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CADENAS DE MARKOV A LA LARGO PLAZO

Limiting Distribution

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix $\boldsymbol{P}$. A *limiting distribution* for the Markov chain is a probability distribution λ with the property that for all i and j,

$$\lim_{n\to\infty} P_{ij}^n = \lambda_j.$$

(i) For any initial distribution, and for all j,

$$\lim_{n\to\infty} P(X_n = j) = \lambda_j.$$

(ii) For any initial distribution $\boldsymbol{\alpha}$,

$$\lim_{n\to\infty} \boldsymbol{\alpha}\boldsymbol{P}^n = \boldsymbol{\lambda}$$

(iii)

$$\lim_{n\to\infty} \boldsymbol{P}^n = \boldsymbol{\Lambda}$$

We interpret λ_j as the long-term probability that the chain hits state j. By the uniqueness of limits, if a Markov chain has a limiting distribution, then that distribution is unique.

Example 3.1 (Two-state Markov chain) The transition matrix for a general two-state chain is

$$\boldsymbol{P} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}, \quad \text{for } 0 \le p, q \le 1$$

If $p+q=1$ $\quad \boldsymbol{P} = \begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix}$ and $\boldsymbol{P}^n = \boldsymbol{P}$ for all $n \ge 1$. Thus, $\lambda = (1-p, p)$ is the limiting distribution

Assume $p+q \ne 1$ $\quad \boldsymbol{P}^n = \dfrac{1}{p+q}\begin{pmatrix} q + p(1-p-q)^n & p - p(1-p-q)^n \\ q - q(1-p-q)^n & p + q(1-p-q)^n \end{pmatrix}$ $\quad |1-p-q| < 1$

$$\lim_{n\to\infty} \boldsymbol{P}^n = \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix} \qquad \lambda = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$$

estados $1, 2, \ldots, p$

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ distrib. Limite

$\sum_{i=1}^{p} \lambda_i = 1$

$\alpha = \pi$

Stationary Distribution $\pi^T = (\pi_1, \pi_2, \pi_3, \ldots, \pi_p) \rightsquigarrow \sum_{i=1}^{p} \pi_i = 1$

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix $\boldsymbol{P}$. A *stationary distribution* is a probability distribution $\boldsymbol{\pi}$, which satisfies

$$\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}. \tag{3.2}$$

That is,

$$\pi_j = \sum_i \pi_i P_{ij}, \quad \text{for all } j.$$

The name *stationary* comes from the fact that if the chain starts in its stationary distribution, then it stays in that distribution. We refer to the *stationary Markov chain* or the Markov chain *in stationarity* for the chain started in its stationary distribution.

Other names for the stationary distribution are *invariant*, *steady-state*, and *equilibrium* distribution. The latter highlights the fact that there is an intimate connection between the stationary distribution and the limiting distribution. If a Markov chain has a limiting distribution then that distribution is a stationary distribution.

$P \to \pi \rightsquigarrow$ simular

$X_0, X_1, X_2, X_3, \ldots, X_n, \ldots$

$\underbrace{3, 4, 3, 5, 6, 7}, \ldots$

muestra aleatoria

variables aleatorias

If we assume that a stationary distribution $\boldsymbol{\pi}$ is the initial distribution, then Equation (3.2) says that the distribution of X_0 is the same as the distribution of X_1.

Since the chain started at $n = 1$ is also a Markov chain with transition matrix $\boldsymbol{P}$, it follows that X_2 has the same distribution as X_1. In fact, all of the X_n have the same distribution, as

$$\boldsymbol{\pi}\boldsymbol{P}^n = (\boldsymbol{\pi}\boldsymbol{P})\boldsymbol{P}^{n-1} = \boldsymbol{\pi}\boldsymbol{P}^{n-1} = (\boldsymbol{\pi}\boldsymbol{P})\boldsymbol{P}^{n-2} = \boldsymbol{\pi}\boldsymbol{P}^{n-2} = \cdots = \boldsymbol{\pi}\boldsymbol{P} = \boldsymbol{\pi}.$$

If $X_0, X_1, X_2, \ldots$ is a stationary Markov chain, then for any $n > 0$, the sequence $X_n, X_{n+1}, X_{n+2}, \ldots$ is also a stationary Markov chain with the same transition matrix and stationary distribution as the original chain.

$S = \{1, 2, \ldots, k\}$

$X_0 = 1, 2, \ldots, k \to P(X_0 = j) \rightsquigarrow$ "α" distrib inicial.

$X_1 = 1, 2, \ldots, k \rightsquigarrow P(X_1 = j) \rightsquigarrow$ "α" distrib

$P(X_0 = 2) = 0.17$

$P(X_1 = 2) = 0.17$

Limiting Distributions are Stationary Distributions

Lemma 3.1. *Assume that $\boldsymbol{\pi}$ is the limiting distribution of a Markov chain with transition matrix $\boldsymbol{P}$. Then, $\boldsymbol{\pi}$ is a stationary distribution.*

Regular Matrices

Lemma 3.1. *Assume that π is the limiting distribution of a Markov chain with transition matrix P. Then, π is a stationary distribution.*

Regular Matrices

A matrix M is said to be *positive* if all the entries of M are positive. We write $M > 0$. Similarly, write $x > 0$ for a vector x with all positive entries.

No es positiva

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

P_4 es positiva.

$$P^4 = \begin{pmatrix} 9/16 & 5/16 & 1/8 \\ 1/4 & 3/8 & 3/8 \\ 1/2 & 5/16 & 3/16 \end{pmatrix}$$

$n=4$

Regular Transition Matrix

A transition matrix P is said to be *regular* if some power of P is positive. That is, $P^n > 0$, for some $n \geq 1$.

P es una matriz de transición regular!

Limit Theorem for Regular Markov Chains

Theorem 3.2. *A Markov chain whose transition matrix P is regular has a limiting distribution, which is the unique, positive, stationary distribution of the chain. That is, there exists a unique probability vector $\pi > 0$, such that*

$$\lim_{n\to\infty} P^n_{ij} = \pi_j, \quad \text{distribución límite}$$

for all i, j, where

$$\sum_i \pi_i P_{ij} = \pi_j. \quad \text{distribución estacionaria}$$

Equivalently, there exists a positive stochastic matrix Π such that

$$\lim_{n\to\infty} P^n = \Pi,$$

where Π has equal rows with common row π, and π is the unique probability vector, which satisfies

$$\pi P = \pi.$$

Example 3.3 Assume that a Markov chain has transition matrix

$$P = \begin{pmatrix} 0 & 1-p & p \\ p & 0 & 1-p \\ 1-p & p & 0 \end{pmatrix},$$

for $0 < p < 1$. Find the limiting distribution.

$$P^2 = \begin{pmatrix} 2p(1-p) & p^2 & (1-p)^2 \\ (1-p)^2 & 2p(1-p) & p^2 \\ p^2 & (1-p)^2 & 2p(1-p) \end{pmatrix}$$

Since $0 < p < 1$, the matrix P^2 is positive. Thus, P is regular. By Theorem 3.2, the limiting distribution is the stationary distribution.

$\pi = (1/3, 1/3, 1/3)$. $\quad \pi P \stackrel{?}{=} \pi$ | π es estacionaria

π es la distribución límite $\to$ P

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \begin{pmatrix} 0 & 1-p & p \\ p & 0 & 1-p \\ 1-p & p & 0 \end{pmatrix} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Here is one way to tell if a stochastic matrix is not regular. If for some power n, all the 0s in P^n appear in the same locations as all the 0s in P^{n+1}, then they will appear in the same locations for all higher powers, and the matrix is not regular.

Example 3.4 A Markov chain has transition matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Determine if the matrix is regular.

p^n

$$P^4 = \begin{pmatrix} \frac{9}{64} & \frac{7}{32} & \frac{1}{8} & \frac{3}{16} & \frac{21}{64} \\ \frac{21}{256} & \frac{11}{128} & \frac{15}{32} & \frac{11}{64} & \frac{49}{256} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{15}{16} & \frac{1}{16} & 0 \\ \frac{35}{256} & \frac{21}{128} & \frac{7}{32} & \frac{13}{64} & \frac{71}{256} \end{pmatrix} \text{ and } P^5 = \begin{pmatrix} \frac{35}{256} & \frac{21}{128} & \frac{7}{32} & \frac{13}{64} & \frac{71}{256} \\ \frac{71}{1024} & \frac{49}{512} & \frac{71}{128} & \frac{33}{256} & \frac{155}{1024} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{31}{32} & \frac{1}{32} & 0 \\ \frac{113}{1024} & \frac{71}{512} & \frac{41}{128} & \frac{47}{256} & \frac{253}{1024} \end{pmatrix}$$

p^{n+1}

La Matriz P es no regular

Obtener la distribución estacionaria (π) $\to$ vector!

Assume that π is a stationary distribution for a Markov chain with transition matrix P. Then,

$\pi = \pi P \leadsto$
$$\sum_i \pi_i P_{ij} = \pi_j, \text{ for all states } j,$$

which gives a system of linear equations. If P is a $k \times k$ matrix, the system has k equations and k unknowns. Since the rows of P sum to 1, the $k \times k$ system will contain a redundant equation.

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

$$\begin{bmatrix} (1-p) & p \\ q & (1-q) \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} \leadsto$$

$$(1-p)\pi_1 + q\pi_2 = \pi_1$$
$$p\pi_1 + (1-q)\pi_2 = \pi_2.$$

Se sabe que: las filas suman 1

Recordar $\to$ $\pi = (\pi_1, \pi_2) \to \pi_1 + \pi_2 = 1$

$p\pi_1 + \cancel{\pi_2} - q\pi_2 = \cancel{\pi_2}$

$p\pi_1 = q\pi_2$!

The equations are redundant and lead to $\pi_1 p = \pi_2 q$. If p and q are not both zero, then together with the condition $\pi_1 + \pi_2 = 1$, the unique solution is

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$p\pi_1 + \cancel{\pi_2} - q\pi_2 = \cancel{\pi_2}$

$p\pi_1 = q\pi_2$

$$\pi = \left(\frac{q}{p+q}, \frac{p}{p+q} \right).$$

Example 3.5 Find the stationary distribution of the weather Markov chain of Example 2.3, with transition matrix

$$P = \begin{array}{c} \\ \text{Rain} \\ \text{Snow} \\ \text{Clear} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Snow} & \text{Clear} \end{array} \\ \begin{pmatrix} 1/5 & 3/5 & 1/5 \\ 1/10 & 4/5 & 1/10 \\ 1/10 & 3/5 & 3/10 \end{pmatrix} \end{array}.$$

Una ecuación es redundante

$$\begin{aligned} (1/5)\pi_1 + (1/10)\pi_2 + (1/10)\pi_3 &= \pi_1 \\ (3/5)\pi_1 + (4/5)\pi_2 + (3/5)\pi_3 &= \pi_2 \\ (1/5)\pi_1 + (1/10)\pi_2 + (3/10)\pi_3 &= \pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

$$\pi = \left(\frac{1}{9}, \frac{3}{4}, \frac{5}{36} \right)$$