More on Markov Chains

1. Structure of Markov Chain Transition Matrices

Which of the following is a Markov chain transition matrix? And why?:

(a)

$$P = \begin{pmatrix} 0.1 & 0.0 & 0.0 & 0.9 & 0.0 \\ 0.0 & 0.4 & 0.5 & 0.0 & 0.6 \\ 0.3 & 0.3 & 0.0 & 0.4 & 0.0 \\ 0.3 & 0.0 & 0.0 & 0.7 & 0.0 \\ 0.0 & 0.7 & 0.0 & 0.0 & 0.3 \end{pmatrix}$$

(b)

$$P = \begin{pmatrix} 0.1 & 0.0 & 0.0 & 0.9 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.0 & 0.6 \\ 0.6 & -0.3 & 0.0 & 0.4 & 0.0 \\ 0.3 & 0.0 & 0.0 & 0.7 & 0.0 \\ 0.0 & 0.7 & 0.0 & 0.0 & 0.3 \end{pmatrix}$$

(c)

$$P = \begin{pmatrix} 0.1 & 0.0 & 0.0 & 0.9 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.0 & 0.6 \\ 0.3 & 0.3 & 0.0 & 0.4 & 0.0 \\ 0.3 & 0.0 & 0.0 & 0.7 & 0.0 \\ 0.0 & 0.7 & 0.0 & 0.0 & 0.3 \end{pmatrix}$$

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What is special about the following Markov chain transition matrices?:

(d)

$$P = \begin{pmatrix} 0.1 & 0.9 & 0.0 & 0.0 & 0.0 \\ 0.6 & 0.4 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.3 & 0.4 & 0.3 \\ 0.0 & 0.0 & 0.3 & 0.7 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.7 & 0.3 \end{pmatrix}$$

(e)

$$P = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.4 & 0.0 & 0.0 & 0.5 \\ 0.3 & 0.3 & 0.0 & 0.4 & 0.0 \\ 0.3 & 0.0 & 0.0 & 0.7 & 0.0 \\ 0.0 & 0.1 & 0.2 & 0.4 & 0.3 \end{pmatrix}$$

(c)

$$P = \begin{pmatrix} 0.1 & 0.0 & 0.0 & 0.9 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.0 & 0.6 \\ 0.3 & 0.3 & 0.0 & 0.4 & 0.0 \\ 0.3 & 0.0 & 0.0 & 0.7 & 0.0 \\ 0.0 & 0.7 & 0.0 & 0.0 & 0.3 \end{pmatrix}$$

2. Classification of States

See Section 4.3.

Concepts:

1. State j is *accessible* from state i if it is possible to get to j from i in some finite number of steps. (notation: $i \rightsquigarrow j$) It does not have to be in a single step; it can be in several steps. In other words, $i \rightsquigarrow j$ if $P_{i,j}^{(n)} > 0$ for some n, i.e., if the n-step transition probability from state i to state j is strictly positive.

2. States i and j *communicate* if both j is accessible from i and i is accessible from j. (notation: $i \leftrightsquigarrow j$)

3. A subset A of states in the Markov chain is *a communication class* if every pair of states in the subset communicate.

4. A communication class A of states in the Markov chain is *closed* if no state outside the class is accessible from a state in the class.

5. A communication class A of states in the Markov chain is *open* if it is not closed; i.e., if it is possible for the Markov chain to leave that communicating class. (Necessarily, it is not possible to return to that set after the chain leaves it.)

6. A Markov chain is *irreducible* if the entire chain is a single communicating class.

7. A Markov chain is *reducible* if there are two or more communication classes in the chain; i.e., if it is not irreducible.

8. A Markov chain transition matrix P is in *canonical form* if the states are re-labelled (re-ordered) so that the states within closed communication classes appear together first, and then afterwards the states in open communicating classes appear together. The recurrent states appear at the top; the transient states appear below. The states within a communication class appear next to each other.

9. State j is a *recurrent state* if, starting in state j, the Markov chain returns to state j with probability 1. (A state is recurrent if and only if it is in a closed communication class.)

10. State j is a *transient state* if, starting in state j, the Markov chain returns to state j with probability < 1; i.e., if the state is *not* recurrent. (A state is transient if and only if it is

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in an open communication class.)

11. State j is a *positive-recurrent state* if the state is recurrent and if, starting in state j, the expected time to return to that state is finite.

12. State j is a *null-recurrent state* if the state is recurrent but, starting in state j, the expected time to return to state j is infinite. (This is possible, e.g., a symmetric simple random walk; see Example 4.18 in §4.3 and §4.5.1, especially the final sentence before Example 4.28.)

3. Canonical Form for a Probability Transition Matrix

Find the canonical form of the following Markov chain transition matrix ((c) above):

(c)
$$P = \begin{pmatrix} 0.1 & 0.0 & 0.0 & 0.9 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.0 & 0.6 \\ 0.3 & 0.3 & 0.0 & 0.4 & 0.0 \\ 0.3 & 0.0 & 0.0 & 0.7 & 0.0 \\ 0.0 & 0.7 & 0.0 & 0.0 & 0.3 \end{pmatrix}$$

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Notice that the sets $\{1, 4\}$ and $\{2, 5\}$ are closed communicating classes containing recurrent states, while $\{3\}$ is an open communicating class containing a transient state.

In general (when the number of states is not too large), we can construct the canonical form by drawing an *transition graph*. This graph contains a node for each state and has a directed arc from node i to node j if $P_{i,j} > 0$. The communication classes are usually easy to identify from this graph. Algorithmically, we can determine accessibility by looking at the sum of powers

$$\Sigma_{i,j} = \sum_{k=1}^{n} P_{i,j}^n, \tag{1}$$

where n is the number of states in the chain. We are using the property that if indeed j is accessible from i, then state j can be reached in at most n steps. Hence, $i \rightsquigarrow j$ if and only if $\Sigma_{i,j} > 0$. (Of course, the matrix Σ is not a probability matrix, because its row sums are greater than 1.)

So you should reorder the states according to the order: $1, 4, 2, 5, 3$. The order $2, 5, 1, 4, 3$ would be OK too, as would $5, 2, 4, 1, 3$. We put the recurrent states first and the transient states last. We group the recurrent states together according to their communicating class. Using the first order - $1, 4, 2, 5, 3$ - you get

canonical form of transition matrix (c):

$$P = \begin{pmatrix} 0.1 & 0.9 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.7 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.4 & 0.6 & 0.0 \\ 0.0 & 0.0 & 0.7 & 0.3 & 0.0 \\ 0.3 & 0.4 & 0.3 & 0.0 & 0.0 \end{pmatrix}$$

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The reason we construct the canonical form is that, from it, it is much easier to see the structure. It is easier to see what kind of chain it is and what will happen in the long run. Notice that the canonical form here has the structure:

$$P = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ R_1 & R_2 & Q \end{pmatrix},$$

where P_1 and P_2 are 2×2 Markov chain transition matrices in their own right, whereas R_i is the one-step transition probabilities from the single transient state to the ith closed set. In this case, $Q \equiv (0)$ is the 1×1 sub-matrix representing the transition probabilities among the transient states. Here there is only a single transient state and the transition probability from that state to itself is 0. The chain leaves that transient state immediately, never to return..

4. Liberating Markov Mouse

We obtain an absorbing Markov chain when we let Markov Mouse escape from his maze, by leaving from new doors out of states 3, 7 and 9; see the last lecture notes.

5. Markov chain models of the weather

We discussed Examples 4.1 and 4.4. Example 4.4 is tricky. We want to construct a Markov chain model. Hence, we must construct a square probability transition matrix. The idea is to make the state by the weather on two consecutive days. The state at step n is $X_n \equiv (w_{n-1}, w_n)$, where w_n is the type of weather on day n. The next state is $X_{n+1} \equiv (w_n, w_{n+1})$, the weather on days n and $n+1$. Note that the weather at day n, w_n, appears in *both* states. We can go from (D, R) to (R, D), but we cannot go from (D, R) to (D, R). Thus half the entries of the transition matrix must be 0. We can solve $\pi = \pi P$ to find the stationary probability vector, but this is a 1×4 probability vector. To get the long-run proportion of dray days, we need to add the probabilities for the two states (D, R) and (D, D) (or, equivalently, (R, D) and (D, D) focusing on the next day).