36. Let X be the number of 1's and Y the number of 2's that occur in n rolls of a fair die. Compute $\text{Cov}(X, Y)$.

37. A die is rolled twice. Let X equal the sum of the outcomes, and let Y equal the first outcome minus the second. Compute $\text{Cov}(X, Y)$.

38. The random variables X and Y have a joint density function given by

$$f(x, y) = \begin{cases} 2e^{-2x}/x & 0 \le x < \infty, 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

Compute $\text{Cov}(X, Y)$.

39. Let $X_1, \ldots$ be independent with common mean μ and common variance σ^2, and set $Y_n = X_n + X_{n+1} + X_{n+2}$. For $j \ge 0$, find $\text{Cov}(Y_n, Y_{n+j})$.

40. The joint density function of X and Y is given by

$$f(x, y) = \frac{1}{y} e^{-(y + x/y)}, \quad x > 0, y > 0$$

Find $E[X]$, $E[Y]$, and show that $\text{Cov}(X, Y) = 1$.

41. A pond contains 100 fish, of which 30 are carp. If 20 fish are caught, what are the mean and variance of the number of carp among the 20? What assumptions are you making?

42. A group of 20 people consisting of 10 men and 10 women is randomly arranged into 10 pairs of 2 each. Compute the expectation and variance of the number of pairs that consist of a man and a woman. Now suppose the 20 people consist of 10 married couples. Compute the mean and variance of the number of married couples that are paired together.

43. Let $X_1, X_2, \ldots, X_n$ be independent random variables having an unknown continuous distribution function F, and let $Y_1, Y_2, \ldots, Y_m$ be independent random variables having an unknown continuous distribution function G. Now order those $n + m$ variables, and let

$$I_i = \begin{cases} 1 & \text{if the } i\text{th smallest of the } n + m \\ & \text{variables is from the } X \text{ sample} \\ 0 & \text{otherwise} \end{cases}$$

The random variable $R = \sum_{i=1}^{n+m} iI_i$ is the sum of the ranks of the X sample and is the basis of a standard statistical procedure (called the Wilcoxon sum-of-ranks test) for testing whether F and G are identical distributions. This test accepts the hypothesis that $F = G$ when R is neither too large nor too small. Assuming that the hypothesis of equality is in fact correct, compute the mean and variance of R.
Hint: Use the results of Example 3e.

44. Between two distinct methods for manufacturing certain goods, the quality of goods produced by method i is a continuous random variable having distribution $F_i, i = 1, 2$. Suppose that n goods are produced by method 1 and m by method 2. Rank the $n + m$ goods according to quality, and let

$$X_j = \begin{cases} 1 & \text{if the } j\text{th best was produced from} \\ & \text{method 1} \\ 2 & \text{otherwise} \end{cases}$$

For the vector $X_1, X_2, \ldots, X_{n+m}$, which consists of n 1's and m 2's, let R denote the number of runs of 1. For instance, if $n = 5, m = 2$, and $X = 1, 2, 1, 1, 1, 1, 2$, then $R = 2$. If $F_1 = F_2$ (that is, if the two methods produce identically distributed goods), what are the mean and variance of R?

45. If X_1, X_2, X_3, and X_4 are (pairwise) uncorrelated random variables, each having mean 0 and variance 1, compute the correlations of

(a) $X_1 + X_2$ and $X_2 + X_3$;

(b) $X_1 + X_2$ and $X_3 + X_4$.

46. Consider the following dice game, as played at a certain gambling casino: Players 1 and 2 roll a pair of dice in turn. The bank then rolls the dice to determine the outcome according to the following rule: Player $i, i = 1, 2$, wins if his roll is strictly greater than the bank's. For $i = 1, 2$, let

$$I_i = \begin{cases} 1 & \text{if } i \text{ wins} \\ 0 & \text{otherwise} \end{cases}$$

and show that I_1 and I_2 are positively correlated. Explain why this result was to be expected.

47. Consider a graph having n vertices labeled $1, 2, \ldots, n$, and suppose that, between each of the $\binom{n}{2}$ pairs of distinct vertices, an edge is independently present with probability p. The degree of vertex i, designated as D_i, is the number of edges that have vertex i as one of their vertices.

(a) What is the distribution of D_i?

(b) Find $\rho(D_i, D_j)$, the correlation between D_i and D_j.

48. A fair die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a 6 and a 5. Find

(a) $E[X]$;

(b) $E[X|Y = 1]$;

(c) $E[X|Y = 5]$.

49. There are two misshapen coins in a box; their probabilities for landing on heads when they are flipped are, respectively, .4 and .7. One of the coins is to be randomly chosen and flipped 10 times. Given that two of the first three flips landed on heads, what is the conditional expected number of heads in the 10 flips?