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2. Class division

Communicating classes, closed classes, absorption, irreducibility.

Class division is a partition of the state space into *communicating classes* generated by transition matrix P. We say that states i and j belong to the same communicating class if $p_{ij}^{(n)} > 0$ and $p_{ji}^{(n')} > 0$ for some $n, n' \geq 0$. This fact is denoted by $i \leftrightarrow j$. If one of these conditions holds, we write $i \to j$ or $j \to i$, and if one of them doesn't, $i \not\to j$ or $j \not\to i$. To check that we have a correctly defined partition, observe that (i) each state communicates with itself ($i \leftrightarrow i$), and (ii) if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$ (as $p_{ik}^{(n_1+n_2)} = \sum\limits_{l \in I} p_{il}^{(n_1)} p_{lk}^{(n_2)} \geq p_{ij}^{(n_1)} p_{jk}^{(n_2)}$ and similarly for $p_{ki}^{(n_1+n_2)}$). Then, because of (i), each state i falls into some class, and beacuse of (ii) any state j falls in no more than one class. States from different classes, of course, do not communicate.

A useful fact is that $i \to j$ iff there exists a sequence of states

$$i_0 = i, i_1, \ldots, i_{n-1}, i_n = j$$

such that $p_{i_l i_{l+1}} > 0$ for each pair (i_l, i_{l+1}), $0 \leq l < n$. In fact, $p_{ij}^{(n)} = \sum\limits_{i-1,\ldots i_{n-1}} p_{ii_1} \cdots p_{i_{n-1}j}$, and the whole sum is > 0 iff there exists at least one summand that is > 0.

A communicating class C is called *closed* if $\forall\ i \in C$ and $j \in I$ such that $i \to j$, state $j \in C$. In other words, you can't escape from a closed class. States forming non-closed communicating classes are often called non-essential: they indeed are not essential in the long run. A state i is called *absorbing* if $p_{ii} = 1$. Equivalently, its communicating class (i) is closed, (ii) consists solely of i.

A Markov chain with a single communicating class (which then must be

closed) is called *irreducible*. The study of a reducible chain, with more than one class, is essentially reduced to studying its 'restrictions' to various classes.

Examples and remarks. 2.1. A particle moves from state $i = 1, \ldots, N-1$ to state $i+1$ with probability p and state $i-1$ with probability $1-p$ where $0 < p < 1$. From states 0 and N it can't move, i.e., once it reaches one of them, it stays there forever. This example describes a match between two players where the winner of a given game gets from the looser one 'score unit', and the match continues until one of the players is left with no unit. Another interpretation is a walk of a drunken man from a pub, where he makes a step towards home (state 0) or a lake (state N). In the first case, the value N is the total number of units of both players before the match: it is obviously preserved in the course of the game. In the second case, it is

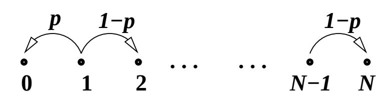

Diagram 2.1.

the distance in steps between the home and the lake (the pub is somewhere in betweeen). The state $i = 0, 1, \ldots, N$ is the number of units in possession of player one or the the distance from home. p and $1-p$ are the probabilities that player one wins or loses a game or that the drunkard makes a step towards home or the lake. Results of different games or directions of different moves are independent.

Here, the transition matrix is $(N+1) \times (N+1)$:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & \dots & 0 & 0 & 0 \\ 0 & 1-p & 0 & p & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & p & 0 \\ 0 & 0 & 0 & 0 & \dots & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}, \tag{2.1}$$

and the diagram is as above. Communicating classes are $\{0\}$, $\{1, \dots, N-1\}$ and $\{N\}$, and classes $\{0\}$ and $\{N\}$ are closed (i.e., states 0 and N absorbing). Thus, states 1, ..., $N-1$ are non-essential, and the game will ultimately end at one of the border states.

2.2. Consider a 6×6 transition matrix on states $\{1, 2, 3, 4, 5, 6\}$, of the form

$$P = \begin{pmatrix} * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & * \\ 0 & * & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

where $*$ stands for a non-zero entry. See Diagram 2.2.

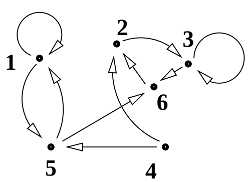

Diagram 2.2.

The communicating classes are $C_1 = \{1, 5\}$, $C_2 = \{2, 3, 6\}$ and $C_3 = \{4\}$, of which only C_2 is closed. The arrows between classes show the dynamics of

the chain. If we start in class C_2, we remain in C_2 forever. If we start in C_3 (i.e., at state 3), we will enter C_2 (in which case we will stay on C_2 forever) or C_1. Intuitively, after sepending some in C_1, we must leave it, i.e., enter C_2. This is what happens in reality, as we will soon discover.

2.3. A Markov chain with a *finite* state space always has at least one closed communicating class. To prove it, consider any class, say C_1. If it is not closed, take the next class you can reach from C_1, say C_2. If it is not closed, you continue. You should end up this process with reaching a closed class.

2.4. The situation with a *countable* state space is more complicated. Here, you may have no closed class. In addition, states from infinite closed classes may also be 'non-essential', in the sense that the chain may visit each of them only finitely many times and will be driwen 'to infinity' (although still

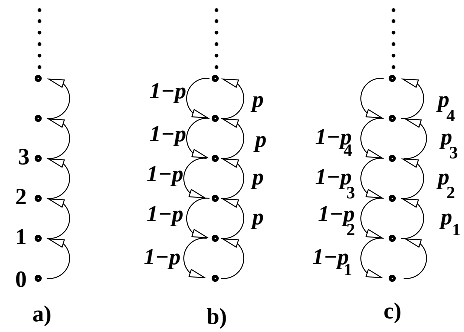

Diagram 2.3.

within the same closed class). Simplest examples are where state space I is the set of non-negative integers $\mathbb{Z}_+ = \{0, 1, 2, ...\}$. Three examples are on

Diagram 2.3; the corresponding transition matrices are:

$$\text{a)}\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \dots \end{pmatrix}, \ \text{b)}\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & \dots \\ 1-p & 0 & p & 0 & \dots & 0 & \dots \\ 0 & 1-p & 0 & p & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \dots \end{pmatrix} \quad (2.3)$$

and

$$\text{c)}\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & \dots \\ 1-p_1 & 0 & p_1 & 0 & \dots & 0 & \dots \\ 0 & 1-p_2 & 0 & p_2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \dots \end{pmatrix}. \quad (2.4)$$

These matrices describe the so-called birth-death processes where state i represents the size of the population, and during a transition a member of the population may die or a new member may be added. In case a) only births are allowed, and the chain is deterministic. Here, every state i forms a non-closed class and is non-essential. In model b) 'death' occurs with the same chance $1-p$ and the birth with the same chance p, regardless of the size i of the population at the given time (unless $i = 0$ of course).

In real life, it may be a queue of 'tasks' served by a 'server' (e.g., clients waiting in a barber shop with a single seat or computer programs subsequently executed by a processor). Then i is the number of tasks in the queue. If, before the hairdresser finishes with a current client, a new client comes, we have jump $i \to i+1$; otherwise from $i \to i-1$; from 0 we can only jump to 1 (although in the 'real time' the hairdrersses may be waiting for a while for this to happen). There are two situations: $p > 1/2$ and $0 < p < 1/2$. Intuitively, if $p > 1/2$, tasks will arrive more often than they are served, and the queue will become eventually infinite (which may rather please our hairdresser). In this situation, as we shal see, each state i will be visited finitely many times and X_n (the size of the queue at time n) will grow indefinitely with n. If $0 < p < 1/2$, the tasks will arrive less often, and the system will be able to reach an 'equilibrium', with some stationary distribution of the queue size.

An often used modification of example b) is where 0 is made an absorbing state, with $p_{00} = 1$.

In the more general case c), the rules of the population dynamics may include chances for every member to die (but one upon the time); e.g., $p_n =$

$\lambda/(\lambda + n\mu)$, $1 - p_n = n\mu/(\lambda + n\mu)$ where $\lambda > 0$ and $\mu > 0$ are positive birth and deats 'rates'; the whole picture becomes more complicated. We will be able to analyse some of these models in detail.

2.5. In general, after a re-enumeration of states, the finite transition matrix P acquires a particular structure: see Diagram 2.4.

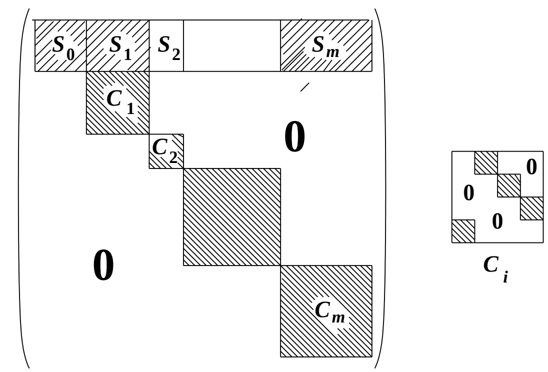

Diagram 2.4.

The top left corner is occupied by the non-essential states (block S_0); inside this block there may be some intricate transitions between and within non-closed classes. Diagonal blocks C_1, ..., C_m represent closed communicationg classes (of various size); they form stochastic sub-matrices. The corresponding Markov chains can be studied individually (which is easier as their size is smaller). Blocks S_1, ..., S_m, to the right of S_0 are non-negative rectangular matrices; some of them (but not all) may be zero. [Of course, the sum of entries along every row of P is always 1.]

The space between blocks is filled with zeros.

When you iterate such a matrix, raising it to large power n, blocks S_0, S_1, ..., S_m will become zeros, and blocks C_1, ..., C_m will simply be raised.

It is not hard to guess that inside each closed communicating class C_i we should have a 'periodic' picture, with a number m_i of smaller square 'cells' of equal size which are cyclically permuted by P: cell 1 is taken to cell 2, and so forth. The space between cells is again filled with zeros. Most of our examples will be where the period equals one, i.e., each class is a single cell. Such chains are called *aperiodic*. Of course, if you raise the sub-matrix corresponding to a closed communicating class to the power equal to its period, it will decompose into stochastic sub-matrices centred on the main diagonal.

The situation with an infinite stochastic matrix is similar; the only difference is that some of blocks, and their number m may be infinite.

2.6. (Math Tripos, Part IB, 1993, Question 501K; (modified)). Consider the stochastic matrix on $\{1, \dots, 7\}$:

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \end{pmatrix}.$$

Find all communicating classes of the associated discrete-time Markov chain.

Solution. See Diagram 2.5. The communicating classes are $\{1, 2, 6, 7\}$, $\{3\}$ and $\{4, 5\}$. The closed classes are $\{1, 2, 6, 7\}$ and $\{4, 5\}$. 3 is a non-essential state. Class $\{4, 5\}$ has a periodic structure. Thus, the limit $p_{i,j}^{(n)}$ does not exists (oscillates) for $i = 3, 4, 5$ and $j = 4, 5$. For class $\{1, 2, 6, 7\}$ we have a transition submatrix

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix},$$

with the symmetry $1 \leftrightarrow 6$ and $2 \leftrightarrow 7$. That is, if we merge 1 with 6 into a single state "0" and 2 with 7 into "1", we get a two-state chain with the matrix

$$\Pi = \begin{pmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

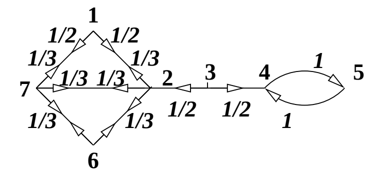

Diagram 2.5.

The last matrix has the characteristic equation

$$\mu^2 - \frac{1}{3}\mu - \frac{2}{3} = 0,$$

with the roots $\mu_0 = 1$, $\mu_1 = -2/3$. Hence, the entries of Π have the form $A + B(-2/3)^n$. Adjusting the constants yields

$$\Pi^n = \begin{pmatrix} 3/5 + 2/5(-2/3)^n & 2/5 - 2/5(-2/3)^n \\ 3/5 - 3/5(-2/3)^n & 2/5 + 3/5(-2/3)^n \end{pmatrix}.$$

Hence,

$$\Pi^n \to \begin{pmatrix} 3/5 & 2/5 \\ 3/5 & 2/5 \end{pmatrix}.$$

Then, by symmetry, for the original $\{1, 2, 6, 7\}$-block, the limiting matrix is

$$\begin{pmatrix} 1/5 & 3/10 & 1/5 & 3/10 \\ 1/5 & 3/10 & 1/5 & 3/10 \\ 1/5 & 3/10 & 1/5 & 3/10 \\ 1/5 & 3/10 & 1/5 & 3/10 \end{pmatrix}.$$

That is, $p_{i1}^{(n)}, p_{i6}^{(n)} \to 1/5$ and $p_{i2}^{(n)}, p_{i7}^{(n)} \to 3/10$, $i = 1, 2, 6, 7$. Probabilities $p_{3j}^{(n)}$ converge to $\frac{1}{2} \lim\limits_{n\to\infty} p_{2j}^{(n)}$, $j = 1, 2, 6, 7$.

It has to stressed that this is not the optimal way of doing this example. Later on, we will learn about much more efficient ways of calculating the limits $\lim_{n\to\infty} p_{ij}^{(n)}$.