Proof: In Theorem 8.2 let $h(x, y) = x + y$. Then

$$\begin{aligned} E(X+Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x,y)\,dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dx\,dy \\ &= E(X)+E(Y). \quad \blacklozenge \end{aligned}$$

Example 8.10 Let X and Y have joint probability density function

$$f(x,y) = \begin{cases} \dfrac{3}{2}(x^2+y^2) & \text{if } 0 < x < 1, \quad 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X^2+Y^2)$.

Solution: By Theorem 8.2,

$$\begin{aligned} E(X^2+Y^2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2+y^2)f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{3}{2}(x^2+y^2)^2\,dx\,dy \\ &= \frac{3}{2}\int_0^1\int_0^1 (x^4+2x^2y^2+y^4)\,dx\,dy = \frac{14}{15}. \quad \blacklozenge \end{aligned}$$

EXERCISES

A

1. Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = \begin{cases} k\left(\dfrac{x}{y}\right) & \text{if } x = 1, 2, \quad y = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Determine (a) the value of the constant k, (b) the marginal probability mass functions of X and Y, (c) $P(X > 1 \mid Y = 1)$, (d) $E(X)$ and $E(Y)$.

2. Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = \begin{cases} c(x+y) & \text{if } x = 1, 2, 3, \quad y = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Determine (a) the value of the constant c, (b) the marginal probability mass functions of X and Y, (c) $P(X \ge 2 \mid Y = 1)$, (d) $E(X)$ and $E(Y)$.

3. Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x, y) = \begin{cases} k(x^2 + y^2) & \text{if } (x, y) = (1, 1), (1, 3), (2, 3) \\ 0 & \text{otherwise.} \end{cases}$$

Determine (a) the value of the constant k, (b) the marginal probability mass functions of X and Y, and (c) $E(X)$ and $E(Y)$.

4. Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x, y) = \begin{cases} \dfrac{1}{25}(x^2 + y^2) & \text{if } x = 1, 2, \quad y = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(X > Y)$, $P(X + Y \leq 2)$, and $P(X + Y = 2)$.

5. Thieves stole four animals at random from a farm that had seven sheep, eight goats, and five burros. Calculate the joint probability mass function of the number of sheep and goats stolen.

6. Two dice are rolled. The sum of the outcomes is denoted by X and the absolute value of their difference by Y. Calculate the joint probability mass function of X and Y and the marginal probability mass functions of X and Y.

7. In a community 30% of the adults are Republicans, 50% are Democrats, and the rest are independent. For a randomly selected person, let

$$X = \begin{cases} 1 & \text{if he or she is a Republican} \\ 0 & \text{otherwise,} \end{cases}$$

$$Y = \begin{cases} 1 & \text{if he or she is a Democrat} \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the joint probability mass function of X and Y.

8. In an area prone to flood, an insurance company covers losses only up to three separate floods per year. Let X be the total number of floods in a random year and Y be the number of the floods that each cause over \$74 million in insured damage. Suppose that, for some constant c, the joint probability mass function of X and Y is

$$p(x, y) = \begin{cases} c(3x + 2y) & 0 \leq x \leq 3,\ 0 \leq y \leq x \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value of the number of floods, in a random year, that cause \$74 million or less in insured damage.

9. From an ordinary deck of 52 cards, seven cards are drawn at random and without replacement. Let X and Y be the number of hearts and the number of spades drawn, respectively.

(a) Find the joint probability mass function of X and Y.

(b) Calculate $P(X \geq Y)$.

10. Let the joint probability density function of random variables X and Y be given by

$$f(x, y) = \begin{cases} 2 & \text{if } 0 \leq y \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Calculate the marginal probability density functions of X and Y, respectively.

(b) Find $E(X)$ and $E(Y)$.

(c) Calculate $P(X < 1/2)$, $P(X < 2Y)$, and $P(X = Y)$.

11. Let the joint probability density function of random variables X and Y be given by

$$f(x, y) = \begin{cases} 8xy & \text{if } 0 \leq y \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Calculate the marginal probability density functions of X and Y, respectively.

(b) Calculate $E(X)$ and $E(Y)$.

12. Let the joint probability density function of random variables X and Y be given by

$$f(x, y) = \begin{cases} \dfrac{1}{2} y e^{-x} & \text{if } x > 0, \quad 0 < y < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the marginal probability density functions of X and Y.

13. Let X and Y have the joint probability density function

$$f(x, y) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Calculate $P(X + Y \leq 1/2)$, $P(X - Y \leq 1/2)$, $P(XY \leq 1/4)$, and $P(X^2 + Y^2 \leq 1)$.

14. Let X be the proportion of customers of an insurance company who bundle their auto and home insurance policies. Let Y be the proportion of customers who insure at least their car with the insurance company. An actuary has discovered that for, $0 \leq x \leq y \leq 1$, the joint distribution function of X and Y is $F(x, y) = x(y^2 + xy - x^2)$. Find the expected value of the proportion of the customers of the insurance company who bundle their auto and home insurance policies.

15. Let F be the joint distribution function of random variables X and Y. For $x_1 < x_2$ and $y_1 < y_2$, in terms of F, calculate $P(x_1 < X \leq x_2,\ y_1 < Y \leq y_2)$.

16. Let F be the joint distribution function of the random variables X and Y. In terms of F, calculate $P(X > t,\ Y > u)$.