(\bar{P}_{t_n})^2 - \mu(\bar{P}_{t_n})\right] \right\} \frac{T}{N} \\ &+ \left\{ R_{\Theta_2}(\Phi_2)(t_n, \bar{Y}_{t_n}) + \frac{(1 + \gamma\alpha(\bar{X}_{t_n}^\Theta))}{\gamma} \sigma(\bar{P}_{t_n}) \right\} \Delta W_n \end{aligned} \tag{3.57}$$

that is an Euler approximation of the continuous process X^Θ with dynamics

$$\begin{aligned} dX_t^\Theta = &\left\{ R_{\Theta_1}(\Phi_1)(t, \bar{Y}_t) + \frac{(1 + \gamma\alpha(X_t^\Theta))}{\gamma} \left[\sigma(\bar{P}_t)^2 - \mu(\bar{P}_t)\right] \right\} dt \\ &+ \left\{ R_{\Theta_2}(\Phi_2)(t, \bar{Y}_t) + \frac{(1 + \gamma\alpha(X_t^\Theta))}{\gamma} \sigma(\bar{P}_t) \right\} dW_t. \end{aligned}$$

Corollary 3.21. *For the expected error of the above approximation we find for a time grid $0 = t_0 < ...t_N = T$ of $[0, T]$ for all $N \in \mathbb{N}$ a constnat $C_\Theta > 0$ such that*

$$\mathbb{E}\left[\left| \bar{f}\left(\bar{Y}_{t_n}, \bar{P}_{t_n}, X_{t_n}^\Theta\right) - \bar{f}\left(\bar{Y}_{t_n}, \bar{P}_{t_n}, \bar{X}_{t_n}^\Theta\right) \right| \right] \le \frac{C_\Theta T}{\sqrt{N}}.$$

Proof. The function $x \mapsto \bar{f}(y, p, x)$ is by Lemma 3.10 Lipschitz continuous for fixed $(y, p) \in [1, \bar{s}] \times \mathbb{R}$ with Lipschitz constant $C_{\bar{f}_1} > 0$ indepdnent of (x, p). Therefore we find by Theorem 3.17 a constant $C_\Theta > 0$ such that

$$\mathbb{E}\left[\left| \bar{f}\left(\bar{Y}_{t_n}, \bar{P}_{t_n}, X_{t_n}^\Theta\right) - \bar{f}\left(\bar{Y}_{t_n}, \bar{P}_{t_n}, \bar{X}_{t_n}^\Theta\right) \right| \right] \le L_{\bar{f}_2} \mathbb{E}\left[\left| X_{t_n}^\Theta - \bar{X}_{t_n}^\Theta \right| \right] \le \frac{L_{\bar{f}_2} C_\Theta T}{\sqrt{N}}.$$

□

Note that the constant $C_\Theta > 0$ in the above Theorem from the Euler approximation of X^Θ depends on the weight vector Θ. Define the process

$$\bar{Z}_{t_n}^\Theta := \left(t_n, \bar{Y}_{t_n}^m, \bar{P}_{t_n}, \bar{X}_{t_n}^\Theta\right).$$

and finally the empirical error

$$R_{\mathcal{SP}}^{N}(h_\Theta) := \frac{T}{NM}\sum_{m=1}^{M}\sum_{n=0}^{N-1}\mathcal{L}\left(h_\Theta\left(\bar{Z}_{t_n}^{\Theta}\right), c\left(\bar{Z}_{t_n}^{\Theta}\right)\right) + \varrho_0\left(-\hat{s} + \frac{1}{1-\lambda}\right), \tag{3.58}$$

where $\hat{s}$ approximates $\bar{s}$. After initializing the weight vector Θ_0 we simulate in the i-th step of the learning algorithm a *minibatch* of M_i independent realisations of $\bar{Z}_N^{\Theta}$ and use the Adam optimizer (ADAM) [24] to update the weight vector to Θ_i by backpropagating the empirical error in (3.58) to the weights Θ_{i-1} and define our learning algorithm for $J \in \mathbb{N}$ steps and $M = \sum_{i=1}^{J} M_i$ observations by

$$\mathcal{A}_{M,N} : (\mathcal{X})^{M,N} \to \mathcal{H}_\alpha, \quad \mathcal{SP} \mapsto h_{\Theta_J}. \tag{3.59}$$

3.4 PAC-Learning Error Bounds

We aim to show that the learning algorithm $\mathcal{A} = (\mathcal{A}_{M,N})_{(M,N)\in\mathbb{N}^2}$ defined in (3.59) is weak PAC-learning. The first ingredient to the proof is the application of Hoeffding's inequality for general bounded random variables [35] to the random variables

$$\hat{f}(\bar{Z}_{t_0}^{\Theta}), ..., \hat{f}(\bar{Z}_{t_N}^{\Theta}), \quad \hat{f} \geq 0 \text{ is bounded.}$$

By the $(i.i.d)$-Learning assumption Hoeffding's inequality reveals a poylynomial rate of the convergence in probability

$$\mathbb{P} - \lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M}\hat{f}(\bar{Z}_{t_n}^{\Theta,m}) = \mathbb{E}\left[\hat{f}(\bar{Z}_{t_n}^{\Theta})\right] \quad \text{for all } 0 \leq n \leq N.$$

Lemma 3.22. *Let $\hat{f} : \mathbb{R}^4 \to [0,\infty]$ be bounded by $C_{\hat{f}} > 0$. Fix $N \in \mathbb{N}$ and a time grid $0 = t_0 < t_1 < ... < t_N = T$ of $[0,T]$. Following the definition of Section 3.3.5 set*

$$\bar{Z}_{t_n}^{\Theta} := \left(t_n, \bar{Y}_{t_n}, \bar{P}_{t_n}, \bar{X}_{t_n}^{\Theta}\right), \quad \text{for all } 0 \leq n \leq N$$

and let $\bar{Z}_N^{\Theta,1}, ..., \bar{Z}_N^{\Theta,M}$ be independent copies of $\bar{Z}_N^{\Theta}$. For any $\epsilon > 0$ and any $\delta > 0$ we find

$$\mathbb{P}\left(\left|\frac{T}{MN}\sum_{m=1}^{M}\sum_{n=0}^{N-1}\hat{f}\left(\bar{Z}_{t_n}^{\Theta,m}\right) - \frac{T}{N}\sum_{n=0}^{N-1}\mathbb{E}\left[\hat{f}\left(\bar{Z}_{t_n}^{\Theta}\right)\right]\right| < \epsilon\right) > 1-\delta \tag{3.60}$$

with

$$M \geq M(\epsilon,\delta) := \left\lceil \frac{C_{\hat{f}}^2 T^2 \ln\left(\frac{2}{\delta}\right)}{2\epsilon^2}\right\rceil + 1.$$

Proof. To ease notation define

$$Z_n := \hat{f}(\bar{Z}_{t_n}^{\Theta}) \quad \text{and} \quad Z := \frac{T}{N}\sum_{n=0}^{N-1} Z_n.$$

and fix $\epsilon > 0$. Hoeffding's inequality for general bounded random variables [35] applied to the independent copies $Z^1, ..., Z^M \in [0, TC_{\bar{f}}]$ of Z yields

$$\mathbb{P}\left(\frac{1}{M}\sum_{m=1}^{M} Z^m - \mathbb{E}\left[Z\right] \geq \epsilon\right) \leq \exp\left(-\frac{2\epsilon^2}{\sum_{m=1}^{M}\left(\frac{C_{\bar{f}}T}{M}\right)^2}\right) = \exp\left(-\frac{2\epsilon^2 M}{C_{\bar{f}}^2 T^2}\right).$$

By the same argument for $-Z^1, ..., -Z^M \in [-C_{\bar{f}}T, 0]$ we find

$$\begin{aligned}\mathbb{P}\left(\left|\frac{1}{M}\sum_{m=1}^{M} Z^m - \mathbb{E}[Z]\right| \geq \epsilon\right) &= \mathbb{P}\left(\frac{1}{M}\sum_{m=1}^{M} Z^m - \mathbb{E}[Z] \geq \epsilon\right) + \mathbb{P}\left(\mathbb{E}[Z] - \frac{1}{M}\sum_{m=1}^{M} Z^m \geq \epsilon\right)\\ &\leq 2\exp\left(-\frac{2\epsilon^2 M}{C_{\bar{f}}^2 T^2}\right).\end{aligned}$$

Hence choosing for any $\epsilon > 0$ and any $\delta > 0$

$$M \geq M(\epsilon, \delta) := \left\lceil \frac{C_{\bar{f}}^2 T^2 \ln\left(\frac{2}{\delta}\right)}{2\epsilon^2} \right\rceil + 1$$

implies with a probability of at least $1 - \delta$

$$\left|\frac{1}{M}\sum_{m=1}^{M} Z^m - \mathbb{E}[Z]\right| \leq C_{\bar{f}}T\sqrt{\frac{\ln\left(\frac{2}{\delta}\right)}{2M}} < \epsilon.$$

□

In order to proof that the learning algorithm $\mathcal{A}$ defined in (3.59) is weak PAC-learning we impose the

Optimization Assumption. *By minimizing with the ADAM optimizer the empirical error defined in* (3.58) *we*

(1) find a weight vector $\Theta^\star$ *such that*

$$\max_{0\leq n\leq N} \mathbb{E}\left[\left|X_{t_n} - X_{t_n}^{\Theta^\star}\right|\right] \leq \frac{C_{E,\Theta^\star}T}{N^{-\frac{\alpha}{2}}} \tag{3.61}$$

for some constant $C_{E,\Theta^\star} > 0$ *and* $\alpha > 0$ *as in* (3.52).

(2) find a weight vector $\Theta^\star$ *such that*

$$\max_{0\leq n\leq N, 0\leq m\leq M} \left|h^2_{\Theta^\star}(t, \bar{Z}^m_{t_n}) - \bar{f}(\bar{Z}^m_{t_n})\right| \leq \frac{1}{N} \tag{3.62}$$

and

$$\left|h^2_{\Theta^\star}(t, y, p, x)\right| \leq C_{\bar{f}} + 1 \quad \textit{for all } (t, y, p, x) \in [0, T] \times [1, \bar{s}] \times \mathbb{R} \times \mathbb{R} \tag{3.63}$$

for neural networks $\Phi_1, \Phi_2 \in \mathcal{N}$ *with a total number of parameters bounded by*

$$\max\{M_p(\Phi_1), M_p(\Phi_2)\} \leq 12 C^3_{\sigma,\tilde{\mu},\tilde{\sigma}} N^3.$$

for some constant $C_{\sigma,\tilde{\mu},\tilde{\sigma}} > 0$

Theorem 3.23. *Assume that the Optimization Assumption holds true for the learning algorithm $\mathcal{A}$ defined in (3.59). Assume further (3.52) for m induced by $S \in \boldsymbol{\mathcal{S}}_{LS,b}$. Under these assumptions, the concept class $\mathcal{C}$ is weak PAC-learnable and $\mathcal{A}$ is weak PAC-learning. To be more precise, for every $\epsilon > 0$ and every $\delta > 0$ for all $c \in \mathcal{C}(S)$ it exists neural networks Φ_1 and Φ_2 with a total number of parameters bounded by*

$$\max\{M_p(\Phi_1), M_p(\Phi_2)\} \leq 12C^3_{\sigma,\tilde{\mu},\tilde{\sigma}}N^3$$

such that

$$\mathbb{P}\left(\left|\hat{R}^N_{\mathcal{SP}}\left(h_{\Theta_J}\right) - R(c)\right| > \epsilon\right) < \delta$$

for any sample path $\mathcal{SP}$ with

$$N \geq N(\epsilon,\delta) := \left\lceil \max\left\{\left(\frac{5T}{\epsilon}\right), \left(\frac{5c_3T}{\epsilon}\right)^{\frac{2}{\alpha}}, \left(\frac{5c_1\bar{s}L_fT}{\epsilon}\right)^2\right\}\right\rceil + 1$$

discretization steps and

$$M \geq M(\epsilon,\delta) := \left\lceil \frac{25(C_{\bar{f}}+1)^2T^2\ln\left(\frac{4}{\delta}\right)}{2\epsilon^2}\right\rceil + 1$$

observations.

Proof. By (3.54) we have *a.s.* the error decomposition

$$\left|\hat{R}_N(h_{\Theta^\star}) - R(c)\right| \leq \underbrace{\left|\hat{R}_N(h_{\Theta^\star}) - \frac{T}{N}\sum_{n=0}^{N-1}\mathbb{E}\left[\bar{f}(\bar{Z}_{t_n})\right]\right|}_{:=I_1} + c_3TN^{-\frac{\alpha}{2}} + c_1\bar{s}L_fTN^{-\frac{1}{2}}.$$

By the Optimization Assumption and the Lipschitz continuity of $\bar{f}$

$$\mathbb{E}\left[\left|\bar{f}(\bar{Y}_{t_n},\bar{P}_{t_n},\bar{X}^{\Theta^\star}_{t_n}) - \bar{f}(\bar{Z}_{t_n})\right|\right] \leq L_{\bar{f}_2}\mathbb{E}\left[\left|\bar{X}_{t_n} - \bar{X}^{\Theta^\star}_{t_n}\right|\right] \leq \frac{L_{\bar{f}}C_{E,\Theta^\star}T}{N^{-\frac{\alpha}{2}}}$$

and hence

$$I_1 \leq \underbrace{\left|\hat{R}_N(h_{\Theta^\star}) - \frac{T}{N}\sum_{n=0}^{N-1}\mathbb{E}\left[\bar{f}(\bar{Y}_{t_n},\bar{P}_{t_n},\bar{X}^{\Theta^\star}_{t_n})\right]\right|}_{=:I_2} + \frac{L_{\bar{f}}C_{E,\Theta^\star}T^2}{N^{-\frac{\alpha}{2}}}.$$

Let $C_{\tilde{\mu}} > 0$ and $c_{min} := \min\{1, c_{\tilde{\sigma}}\}$ from Lemma 3.10 and define

$$C_{\bar{f}} := \left[\frac{C_{\tilde{\mu}}}{c_{min}}\right]^2$$

such that

$$\bar{f}(y,p,x) \leq C_{\bar{f}} \quad \textit{for all } (y,p,x) \in [1,\bar{s}] \times (0,\infty) \times \mathbb{R}$$

and

$$\left|h^2_{\Theta^\star}(t,y,p,x)\right| \leq C_{\bar{f}} + 1 \quad \textit{for all } (t,y,p,x) \in [0,T] \times [1,\bar{s}] \times (0,\infty) \times \mathbb{R}$$

by (3.63) of the Optimization Assumption. Fix any $\epsilon > 0$ and any $\delta > 0$ and choose

$$N \geq N(\epsilon, \delta) := \left\lceil \max\left\{ \left(\frac{5T}{\epsilon}\right), \left(\frac{5c_3 T}{\epsilon}\right)^{\frac{2}{\alpha}}, \left(\frac{5c_1 \bar{s} L_f T}{\epsilon}\right)^2 \right\} \right\rceil + 1$$

as the number of discretization steps and

$$M \geq M(\epsilon, \delta) := \left\lceil \frac{25(C_{\bar{f}}+1)^2 T^2 \ln\left(\frac{4}{\delta}\right)}{2\epsilon^2} \right\rceil + 1$$

Define

$$A := \left\{ \left| \frac{T}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{M} \bar{f}\left(\bar{Y}^m_{t_n}, \bar{P}^m_{t_n}, \bar{X}^{\Theta^\star,m}_{t_n}\right) - \frac{T}{N} \sum_{n=0}^{N-1} \mathbb{E}\left[\bar{f}\left(\bar{Y}_{t_n}, \bar{P}_{t_n}, \bar{X}^{\Theta^\star}_{t_n}\right)\right] \right| \geq \frac{\epsilon}{5} \right\}$$

and

$$B := \left\{ \left| \frac{T}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{M} h^2_{\Theta^\star}(t_n, \bar{Y}^m_{t_n}, \bar{P}^m_{t_n}, \bar{X}^{\Theta^\star,m}_{t_n}) - \hat{R}_N(h_{\Theta^\star}) \right| \geq \frac{\epsilon}{5} \right\}$$

to conclude by Lemma 3.22

$$\mathbb{P}(A) \leq \frac{\delta}{2} \quad \textit{and} \quad \mathbb{P}(B) \leq \frac{\delta}{2}.$$

Hence

$$\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B) \leq \delta$$

and

$$\mathbb{P}(\overline{A \cup B}) \geq 1 - \delta.$$

Therefore with probability of at least $1 - \delta$ by the triangle inequality and (3.62) of the Optimization Assumption

$$I_2 \leq \frac{T}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M} \left| h^2_{\Theta^\star}(t_n, \bar{Y}^m_{t_n}, \bar{P}^m_{t_n}, \bar{X}^{\Theta^\star,m}_{t_n}) - \bar{f}(\bar{Y}^m_{t_n}, \bar{P}^m_{t_n}, \bar{X}^{\Theta^\star,m}_{t_n}) \right| + \frac{2\epsilon}{5} \leq \frac{T}{N} + \frac{2\epsilon}{5}.$$

In total using (2.4)

$$\begin{aligned} \left|\hat{R}_N(h_{\Theta^\star}) - R(c)\right| &\leq I_2 + I_1 + c_3 T N^{-\frac{\alpha}{2}} + c_1 \bar{s} L_f T N^{-\frac{1}{2}} \\ &\leq \frac{T}{N} + \frac{2\epsilon}{5} + c_3 T N^{-\frac{\alpha}{2}} + c_1 \bar{s} L_f T N^{-\frac{1}{2}} \\ &< \frac{\epsilon}{5} + \frac{2\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} = \epsilon \end{aligned}$$

completes the proof. □

Chapter 4

Experimental Results

We deploy the developed method in the Black-Scholes model (B, P) driven by a geometric Brownian motion

$$P_t = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right), \quad P_0 = 1$$

with $\sigma^2 > \mu > 0$, a constant bank account $B \equiv 1$ and proportional transaction costs with transaction cost parameter $\gamma = 0.01$. Define $S := (1+\gamma)P$ and note that by Theorem 2.10 we find that $S \in \boldsymbol{\mathcal{S}}_{LS,b}$ induces a shadow price process

$$(1-\lambda)S \leq \tilde{S} \leq S$$

following

$$d\tilde{S}_t = S_t\tilde{\mu}(Y_t)dt + S_t\tilde{\sigma}(Y_t)dW_t, \quad \tilde{S}_0 = (1+\gamma).$$

Morever, using the definitions of Section 3.3.2, the hypothesis set $\mathcal{H}_\alpha$ approximates the target concept

$$c := \frac{\hat{\mu}}{\hat{\sigma}}$$

arbitrarily well from above by Remark 3.9, Lemma 3.12 and Theorem 3.11. Finally, under the Optimization Assumption the proposed learning algorithm $\mathcal{A}$ defined in (3.59) is weak-PAC learning by Theorem 3.23.

4.1 Application of the Learning Algorithm

Following Belak and Sass [4], the optimal risky fraction $\pi^\star$ in the financial model (B, P) moves freely within a *no trade region* reflecting an inactive trader who only sells shares of the risky asset when $\pi^\star$ exceeds a certain *sell boundary* and only buys shares of the risky asset when $\pi^\star$ falls below a certain *buy boundary*. See Figure E.14 for the trading regions in (B, P) for $\mu = \frac{\sigma^2}{2}$ calculated with the code from Belak and Sass [4]. To define our *baseline* method we fix $\Phi_1 = \Phi_2 \in \mathcal{N}$ with four hidden layers and eight neurons in each layer. In the *ExplicitBaseline* method we simulate the process $\bar{X}^\Theta_N$ according to (3.57) and in *UnExplicitBaseline* we simulate the process $\bar{X}^\Theta_N$ simply by

$$d\bar{X}^\Theta_{t_{n+1}} = \bar{X}^\Theta_{t_n} + R_{\Theta_1}\left(\Phi_1\right)\left(t_n, \bar{Y}_{t_n}\right)\Delta t + R_{\Theta_2}\left(\Phi_2\right)\left(t_n, \bar{Y}_{t_n}\right)\Delta W_n \tag{4.1}$$

assuming that the explicit form of (3.57) is learned as well.

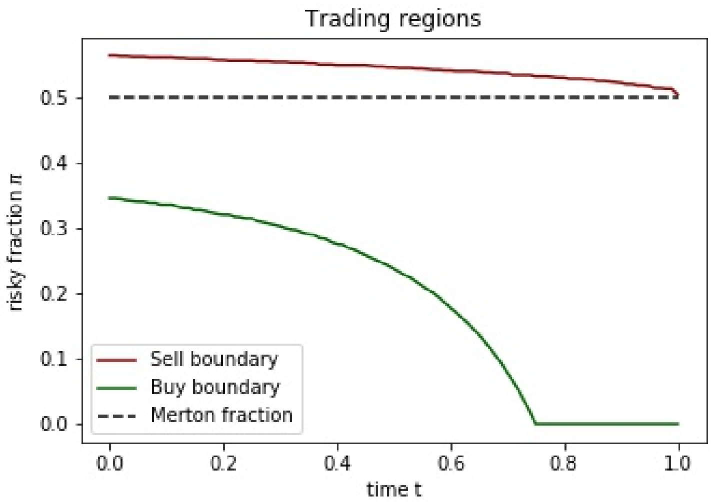

FIGURE 4.1: Trading regions for $\mu = \frac{\sigma^2}{2}$ calculated with the code of Belak and Sass [4]. The dashed line is the Merton fraction $\frac{\mu}{\sigma^2} = \frac{1}{2}$, that is the log-optimal fraction in the financial model (B, P) without transaction costs. We aim to determine the sell boundary and the buy boundary, but also aim to determine the behavior of $\pi^\star$ after one of the boundaries is touched.

In both methods, (3.21) is used to simulate the process $\bar{S}_N^\alpha$. In the *oracle* method we use (A.17) to simualte $\bar{S}_N^\alpha$ and

$$d\bar{X}_{t_{n+1}}^{\Theta} = \bar{X}_{t_n}^{\Theta} + \bar{P}_{t_n} R_{\Theta_1}(\Phi_1)(t_n, \bar{Y}_{t_n})\Delta t + \bar{P}_{t_n} R_{\Theta_2}(\Phi_2)(t_n, \bar{Y}_{t_n})\Delta W_n,$$

to simulate $\bar{X}_N^\Theta$, but using the form (4.1) to calculate $\bar{S}_{t_{n+1}}^\alpha$ without adjusting $\bar{S}_N^\alpha$ to

$$\begin{aligned} d\bar{S}_{t_{n+1}}^{\alpha} = \bar{S}_{t_{n+1}}^{\alpha} &\left\{ \mu + \bar{P}_{t_n} \frac{\alpha'\left(\bar{X}_{t_n}^{\Theta}\right)}{1 + \gamma\alpha\left(\bar{X}_{t_n}^{\Theta}\right)} \left[R_{\Theta_1}(\Phi_1)(t_n, \bar{Y}_{t_n}) + \sigma R_{\Theta_2}(\Phi_2)(t_n, \bar{Y}_{t_n}) + \frac{1}{2\gamma}\frac{\alpha}{\,} \right] \right\} \frac{1}{\bar{S}_{t_n}^{\alpha}} \Delta t \\ &+ \bar{S}_{t_{n+1}}^{\alpha} \bar{P}_{t_n}^2 \frac{1}{2\gamma} \frac{\alpha''((\bar{X}_{t_n}^{\Theta}))}{1 + \gamma\alpha\left(\bar{X}_{t_n}^{\Theta}\right)} R_{\Theta_2}^2(\Phi_2)(t_n, \bar{Y}_{t_n}) \frac{1}{\bar{S}_{t_n}^{\alpha}} \\ &+ \bar{S}_{t_{n+1}}^{\alpha} \left\{ \sigma + \bar{P}_{t_n} \frac{\alpha'\left(\bar{X}_{t_n}^{\Theta}\right)}{1 + \gamma\alpha\left(\bar{X}_{t_n}^{\Theta}\right)} R_{\Theta_2}(\Phi_2)(t_n, \bar{Y}_{t_n}) \right\} \frac{1}{\bar{S}_{t_n}^{\alpha}} \Delta W_n \end{aligned}$$

Hence, we disregard in the simulation of $\bar{S}_N^\alpha$ the multiplication of the increment of $\bar{X}_{t_{n+1}}^\Theta$ with $\bar{P}_{t_n}$ in every time step. In the authors opinion this is contra-intuitive and simulates a different process as needed, but it turns out that this method achieves one of the best performances. In Table 4.3 we define 14 methods with different choices of α, α' and α''. The resulting approximated value of the expected optimal log-utility of the terminal wealth (EOTW) is summarized in Table 4.2 and a visual summary of the learning procedure is provided in Appendix E.1. Note that we aim for a low value of EOTW, as it is the generalization error of our learning problem. In Table 4.2 we observe that the standard deviation over different predictions is small, since we already use 1000 *HardtanhExplicit* has the worst performance. Surprisingly, *Oracle* reaches a lower EOTW as the corresponding methods *Explicit* and *Unexplicit*. The method *OracleSmooth* has the lowest EOTW.

name of the method	α	α'	α''
HardtanhExplicit	hardtanh	$\mathbb{1}_{[-1,1]}$	$\alpha'' \equiv 0$
HardtanhUnExplicit	hardtanh	$\mathbb{1}_{[-1,1]}$	$\alpha'' \equiv 0$
ExplicitBaseline	hardtanh	$\tanh'$	$\alpha'' \equiv 0$
UnExplicitBaseline	hardtanh	$\tanh'$	$\alpha'' \equiv 0$
OracleExplicit	hardtanh	$\tanh'$	$\alpha'' \equiv 0$
OracleUnExplicit	hardtanh	$\tanh'$	$\alpha'' \equiv 0$
SmoothExplicit	tanh	$\tanh'$	$\alpha'' \equiv 0$
SmoothUnExplicit	tanh	$\tanh'$	$\alpha'' \equiv 0$
OracleSmoothExplicit	tanh	$\tanh'$	$\alpha'' \equiv 0$
OracleSmoothUnExplicit	tanh	$\tanh'$	$\alpha'' \equiv 0$
TanhExplicit	tanh	$\tanh'$	$\tanh''$
TanhUnExplicit	tanh	$\tanh'$	$\tanh''$
TanhOracleExplicit	tanh	$\tanh'$	$\tanh''$
TanhOracleUnExplicit	tanh	$\tanh'$	$\tanh''$

TABLE 4.1: Choices of α,α' and α'' for the different methods.

name of the method	empirical mean of predicted EOTW	empirical std of predicted EOTW
HardtanhExplicit	$0.01844 + c(x,T)$	$\pm\, 3.62823e-05$
HardtanhUnExplicit	$0.01719 + c(x,T)$	$\pm\, 7.43856e-05$
BaselineExplicit	$0.01331 + c(x,T)$	$\pm\, 8.21032e-05$
BaselineUnExplicit	$0.0131 + c(x,T)$	$\pm\, 8.04659e-05$
SmoothExplicit	$0.01656 + c(x,T)$	$\pm\, 5.69418e-05$
SmoothUnExplicit	$0.01311 + c(x,T)$	$\pm\, 9.62231e-05$
TanhExplicit	$0.01575 + c(x,T)$	$\pm\, 6.01526e-05$
TanhUnExplicit	$0.0151 + c(x,T)$	$\pm\, 5.33738e-05$
OracleBaseline	$\mathbf{0.01231} + c(x,T)$	$\pm\, 0.00013e-05$
AdjustedOracleBaseline	$0.01532 + c(x,T)$	$\pm\, 4.04382e-05$
OracleSmooth	$\mathbf{0.01237} + c(x,T)$	$\pm\, 0.00016e-05$
AdjustedOracleSmooth	$\mathbf{0.0128} + c(x,T)$	$\pm\, 9.88652e-05$
OracleTanh	$0.01431 + c(x,T)$	$\pm\, 0.00011$
AdjustedOracleTanh	$0.01548 + c(x,T)$	$\pm\, 3.63309e-05$

TABLE 4.2: Empirical mean (mean) and empirical standard deviation (std) over 10×1000 predictions of the EOTW with initial endowment $x = 100$. To increase readability and distinguishability of the results, we normalize by $c(x,T) = \ln\left(\frac{x}{T+1}\right) = \ln(50)$ and round to five digits.

Optimizer	learning rate	Opimization steps	Over the weights
ADAM	$1e-03$	10^3	$R_1, R_2, \hat{s}$

TABLE 4.3: Hyperparameters used for the training of the 14 methods.

4.2 Comparison of Different Activations

In the setting of Section 4.1 we adjust the *oracle* method such that the two neural networks receive as input $\bar{Y}_N$ and $\hat{m}_N$ instead of only $\bar{Y}_N$. Denote by *ReluOracleTanh* the method with the same underlying ReLU-activation as in *OracleTanh*. Inspired by the architecture Kidger et al. [23] we define a second method *NonReluOracleTanh*: Φ_1 uses *tanh*-activation, the first three layers Φ_2 use $softplus$-activation and in the last layer of Φ_2 a $sigmoid$-activation is applied. We use Table 4.4 as hyperparameters during training. For the resulting approximate optimal trading strategy and the resulting approximate optimal risky fraction see Figure 4.2 and Figure 4.3. For a summary of all processes involved in the approximation see Figure 4.5 and Figure 4.4. Observe in Table 4.5 that *NonReluOracleTanh* performes better than *ReluOracleTanh*, but not as good as *OracleBaselnie*. Nevertheless, the buy boundary seems to be learned very well.

Optimizer	learning rate	Opimization steps	Over the weights
ADAM	$1e-03$	3×10^3	$R_1, R_2, \hat{s}$

TABLE 4.4: Hyperparameters used for the activation comparison.

name of the method	empirical mean of predicted EOTW	empirical std of predicted EOTW
ReluOracleTanh	$0.01388 + c(x, T)$	$\pm$ 0.00013
NonReluOracleTanh	$0.01379 + c(x, T)$	$\pm$ 0.00016

TABLE 4.5: Empirical mean (mean) and empirical standard deviation (std) over 10×1000 predictions of the EOTW with initial endowment $x = 100$.

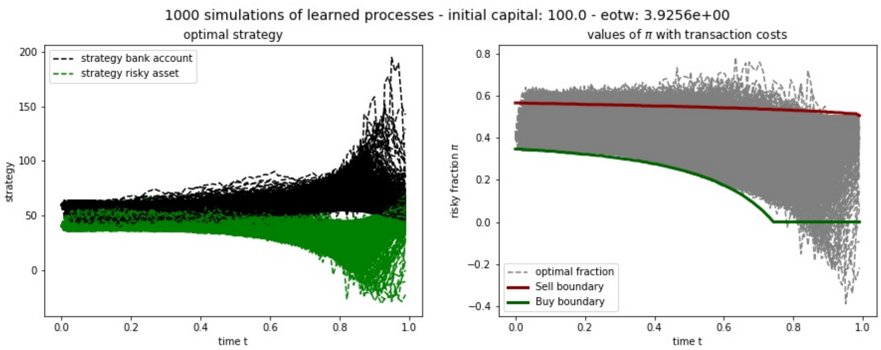

FIGURE 4.2: Optimal strategy and optimal risky fraction for relu activation

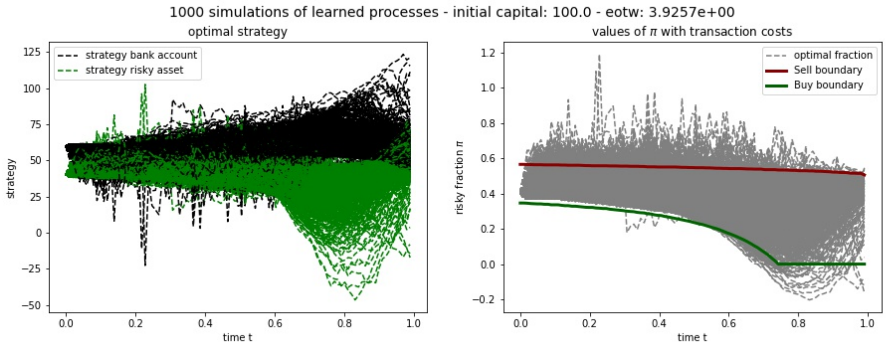

FIGURE 4.3: Optimal strategy and optimal risky fraction for non-relu activation

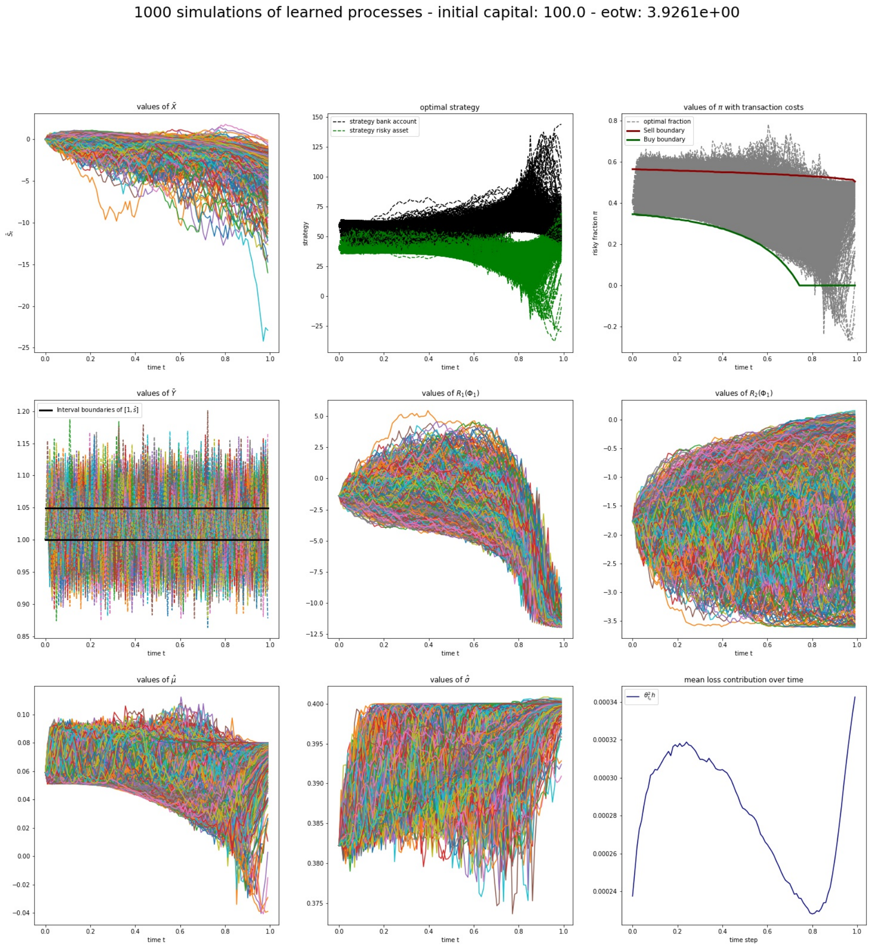

FIGURE 4.4: Summary of the learned processes for relu activation.

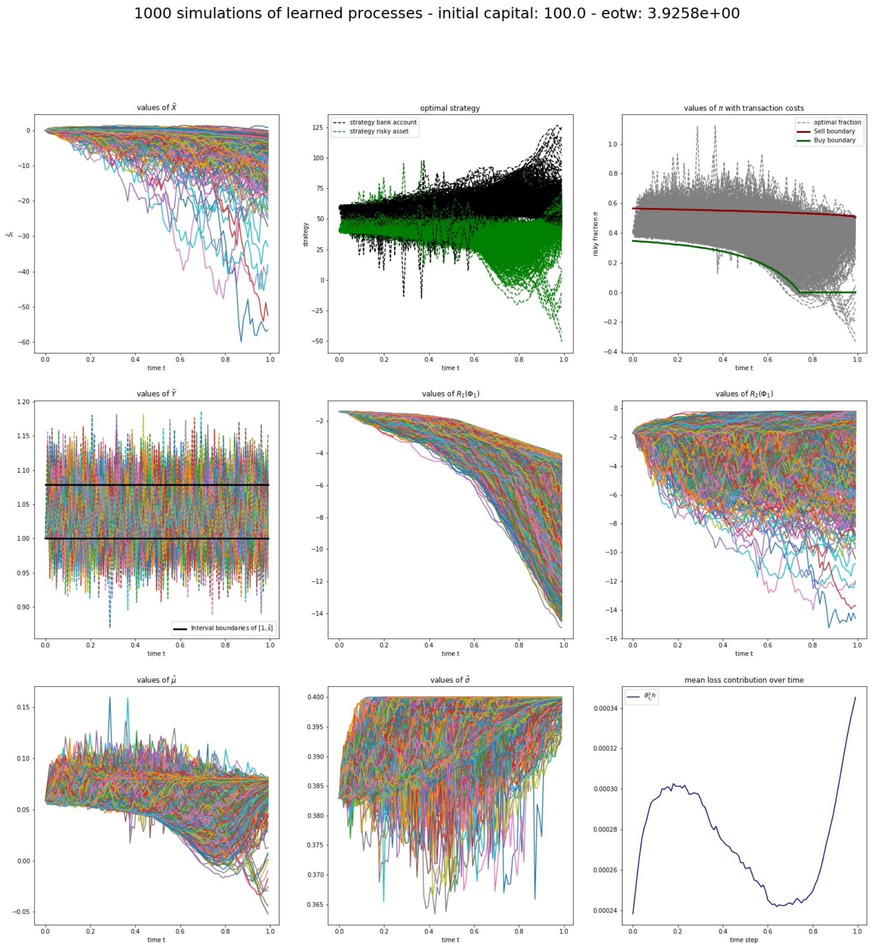

FIGURE 4.5: Summary of the learned processes for non-relu activation.

Chapter 5

Discussion and Outlook

To show Assumption 2 in Chapter 2 for general recurrent Markov processes, the missing part is the existence of a smooth pasting function g that is shown for geometric Brownian motion in Gerhold et al. [15]. Hence, one needs to show the existence of such a smooth pasting function for all processes in $\boldsymbol{\mathcal{S}}_r^{MP}$ in order to follow the presented arguments to justify an application of the proposed method in a financial model driven by any price process in $\boldsymbol{\mathcal{S}}_r^{MP}$ with bounded coefficient functions.

The most restrictive and not well motivated assumption in the theoretical justification of the steps described in Section 3.3.2 to Section 3.3.4 is the assumption of the upper bound in Equation (3.52) for the approximation $\hat{m}_N$ of m_N resulting from Algorithm 1. To truly justify the proposed method, an additional argument is needed.

The Optimization assumption on the optimization procedure ADAM lacks also motivation, but is in the author's opinion hard to bypass as the optimization of the weights of a neural network is a highly non-convex optimization problem. However, we assume in addition that the unknown real number $\bar{s}$ is well approximated by $\hat{s}$, an assumption that might be replaceable by an argument.

The evaluation of the approximate strategies and the approximate risky fractions in Chapter 4 remains quite vague and needs to be revised in future work.

The simple architecture of the used neural networks in Chapter 4 indicates the potential that the proposed method may also be applied in a financial model with an underlying higher-dimensional price process S. To justify the application of the method in higher dimensions, a generalization of the shadow price approach to higher dimensions is needed.

Since the existence of a shadow price is proven in Czichowsky and Schachermayer [10] for a financial model driven by a fractional Brownian motion, it might even be possible to extend the proposed method to approximate the optimal trading strategy in a financial model based on fractional Brownian motion with Hurst parameter $H \in (0,1) \setminus \{\frac{1}{2}\}$. Lemma 5.1 in Czichowsky and Schachermayer [10] states that the induced shadow process in an Itô process. Note however, that the existence result of Czichowsky and Schachermayer [10] does not include a logarithmic utility function, used in the derivation of $(B.5)$.

Appendix A

Proofs and Calculations

A.1 Proof of Lemma 2.6

Proof.

(i) We proof the claim by induction over $k \in \mathbb{N}_0$. Since $\varrho_0 \equiv 0$, the claim holds for $k = 0$. Assume now that ϱ_{k-1} is an *a.s.* finite $\{\mathcal{F}_t\}$-stopping time.

Claim. *τ_k and ϱ_k are a.s. finite $\{\mathcal{F}_t\}$-stopping times*

By definition, S is adapted to the complete filtration $\{\mathcal{F}_t\}$ and a.s.-continuous. Since the infimum of a $\{\mathcal{F}_t\}$-adapted, continuous process on a complete filtration is $\{\mathcal{F}_t\}$-measurable, the process

$$\hat{Y}^1_t = \frac{S_t}{\inf_{\varrho_{k-1} \leq u \leq t} S_u}$$

is $\{\mathcal{F}_t\}$-adapted and continuous on $]]\varrho_{k-1}, t]]$. Therefore

$$\{\tau_k \leq t\} = \left(\bigcap_{n \in \mathbb{N}} \bigcup_{s \in \mathbb{Q} \cap [0,t]} \left(\left\{ \hat{Y}^1_s \geq \bar{s} + n^{-1} \right\} \cap \{\varrho_{k-1} \leq s\} \right) \right) \in \mathcal{F}_t$$

where we use that $\{\varrho_{k-1} \leq t\} \in \mathcal{F}_t$ by assumption. Using that the supremum of a $\{\mathcal{F}_t\}$-adapted, continuous process on a complete filtration is $\{\mathcal{F}_t\}$-measurable, the process

$$\hat{Y}^2_t = \frac{S_t}{\sup_{\tau_k \leq u \leq t} S_u}$$

is $\{\mathcal{F}_t\}$-adapted and continuous on $]]\tau_k, t]]$. Hence

$$\{\varrho_k \leq t\} = \left(\bigcap_{n \in \mathbb{N}} \bigcup_{s \in \mathbb{Q} \cap [0,t]} \left(\left\{ \hat{Y}^2_s \leq \frac{1}{\bar{s}} - n^{-1} \right\} \cap \{\tau_k \leq s\} \right) \right) \in \mathcal{F}_t.$$

Towards

$$\mathbb{P}(\tau_k = \infty) = 0$$

define the process

$$\underline{Y}_t := \frac{S_{\varrho_{k-1}+t}}{m_{\varrho_{k-1}}}$$

and the stopping time

$$\underline{T}_{\bar{s}} := \inf\{t \geq 0 \mid \underline{Y}_t \geq \bar{s}\}.$$

For $t \leq \tau_k$ we have $m_{\varrho_{k-1}} \geq m_{\varrho_{k-1}+t}$ and hence $\underline{Y}_t \leq Y_{\varrho_{k-1}+t}$. This implies

$$\mathbb{P}(\underline{T}_{\bar{s}} < \infty) \leq \mathbb{P}(\tau_k < \infty). \tag{A.1}$$

Let $(\mathbb{P}_{s0})_{s0>0}$ be the family of probability measures associated to the strong Markov process S. Note that $\mathbb{P} = \mathbb{P}_{s0}$. The process S is by assumption recurrent under $\mathbb{P}_{S_{\varrho_{k-1}}}$ with $\mathbb{P}_{S_{\varrho_{k-1}}}(S_0 = S_{\varrho_{k-1}}) = 1$. Hence for the stopping time

$$T_{\bar{s}} := \inf\{t \geq 0 \mid S_t = \bar{s} S_{\varrho_{k-1}}\}$$

we have

$$\mathbb{P}_{S_{\varrho_{k-1}}}(T_{\bar{s}} < \infty) = 1. \tag{A.2}$$

By the strong Markov property for all $A \in \mathcal{F}$ using $\mathbb{P}_{s0}(\varrho_{k-1} < \infty) = 1$

$$\mathbb{P}_{s_0}(S_{\varrho_{k-1}+t} \in A) = \mathbb{P}_{S_{\varrho_{k-1}}}(S_t \in A) = \mathbb{P}_{S_{\varrho_{k-1}}}(S_t \in A) \quad \mathbb{P}_{s0} - a.s. \tag{A.3}$$

Hence

$$\begin{aligned}
\mathbb{P}_{S_{\varrho_{k-1}}}(T_{\bar{s}} < \infty) &= \mathbb{P}_{S_{\varrho_{k-1}}}\left(\bigcup_{n\in\mathbb{N}} \{T_{\bar{s}} < n\}\right) \\
&= \mathbb{P}_{S_{\varrho_{k-1}}}\left(\bigcup_{n\in\mathbb{N}} \{\exists t < n : S_t \geq \bar{s} S_{\varrho_{k-1}}\}\right) \\
&\overset{(A.3)}{=} \mathbb{P}_{s_0}\left(\bigcup_{n\in\mathbb{N}} \{\exists t < n : S_{\varrho_{k-1}+t} \geq \bar{s} S_{\varrho_{k-1}}\}\right) \\
&= \mathbb{P}_{s_0}\left(\bigcup_{n\in\mathbb{N}} \{\exists t < n : \underline{Y}_t \geq \bar{s}\}\right) \\
&= \mathbb{P}_{s_0}\left(\bigcup_{n\in\mathbb{N}} \{\underline{T}_{\bar{s}} < n\}\right) \\
&= \mathbb{P}\left(\underline{T}_{\bar{s}} < \infty\right)
\end{aligned} \tag{A.4}$$

Combined with $(A.2)$ and $(A.1)$ this leads to

$$\mathbb{P}(\tau_k < \infty) = 1.$$

To show

$$\mathbb{P}(\varrho_k < \infty) = 1$$

define the process

$$\overline{Y}_t = \frac{S_{\tau_k+t}}{m_{\tau_k}}$$

and the stopping time

$$\overline{T}_{\bar{s}} := \inf\left\{t \geq 0 \mid \overline{Y}_t \leq \frac{1}{\bar{s}}\right\}.$$

Note that for $t \leq \varrho_k$ we have $m_{\varrho_{k-1}} \leq m_t$ and hence $\overline{Y}_t \geq Y_t$. By the same arguments as above

$$1 = \mathbb{P}(\overline{T} < \infty) \leq \mathbb{P}(\varrho < \infty).$$

Hence τ_k and ϱ_k are $a.s.$ finite $\{\mathcal{F}_t\}$-stopping times.

(ii) Assume

$$\mathbb{P}(\Omega_0) > 0, \quad \Omega_0 := \{(\tau_k)_{k\in\mathbb{N}} \subset [0,T]\}. \tag{A.5}$$

We argue on Ω_0. Set $a_{2k} := \varrho_k$ and $a_{2k+1} = \tau_k$. The sequence $a = (a_k)_{k\in\mathbb{N}_0}$ is increasing and bounded and hence converges towards some $\hat{T} \leq T$. Therefore, we find for the subsequences

$$\lim_{k\to\infty} \tau_k = \hat{T} = \lim_{k\to\infty} \varrho_k.$$

Following the arguments in the proof of (ii) from Lemma 2.7 we know that m is $a.s.$ continuous on $\bigcup_{k\in\mathbb{N}}]]\varrho_{k-1}, \tau_k]]\cup]]\tau_k, \varrho_k]] \subset [0,T]$. Since S is $a.s.$ continuous on $[0,T]$ we find

$$\lim_{k\to\infty} \left(\frac{S_{\tau_k}}{m_{\tau_k}} - \frac{S_{\varrho_{k-1}}}{m_{\varrho_{k-1}}}\right) = \left(\frac{S_{\hat{T}}}{m_{\hat{T}}} - \frac{S_{\hat{T}}}{m_{\hat{T}}}\right) = 0$$

in contradiction to

$$\left(\frac{S_{\tau_k}}{m_{\tau_k}} - \frac{S_{\varrho_{k-1}}}{m_{\varrho_{k-1}}}\right) \geq \left(\frac{1}{\bar{s}} - 1\right) \geq \lambda > 0.$$

Hence

$$\mathbb{P}(\Omega_0) = 0$$

and similar

$$\mathbb{P}((\varrho_{k-1})_{k\in\mathbb{N}} \subset [0,T]) = 0$$

holds true.

Since $0 = \varrho_0 < \tau_1 < \varrho_1 < \tau_2 < \varrho_2 < ...$ is a partition of $[0, T]$

$$\sum_{k \in \mathbb{N}} \left(\frac{S_{\tau_k}}{m_{\tau_k}} - \frac{S_{\varrho_{k-1}}}{m_{\varrho_{k-1}}} \right)^2 \geq \sum_{k \in \mathbb{N}} \left(\frac{1}{\bar{s}} - 1 \right)^2 \geq \sum_{k \in \mathbb{N}} (\lambda)^2 = \infty$$

in contradiction to

$$\sum_{k \in \mathbb{N}} \left(\frac{S_{\tau_k}}{m_{\tau_k}} - \frac{S_{\varrho_{k-1}}}{m_{\varrho_{k-1}}} \right)^2 \leq \sum_{k \in \mathbb{N}} \left(\frac{S_{\tau_k}}{\inf_{0 \leq u \leq T} S_u} - \frac{S_{\varrho_{k-1}}}{\sup_{0 \leq u \leq T} S_u} \right)^2$$

$$\langle S, S \rangle_T \geq \sum_{k \in \mathbb{N}} \left(S_{\tau_k} - S_{\varrho_{k-1}} \right)^2 = \sum_{k \in \mathbb{N}} \left(\bar{s} m_{\tau_k} - m_{\varrho_{k-1}} \right)^2$$

(iii) By (ii) we find $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that

$$\bigcup_{k \in \mathbb{N}}]\varrho_{k-1}(\omega), \varrho_k(\omega)] = (0, \infty)$$

for all $\omega \in \tilde{\Omega}$. Since

$$]\varrho_{k-1}(\omega), \varrho_k(\omega)] =]\varrho_{k-1}(\omega), \tau_k(\omega)] \cup]\tau_k(\omega), \varrho_k(\omega)]$$

we have

$$\bigcup_{k \in \mathbb{N}} (]]\varrho_{k-1}, \tau_k]] \cup]]\tau_k, \varrho_k]]) \cap [0, T] = (0, \infty) \cap [0, T] = (0, T] \quad a.s.$$

□

A.2 Proof of Lemma 2.7

Proof.

(i) Fix $t \in [0, T]$. By (iii) of Lemma 2.6 we find $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that for every $\omega \in \tilde{\Omega}$ we have $t \in]\varrho_{k-1}(\omega), \tau_k(\omega)] \cup]\tau_k(\omega), \varrho_k(\omega)]$ for some $k \in \mathbb{N}$. Hence either

$$Y_t(\omega) = \frac{S_t(\omega)}{m_t(\omega)} \leq \bar{s}$$

or

$$Y_t(\omega) = \frac{\bar{s} S_t(\omega)}{M_t(\omega)} \geq \bar{s} \frac{1}{\bar{s}} = 1.$$

Therefore $Y \in [1, \bar{s}]$ $a.s.$ holds true.

(ii) Towards continuity we need to show that

$$\mathbb{P}(\lim_{s \to t} m_s = m_t \quad \forall s \in [0, T]) = 1.$$

By [7, Theorem 17.2.3] the process S has *a.s.* continuous paths. By (iii) of Lemma 2.6 either $t \in]]\varrho_{k-1}, \tau_k]]$ or $t \in]]\tau_k, \varrho_k]]$ for some $k \in \mathbb{N}$ and any $t \in [0, T]$. Therefore, we find $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that $t \mapsto S_t(\omega)$ is continuous for all $\omega \in \tilde{\Omega}$ and for all $t \in [0, T]$ we find $k \in \mathbb{N}$ such that $t \in]\varrho_{k-1}(\omega), \tau_k(\omega)] \cup]\tau_k(\omega), \varrho_k(\omega)]$. By the continuity of S we find for $t > \varrho_{k-1}$

$$\begin{aligned}
\lim_{u \searrow t} \inf_{v \in]\varrho_{k-1}(\omega), u]} S_v(\omega) &= \min\{\inf_{v \in]\varrho_{k-1}(\omega), t]} S_v, \lim_{u \searrow t} \inf_{v \in]t, u]} S_v(\omega)\} \\
&= \min\{\inf_{v \in]\varrho_{k-1}(\omega), t]} S_v, S_t(\omega)\} = \inf_{v \in]\varrho_{k-1}(\omega), t]} S_v \\
&= \min\{\inf_{v \in]\varrho_{k-1}(\omega), t]} S_v, S_t(\omega)\} = \min\{\inf_{v \in]\varrho_{k-1}(\omega), t]} S_v, \lim_{u \nearrow t} \inf_{v \in]u, t]} S_v(\omega)\} \\
&= \lim_{u \nearrow t} \min\{\inf_{v \in]\varrho_{k-1}(\omega), u]} S_v, \inf_{v \in]u, t]} S_v(\omega)\} \\
&= \lim_{u \nearrow t} \inf_{v \in]\varrho_{k-1}(\omega), u]} S_v(\omega)
\end{aligned} \tag{A.6}$$

and similar for $t > \tau_k$

$$\lim_{u \searrow t} \sup_{v \in]\tau_k(\omega), u]} S_v(\omega) = \sup_{v \in]\tau_k(\omega), t]} S_v(\omega) = \lim_{u \nearrow t} \sup_{v \in]\tau_k(\omega), u]} S_v(\omega). \tag{A.7}$$

Fix $t \in [0, T]$. The following equations hold for all $\omega \in \tilde{\Omega}$. We distinguish four cases.

$\underline{\varrho_{k-1} < t < \tau_k}$:

$$\lim_{u \searrow t} m_u = \lim_{u \searrow t} \inf_{\varrho_{k-1} \leq v \leq u} S_v \overset{(A.6)}{=} \lim_{u \nearrow t} \inf_{\varrho_{k-1} \leq v \leq u} S_v = \lim_{u \nearrow t} m_u$$

$\underline{\tau_k < t < \varrho_k}$:

$$\lim_{u \searrow t} m_u = \frac{1}{\bar{s}} \lim_{u \searrow t} \sup_{\tau_k \leq v \leq u} S_v \overset{(A.7)}{=} \frac{1}{\bar{s}} \lim_{u \nearrow t} \sup_{\tau_k \leq v \leq u} S_v = \lim_{u \nearrow t} m_u$$

$\underline{t = \varrho_{k-1}}$:

$$\lim_{u \searrow t} m_u = \lim_{u \searrow t} \inf_{\varrho_{k-1} \leq v \leq u} S_v = S_{\varrho_{k-1}} = \frac{1}{\bar{s}} \sup_{\tau_{k-1} \leq u \leq \varrho_{k-1}} S_u = \lim_{u \nearrow t} \frac{1}{\bar{s}} \sup_{\tau_{k-1} \leq v \leq u} S_v = \lim_{u \nearrow t} m_u$$

$\underline{t = \tau_k}$:

$$\lim_{u \searrow t} m_u = \lim_{u \searrow t} \frac{1}{\bar{s}} \sup_{\tau_{k-1} \leq v \leq u} S_v = \frac{1}{\bar{s}} S_{\tau_k} = \frac{1}{\bar{s}} \bar{s} \inf_{\varrho_{k-1} \leq u \leq \tau_k} S_u = \lim_{u \nearrow t} \inf_{\varrho_{k-1} \leq v < u} S_v = \lim_{u \nearrow t} m_u.$$

Hence, $t \mapsto m_t$ is continuous on $\tilde{\Omega}$ for all $t \in [0, T]$.

(iii) Fix $0 \leq t_n < t_{n+1} \leq T$ and define

$$\mathcal{I}^n(\omega) := \left\{ [\varrho_{k-1}(\omega), \tau_k(\omega)] \cap [t_n, t_{n+1}] \,\middle|\, k \in \mathbb{N},\ [\varrho_{k-1}(\omega), \tau_k(\omega)] \cap [t_n, t_{n+1}] \neq \emptyset \right\}$$

$$\mathcal{J}^n(\omega) := \left\{ [\tau_k(\omega), \varrho_k(\omega)] \cap [t_n, t_{n+1}] \,\middle|\, k \in \mathbb{N},\ [\tau_k(\omega), \varrho_k(\omega)] \cap [t_n, t_{n+1}] \neq \emptyset \right\}$$

for every $\omega \in \Omega$. Note that m is decreasing on $[\varrho, \tau] \in \mathcal{I}^n$ and increasing on $[\tilde{\tau}, \tilde{\varrho}] \in \mathcal{J}^n$. Hence

$$|m_\varrho - m_\tau| \leq m_\varrho \quad \textit{for } [\varrho, \tau] \in \mathcal{I}^n$$

and

$$|m_{\tilde{\tau}} - m_{\tilde{\varrho}}| \leq m_{\tilde{\varrho}} \quad \textit{for } [\tilde{\tau}, \tilde{\varrho}] \in \mathcal{J}^n.$$

Using the triangle inequality

$$\begin{aligned} |m_{t_{n+1}} - m_{t_n}| &\leq \sum_{[\varrho,\tau]\in\mathcal{I}^n} |m_\varrho - m_\tau| + \sum_{[\tilde{\tau},\tilde{\varrho}]\in\mathcal{J}^n} |m_{\tilde{\tau}} - m_{\tilde{\varrho}}| \\ &\leq \sum_{[\varrho,\tau]\in\mathcal{I}^n} m_\varrho + \sum_{[\tilde{\tau},\tilde{\varrho}]\in\mathcal{J}^n} m_{\tilde{\varrho}} \leq 2\sum_{\varrho\leq T} m_\varrho. \end{aligned}$$

Define the set

$$\mathcal{I}(\omega) := \{\varrho_k(\omega) \mid \varrho_k(\omega) \leq T,\ k \in \mathbb{N}_0\}$$

and

$$K(\omega) := |\mathcal{I}(\omega)|.$$

By (ii) of Lemma 2.6

$$\mathbb{P}(K < \infty) = 1.$$

For any time grid $0 = t_0 < t_1 < t_N = T$ it exists $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that

$$\sum_{n=0}^{N-1} |m_{t_{n+1}} - m_{t_n}| \leq 2\sum_{\varrho\leq T} m_\varrho = 2\sum_{\varrho\in\mathcal{I}} m_\varrho \leq 2K \max_{\varrho_0,\ldots,\varrho_K}\{m_\varrho\} < \infty$$

holds true for all $\omega \in \tilde{\Omega}$. Since $2K\max_{\varrho_0,\ldots,\varrho_K}\{m_\varrho\}$ does not depend on the choice of the time grid we have

$$\sup_{0=t_0<\ldots<t_n=T} \left(\sum_{n=0}^{N-1} |m_{t_{n+1}} - m_{t_n}|\right) < \infty \quad a.s.$$

Hence, m is of finite variation.

(iv) The Lipschitz continuity of μ and (2.17) leads to

$$\begin{aligned} I_1 &:= \mathbb{E}\left[\int_{t_n}^{t_{n+1}} \left|\mu(S_u)\frac{S_u}{m_u}\right| du\right] \leq \bar{s}\mathbb{E}\left[\int_{t_n}^{t_{n+1}} |\mu(S_u)|\, du\right] \\ &\leq \bar{s}\mathbb{E}\left[K\int_{t_n}^{t_{n+1}} S_u du\right] + |\mu(0)|(t_{n+1} - t_n) \\ &\leq \bar{s}K\mathbb{E}\left[\int_{t_n}^{t_{n+1}} S_u^2 du\right] + (\bar{s}K + |\mu(0)|)\,(t_{n+1} - t_n) \\ &\leq \bar{s}\,\underbrace{(KC_S + |\mu(0)|)}_{:=C_{dt}}(t_{n+1} - t_n). \end{aligned} \tag{A.8}$$

Towards a bound for I_2 let $\mathcal{I}^n$ and $\mathcal{J}^n$ be the sets of stochastic intervals from the proof of (iii). Define

$$\underline{X}_u := \inf_{\varrho\leq v\leq u}\left\{\left(\int_0^v \mu(S_r) - \frac{1}{2}\sigma^2(S_r)\right) dr + \int_0^v \sigma(S_r)dW_r\right\}$$

and

$$\overline{X}_u := \sup_{\tilde{\tau} \le v \le u} \left\{ \left(\int_0^v \mu(S_r) - \frac{1}{2}\sigma^2(S_r) \right) dr + \int_0^v \sigma(S_r) dW_r \right\}$$

to calculate

$$\begin{aligned}
I_2 &:= \mathbb{E}\left[\int_{t_n}^{t_{n+1}} \left| \frac{S_u}{m_u} \right| |d\ln(m_u)| \right] \\
&= \mathbb{E}\left[\sum_{[\varrho,\tau]\in\mathcal{I}^n} \int_\varrho^\tau \left| \frac{S_u}{m_u} \right| \left| d\ln\left(\inf_{\varrho\le v\le u} S_v \right) \right| \right] + \mathbb{E}\left[\sum_{[\tilde{\tau},\tilde{\varrho}]\in\mathcal{J}^n} \int_{\tilde{\tau}}^{\tilde{\varrho}} \left| \frac{\bar{s}S_u}{M_u} \right| \left| d\ln\left(\frac{1}{\bar{s}} \sup_{\varrho\le v\le u} S_v \right) \right| \right] \\
&\le \bar{s} \sum_{[\varrho,\tau]\in\mathcal{I}^n} \mathbb{E}\left[\int_\varrho^\tau \left| d\ln\left(\inf_{\varrho\le v\le u} S_v \right) \right| \right] + \bar{s} \sum_{[\tilde{\tau},\tilde{\varrho}]\in\mathcal{J}^n} \mathbb{E}\left[\int_{\tilde{\tau}}^{\tilde{\varrho}} \left| d\ln\left(\sup_{\varrho\le v\le u} S_v \right) \right| \right] \\
&\le \bar{s} \sum_{[\varrho,\tau]\in\mathcal{I}^n} \mathbb{E}\left[\int_\varrho^\tau |d\underline{X}_u| \right] + \bar{s} \sum_{[\tilde{\tau},\tilde{\varrho}]\in\mathcal{J}^n} \mathbb{E}\left[\int_{\tilde{\tau}}^{\tilde{\varrho}} |d\overline{X}_u| \right] \le \bar{s}\mathbb{E}\left[\int_{t_n}^{t_{n+1}} |d\underline{X}_u| \right] + \bar{s}\mathbb{E}\left[\int_{t_n}^{t_{n+1}} |d\overline{X}_u| \right] \\
&= \bar{s}\mathbb{E}\left[\left| \underline{X}_{t_{n+1}} - \underline{X}_{t_n} \right| \right] + \bar{s}\mathbb{E}\left[|\overline{X}_{t_{n+1}} - \overline{X}_{t_n}| \right] = \bar{s}\mathbb{E}\left[\underline{X}_{t_n} - \underline{X}_{t_{n+1}} \right] + \bar{s}\mathbb{E}\left[\overline{X}_{t_{n+1}} - \overline{X}_{t_n} \right]
\end{aligned} \tag{A.9}$$

Note

$$\underline{X}_{t_n} - \underline{X}_{t_{n+1}} = \sup_{0\le u\le t_{n+1}-t_n} \int_{t_n}^{t_n+u} \left(\mu(S_r) - \frac{1}{2}\sigma^2(S_r) \right) dr + \int_{t_n}^{t_n+u} \sigma(S_r) W_r$$

and

$$\overline{X}_{t_{n+1}} - \overline{X}_{t_n} = \sup_{0\le u\le t_{n+1}-t_n} \int_{t_n}^{t_n+u} \left(\mu(S_r) - \frac{1}{2}\sigma^2(S_r) \right) dr + \int_{t_n}^{t_n+u} \sigma(S_r) W_r.$$

By the Lipschitz continuity of μ

$$\overline{X}_{t_{n+1}} - \overline{X}_{t_n} \le K \int_{t_n}^{t_{n+1}} S_r^2 dr + (K + |\mu(0)|)(t_{n+1} - t_n) + \sup_{0\le u\le t_{n+1}-t_n} \int_{t_n}^{t_n+u} \sigma(S_r) W_r$$

Using (2.17), the BDG inequality and Jensens inequality leads to

$$\mathbb{E}[\overline{X}_{t_{n+1}} - \overline{X}_{t_n}] \le (KC_s + K + |\mu(0))|(t_{n+1} - t_n) + \tilde{C}_{BDG} \left(\mathbb{E}\left[\left(\int_{t_n}^{t_{n+1}} \sigma^2(S_r) dr \right) \right] \right)^{\frac{1}{2}}.$$

Since

$$\mathbb{E}\left[\int_{t_n}^{t_{n+1}} \sigma^2(S_r) dr \right] \le K\mathbb{E}\left[\int_{t_n}^{t_{n+1}} (1 + S_r^2) dr \right] \le K(1 + C_s)(t_{n+1} - t_n)$$

by the growth bound of σ, we arrive at

$$\begin{aligned}
\mathbb{E}[\overline{X}_{t_{n+1}} - \overline{X}_{t_n}] &\le (KC_s + K + |\mu(0)|)(t_{n+1} - t_n) + \tilde{C}_{BDG} \sqrt{K(1 + C_s)(t_{n+1} - t_n)} \\
&\le \left(KC_s + K + |\mu(0)| + \tilde{C}_{BDG} \sqrt{K(1 + C_s)} \right) \sqrt{(t_{n+1} - t_n)}.
\end{aligned}$$

Hence by (A.9)

$$I_2 \le 2\bar{s}\underbrace{\left(KC_s + K + |\mu(0)| + \tilde{C}_{BDG}\sqrt{K(1+C_s)}\right)}_{:=C_m}\sqrt{(t_{n+1}-t_n)}.$$

(v) The process S is a continuous semimartingale. The process m is continuous by (ii) and of finite variation by (iii), hence a continuous semimartingale as well. By Itôs rule $Y = h(S,m)$ for $h(x,y) = \frac{x}{y}$ is a continuous semimartingale. Applying Itôs product rule to Y leads to

$$\begin{aligned} d\left(\frac{S_t}{m_t}\right) &= \frac{1}{m_t}dS_t + S_t d\left(\frac{1}{m_t}\right) + d\left\langle S, \frac{1}{m_t}\right\rangle_t \\ &= \mu(S_t)\frac{S_t}{m_t}dt + \sigma(S_t)\frac{S_t}{m_t}dW_t + S_t d\left(\frac{1}{m_t}\right) + d\left\langle S, \frac{1}{m_t}\right\rangle_t \\ &= \underbrace{\mu(S_t)\frac{S_t}{m_t}dt + \frac{S_t}{m_t}d\ln(m_t)}_{:=A_t} + \underbrace{\sigma(S_t)\frac{S_t}{m_t}dW_t}_{:=N_t} \end{aligned}$$

Define

$$A_t^1 := \int_0^t \mu(S_u)\frac{S_u}{m_u}du \quad and \quad A_t^2 := \int_0^t \frac{S_u}{m_u}d\ln(m_u).$$

For any time grid $0 = t_0 < t_1 < ... < t_N = T$ observe

$$\sum_{n=0}^{N-1}|A_{t_{n+1}}^1 - A_{t_n}^1| \le \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\left|\mu(S_u)\frac{S_u}{m_u}\right|du = \int_0^T\left|\mu(S_u)\frac{S_u}{m_u}\right|du$$

Hence we find by (2.19)

$$\mathbb{E}\left[\sup_{0=t_0<...<t_N=T}\sum_{n=0}^{N-1}\left|A_{t_{n+1}}^1 - A_{t_n}^1\right|\right] \le \mathbb{E}\left[\int_0^T\left|\mu(S_u)\frac{S_u}{m_u}\right|du\right] \le \bar{s}C_{dt}T.$$

Therefore A^1 is of finite variation, since the above equation implies

$$\mathbb{P}\left(\sup_{0=t_0<...<t_N=T}\sum_{n=0}^{N-1}\left|A_{t_{n+1}}^1 - A_{t_n}^1\right| < \infty\right) = 1.$$

Similar A^2 is of finite variation, since by (2.20)

$$\mathbb{E}\left[\sup_{0=t_0<...<t_N=T}\sum_{n=0}^{N-1}|A_{t_{n+1}}^2 - A_{t_n}^2|\right] \le \mathbb{E}\left[\int_0^T\left|\frac{S_u}{m_u}\right||d\ln(m_u)|\right] \le 2\bar{s}C_m\sqrt{T}.$$

Therefore

$$A = A^1 + A^2$$

is of finite variation. The process N is a local martingale. Moreover,

$$\mathbb{E}\left[\int_0^T \left|\sigma(S_t)\frac{S_t}{m_t}\right|^2 d\langle W\rangle_t\right] \le \bar{s}^2\mathbb{E}\left[\int_0^T |\sigma^2(S_t)|\,dt\right] \le \bar{s}^2K\mathbb{E}\left[\int_0^T (1+S_t^2)\,dt\right]$$
$$\le \bar{s}^2K + \bar{s}^2K\mathbb{E}\left[\int_0^T S_t^2dt\right] \le \bar{s}^2K\left(1+C_S\right).$$

Hence N is a true martingale, since $|\sigma^2(S_u)\frac{S_u}{m_u}| \in L^2(W)$.

□

A.3 Proof of Lemma 2.8

Proof. We closely follow the proof from [15, Proposition 2.1]. For fixed but arbitrary $T > 0$ we show that the SDE (2.24) holds true on the stochastic interval $[[0, \sigma_1 \wedge T]]$. Fix $0 < \epsilon < \bar{s} - 1$. Define inductively the sequences of stopping times $(\zeta_k)_{k\in\mathbb{N}_0}$ and $(\eta_k)_{k\in\mathbb{N}}$ by $\zeta_0 = 0$ and for or $k \in \mathbb{N}$

$$\eta_k = \inf\left\{t \mid \zeta_{k-1} < t \le \tau_1, \frac{S_t}{m_t} \ge 1+\epsilon\right\} \wedge T,$$

$$\zeta_k = \inf\left\{t \mid \eta_k < t \le \tau_1, \frac{S_t}{m_t} \le 1+\frac{\epsilon}{2}\right\} \wedge T.$$

Note that $\frac{S_{\tau_1}}{m_{\tau_1}} \ge \bar{s} \ge 1+\epsilon$ holds true. Therefore

$$(\eta_k)_{k\in\mathbb{N}_0} \nearrow \tau_1 \wedge T \ a.s. \quad and \quad (\zeta_k)_{k\in\mathbb{N}} \nearrow \tau_1 \wedge T \ a.s.,$$

since otherwise $\mathbb{P}\left(\tau_1 = \infty\right) > 0$ in contradiction to Lemma 2.6. Define the partitions

$$L^\epsilon = \bigcup_{k\in\mathbb{N}} [[\zeta_{k-1}, \eta_k]], \quad R^\epsilon = \bigcup_{k\in\mathbb{N}} [[\eta_k, \zeta_k]]$$

of $]]0, \tau_1 \wedge T]]$ into $L^\epsilon \cup R^\epsilon$. Moreover, the set R^ϵ is predictable and the stochastic integral $\int_0^\cdot \mathbb{1}_{R^\epsilon}(u)dm_u$ is well defined. By definition, the process m is constant on R^ϵ, such that

$$\int_0^t \mathbb{1}_{\mathbb{R}}(u)dm_u = \sum_{k=1}^\infty \int_0^t \mathbb{1}_{]]\eta_k,\zeta_k]]}(u)dm_u = 0 \quad for\ 0 \le t \le \tau_1 \wedge T.$$

Assumption 1 and Itôs rule applied to $\tilde{S} = mg\left(\frac{S}{m}\right)$ gives

$$\int_0^t \mathbb{1}_{\mathbb{R}}(u)d\tilde{S}_u = \int_0^t \mathbb{1}_{\mathbb{R}}(u)\left[g'\left(\frac{S_u}{m_u}\right)dS_u + \frac{1}{2m_u}g''\left(\frac{S_u}{m_u}\right)d\langle S,S\rangle_u\right]. \tag{A.10}$$

We aim to show

$$\mathbb{P} - \lim_{\epsilon\to 0}\sup_{0\le t\le\tau_1}\left|\int_0^t \mathbb{1}_{L^\epsilon}(u)(d\tilde{S}_u - dS_u)\right| = 0. \tag{A.11}$$

Fix $\omega \in \Omega$ and $k \ge 1$ such that $\eta_{k+1}(\omega) < \tau_1(\omega) \wedge T$
The second order Taylor expansion of g in 1 leads with $g(1) = g'(1) = 1$ to

$$g\left(1+\frac{\epsilon}{2}\right) = g(1) + g'(1)\frac{\epsilon}{2} + g''(\xi)\frac{\epsilon^2}{4} \le 1 + \frac{\epsilon}{2} + g''(\xi)\epsilon^2 \tag{A.12}$$

for some $\xi \in (1, 1+\frac{\epsilon}{2})$. Using Assumption 1, the definition of τ_k and the second order Taylor expansion from above

$$\begin{aligned}
\left|S_{\zeta_k}(\omega) - \tilde{S}_{\zeta_k}(\omega)\right| &= \left|S_{\zeta_k}(\omega) - m_{\zeta_k}(\omega) g\left(\frac{S_{\zeta_k}(\omega)}{m_{t_k}(\omega)}\right)\right| \\
&= \left|S_{\zeta_k}(\omega) - \frac{S_{\zeta_k}(\omega)}{1+\frac{\epsilon}{2}} g\left(1+\frac{\epsilon}{2}\right)\right| \\
&= \left|S_{\zeta_k}(\omega)\left(1 - \frac{1}{1+\frac{\epsilon}{2}} g\left(1+\frac{\epsilon}{2}\right)\right)\right| \\
&\leq \left|S_{\zeta_k}(\omega)\left(1 - \frac{1+\frac{\epsilon}{2}}{1+\frac{\epsilon}{2}} - \frac{g''(\xi)\epsilon^2}{1+\frac{\epsilon}{2}}\right)\right| \\
&\leq S_{\zeta_k}(\omega)\left|g''(\xi)\right|\epsilon^2 \leq \underbrace{\max_{0\leq t\leq T} S_t(u) L_{g'}}_{C(\omega)} \epsilon^2,
\end{aligned}$$

where $L_{g'}$ exists by assumption 1. The constant $C(\omega)$ does not depend on ϵ and k. We even find for fixed k

$$|S_t(\omega) - \tilde{S}_t(\omega)| \leq C(\omega)\epsilon^2 \quad \text{for } \zeta_k(\omega) \leq t \leq \eta_k(\omega). \tag{A.13}$$

Define the random number

$$N^\epsilon := \sup\{k \in \mathbb{N} \mid \zeta_k < \tau_1 \wedge T\}$$

of upcrossings from below $1+\frac{\epsilon}{2}$ to above $1+\epsilon$ and observe by (A.13)

$$\sup_{0\leq t\leq \tau_1}\left|\int_0^t \mathbb{1}_{L^\epsilon}(u)(d\tilde{S}_u - dS_u)\right| \leq (N^\epsilon + 1)\, C\epsilon^2. \tag{A.14}$$

In the notation of [30] we have

$$U\left(Y, 1+\frac{\epsilon}{2}, 1+\epsilon\right) = N^\epsilon.$$

By (v) of Lemma 2.7 we know that Y is a continuous semimartingale with decomposition

$$Y = A + N$$

for a process of finite variation A and a martingale N. Hence, we may apply the main theorem from [30] to Y and deduce

$$\mathbb{E}\left[\left|\sup_{a\in[1,\bar{s}]} U(Y, a, a+\frac{\epsilon}{2})\right|\right] \leq \frac{2}{\epsilon} c_1^2\, \underbrace{\mathbb{E}\left[\left|\sup_{t\in[0,T]} N_t\right| + \left|\int_0^T |dA_s|\right|\right]}_{=:C_N} = \frac{2}{\epsilon} c_1 C_N$$

and hence

$$\begin{aligned}
\lim_{\epsilon\to 0}\mathbb{E}\left[\left|(N^\epsilon+1)C\epsilon^2\right|\right] &= \lim_{\epsilon\to 0}\epsilon^2\left(\mathbb{E}\left[|N^\epsilon C|\right]+\mathbb{E}\left[|C|\right]\right)\\
&\leq \lim_{\epsilon\to 0}\epsilon^2\left(\mathbb{E}\left[N^\epsilon C\right]+\left(1+\mathbb{E}\left[\max_{t\in[0,T]}S_t^2\right]\right)\right)\\
&\leq \lim_{\epsilon\to 0}\epsilon^2\left(\mathbb{E}\left[N^\epsilon C\right]+(1+C_S)\right)\\
&= \lim_{\epsilon\to 0}\epsilon^2\mathbb{E}\left[N^\epsilon C\right]\\
&\leq \lim_{\epsilon\to 0}\epsilon^2\mathbb{E}\left[N^\epsilon\right]\left\|\max_{t\in[0,T]}S_t^2\right\|_{L^\infty}\\
&\overset{(A.3)}{\leq} \lim_{\epsilon\to 0}\epsilon c_1 C(\omega)\\
&= 0
\end{aligned}$$

where we used Hölder's inequality. In particular combined with (A.14)

$$\mathbb{P}-\lim_{\epsilon\to 0}\sup_{0\leq t\leq\tau_1}\left|\int_0^t \mathbb{1}_{L^\epsilon}(u)(d\tilde{S}_u-dS_u)\right| \leq \mathbb{P}-\lim_{\epsilon\to 0}\left(N^\epsilon+1\right)C\epsilon^2=0,$$

since convergence in $\boldsymbol{L}^1$ implies convergence in probability and hence (A.11). Moreover for $t>0$ we find

$$\begin{aligned}
\lim_{\epsilon\to 0}\left(Leb\otimes\mathbb{P}\right)\left(L^\epsilon\right) &= \lim_{\epsilon\to 0}\left(Leb\otimes\mathbb{P}\right)\left(\left\{(t,\omega)\mid \frac{S_t}{m_t}\in\left[1+\frac{\epsilon}{2},1+\epsilon\right]\right\}\right)\\
&= Leb\otimes\mathbb{P}\left(\bigcap_{n\in\mathbb{N}}\left\{(t,\omega)\mid\frac{S_t}{m_t}\in\left[1+\frac{1}{2n},1+\frac{1}{n}\right]\right\}\right)\\
&= Leb\otimes\mathbb{P}\left(\left\{(t,\omega)\mid S_t=\inf_{0\leq u\leq t}S_u\right\}\right)\\
&= 0.
\end{aligned}\tag{A.15}$$

This implies combined with (A.3)

$$\mathbb{P}-\lim_{\epsilon\to 0}\sup_{0\leq t\leq\tau_1}\left|\int_0^t \mathbb{1}_{L^\epsilon}(u)dS_u\right| = \mathbb{P}-\lim_{\epsilon\to 0}\sup_{0\leq t\leq\tau_1}\left|\int_0^t \mathbb{1}_{L^\epsilon}(u)d\tilde{S}_u\right|=0\tag{A.16}$$

For $0\leq t\leq\tau_1\wedge T$ the claim follows by

$$\begin{aligned}
\tilde{S}_t &= 1+\int_0^t \mathbb{1}_{[[0,\tau_1\wedge T]]}d\tilde{S}_u+1+\lim_{\epsilon\to 0}\int_0^t\left(\mathbb{1}_{L^\epsilon}+\mathbb{1}_{R^\epsilon}\right)d\tilde{S}_u \overset{(A.16)}{=} 1+\lim_{\epsilon\to 0}\int_0^t\mathbb{1}_{R^\epsilon}d\tilde{S}_u\\
&\overset{(A.10)}{=} 1+\lim_{\epsilon\to 0}\int_0^t\mathbb{1}_{R^\epsilon}(u)\left[g'\left(\frac{S_u}{m_u}\right)dS_u+\frac{1}{2m_u}g''\left(\frac{S_u}{m_u}\right)d\langle S,S\rangle_u\right]\\
&\overset{(A.15)}{=} 1+\int_0^t\left(\mathbb{1}_{R^\epsilon}(u)+\mathbb{1}_{L^\epsilon}(u)\right)\left[g'\left(\frac{S_u}{m_u}\right)dS_u+\frac{1}{2m_u}g''\left(\frac{S_u}{m_u}\right)d\langle S,S\rangle_u\right]\\
&= 1+\int_0^t\left[g'\left(\frac{S_u}{m_u}\right)dS_u+\frac{1}{2m_u}g''\left(\frac{S_u}{m_u}\right)d\langle S,S\rangle_u\right].
\end{aligned}$$

In order to show (2.24) on $\tau_1 \wedge T \leq t \leq \varrho_1 \wedge T$ define inductively for $0 < \epsilon < \bar{s} - 1$ the sequences of stopping times $(\hat{\zeta}_k)_{k\in\mathbb{N}_0}$ and $(\hat{\eta}_k)_{k\in\mathbb{N}}$ by $\hat{\zeta}_0 = \tau_1$ and for or $k \in \mathbb{N}$

$$\hat{\eta}_k = \inf\left\{t \mid \hat{\zeta}_{k-1} < t \leq \varrho_1, \frac{S_t}{m_t} \leq \bar{s} - \epsilon\right\} \wedge T,$$

$$\hat{\zeta}_k = \inf\left\{t \mid \hat{\eta}_k < t \leq \varrho_1, \frac{S_t}{m_t} \geq \bar{s} - \epsilon\right\} \wedge T.$$

The claim follows by the same arguments using $g(\bar{s}) = (1-\lambda)\bar{s}$ and $g'(\bar{s}) = 1 - \lambda$ in the second order Taylor expansion. Therefore (2.24) holds true for $\tau_1 \wedge T \leq t \leq \varrho_1 \wedge T$. By the strong Markov property of S a repetition of the argument for all $k \in \mathbb{N}$ shows (2.24) on

$$\bigcup_{k\in\mathbb{N}} (]]\varrho_{k-1}, \tau_k]]\cup]]\tau_k, \varrho_k]]) \cap [0, T], \quad a.s.$$

The claim follows by Lemma 2.6. □

A.4 Derivation of the Constant L_f

Fix $s \in [0, \bar{s}]$. The smooth pasting conditions (2.22) and (2.22) imply

$$|g'(s)| \leq L_g$$

for some $L_g > 0$. Moreover, define $g'_{\min} := \min_{s\in[0,\bar{s}]}\{g(s)\} > 0$. We find

$$|\mu(s)| \leq \sigma^2 \bar{s} L_g^2 =: \tilde{\mu}_{max}$$

and

$$|\tilde{\sigma}(s)| \leq \sigma \bar{s} L_g =: \tilde{\sigma}_{max}.$$

By lemma 4.3 in [15]

$$g''(s) = \frac{2g'(s)^2}{c + g(s)} - \frac{2\theta g'(s)}{s}$$

and hence

$$|g''(s)| \leq \frac{2g'(s)^2}{c + g(s)} + \frac{2\theta g'(s)}{s} \leq 2L_g^2 + 2L_g =: L_{g'}.$$

We calculate

$$\begin{aligned}
|\tilde{\mu}'(s)| &\leq \left|\left[\sigma^2\left(2g'(s)g''(s)s + g'(s)^2\right)\right] g(s)(1 + g(s)) - \sigma^2 g'(s)^2 s\left[g'(s)c + 2g(s)g'(s)\right]\right| \\
&\leq (\bar{s} + \bar{s}^2)\left[2\sigma^2 L_g L_{g'}\bar{s} + \sigma^2 L_g^2\right] + \sigma^2 L_g^2 \bar{s}\left[L_g c + 2\bar{s} L_g\right] =: L_{\tilde{\mu}}
\end{aligned}$$

and

$$|\tilde{\sigma}'(s)| \leq \left|\left[\sigma g''(s)s + \sigma g'(s)\right] g(s) - \sigma g'(s) s g'(s)\right| \leq \bar{s}^2 \sigma L_{g'} + \sigma L_g \bar{s} + \sigma \bar{s} L_g^2 =: L_{\tilde{\sigma}}.$$

In total

$$|f'(s)| = 2\left|\frac{\tilde{\mu}(s)\tilde{\mu}'(s)\tilde{\sigma}^2(s) - \tilde{\mu}^2(s)\tilde{\sigma}(s)\tilde{\sigma}'(s)}{\tilde{\sigma}^4(s)}\right| \leq \frac{2\tilde{\mu}_{max} L_{\tilde{\mu}} \tilde{\sigma}_{max}^2}{\sigma^4 g'_{min}} + \frac{2\tilde{\mu}_{max}^2 \tilde{\sigma}_{max} L_{\tilde{\sigma}}}{\sigma^4 g'_{min}} =: L_f.$$

A.5 Processes Dynamics Towards a Hypothesis Set

In the setting of Section 3.3.1 we calculate the dynamics of the process $\frac{1}{P}$

$$
\begin{aligned}
d\left(\frac{1}{P_t}\right) &= -\frac{1}{P_t^2}P_t\mu(P_t)dt - \frac{1}{P_t^2}P_t\sigma(P_t)dW_t + \frac{1}{2}\frac{2}{P_t^3}P_t^2\sigma(P_t)^2dt \\
&= \left\{-\frac{1}{P_t}\mu(P_t) + \frac{1}{P_t}\sigma(P_t)^2\right\}dt - \frac{1}{P_t}\sigma(P_t)dW_t,
\end{aligned}
$$

the dynamics of then process X

$$
\begin{aligned}
dX_t &= d\left(\frac{1}{\gamma}\frac{\tilde{S}_t}{P_t} - 1\right) = \frac{1}{\gamma}d\left(\frac{\tilde{S}_t}{P_t}\right) = \frac{1}{\gamma}\left(\frac{1}{P_t}\right)d\tilde{S}_t + \frac{1}{\gamma}\tilde{S}_t d\left(\frac{1}{P_t}\right) + \frac{1}{\gamma}d\left\langle \tilde{S}, \frac{1}{P}\right\rangle_t \\
&= \frac{S_t}{\gamma P_t}\left[\tilde{\mu}\left(Y_t\right)dt + \tilde{\sigma}\left(Y_t\right)dW_t\right] \\
&+ \frac{1}{\gamma}\tilde{S}_t\left[-\frac{1}{P_t}\mu(P_t) + \frac{1}{P_t}\sigma(P_t)^2dt - \frac{1}{P_t}\sigma(P_t)dW_t\right] - \frac{1}{\gamma}S_t\tilde{\sigma}\left(Y_t\right)\frac{1}{P_t}\sigma(P_t)dt \\
&= \frac{1}{\gamma}\underbrace{(1+\gamma)\left\{\tilde{\mu}\left(Y_t\right) - \tilde{\sigma}\left(Y_t\right)\sigma(P_t)\right\}}_{=:\tilde{a}_1(Y_t,P_t)}dt + \frac{1}{\gamma}\underbrace{\frac{\tilde{S}_t}{P_t}\left[\sigma(P_t)^2 - \mu(P_t)\right]}_{=:\tilde{a}_2\left(Y_t,P_t,\tilde{S}_t\right)}dt \\
&+ \frac{1}{\gamma}\underbrace{(1+\gamma)\tilde{\sigma}\left(Y_t\right)}_{:=\tilde{b}_1(Y_t)}dW_t - \frac{1}{\gamma}\underbrace{\frac{\tilde{S}_t}{P_t}\sigma(P_t)}_{:=\tilde{b}_2(P_t,\tilde{S}_t)}dW_t \\
&= \frac{1}{\gamma}\tilde{a}\left(Y_t, P_t, \tilde{S}_t\right)dt + \frac{1}{\gamma}\tilde{b}\left(Y_t, P_t, \tilde{S}_t\right)dW_t.
\end{aligned}
$$

The dynamics of the process $\tilde{S}^\alpha$ are given by

$$
\begin{aligned}
d\tilde{S}_t^\alpha &= \frac{\partial g_\alpha}{\partial p}(P_t, X_t)dP_t + \frac{\partial g_\alpha}{\partial x}(P_t, X_t)dX_t \\
&+ \frac{1}{2}\frac{\partial^2 g_\alpha}{\partial x \partial p}(P_t, X_t)d\langle P, X\rangle_t + \frac{1}{2}\frac{\partial^2 g_\alpha}{\partial p \partial x}(P_t, X_t)d\langle X, P\rangle_t + \frac{1}{2}\frac{\partial^2 g_\alpha}{\partial^2 x}(P_t, X_t)\langle X, X\rangle_t \\
&= \frac{\partial g_\alpha}{\partial p}(P_t, X_t)P_t\mu(P_t)dt + \frac{\partial g_\alpha}{\partial p}(P_t, X_t)P_t\sigma(P_t)dW_t \\
&+ \frac{\partial g_\alpha}{\partial x}(P_t, X_t)\tilde{a}\left(Y_t, P_t, \tilde{S}_t\right)dt + \frac{\partial g_\alpha}{\partial x}(P_t, X_t)\tilde{b}\left(Y_t, P_t, \tilde{S}_t\right)dW_t \\
&+ \frac{\partial^2 g_\alpha}{\partial x \partial p}(P_t, X_t)P_t\sigma(P_t)\tilde{b}\left(Y_t, P_t, \tilde{S}_t\right)dt + \frac{1}{2}\frac{\partial^2 g_\alpha}{\partial^2 x}(P_t, X_t)\tilde{b}^2\left(Y_t, P_t, \tilde{S}_t\right)dt \\
&\hat{S}_t^\alpha \left\{\frac{1}{\tilde{S}_t^\alpha}\frac{\partial g_\alpha}{\partial p}(P_t, X_t)P_t\mu(P_t) + \frac{1}{\tilde{S}_t^\alpha}\frac{\partial g_\alpha}{\partial x}(P_t, X_t)\tilde{a}\left(Y_t, P_t, \tilde{S}_t\right)\right\}dt \\
&+ \hat{S}_t^\alpha \left\{\frac{1}{\tilde{S}_t^\alpha}\frac{\partial^2 g_\alpha}{\partial x \partial p}(P_t, X_t)P_t\sigma(P_t)\tilde{b}\left(Y_t, P_t, \tilde{S}_t\right) + \frac{1}{\tilde{S}_t^\alpha}\frac{1}{2}\frac{\partial^2 g_\alpha}{\partial^2 x}(P_t, X_t)\tilde{b}^2\left(Y_t, P_t, \tilde{S}_t\right)\right\}dt \\
&+ \hat{S}_t^\alpha \left\{\frac{1}{\tilde{S}_t^\alpha}\frac{\partial g_\alpha}{\partial p}(P_t, X_t)P_t\sigma(P_t) + \frac{1}{\tilde{S}_t^\alpha}\frac{\partial g_\alpha}{\partial x}(P_t, X_t)\tilde{b}\left(Y_t, P_t, \tilde{S}_t\right)\right\}dW_t \qquad \text{(A.17)} \\
&= \hat{S}_t^\alpha \underbrace{\left\{\mu(P_t) + \frac{\alpha'(X_t)}{1+\gamma\alpha(X_t)}\tilde{a}(Y_t, P_t, \tilde{S}_t)\right\}}_{=:\hat{\mu}_1\left(Y_t, P_t, \tilde{S}_t, X_t\right)} dt \\
&+ \hat{S}_t^\alpha \underbrace{\left\{\frac{\alpha'(X_t)}{1+\gamma\alpha(X_t)}\sigma(P_t)\tilde{b}(Y_t, P_t, \tilde{S}_t) + \frac{1}{2\gamma}\frac{\alpha''(X_t)}{1+\gamma\alpha(X_t)}\tilde{b}^2(Y_t, P_t, \tilde{S}_t)\right\}}_{=:\hat{\mu}_2\left(Y_t, P_t, \tilde{S}_t, X_t\right)} dt \\
&+ \hat{S}_t^\alpha \underbrace{\left\{\sigma(P_t) + \frac{\alpha'(X_t)}{1+\gamma\alpha(X_t)}\tilde{b}(Y_t, P_t, \tilde{S}_t)\right\}}_{=:\hat{\sigma}\left(Y_t, P_t, \tilde{S}_t, X_t\right)} dW_t
\end{aligned}
$$

where we used $\tilde{S}^\alpha > 0$ to insert a multiplicative one.

Appendix B

Optimal Trading in a Frictionless Market

B.1 The General Setting for Semimartingales

Let $\tilde{S} = (\tilde{S}_t)_{t\in[0,T]}$ be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0,T}, \mathbb{P})$ taking values in the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. A *trading strategy* is a predictable and S-integrable process $\tilde{\varphi}^1$ specifying the amount of shares hold in $\tilde{S}$. For a given *initial endowment* $x > 0$ and a trading strategy $\tilde{\varphi}^1$ the corresponding *wealth process* is defined by

$$\tilde{V}_t^{\tilde{\varphi}^1} := x + \int_0^t \tilde{\varphi}_u^1 d\tilde{S}_u, \quad 0 \le t \le T. \tag{B.1}$$

A *self financing* trading strategy is a trading strategy such that $\tilde{V}_0^{\tilde{\varphi}^1} = x$. A self financing trading strategy is called *admissible* if the wealth process $\tilde{V}^{\tilde{\varphi}^1} \ge 0$ is non-negative. For a given initial endowment $x > 0$ denote by

$$\mathcal{A}(x) := \left\{ \tilde{\varphi}^1 \textit{ is a trading strategy} \mid \tilde{V}_0^{\tilde{\varphi}^1} = x \textit{ and } \tilde{V}^{\tilde{\varphi}^1} \ge 0 \right\}$$

the set of self-financing and admissible trading strategies. Define

$$\tilde{\mathcal{C}}(x) := \left\{ \tilde{V}_T^{\tilde{\varphi}^1} \mid \tilde{\varphi}^1 \in \mathcal{A}(x) \right\}$$

the set of set *of admissible terminal wealth*. For a utility function $U : (0, \infty) \to \mathbb{R}$ the *value function* of the *expected terminal wealth* is defined by

$$\tilde{u}(x) := \sup_{\tilde{V}_T \in \tilde{\mathcal{C}}} \mathbb{E}\left[U(\tilde{V}_T)\right]. \tag{B.2}$$

A self-financing and admissible trading strategy $\tilde{\varphi}^1$ is called *optimal* if the supremum in (B.2) is attained for the associated wealth process [27].

B.2 The Log-Optimal Strategy for Itô Processes

Let $\tilde{S}$ be a stochastic process on the complete and filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]})$ equipped with a filtration generated by Brownian motion $W = (W_t)_{t\in[0,T]}$. Assume further that the coefficient processes of $\tilde{S}$ are $\{\mathcal{F}_t\}_{t\in[0,T]}$ progressive measurable and uniformly bounded in (t, ω) permitting a representation of $\tilde{S}$ as the unique solution of the stochastic differential equation

$$d\tilde{S}_t = \tilde{S}_t \tilde{\mu}_t dt + \tilde{S}_t \tilde{\sigma}_t dW_t.$$

Consider a financial model $(B, \tilde{S})$ with a constant *bank account* $B \equiv 1$. Following the theory of optimal investment in the absence of transaction costs [26], a *trading strategy* $\tilde{\varphi} = (\tilde{\varphi}_t^0, \tilde{\varphi}_t^1)_{t \in [0,T]}$ is an $\mathbb{R}^2$-valued $\{\mathcal{F}_t\}_{t \in [0,T]}$ progressive measurable process with

$$\int_0^T |\tilde{\varphi}_t^0| dt < \infty \ \mathbb{P} - a.s. \quad and \quad \int_0^T \left(\tilde{\varphi}_t^1 \bar{S}_t\right)^2 dt < \infty \ \mathbb{P} - a.s.$$

The corresponding *wealth process* is defined by

$$\tilde{V}_t^{\tilde{\varphi}^1} := \tilde{\varphi}_t^0 B_t + \tilde{\varphi}_t^1 \tilde{S}_t = \tilde{\varphi}_t^0 + \tilde{\varphi}_t^1 \tilde{S}_t.$$

For a given initial endowment $x > 0$ a trading strategy is called *self-financing* if $\tilde{V}^{\tilde{\varphi}^1}$ satisfies (B.1) and $\tilde{V}_0^{\tilde{\varphi}^1} = x$. The trader with initial capital $x > 0$ decides on a self-financing and admissible trading strategy $\tilde{\varphi} = (\tilde{\varphi}_t^0, \tilde{\varphi}_t^1)_{t \in [0,T]} \in \mathcal{A}(x)$. Such a self-financing trading strategy $\tilde{\varphi}$ is equivalently characterized by the *portfolio process* [26]

$$\tilde{\pi}_t = \frac{\tilde{\varphi}_t^1 \tilde{S}_t}{\tilde{V}_t^{\tilde{\varphi}}}.$$

Assuming logarithmic utility $U = \ln = \log_e$ enables to explicit calculate the optimal trading strategy [26]. The optimal terminal wealth is given by

$$\tilde{V}_T^{\tilde{\varphi}^1} = \frac{x}{(T+1)H_T},$$

with

$$H_t := \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right) \quad and \quad \theta_t := \frac{\tilde{\mu}_t}{\tilde{\sigma}_t}. \tag{B.3}$$

The optimal risky fraction is [26]

$$\pi_t^\star = \frac{\tilde{\mu}_t}{\tilde{\sigma}_t^2}. \tag{B.4}$$

Using $\mathbb{E}\left[\int_0^T \theta_s dW_s\right] = 0$ we deduce

$$\tilde{u}(x) = \mathbb{E}\left[\ln(\tilde{V}_T^{\tilde{\varphi}^1})\right] = \frac{1}{2}\mathbb{E}\left[\int_0^T \theta_s^2 ds\right] + \ln\left(\frac{x}{T+1}\right). \tag{B.5}$$

Appendix C

Additional Explanations

C.1 Self-financing Trading Strategies

The integral wrt. to a process of finite variation is defined as a ω-pathwise Stieltjes integral [7]. In the RHS of (2.1) the stochastic integrand or stochastic path $u \mapsto S_u(\omega)$ is integrated against the random signed measure $d\varphi^1 = d\varphi^{1,\uparrow} - d\varphi^{1,\downarrow}$ associated to the Jordan-Hahn decomposition $\varphi^1 = \varphi^1_{0-} + \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ [7]. By definition of this signed measure we find

$$d\varphi^{1,\uparrow}(]s,t]) = \varphi_t^{1,\uparrow} - \varphi_s^{1,\uparrow}$$

and using that the limit from the left side exists

$$d\varphi^{1,\uparrow}(\{s\}) = d\varphi^{1,\uparrow}\left(\cap_{n\in\mathbb{N}}\left(s-\frac{1}{n},s\right]\right) = \lim_{u\nearrow s}\varphi_s^{1,\uparrow} - \varphi_u^{1,\uparrow} = \varphi_s^{1,\uparrow} - \varphi_{s-}^{1,\uparrow}$$

by the continuity of a measure. Hence

$$d\varphi^{1,\uparrow}([s,t]) = d\varphi^{1,\uparrow}(\{s\}) + d\varphi^{1,\uparrow}(]s,t]) = \varphi_t^{1,\uparrow} - \varphi_{s-}^{1,\uparrow}$$

and similar

$$d\varphi^{1,\downarrow}([s,t]) = \varphi_t^{1,\downarrow} - \varphi_{s-}^{1,\downarrow}.$$

The LHS of (2.1) is as well a ω-pathwise Stieltjes integral wrt. the process of finite variation φ^0 with

$$\int_s^t d\varphi^0 = d\varphi^0([s,t]) = \varphi_t^0 - \varphi_{s-}^0.$$

In absence of transaction costs $\lambda = 0$ a self-financing strategy satisfies

$$\varphi_t^0(\omega) \leq \varphi_{s-}^0(\omega) - \int_0^t S_u d\varphi_u^1.$$

For the wealth process $V_t = \varphi_t^0 + \varphi_t^1 S_t$ we therefore find

$$\begin{aligned} dV_t &= d\varphi_t^0 + \varphi_t^1 dS_t + S_t d\varphi_t^1 + d\langle \varphi^1, S\rangle_t \\ &= \varphi_t^0 - \varphi_{s-}^0 + S_u d\varphi_t^1 + \varphi_t^1 dS_t \\ &\leq \varphi_t^1 dS_t \end{aligned}$$

Hence

$$V_t \leq V_0 + \int_0^t \varphi_t^1 dS_t, \quad 0 \leq t \leq T$$

ensures that during the whole trading period no money is added to the amount of initial investment $V_0 > 0$. A withdraw of money is not excluded by this inequality. Requiring as well no withdraw of money during the whole trading period reduces the condition (2.1) for initial endowment $V_0 > 0$ to the definition of a self-financing strategy in a market without transaction costs

$$V_t = V_0 + \int_0^t \varphi_t^1 dS_t, \quad 0 \leq t \leq T,$$

defined in Appendix B.1.

C.2 Initial Investment

Following the arguments of Appendix C.1 we deduce by the right continuity of φ^0

$$\lim_{t \searrow 0} \int_0^t d\varphi_u^0 = \lim_{t \searrow 0} d\varphi^0([0,t]) = \lim_{t \searrow 0} \varphi_t^0 - \varphi_{0-}^0 = \varphi_0^0 - \varphi_{0-}^0$$

By the right continuity of $\varphi^{1,\uparrow}$ and $\varphi^{1,\downarrow}$

$$\lim_{t \searrow 0} \int_0^t S_u d\varphi_u^{1,\uparrow} = S_0(\varphi_0^{1,\uparrow} - \varphi_{0-}^{1,\uparrow}) = S_0 \left(\varphi_0^1 - \varphi_{0-}^1\right)^+ = S_0 \left(\varphi_0^1\right)^+$$

and similar

$$\lim_{t \searrow 0} \int_0^t (1-\lambda) S_u d\varphi_u^{1,\downarrow} = (1-\lambda) S_0 \left(\varphi_0^1\right)^-$$

Hence (2.1) implies

$$\varphi_0^0 - \varphi_{0-}^0 \leq -S_0 \left(\varphi_0^1\right)^+ + (1-\lambda) S_0 \left(\varphi_0^1\right)^- .$$

For an initial endowment $(\varphi_{0-}^0, \varphi_{0-}^1) = (x, 0)$ we deduce

$$x \geq \varphi_0^0 + S_0 \left(\varphi_0^1\right)^+ - (1-\lambda) S_0 \left(\varphi_0^1\right)^- .$$

C.3 The Augmented Natural Filtration

We summarize the properties of filtration given in Cohen and J.Elliott [7].

Definition 22 (Complete filtration). *A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete if*

$$\forall B \in \mathcal{F} \wedge \mathbb{P}(B) = 0 : A \subset B \Rightarrow A \in \mathcal{F}$$

i.e. if subsets of $\mathbb{P}$-nullsets are $\mathcal{F}$-measurable. In particular

$$\mathbb{P}(B) = 0 \wedge A \subset B \Rightarrow \mathbb{P}(A) = 0.$$

A filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is called complete, if $(\Omega, \mathcal{F}_t, \mathbb{P})$ is complete for every $t \in [0, T]$.

Definition 23 (Right-continuous filtration). *[7, 3.1.4] For a filtration $(\mathcal{F}_t)_{t \geq 0}$ define*

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$$

the σ-algebra of events immediately after t. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is called right-continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$.

Definition 24 (Augmented natural filtration). *Let $X = (X_t)_{t\geq 0}$ be a stochastic process on the probability space $(\Omega, \mathcal{F}^0, \mathbb{P})$ and $\mathcal{F}$ the completion of $\mathcal{F}^0$. We say that X is a stochastic process defined on the completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The natural filtration $(\mathcal{F}_t^X)_{t\geq 0}$ of X is defined by $\mathcal{F}_t^X = \sigma\{X_s : s \leq t\}$. Let $\tilde{\mathcal{F}}_t$ be the σ-algebra generated by $\mathcal{F}_t^X$ and the $\mathbb{P}$-null sets of $\mathcal{F}$ and $\mathcal{F}_t := \tilde{\mathcal{F}}_{t+}$. We call $(\mathcal{F}_t)_{t\geq 0}$ the augmented natural filtration of X*

Definition 25 (stopping time). *[7, 3.1.6] Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a filtered measurable space. A random variable $T : \Omega \to [0, \infty]$ is called a random time. A random time is said to be a $\mathcal{F}_t$-stopping time if for every $t \in [0, \infty)$*

$$\{T \leq t\} = \{\omega \in \Omega \,|\, T(\omega) \leq t\} \in \mathcal{F}_t.$$

Definition 26 (predictable stopping time). *[7, 3.1.6] A stopping time T is called predictable, if there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times such that*

- *(i) $(T_n)_{n\in\mathbb{N}}$ is almost surely a nondecreasing sequence in $[0, \infty)$ and $\lim_{n\to\infty} T_n(\omega) = T(\omega)$ a.s.*
- *(ii) on the set $\{T > 0\}$, $T_n(\omega) < T(\omega)$ a.s. for all $n \in \mathbb{N}$.*

The sequence $(T_n)_{n\in\mathbb{N}}$ is said to announce T.

Definition 27 (optional σ-Algebra). *Let $\mathcal{T}$ be the set of all stopping times and $\mathcal{T}_p$ the set of all predictable stopping times. Define*

$$\mathcal{E} := \{[[S, T[[\,|\, S, T \in \mathcal{T}\}$$

and

$$\mathcal{E}_p := \{[[S, T[[\,|\, S, T \in \mathcal{T}_p\}.$$

The smallest σ-algebra Σ containing $\mathcal{E}$ is called the optional σ-Algebra and the smallest σ-algebra Σ_p containing $\mathcal{E}_p$ is called the predictable σ-Algebra.

Remark C.1. *Note that $\Sigma \subset \Sigma_p$ (+progessiv). Moreover, Σ is generated by the adapted càdlàg processes and Σ_p is generated by the left-continuous and adapted processes, even by the continuous and adapted processes.*

Appendix D

Simulation of the Final, Minimal and Maximal Value of Geometric Brownian Motion

We summarize the notation of Becker [3] to simulate the final, minimal and maximal values of a geometric Brownian motion. Let $W = W_{t\in[0,T]}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A Brownian motion with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$ is defined by

$$W_{x,t}^{\mu,\sigma} := x + \mu t + \sigma W_t \quad \textit{for all } t \in [0,T].$$

The *running minimum* and the *running maximum* of $W_x^{\mu,\sigma}$ is

$$m_{x,T}^{\mu,\sigma} := \inf_{t\in[0,T]} W_{x,t}^{\mu,\sigma} \quad \textit{and} \quad M_{x,T}^{\mu,\sigma} := \sup_{t\in[0,T]} W_{x,t}^{\mu,\sigma}.$$

Following [3] to simulate

$$\mathcal{W}_{x,T}^{\mu,\sigma} := \left(W_{x,T}, m_{x,T}^{\mu,\sigma}, M_{x,T}^{\mu,\sigma}\right)$$

it suffices to simulate $\mathcal{W}_{0,1}^{\tilde{\mu},1}$ for an arbitrary $\tilde{\mu} \in \mathbb{R}$, since

$$W_{x,t}^{\mu,\sigma} \overset{d}{=} x + \sigma\sqrt{T} W_{0,\frac{1}{T}}^{\sqrt{T}\frac{\mu}{\sigma}}$$

implies

$$\mathcal{W}_{x,T}^{\mu,\sigma} = \left(x + \sigma\sqrt{T} W_{0,1}^{\sqrt{T}\frac{\mu}{\sigma},1}, x + \sigma\sqrt{T} m_{0,1}^{\sqrt{T}\frac{\mu}{\sigma},1}, x + \sigma\sqrt{T} M_{0,1}^{\sqrt{T}\frac{\mu}{\sigma},1}\right).$$

Let $0 = t_0 < t_1 < ... < t_N = T$ be an equidistant time grid on $[0,T]$. To ease notation we fix $\mu \in \mathbb{R}$ and $\sigma > 0$. Let S be a geometric Brownian motion with

$$S_t = (1+\gamma)\exp\left(W_{0,t}^{\tilde{\mu},\sigma}\right) \quad \textit{with} \quad \tilde{\mu} = \mu - \frac{\sigma^2}{2}.$$

Set $X_t := W_{0,t}^{\tilde{\mu},\sigma}$ for all $t \in [0,T]$. To approximate the process m from Definition 8 we use the method proposed in Becker [3] to simulate the discrete process

$$\mathcal{W}_{n+1} := \left(X_{t_{n+1}}, m_{X_{t_n},\frac{T}{N}}, M_{X_{t_n},\frac{T}{N}}\right) \quad \textit{and} \quad \mathcal{W}_0 := (0,0,0) \tag{D.1}$$

that approximates for a given state X_{t_n} of $W_x^{\tilde{\mu},\sigma}$ at time t_n the subsequent state $X_{t_{n+1}}$,

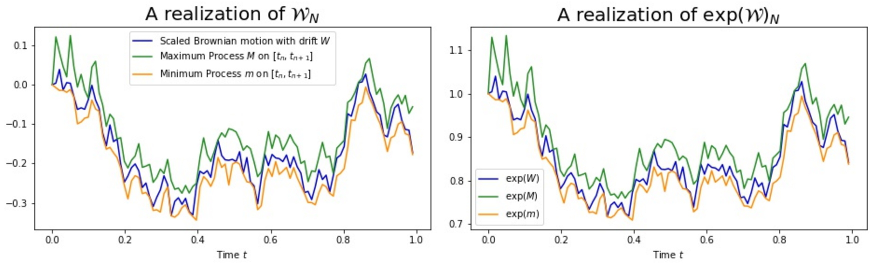

FIGURE D.1: The LHS shows a realizations of $\mathcal{W}$ on $[0, 1]$ over $N = 100$ time steps with $\mu = 0.08$ and $\sigma = 0.4$ resulting in a Brownian motion W with drift $\tilde{\mu} = 0$. On the RHS is the associated realization for geometric Brownian motion $S = \exp(W)$.

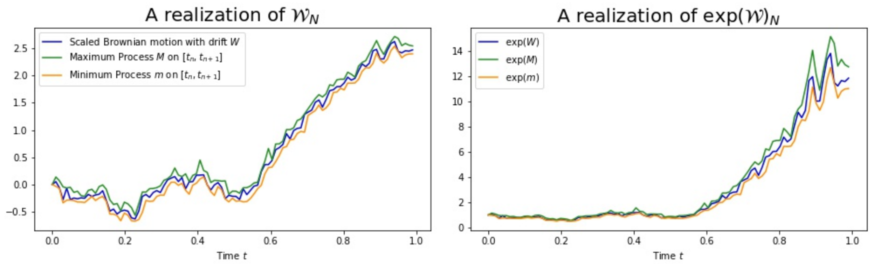

FIGURE D.2: The LHS shows a realizations of $\mathcal{W}$ on $[0, 1]$ over $N = 100$ time steps with $\mu = 4$ and $\sigma = 1$ such that W is a Brownian motion with drift $\tilde{\mu} = 3$. On the RHS is the associated realization for geometric Brownian motion $S = \exp(W)$.

the infimum and the supremum of $W_x^{\tilde{\mu},\sigma}$ on $[t_n, t_{n+1}]$. By the strict monotonicity of the exponential function we compute the triple

$$\exp\left(\mathcal{W}\right)_{n+1} := \left(\exp\left(X_{t_{n+1}}\right), \exp\left(m_{X_{t_n}, \frac{T}{N}}\right), \exp\left(M_{X_{t_n}, \frac{T}{N}}\right)\right) \tag{D.2}$$

that approximates the subsequent state $P_{t_{n+1}}$, the infimum and the supremum of P on $[t_n, t_{n+1}]$ such that approximately

$$\exp\left(m_{X_{t_n}, \frac{T}{N}}\right) \sim \mathbb{P} \circ \left(\inf_{u \in [t_n, t_{n+1}]} P_u\right)^{-1}$$

and

$$\exp\left(M_{X_{t_n}, \frac{T}{N}}\right) \sim \mathbb{P} \circ \left(\sup_{u \in [t_n, t_{n+1}]} P_u\right)^{-1}$$

for all $0 \leq n \leq N$. See Figure D.1 and Figure D.2 for realisations of $\mathcal{W}$ and $\exp(\mathcal{W})$. For the code of the implementation see `https://github.com/GabrielNobis/Thesis` and in particular `Processes/showcase_ItoProcesses.ipynb`.

Appendix E

Learning Summaries

E.1 Results for different choices of α, α' and α''

We summarize the results from the methods defined in Table 4.3.

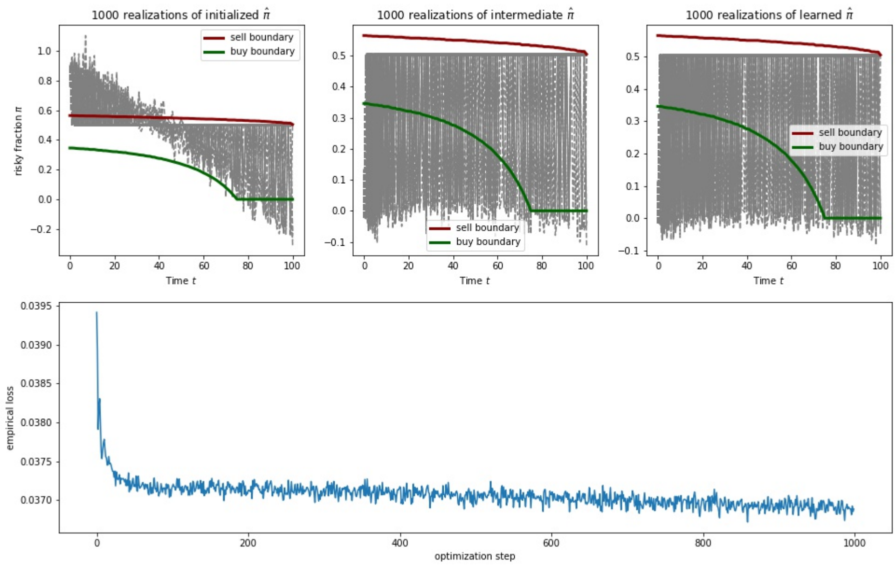

FIGURE E.1: Learning summary for *HardtanhExplicit*

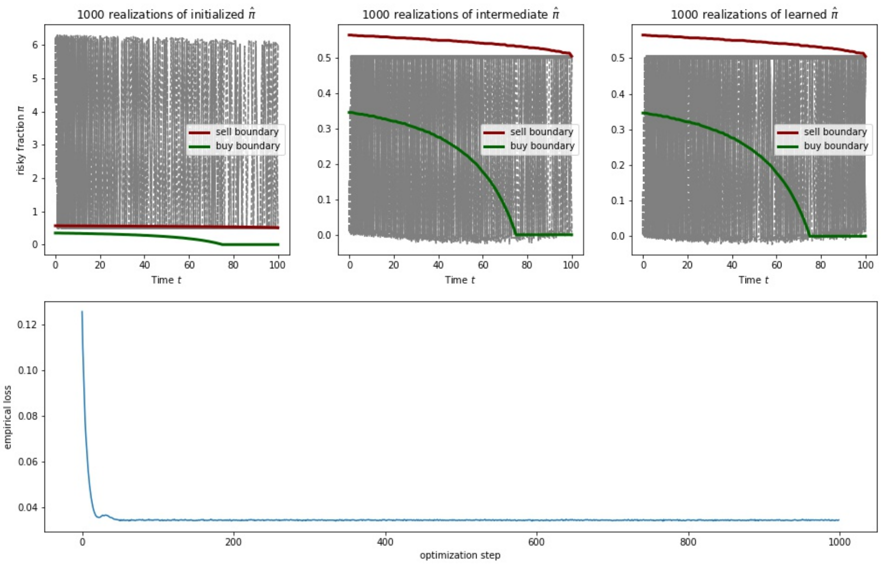

FIGURE E.2: Learning summary for *HardtanhUnExplicit*

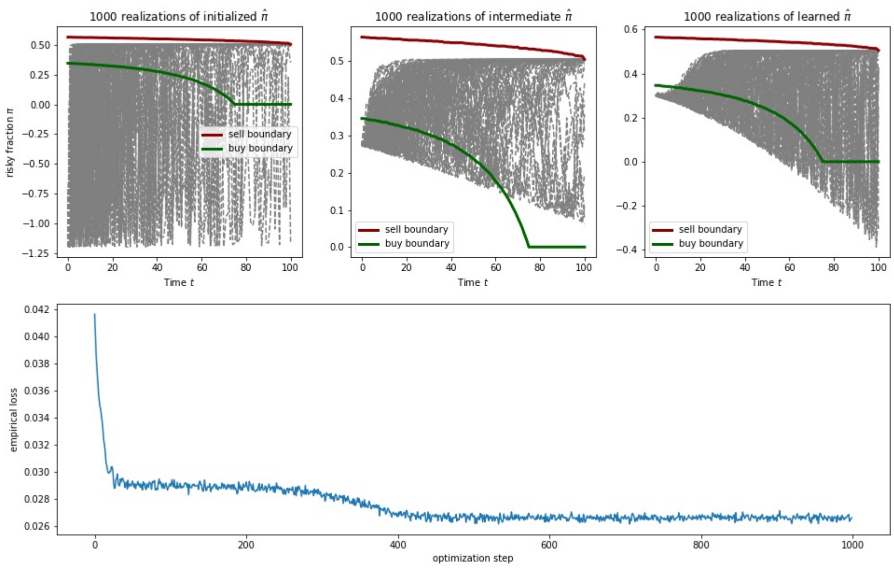

FIGURE E.3: Learning summary for *BaselineExplicit*

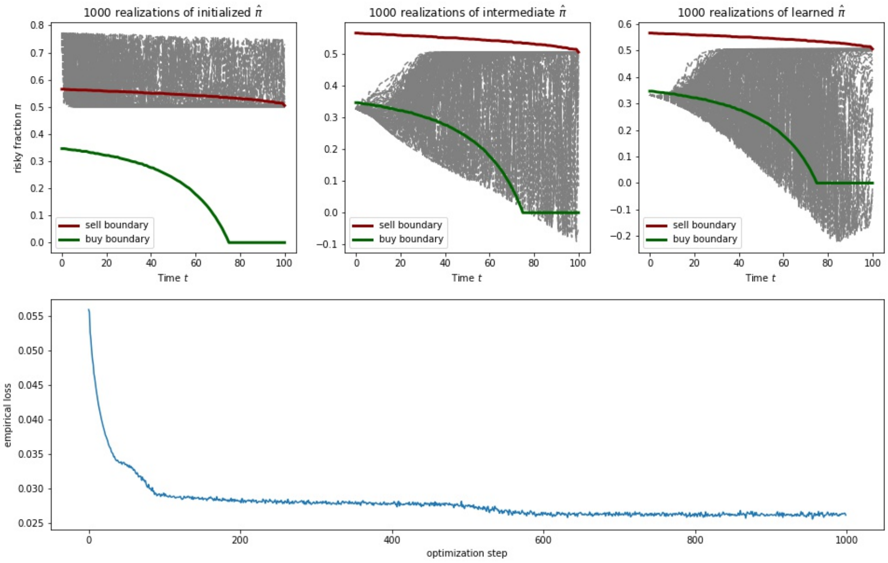

FIGURE E.4: Learning summary for *BaselineUnExplicit*

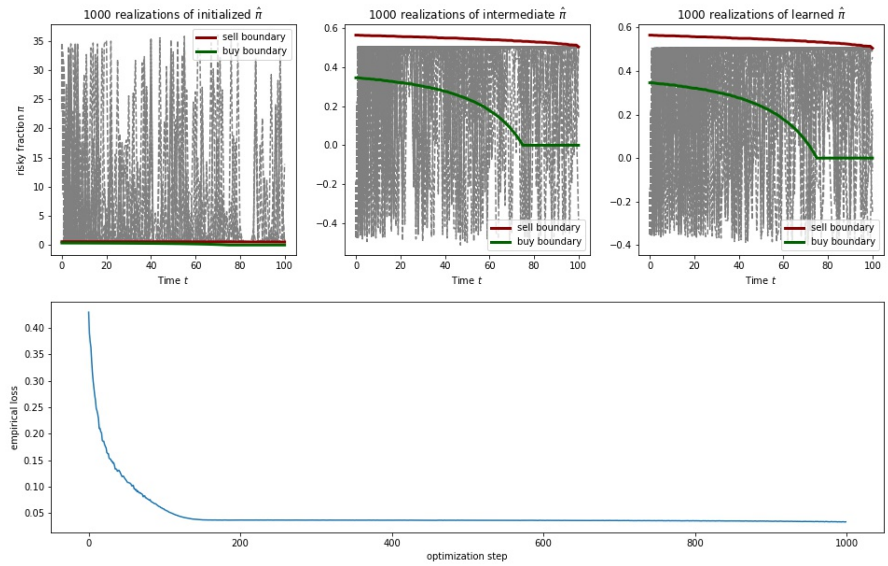

FIGURE E.5: Learning summary for *SmoothExplicit*

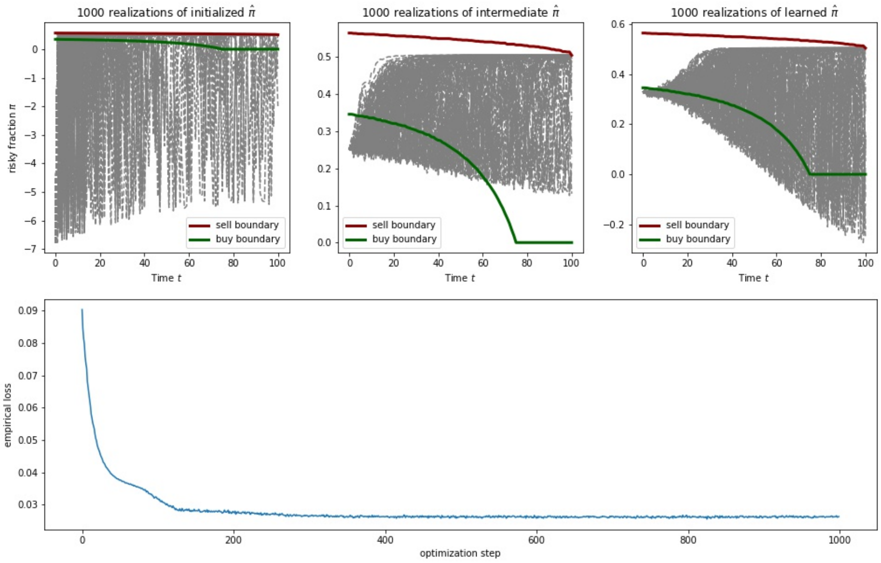

FIGURE E.6: Learning summary for *SmoothUnExplicit*

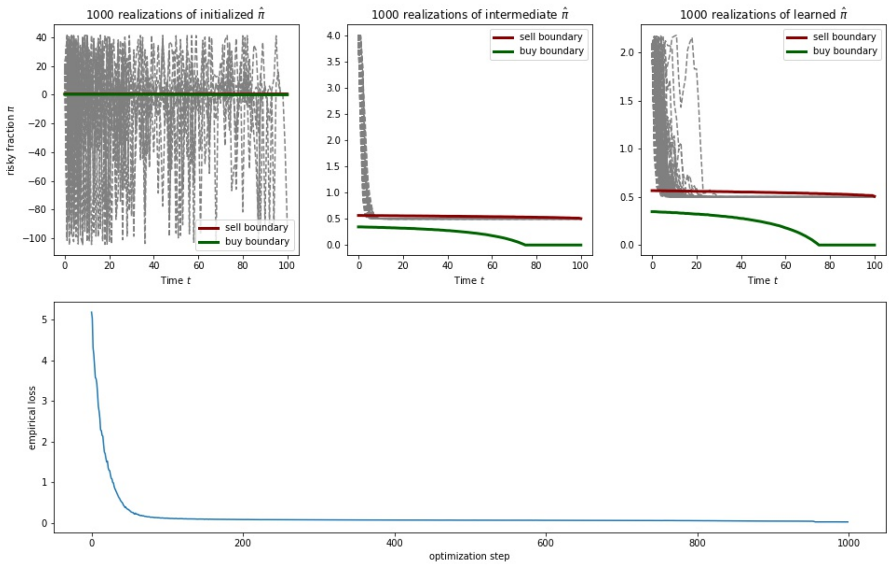

FIGURE E.7: Learning summary for *TanhExplicit*

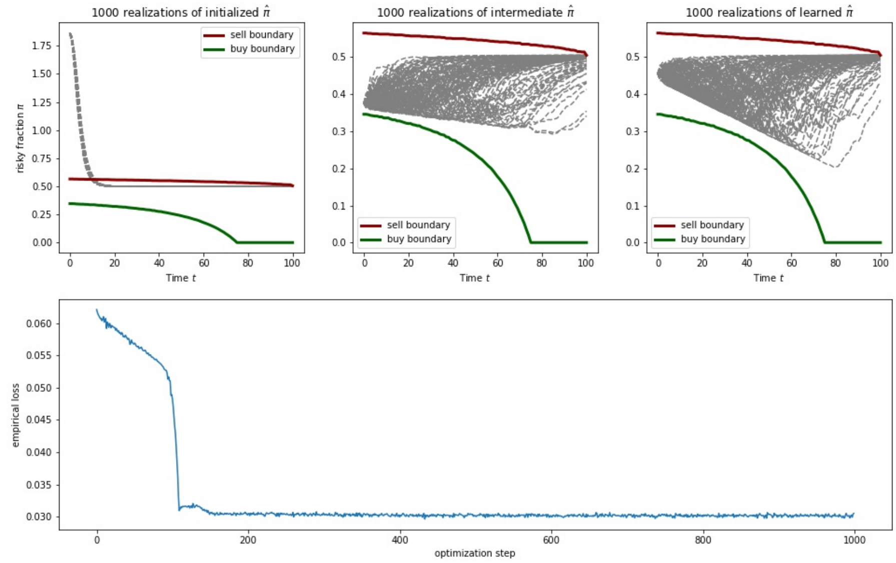

FIGURE E.8: Learning summary for *TanhUnExplicit*

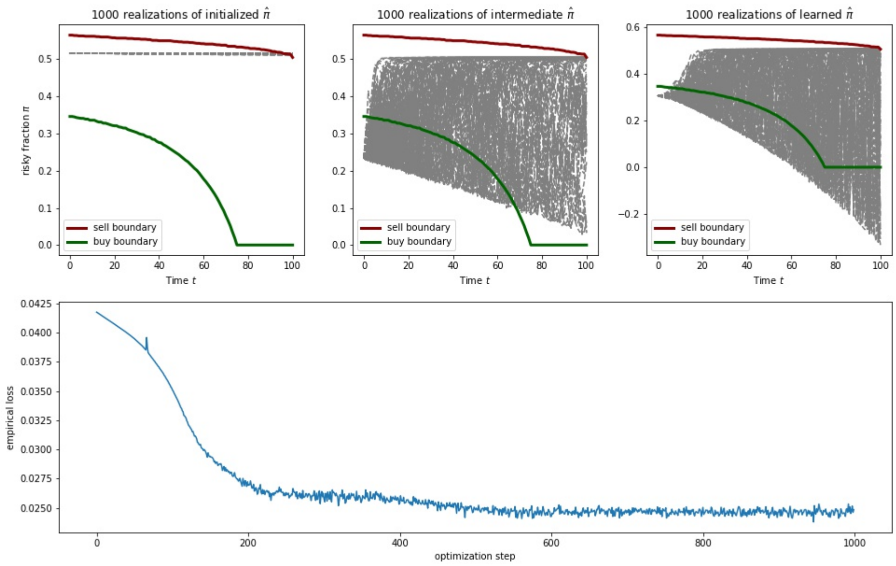

FIGURE E.9: Learning summary for *OracleBaseline*

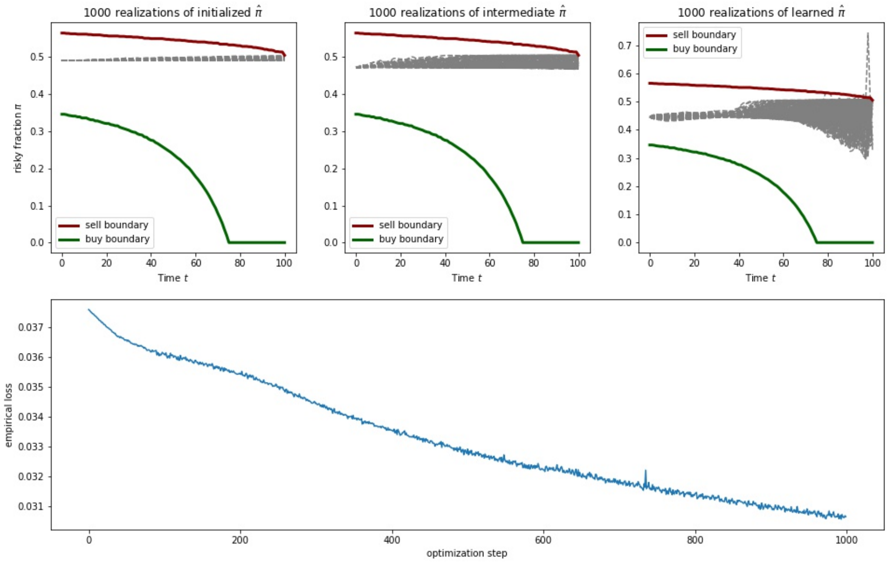

FIGURE E.10: Learning summary for *AdjustedOracleBaseline*

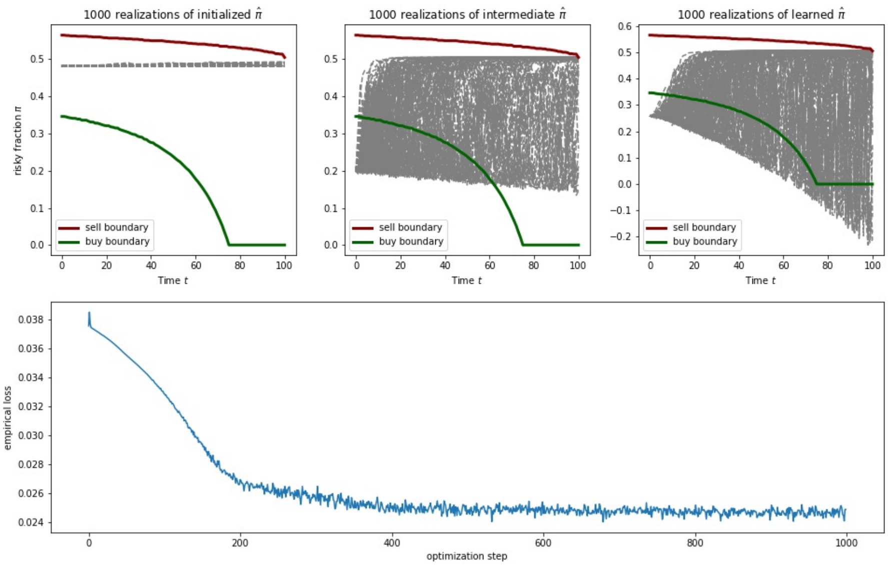

FIGURE E.11: Learning summary for *OracleSmooth*

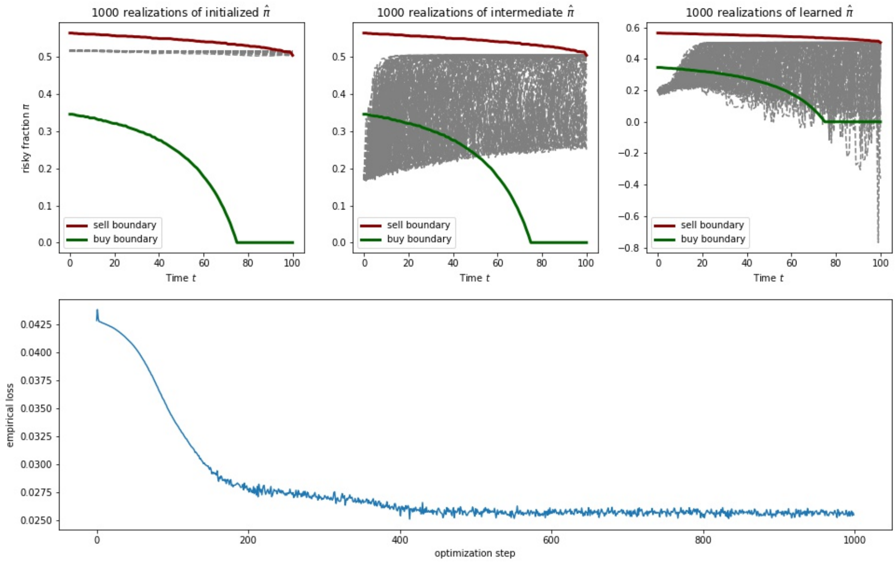

FIGURE E.12: Learning summary for *AdjustedOracleSmooth*

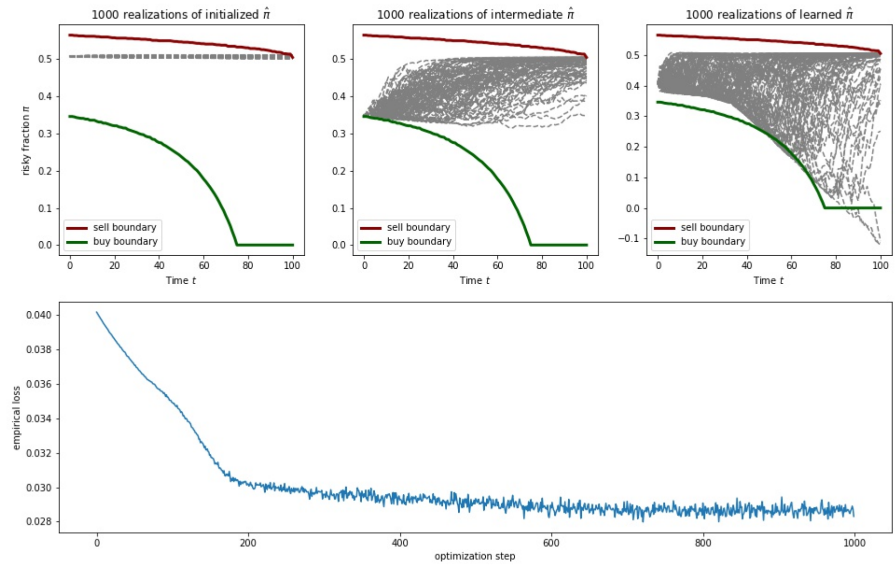

FIGURE E.13: Learning summary for *OracleTanh*

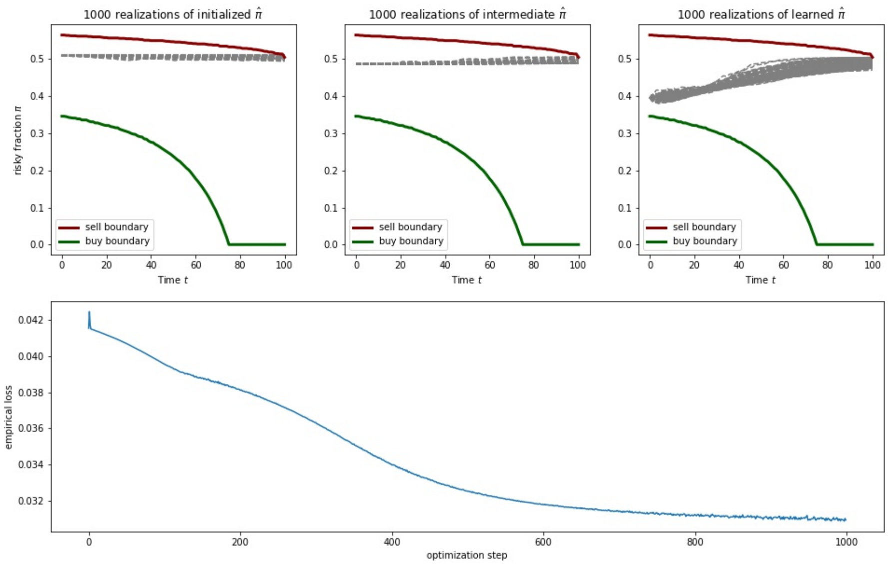

FIGURE E.14: Learning summary for *AdjustedOracleTanh*

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