Example 2.8 (Random walk on a graph) A *graph* is a set of vertices and a set of edges. Two vertices are *neighbors* if there is an edge joining them. The *degree* of vertex v is the number of neighbors of v. For the graph in Figure 2.1, $\deg(a) = 1$, $\deg(b) = \deg(c) = \deg(d) = 4$, $\deg(e) = 3$, and $\deg(f) = 2$.

$\{a, b, c, d, e, f\}$

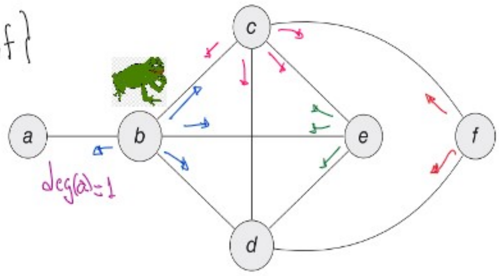


Figure 2.1 Graph on six vertices.

Let X_n be the frog's location after n hops. The sequence $X_0, X_1, \ldots$ is a Markov chain. Given a graph G such a process is called *simple random walk on* G.

For vertices i and j, write $i \sim j$ if i and j are neighbors. The one-step transition probabilities are

$$P(X_1 = j | X_0 = i) = \begin{cases} \dfrac{1}{\deg(i)}, & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix} \end{array}$$

(a) The *cycle graph* on nine vertices is shown in Figure 2.2. Simple random walk on the cycle moves left or right with probability 1/2. Each vertex has degree two. The transition matrix is defined using clock arithmetic. For a cycle with k vertices, $P(X_1 = j \mid X_0 = i)$

$$P_{ij} = \begin{cases} 1/2, & \text{if } j \equiv i \pm 1 \mod k, \\ 0, & \text{otherwise.} \end{cases}$$

(b) In the *complete graph*, every pair of vertices is joined by an edge. The complete graph on five vertices is shown in Figure 2.2. The complete graph on k vertices has $\binom{k}{2}$ edges. Each vertex has degree $k - 1$. The entries of the transition matrix are

$$P_{ij} = \begin{cases} 1/(k-1), & \text{if } i \neq j \\ 0, & \text{if } i = j. \end{cases}$$

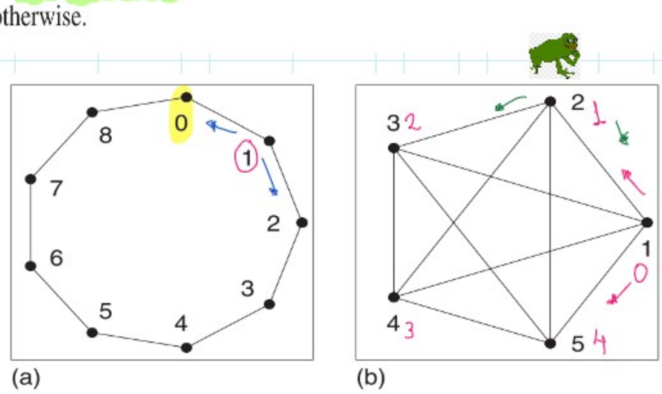


Figure 2.2 (a) Cycle graph on nine vertices. (b) Complete graph on five vertices.

Consider random walk on the cycle graph consisting of five vertices $\{0, 1, 2, 3, 4\}$. Describe the six-step transition probabilities of the chain.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{pmatrix} \end{array}$$

$$P^6 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 5/16 & 7/64 & 15/64 & 15/64 & 7/64 \\ 7/64 & 5/16 & 7/64 & 15/64 & 15/64 \\ 15/64 & 7/64 & 5/16 & 7/64 & 15/64 \\ 15/64 & 15/64 & 7/64 & 5/16 & 7/64 \\ 7/64 & 15/64 & 15/64 & 7/64 & 5/16 \end{pmatrix} \end{array}$$

Example 2.14 For gambler's ruin, assume that the gambler's initial stake is \$3 and the gambler plays until either gaining \$8 or going bust. At each play the gambler wins \$1, with probability 0.6, or loses \$1, with probability 0.4. Find the gambler's expected fortune after four plays.

$N = 8$; $k = 3$

$x_i = \begin{cases} +1 ; & 0.6 \\ -1 ; & 0.4 \end{cases}$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Inicio

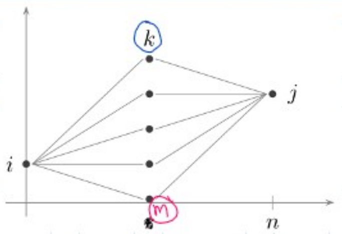
$X_0 = 3$

$E[X_4 \mid X_0 = 3] = \sum_{x_4} x_4\, p(x_4 \mid x_0 = 3)$

$$P^4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.496 & 0.115 & 0 & 0.259 & 0 & 0.130 & 0 & 0 & 0 \\ 0.237 & 0 & 0.288 & 0 & 0.346 & 0 & 0.130 & 0 & 0 \\ 0.064 & 0.115 & 0 & 0.346 & 0 & 0.346 & 0 & 0.130 & 0 \\ 0.026 & 0 & 0.154 & 0 & 0.346 & 0 & 0.346 & 0 & 0.130 \\ 0 & 0.026 & 0 & 0.154 & 0 & 0.346 & 0 & 0.259 & 0.216 \\ 0 & 0 & 0.026 & 0 & 0.154 & 0 & 0.288 & 0 & 0.533 \\ 0 & 0 & 0 & 0.026 & 0 & 0.115 & 0 & 0.115 & 0.744 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{aligned} E(X_4 \mid X_0 = 3) &= \sum_{j=0}^{8} jP(X_4 = j \mid X_0 = 3) = \sum_{j=0}^{8} jP^4_{3,j} \\ &= 0(0.064) + 1(0.115) + 3(0.346) + 5(0.346) + 7(0.130) \\ &= \$3.79. \end{aligned}$$

Ecuación de Chapman-Kolmogorov



For $m, n \geq 0$, the matrix identity $\boldsymbol{P}^{m+n} = \boldsymbol{P}^m \boldsymbol{P}^n$ gives

$$\boxed{P^{m+n}_{ij} = \sum_k P^m_{ik} P^n_{kj}, \quad \text{for all } i, j.}$$

$$P(X_{n+m} = j \mid X_0 = i) = \sum_k P(X_{n+m} = j \mid X_m = k)\, P(X_m = k \mid X_0 = i)$$

Estados → $S = \{0, 1, 2, 3, \ldots, 8\}$

$X_n = j$

X_7	$P(X_7 = j)$
0	$P(X_7 = 0)$
1	$P(X_7 = 1)$

$X_7 = 3$ → $X_7 = 0, 1, 2, \ldots, 8$

$p(X_7)$ → distribución

Distribution of X_n

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix $\boldsymbol{P}$ and initial distribution $\boldsymbol{\alpha}$. For all $n \geq 0$, the distribution of X_n is $\boldsymbol{\alpha}\boldsymbol{P}^n$. That is,

$$P(X_n = j) = (\boldsymbol{\alpha}\boldsymbol{P}^n)_j, \quad \text{for all } j.$$

Example 2.3 (Chained to the weather) Some winter days in Minnesota it seems like the snow will never stop. A Minnesotan's view of winter might be described by the following transition matrix for a weather Markov chain, where r, s, and c denote rain, snow, and clear, respectively.

$$\boldsymbol{P} = \begin{array}{c} \\ r \\ s \\ c \end{array} \begin{array}{c} \begin{array}{ccc} r & s & c \end{array} \\ \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.6 & 0.3 \end{pmatrix} \end{array}$$

$X_2 = r, s, c$

$$\boldsymbol{P}^2 = \begin{array}{c} \\ r \\ s \\ c \end{array} \begin{array}{c} \begin{array}{ccc} r & s & c \end{array} \\ \begin{pmatrix} 0.12 & 0.72 & 0.16 \\ 0.11 & 0.76 & 0.13 \\ 0.11 & 0.72 & 0.17 \end{pmatrix} \end{array}$$

$$\boldsymbol{\alpha}\boldsymbol{P}^2 = (0.5, 0.5, 0) \begin{pmatrix} 0.12 & 0.72 & 0.16 \\ 0.11 & 0.76 & 0.13 \\ 0.11 & 0.72 & 0.17 \end{pmatrix} = (0.115, 0.74, 0.145).$$

x_2	$p(x_2)$
r	0.115
s	0.74
c	0.145

Joint Distribution

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix $\boldsymbol{P}$ and initial distribution $\boldsymbol{\alpha}$. For all $0 \leq n_1 < n_2 < \cdots < n_{k-1} < n_k$ and states $i_1, i_2, \ldots, i_{k-1}, i_k$,

$$\begin{aligned} &P(X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_{k-1}} = i_{k-1}, X_{n_k} = i_k) \\ &\quad = (\boldsymbol{\alpha}\boldsymbol{P}^{n_1})_{i_1} (\boldsymbol{P}^{n_2 - n_1})_{i_1 i_2} \cdots (\boldsymbol{P}^{n_k - n_{k-1}})_{i_{k-1} i_k}. \end{aligned} \tag{2.5}$$

Example 2.16 Danny's daily lunch choices are modeled by a Markov chain with transition matrix

$$\boldsymbol{P} = \begin{array}{r} \\ \text{b Burrito} \\ \text{f Falafel} \\ \text{p Pizza} \\ \text{s Sushi} \end{array} \begin{array}{c} \begin{array}{cccc} \text{b} & \text{f} & \text{p} & \text{s} \\ \text{Burrito} & \text{Falafel} & \text{Pizza} & \text{Sushi} \end{array} \\ \begin{pmatrix} 0.0 & 0.5 & 0.5 & 0.0 \\ 0.5 & 0.0 & 0.5 & 0.0 \\ 0.4 & 0.0 & 0.0 & 0.6 \\ 0.0 & 0.2 & 0.6 & 0.2 \end{pmatrix} \end{array}.$$

$n_1 = 3 < ;\ n_2 = 5 < ;\ n_3 = 6$

$n_2 - n_1 = 5 - 3 = 2$

$n_3 - n_2 = 6 - 5 = 1.$

$P(X_{\cdot} = s, X_{\cdot} = s, X_{\cdot} = p) = (\alpha P^3)\, P^2\, P^1$

$$P = \begin{array}{r} \text{Falafel} \\ \text{Pizza} \\ \text{Sushi} \end{array} \begin{pmatrix} 0.5 & 0.0 & 0.5 & 0.0 \\ 0.4 & 0.0 & 0.0 & 0.6 \\ 0.0 & 0.2 & 0.6 & 0.2 \end{pmatrix}.$$

On Sunday, Danny chooses lunch uniformly at random. Find the probability that he chooses sushi on the following Wednesday and Friday, and pizza on Saturday.

Do — Lu — Ma — Mie — Jue — Vie — Sabado

0 — 1 — 2 — 3 — 4 — 5 — 6

$$P^2 = \begin{array}{c|cccc} & b & f & p & s \\ \hline b & 0.45 & 0.00 & 0.25 & 0.30 \\ f & 0.20 & 0.25 & 0.25 & 0.30 \\ p & 0.00 & 0.32 & 0.56 & 0.12 \\ s & 0.34 & 0.04 & 0.22 & 0.40 \end{array}, \quad P^3 = \begin{array}{c|cccc} & b & f & p & s \\ \hline b & 0.100 & 0.285 & 0.405 & 0.210 \\ f & 0.225 & 0.160 & 0.405 & 0.210 \\ p & 0.384 & 0.024 & 0.232 & 0.360 \\ s & 0.108 & 0.250 & 0.430 & 0.212 \end{array}.$$

$n_1 = 3 < n_2 = 5 < n_3 = 6$

$n_2 - n_1 = 5 - 3 = 2$

$n_3 - n_2 = 6 - 5 = 1$

$$P(X_3 = s, X_5 = s, X_6 = p) = (\alpha P^3)_s P^2_{ss} P^1_{sp}$$

$$\alpha = (1/4, 1/4, 1/4, 1/4)$$

$$(\alpha P^3)_s P^2_{ss} P_{sp} = (0.248)(0.40)(0.60) = 0.05952.$$

$$\underset{1\times 4}{(1/4, 1/4, 1/4, 1/4)} \begin{bmatrix} 0.45 & 0 & 0.25 & 0.3 \\ 0.20 & 0.25 & 0.25 & 0.3 \\ \vdots & \vdots & \vdots & 0.12 \\ \vdots & \vdots & \vdots & 0.40 \end{bmatrix}_{4\times 4}$$

$(\quad , \quad , \quad , \quad)$

$\frac{1}{4}(0.210) + \frac{1}{4}(0.210) + \frac{1}{4}(0.36) + \frac{1}{4}(0.212)$

$=$