EXERCISES

A

1. Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x, y) = \begin{cases} \dfrac{1}{25}(x^2 + y^2) & \text{if } x = 1, 2, \quad y = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find $p_{X|Y}(x|y)$, $P(X = 2 \mid Y = 1)$, and $E(X \mid Y = 1)$.

2. Regions A and B are prone to dust storms. An actuary has calculated that the number of dust storms that strike region A, in a year, is binomial with parameters 4 and 0.7. Furthermore, she has observed that if, in a year, n dust storms strike region A, then the number of dust storms striking region B is n with probability 2/3 and $n + 1$ with probability 1/3. What is the expected number of dust storms in region A in a year in which 3 dust storms strike region B.

3. Let the joint probability density function of continuous random variables X and Y be given by

$$f(x, y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find $f_{X|Y}(x|y)$.

4. An unbiased coin is flipped until the sixth head is obtained. If the third head occurs on the fifth flip, what is the probability mass function of the number of flips?

5. Let the conditional probability density function of X given that $Y = y$ be given by

$$f_{X|Y}(x|y) = \frac{3(x^2 + y^2)}{3y^2 + 1}, \qquad 0 < x < 1, \quad 0 < y < 1.$$

Find $P(1/4 < X < 1/2 \mid Y = 3/4)$.

6. Let X and Y be independent discrete random variables. Prove that for all y, $E(X \mid Y = y) = E(X)$. Do the same for continuous random variables X and Y.

7. Let X and Y be continuous random variables with joint probability density function

$$f(x, y) = \begin{cases} x + y & \text{if } 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Calculate $f_{X|Y}(x|y)$.

8. Let X and Y be continuous random variables with joint probability density function given by

$$f(x, y) = \begin{cases} e^{-x(y+1)} & \text{if } x \geq 0, \quad 0 \leq y \leq e - 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Calculate $E(X \mid Y = y)$.

9. A box contains 5 blue, 10 green, and 5 red chips. We draw 4 chips at random and without replacement. If exactly one of them is blue, what is the probability mass function of the number of green balls drawn?

10. First a point Y is selected at random from the interval $(0, 1)$. Then another point X is selected at random from the interval $(Y, 1)$. Find the probability density function of X.

11. Let (X, Y) be a random point from a unit disk centered at the origin. Find $P(0 \leq X \leq 4/11 \mid Y = 4/5)$.

12. An actuary working for an insurance company has calculated that the time that it will take for an insured driver to report an accident he or she has been involved in, in days, is a continuous random variable X, with the probability density function

$$f(x) = \begin{cases} \dfrac{196 - x^2}{579} & 0 < x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the actuary has discovered that if the accident is reported x days after its occurrence, and if the insured is entitled to claim payment, the time that it will take until he or she receives payment is a uniform random variable between $x + 7$ and 21 days. What is the probability that a driver involved in an accident and entitled to claim payment will receive his or her payment within 14 days from the time of the accident?

13. The joint probability density function of X and Y is given by

$$f(x, y) = \begin{cases} c\,e^{-x} & \text{if } x \geq 0, \quad |y| < x \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the constant c.

(b) Find $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

(c) Calculate $E(Y \mid X = x)$ and $\text{Var}(Y \mid X = x)$.

14. Leon leaves his office every day at a random time between 4:30 P.M. and 5:00 P.M. If he leaves t minutes past 4:30, the time it will take him to reach home is a random number between 20 and $20 + (2t)/3$ minutes. Let Y be the number of minutes past 4:30 that Leon leaves his office tomorrow and X be the number of minutes it takes him to reach home. Find the joint probability density function of X and Y.

15. Show that if $\{N(t) : t \geq 0\}$ is a Poisson process, the conditional distribution of the first arrival time given $N(t) = 1$ is uniform on $(0, t)$.