12-JULIO

lunes, 12 de julio de 2021 10:56

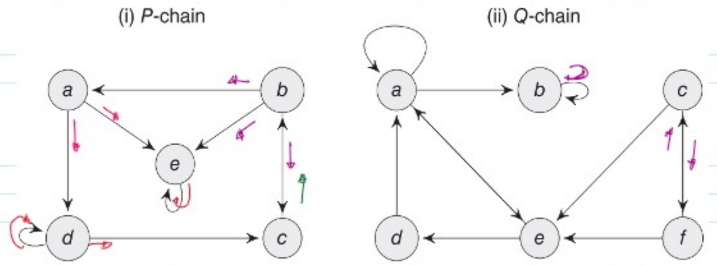
Say that state $j$ is *accessible* from state $i$, if $P^n_{ij} > 0$, for some $n \geq 0$. That is, there is positive probability of reaching $j$ from $i$ in a finite number of steps. States $i$ and $j$ *communicate* if $i$ is accessible from $j$ and $j$ is accessible from $i$.

Communication is an equivalence relation, which means that it satisfies the following three properties.

1. *(Reflexive)* Every state communicates with itself.
2. *(Symmetric)* If $i$ communicates with $j$, then $j$ communicates with $i$.
3. *(Transitive)* If $i$ communicates with $j$, and $j$ communicates with $k$, then $i$ communicates with $k$.

**Example 3.11** Find the communication classes for the Markov chains with these transition matrices.

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/6 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad Q = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1/6 & 1/3 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/4 & 1/4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 4/5 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{pmatrix} \end{array}.$$




Para la matriz P se tiene

$\underbrace{\{a,b,c,d\}}_{1°}, \underbrace{\{e\}}_{2°}$

Matriz Q

$\{b\}, \{c,f\}, \{a,d,e\}$

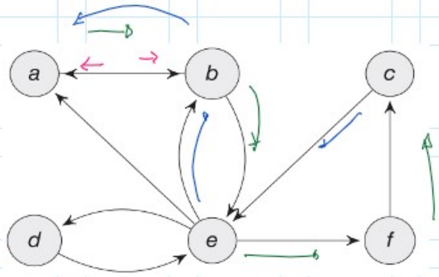
el agrupamiento fue debido a la comunicación entre estados

**Irreducibility**

A Markov chain is called *irreducible* if it has exactly one communication class. That is, all states communicate with each other.

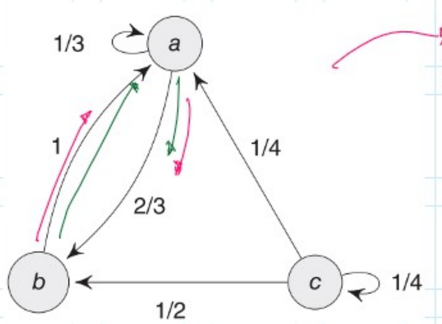
**Example 3.12** The Markov chain with transition matrix

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$




La cadena de Markov es irreducible!

# Recurrencia y transitoriedad




$$P = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1/3 & 2/3 & 0 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \end{array}$$

$\{a,b\}$ ; $\{c\}$

Given a Markov chain $X_0, X_1, \ldots$, let $T_j = \min\{n > 0 : X_n = j\}$ be the *first passage time* to state $j$. If $X_n \neq j$, for all $n > 0$, set $T_j = \infty$. Let

! tiempo de primera visita !

$$f_j = P(T_j < \infty | X_0 = j)$$

**Recurrent and Transient States**

State $j$ is said to be *recurrent* if the Markov chain started in $j$ eventually revisits $j$. That is, $f_j = 1$.

State $j$ is said to be *transient* if there is positive probability that the Markov

**Recurrence, Transience**

(i) State $j$ is recurrent if and only if

$$\sum^{\infty} P^n_{jj} = \infty.$$

State $j$ is said to be *recurrent* if the Markov chain started in $j$ eventually revisits $j$. That is, $f_j = 1$.

State $j$ is said to be *transient* if there is positive probability that the Markov chain started in $j$ never returns to $j$. That is, $f_j < 1$.

(i) State $j$ is recurrent if and only if

$$\sum_{n=0}^{\infty} P_{jj}^n = \infty.$$

(ii) State $j$ is transient if and only if

$$\sum_{n=0}^{\infty} P_{jj}^n < \infty.$$

**Recurrence and Transience are Class Properties**

**Theorem 3.3.** *The states of a communication class are either all recurrent or all transient.*

**Corollary 3.4.** *For a finite irreducible Markov chain, all states are recurrent.*

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# Cadena reversible

**Time Reversibility**

An irreducible Markov chain with transition matrix $\boldsymbol{P}$ and stationary distribution $\boldsymbol{\pi}$ is *reversible*, or *time reversible*, if

$$\pi_i P_{ij} = \pi_j P_{ji}, \text{ for all } i, j. \qquad (3.8)$$

$P(X_0 = i, X_1 = j) = P(X_0 = j, X_1 = i)$, for all $i, j$.

$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_0 = i_n, X_1 = i_{n-1}, \ldots, X_n = i_0)$,

for all $i_0, i_1, \ldots, i_n$.

**Example 3.22** A Markov chain has transition matrix

$\boldsymbol{\pi} = (1/3, 4/15, 2/5)$

$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 2/5 & 3/5 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 1/6 & 1/3 \end{pmatrix} \end{array}.$$

Determine if the chain is reversible.

$$\pi_1 P_{12} = \left(\frac{1}{3}\right)\left(\frac{2}{5}\right) = \frac{2}{15} = \left(\frac{4}{15}\right)\left(\frac{1}{2}\right) = \pi_2 P_{21},$$

$$\pi_1 P_{13} = \left(\frac{1}{3}\right)\left(\frac{3}{5}\right) = \frac{1}{5} = \left(\frac{2}{5}\right)\left(\frac{1}{2}\right) = \pi_3 P_{31},$$

$$\pi_2 P_{23} = \left(\frac{4}{15}\right)\left(\frac{1}{4}\right) = \frac{1}{15} = \left(\frac{2}{5}\right)\left(\frac{1}{6}\right) = \pi_3 P_{32}.$$

Con $\pi = (1/3, 4/15, 2/5)$ la cadena es de tiempo reversible.

If the stationary distribution of a Markov chain is uniform, it is apparent from Equation (3.8) that the chain is reversible if the transition matrix is symmetric.

Si $\pi = (\pi_1, \ldots, \pi_k)$ es uniforme

$\pi_i = 1/k$ ; $i = 1, 2, \ldots, k$

$\pi_i P_{ij} = \pi_j P_{ji}$

$\cancel{\tfrac{1}{k}} P_{ij} = \cancel{\tfrac{1}{k}} P_{ji} \sim P_{ij} = P_{ji}$ ; $\forall\, i, j$

$\hookrightarrow$ P es simétrica!