19 julio

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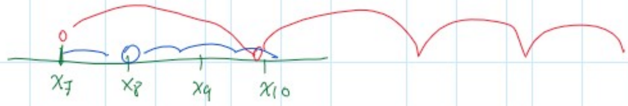


Example 5.2 Power-law distributions are positive probability distributions of the form $\pi_i \propto i^s$, for some constant S. Unlike distributions with exponentially decaying tails (e.g., Poisson, geometric, exponential, normal), power-law distributions have *fat tails*, and thus are often used to model skewed data. Let

iid

target

$$\pi_i = \frac{i^{-3/2}}{\sum_{k=1}^{\infty} k^{-3/2}}, \text{ for } i = 1, 2, \ldots \rightarrow X_0, X_1, X_2, \ldots, X_n$$

$1, 2, 3, 8, 10, \ldots$

$\sim \pi_i \propto i^{-3/2}$, $i = 1, 2, 3, \ldots$

Implement a Metropolis–Hastings algorithm to simulate from $\boldsymbol{\pi}$.

"n" "n+1"

"i" $\longrightarrow$ "j"

$\hookrightarrow a(i,j) = \dfrac{\pi_j T_{ji}}{\pi_i T_{ij}}$

1. Choose a new state according to $\boldsymbol{T}$. That is, choose j with probability T_{ij}. State j is called the *proposal state*.
2. Decide whether to accept j or not. Let

$$a(i,j) = \frac{\pi_j T_{ji}}{\pi_i T_{ij}}.$$

$\pi_j = \pi(j) = j^{-3/2}$

$T \rightarrow$ paseo aleatorio (caminata aleatoria)

$\hookrightarrow \{1, 2, 3, 4, \ldots\}$

The function a is called the *acceptance function*. If $a(i,j) \geq 1$, then j is accepted as the next state of the chain. If $a(i,j) < 1$, then j is accepted with probability $a(i,j)$. If j is not accepted, then i is kept as the next step of the chain.

In other words, assume that $X_n = i$. Let U be uniformly distributed on $(0, 1)$. Set

$$X_{n+1} = \begin{cases} j, & \text{if } U \leq a(i,j), \\ i, & \text{if } U > a(i,j). \end{cases} \quad U \sim U(0,1)$$

$$T_{ij} = \begin{cases} 1/2, & \text{if } j = i \pm 1 \text{ for } i > 1, \\ 1, & \text{if } i = 1 \text{ and } j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

$i > 1$

$$a(i,j) = a(i, i+1) = \frac{\pi_{i+1} T_{(i+1)i}}{\pi_i T_{i(i+1)}} = \frac{(i+1)^{-3/2} \cdot 1/2}{i^{-3/2} \cdot 1/2} = \frac{(i+1)^{-3/2}}{i^{-3/2}} = \left(\frac{i}{i+1}\right)^{3/2}$$

$$a(1,2) = \frac{\pi_2 T_{21}}{\pi_1 T_{12}} = \frac{2^{-3/2}}{1^{-3/2}} \cdot \frac{1/2}{1} = \frac{1}{2^{3/2}} \cdot \frac{1}{2} = \frac{1}{2^{5/2}}$$

$$a(i, i+1) = \left(\frac{i}{i+1}\right)^{3/2}, \text{ and } \quad a(i+1, i) = \left(\frac{i+1}{i}\right)^{3/2}, \text{ for } i \geq 2,$$

$i > 1$

with

$$a(1,2) = \frac{\pi_2 T_{21}}{\pi_1 T_{12}} = \left(\frac{1}{2}\right)^{3/2} \frac{1}{2} = \left(\frac{1}{2}\right)^{5/2} \quad \text{and} \quad a(2,1) = 2^{5/2}.$$

ALGORITMO METROPOLIS HASTINGS: ESPACIO DE ESTADOS CONTINUO

Example 5.5 Using only a uniform random number generator, simulate a standard normal random variable using MCMC.

$f(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} z^2}$; $z \sim N(0,1)$ } espacio de estados continuos

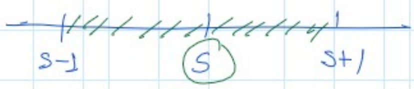
$\hookrightarrow \pi$ $\quad z \in (-\infty; +\infty)$

$T \rightarrow$ función de densidad p proponer candidato para las variables aleatoria

Propuesta: $T \sim U(i-1, i+1) \rightarrow$ longitud es 2 $(i+1) - (i-1) = 2$

$$T = \begin{cases} \frac{1}{2} ; & i^* \in (i+1, i-1) \\ 0, & c.c. \end{cases}$$

estado actual estado futuro $U(s-1, s+1)$



estado actual $\quad$ estado futuro $\quad U(s-1, s+1)$

$s \longrightarrow t$

$s-1 \quad s \quad s+1$

↳ aceptado ó rechazado

$s \to t$

$$a(s,t) = \frac{\pi_t \cancel{T_{ts}}^{\,1/2}}{\pi_s \cancel{T_{st}}^{\,1/2}} = \frac{\pi_t}{\pi_s} = \frac{e^{-\frac{1}{2}t^2}}{e^{-\frac{1}{2}s^2}} = e^{-\frac{1}{2}(t^2+s^2)}$$

MUESTREADOR DE GIBBS (GIBBS SAMPLER OR GIBBS SAMPLING)

In the Gibbs sampler, the target distribution $\boldsymbol{\pi}$ is an m-dimensional joint density

$$\boldsymbol{\pi}(\boldsymbol{x}) = \pi(x_1, \ldots, x_m).$$

A multivariate Markov chain is constructed whose limiting distribution is $\boldsymbol{\pi}$, and which takes values in an m-dimensional space. The algorithm generates elements of the form

$$\begin{aligned}
&\boldsymbol{X}^{(0)}, \boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}, \ldots \\
&\quad = (X_1^{(0)}, \ldots, X_m^{(0)}), (X_1^{(1)}, \ldots, X_m^{(1)}), (X_1^{(2)}, \ldots, X_m^{(2)}), \ldots
\end{aligned}$$

by iteratively updating each component of an m-dimensional vector conditional on the other $m-1$ components. We show that the Gibbs sampler is a special case of the Metropolis–Hastings algorithm, with a particular choice of proposal distribution.

The Gibbs sampler is a special case of the Metropolis–Hastings algorithm. To see this, assume that $\boldsymbol{\pi}$ is an m-dimensional joint distribution. To avoid excessive notation and simplify the presentation, we just consider one step of the Gibbs sampler when the first component is being updated.

Assume that $\boldsymbol{i} = (x_1, x_2, \ldots, x_m)$ is the current state, and $\boldsymbol{j} = (x_1', x_2, \ldots, x_m)$ is the proposed state. The proposal distribution T is the conditional distribution of X_1 given $X_2, \ldots, X_m$. The acceptance function is $a(\boldsymbol{i}, \boldsymbol{j}) = \pi_{\boldsymbol{j}} T_{\boldsymbol{ji}} / \pi_{\boldsymbol{i}} T_{\boldsymbol{ij}}$.

We have that

$$\begin{aligned}
\pi_{\boldsymbol{j}} T_{\boldsymbol{ji}} &= \pi(x_1', x_2, \ldots, x_m) f_{X_1|X_2,\ldots,X_m}(x_1|x_2, \ldots, x_m) \\
&= \pi(x_1', x_2, \ldots, x_m) \left(\frac{\pi(x_1, x_2, \ldots, x_m)}{\int \pi(x, x_2, \ldots, x_m)\, dx} \right) \\
&= \pi(x_1, x_2, \ldots, x_m) \left(\frac{\pi(x_1', x_2, \ldots, x_m)}{\int \pi(x, x_2, \ldots, x_m)\, dx} \right) \\
&= \pi(x_1, x_2, \ldots, x_m) f_{X_1|X_2,\ldots,X_m}(x_1'|x_2, \ldots, x_m) \\
&= \pi_{\boldsymbol{i}} T_{\boldsymbol{ij}}.
\end{aligned}$$

EJEMPLO: Condicional de la normal bivariada

Distribución normal bivariada. Se dice que las variables aleatorias continuas X y Y tienen una distribución normal bivariada si su función de densidad conjunta es

$$\begin{aligned}
f(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \\
&\quad \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right),
\end{aligned}$$

para cualesquiera valores reales de x y y, y en donde $-1 < \rho < 1$, $\sigma_1 > 0$, $\sigma_2 > 0$, y μ_1, μ_2 son dos constantes reales sin restricción. Se escribe entonces $(X,Y) \sim \mathrm{N}(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho)$. Cuando $\mu_1 = \mu_2 = 0$, y $\sigma_1 = \sigma_2 = 1$, la distribución se llama normal bivariada *estándar*, y su gráfica se muestra en la Figura 3.11 cuando $\rho = 0$. En el siguiente ejercicio se enuncian algunas propiedades de esta distribución.

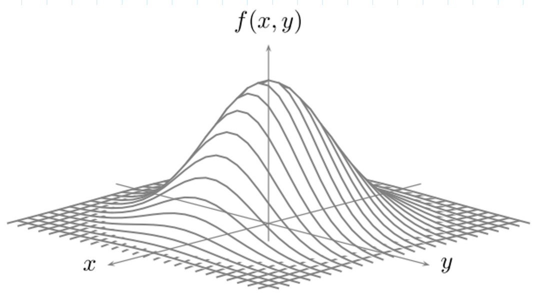


Figura 3.11: Función de densidad normal bivariada estándar.

Sea (X, Y) un vector con distribución normal $\mathrm{N}(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho)$. Demuestre que la distribución condicional de Y dado que $X = x$ es normal con media $\mu_2 + \rho(x - \mu_1)\sigma_2/\sigma_1$ y varianza $\sigma_2^2(1 - \rho^2)$, y que la distribución condicional de X dado que $Y = y$ es normal con media $\mu_1 + \rho(y - \mu_2)\sigma_1/\sigma_2$ y varianza $\sigma_1^2(1 - \rho^2)$.