AMME 2301/9301: *Mechanics of Solids*

-WK11&12: *Deflection of Beams*

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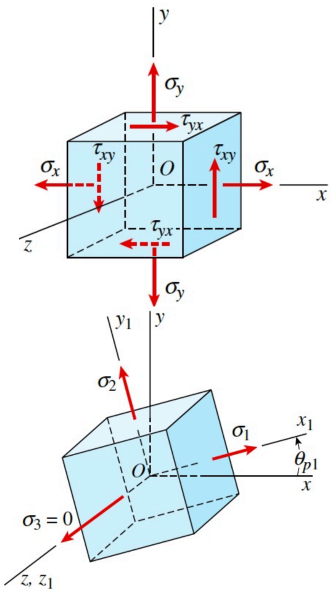
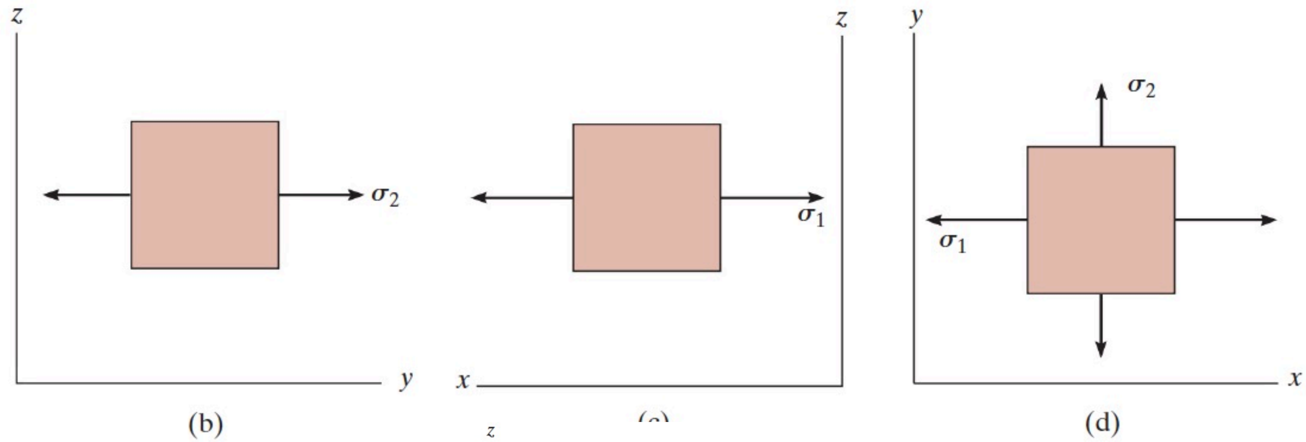
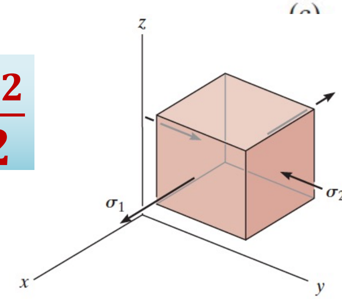
Absolute Maximum Shear Stress



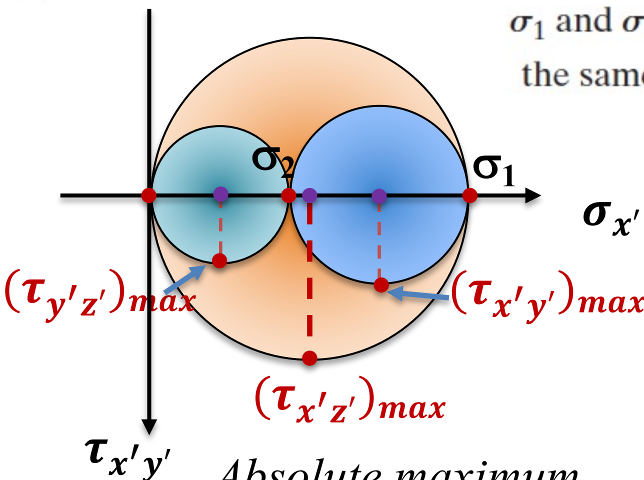




x–y plane stress

$$\tau_{\substack{abs\\max}} = \pm\frac{\sigma_1}{2}, \pm\frac{\sigma_2}{2}$$

σ_1 and σ_2 have the same sign

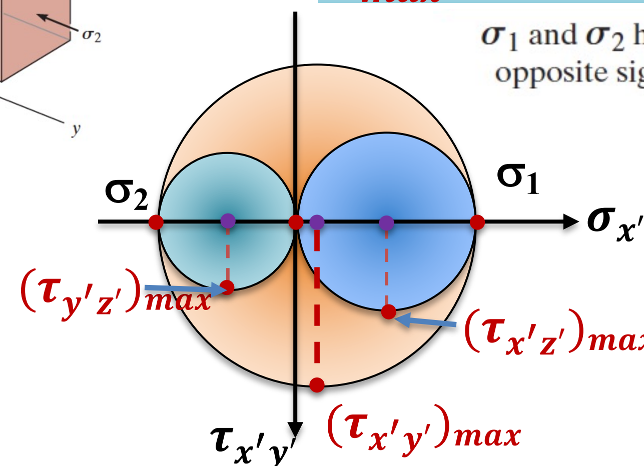


Maximum in-plane shear stress

Absolute maximum shear stress

$$\tau_{\substack{abs\\max}} = \pm\frac{\sigma_1 - \sigma_2}{2}$$

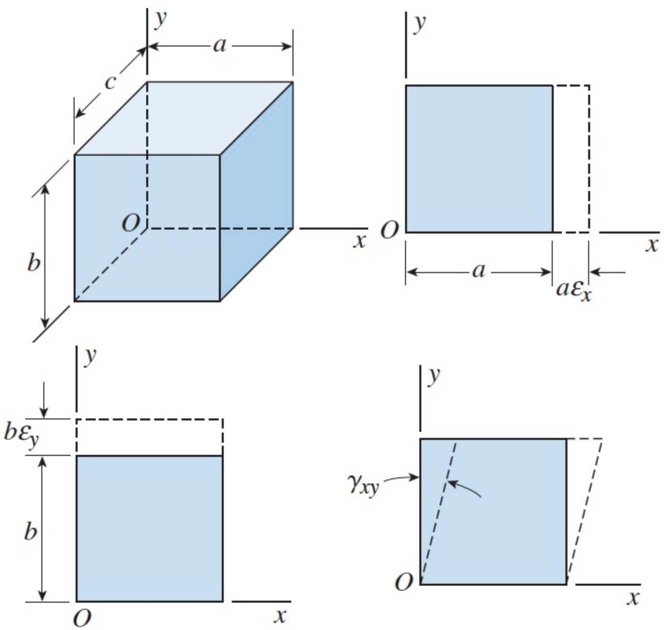
σ_1 and σ_2 have opposite signs



Maximum in-plane and absolute maximum shear stress

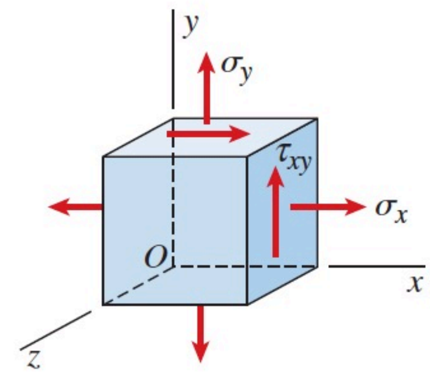
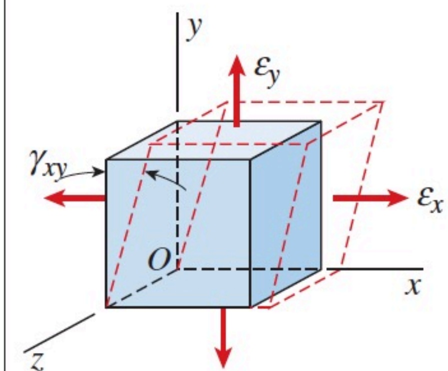
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson. **J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.**

Plain Strain



Plain Strain

$$\varepsilon_z = 0$$
$$\gamma_{xz} = 0$$
$$\gamma_{yz} = 0$$

	Plane stress	**Plane strain**
		
Stresses	$\sigma_z = 0$ $\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, and τ_{xy} may have nonzero values	$\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values
Strains	$\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ε_x, ε_y, ε_z, and γ_{xy} may have nonzero values	$\varepsilon_z = 0$ $\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ε_x, ε_y, and γ_{xy} may have nonzero values

It should not be inferred from the similarities in the definitions of plane stress and plane strain that both occur simultaneously

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.
J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Transformation Equation for Plain Strain

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$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} cos2\theta + \frac{\gamma_{xy}}{2} sin2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} sin2\theta + \frac{\gamma_{xy}}{2} cos2\theta$$

$$\varepsilon_{x'} + \varepsilon_{y'} = \varepsilon_x + \varepsilon_y$$

Principal Strain

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{(\frac{\varepsilon_x - \varepsilon_y}{2})^2 + (\frac{\gamma_{xy}}{2})^2}$$

$$tan2\theta_p = \frac{\gamma_{xy}}{(\varepsilon_x - \varepsilon_y)}$$

- The shear strain are zero on the principal planes.
- At a given point in a stressed body, the principal strains and principal stresses occur in the same directions.

Stresses	Strains
σ_x	ε_x
σ_y	ε_y
τ_{xy}	$\frac{\gamma_{xy}}{2}$
$\sigma_{x'}$	$\varepsilon_{x'}$
$\tau_{x'y'}$	$\frac{\gamma_{x'y'}}{2}$

Maximum In-Plane Shear Strain

$$\frac{\gamma_{max}}{2} = \sqrt{(\frac{\varepsilon_x - \varepsilon_y}{2})^2 + (\frac{\gamma_{xy}}{2})^2}$$

$$\varepsilon_{avg} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

$$tan2\theta_s = -\frac{(\varepsilon_x - \varepsilon_y)}{\gamma_{xy}}$$

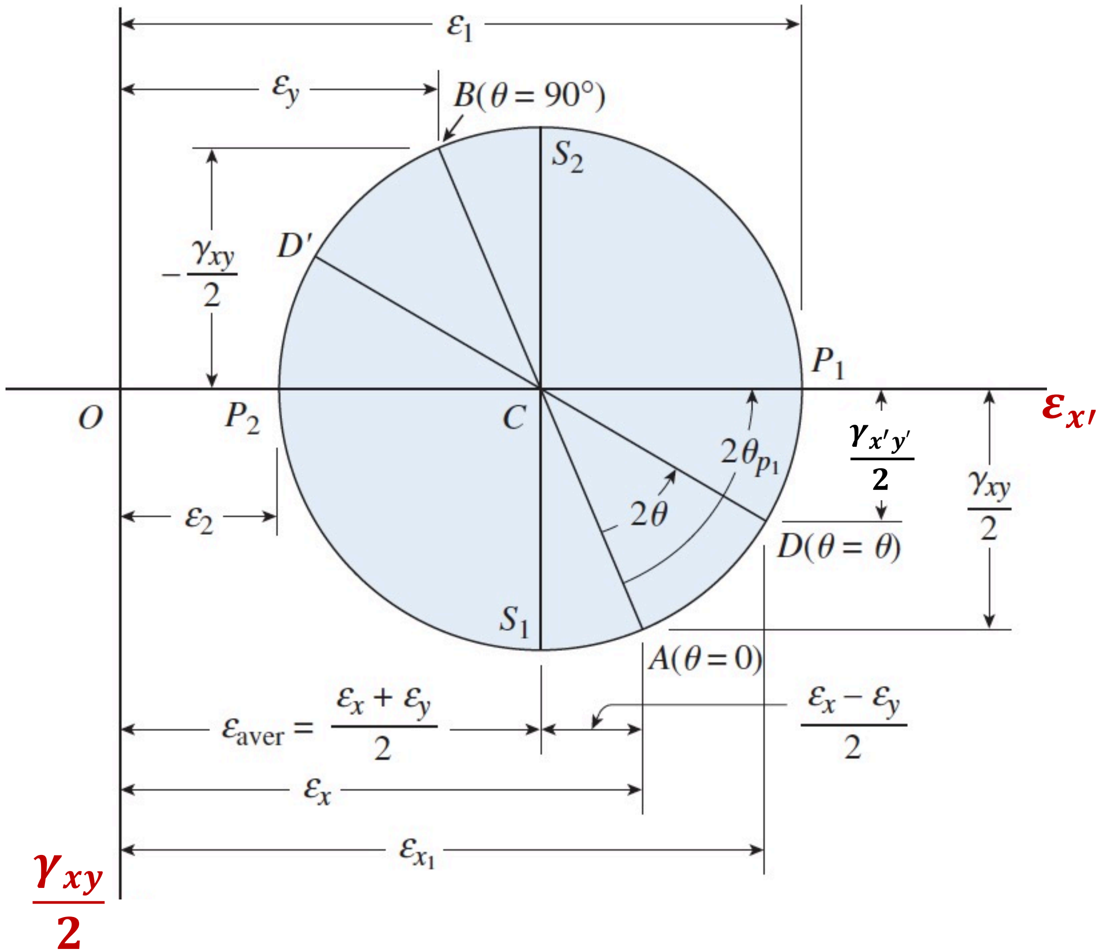
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson. J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Mohr's Circle for Plane strain

$$(\varepsilon_{x'}-\varepsilon_{avg})^2+\left(\frac{\gamma_{x'y'}}{2}\right)^2=R^2$$

$$\varepsilon_{avg}=\frac{\varepsilon_x+\varepsilon_y}{2}$$

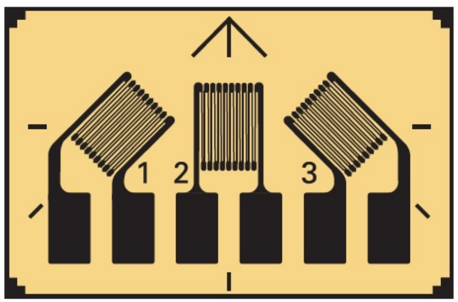
$$R=\sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2}$$



R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson. J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

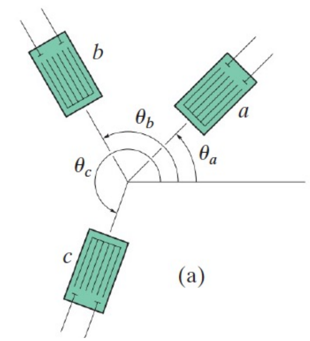
Strain Measurement

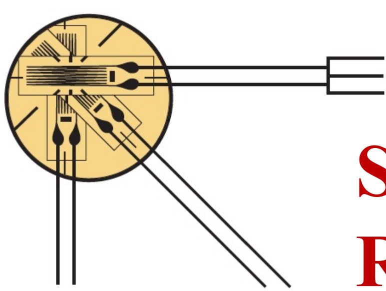
- An electrical-resistance strain gage is a device for measuring normal strains on the surface of a stressed object.
- Each gage consists of a fine metal grid that is stretched or shortened when the object is strained at the point where the gage is attached.
- Since each gage measures the normal strain in only one direction, and since the directions of the principal stresses are usually unknown, it is necessary to use three gages in combination, with each gage measuring the strain in a different direction.



(a) 45° strain gages three-element rosette

$$\begin{aligned}
\epsilon_a &= \epsilon_x \cos^2\theta_a + \epsilon_y \sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a \\
\epsilon_b &= \epsilon_x \cos^2\theta_b + \epsilon_y \sin^2\theta_b + \gamma_{xy}\sin\theta_b\cos\theta_b \\
\epsilon_c &= \epsilon_x \cos^2\theta_c + \epsilon_y \sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c
\end{aligned}$$





(b) Three-element strain-gage rosettes prewired

Strain Rosette

$$\theta_a = 0^\circ \quad \theta_b = 45^\circ \quad \theta_c = 90^\circ$$

$$\begin{aligned}
\epsilon_x &= \epsilon_a \\
\epsilon_y &= \epsilon_c \\
\gamma_{xy} &= 2\epsilon_b - (\epsilon_a + \epsilon_c)
\end{aligned}$$

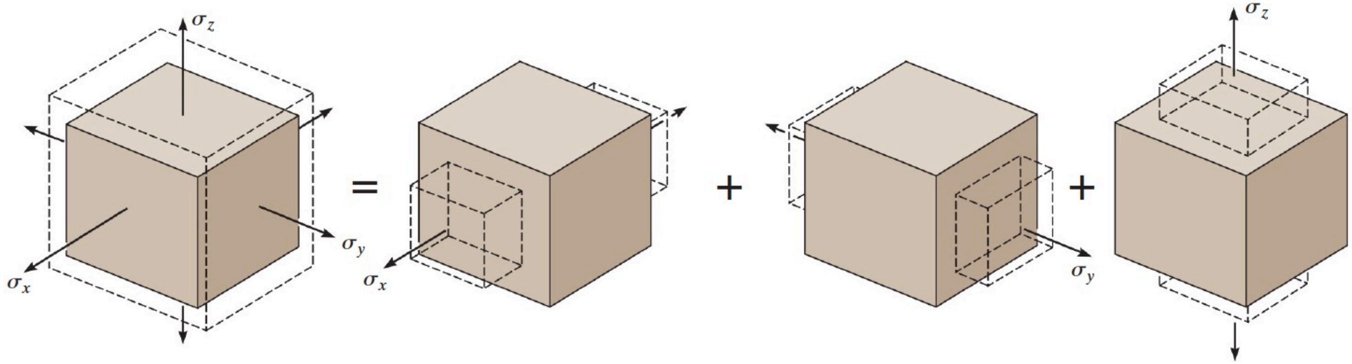
- The stress in the material that causes these strains can then be determined using Hooke’s law

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Materials-Property Relationship

Generalised Hooke's Law



$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

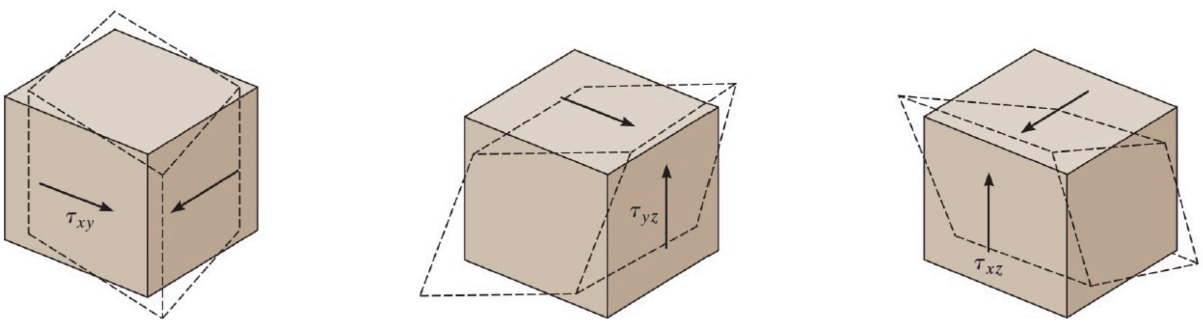
$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x)$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)]$$



$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz} \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz}$$

Relationship Involving E, ν, and G.

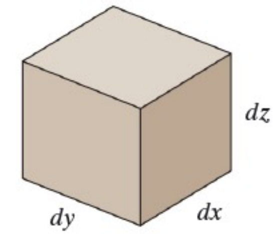
$$G = \frac{E}{2(1+\nu)}$$

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Materials-Property Relationship

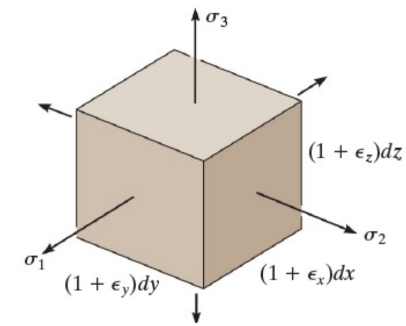
Dilation *e*

$$\delta V = (1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z)\, dx\, dy\, dz - dx\, dy\, dz$$



Neglecting the products of the strains, since the strains are very small,

$$\delta V = (\epsilon_x + \epsilon_y + \epsilon_z)\, dx\, dy\, dz$$



***The change in volume per unit volume is called the "volumetric strain" or the* dilatation *e*.**

$$e = \frac{\delta V}{dV} = \epsilon_x + \epsilon_y + \epsilon_z$$

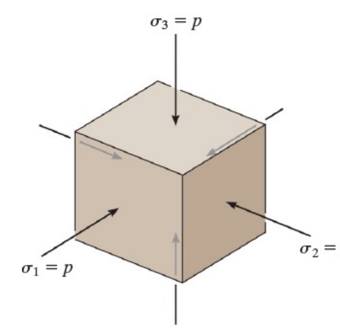
$$\boxed{e = \frac{1 - 2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)}$$

Bulk Modulus *k*

When a volume element of material is subjected to a uniform pressure p caused by a static fluid, the pressure will be the same in all directions. This state of "hydrostatic" loading therefore requires $\sigma_1 = \sigma_2 = \sigma_3 = -p$

$$\frac{p}{e} = -\frac{E}{3(1 - 2\nu)}$$

The term on the right is called the volume modulus of elasticity or the bulk modulus



Hydrostatic stress

$$\boxed{k = \frac{E}{3(1 - 2\nu)}}$$

For most metals $\nu \approx 1/3$ and so $k \approx E$. However, if we assume the material did not change its volume when loaded, then $e = 0$, and k would be infinite. As a result, the theoretical maximum value for ν to be $\nu = 0.5$.

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Ductile materials: Maximum-Shear-Stress (Tresca) Theory

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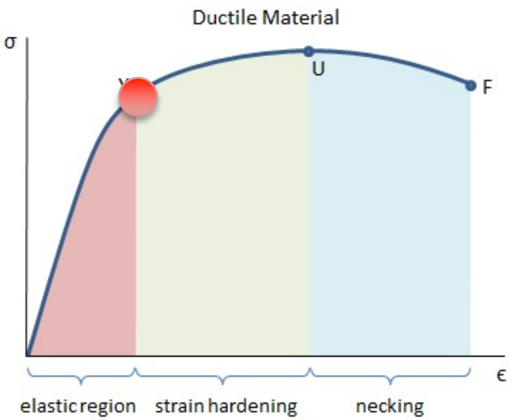
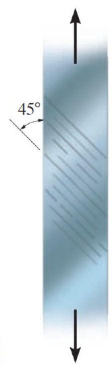
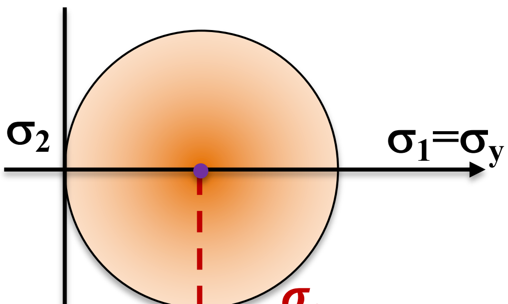
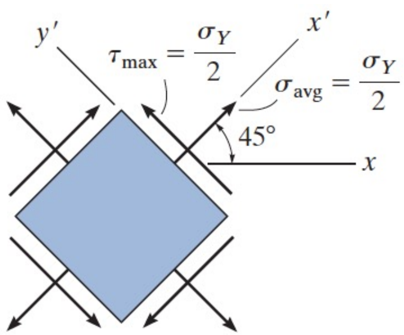


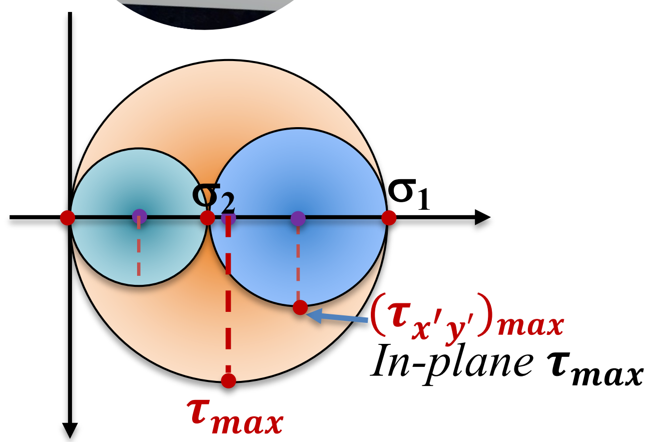


Regardless of the loading, yielding occurs when the absolute τ_{max} reaches the shear stress at yielding in a uniaxial tensile test.



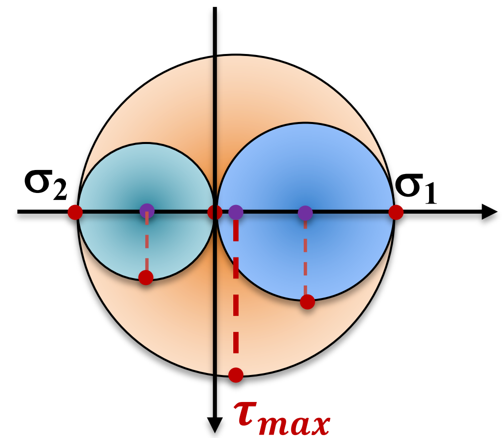
$$\tau_{max} = \frac{\sigma_y}{2}$$





$$\tau_{max} = \begin{cases} |\sigma_1|/2 \\ |\sigma_2|/2 \end{cases}$$

$(\sigma_1, \sigma_2$ *have same sign*$)$



$$\tau_{max} = (\sigma_1 - \sigma_2)/2$$

$(\sigma_1, \sigma_2$ *have opposite sign*$)$

$$\left.\begin{matrix} |\sigma_1| = \sigma_y \\ |\sigma_2| = \sigma_y \end{matrix}\right\} \qquad |\sigma_1 - \sigma_2| = \sigma_y$$

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

https://mechanicalc.com/reference/mechanical-properties-of-materials

Ductile materials: Maximum-Shear-Stress (Tresca) Theory

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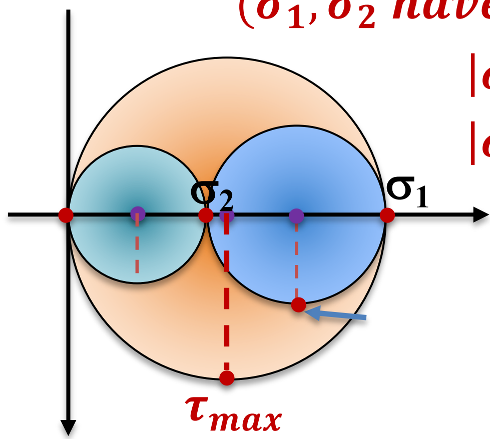
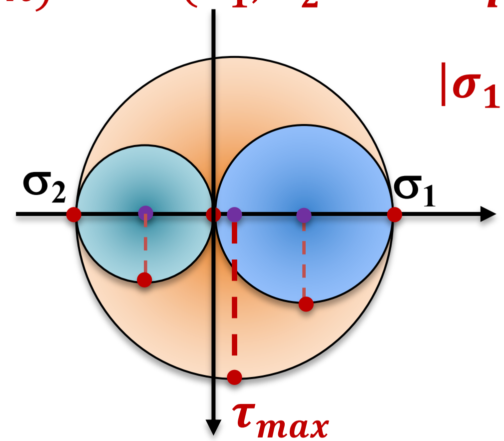
$(\sigma_1, \sigma_2 \text{ have same sign})$



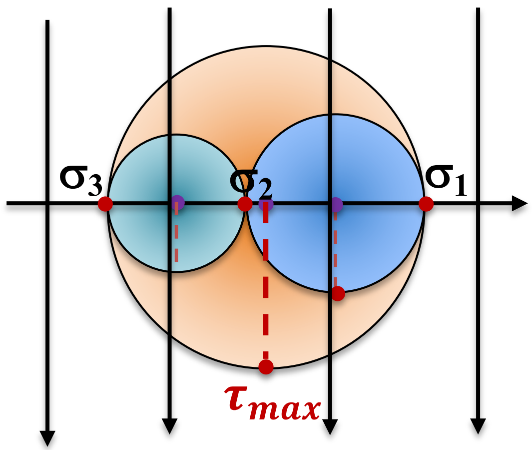
$$\left.\begin{array}{l}|\sigma_1| = \sigma_y \\ |\sigma_2| = \sigma_y\end{array}\right\}$$

$(\sigma_1, \sigma_2 \text{ have opposite sign})$



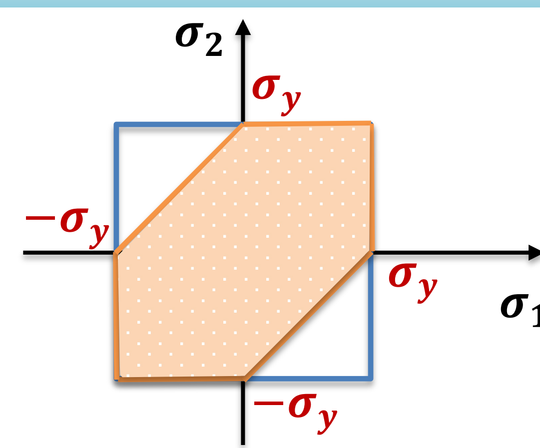
$$|\sigma_1 - \sigma_2| = \sigma_y$$

Yield surface: the representation of a failure theory in the principal stress space



$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

$$\boxed{\sigma_1 - \sigma_3 = \sigma_y}$$



$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

If any point of the material is subjected to plane stress that fall on the boundary or outside the shaded area, the material will yield at the point

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

https://mechanicalc.com/reference/mechanical-properties-of-materials

Triaxial stress

- Hydrostatic stresses do not cause yielding in ductile materials
- Deviatoric stresses causes yielding

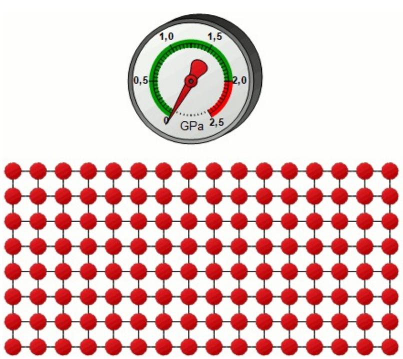
Stress causing a change in volume

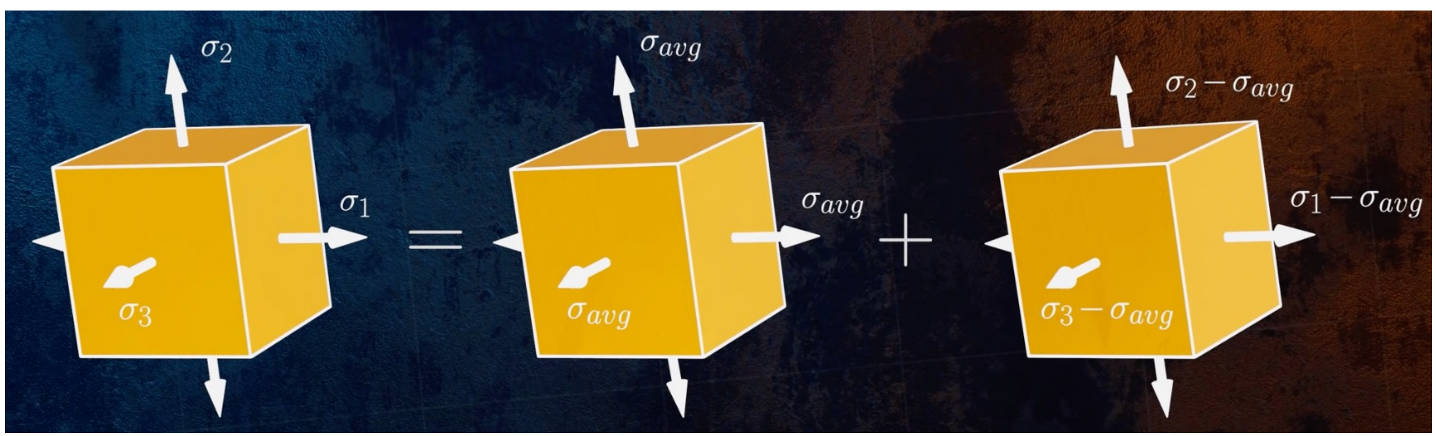
Stress causing a distortion of shape

$$\sigma_{avg} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

Hydrostatic stress component

Deviatoric stress component





https://www.tec-science.com/material-science/ductility-of-metals/fundamentals-of-deformation/

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

https://https://www.youtube.com/watch?v=xkbQnBAOFEg&t=9s

Ductile mater.: Maximum-Distortion-Energy (Von Mises) Theory

Yielding occurs when the *maximum distortion energy* is equal to the *distortion energy at yielding* in a uniaxial tensile test.

$(u_d = u_{d,y})$

$$u_d = \frac{1+\nu}{6E}\left[(\sigma_1 - \sigma_2{}^2) + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

$$u_{d,y} = \frac{1+\nu}{3E}\sigma_y^2$$

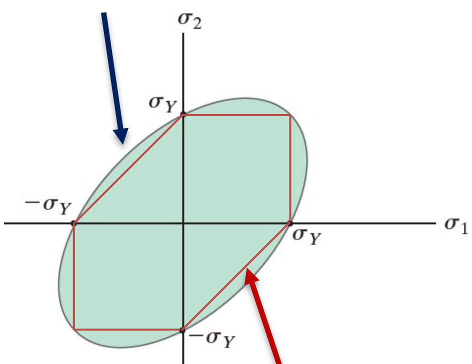
$$\underbrace{\sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2{}^2) + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}}_{\text{Equivalent von Mises stress } \sigma_{eq}} = \sigma_y$$

$$u_d = \frac{1+\nu}{3E}\left(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2\right)$$

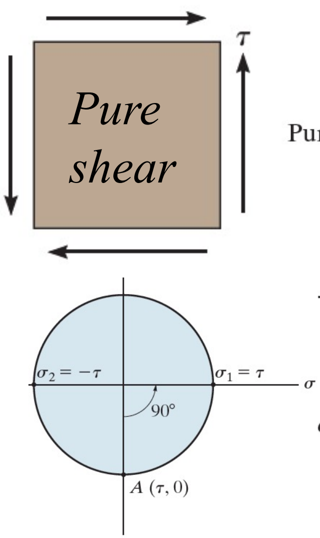
$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_y^2$$

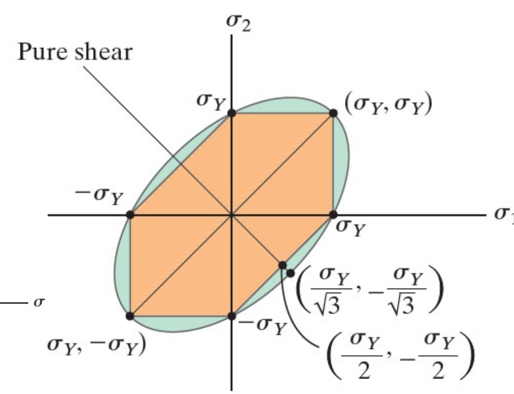
Plane stress

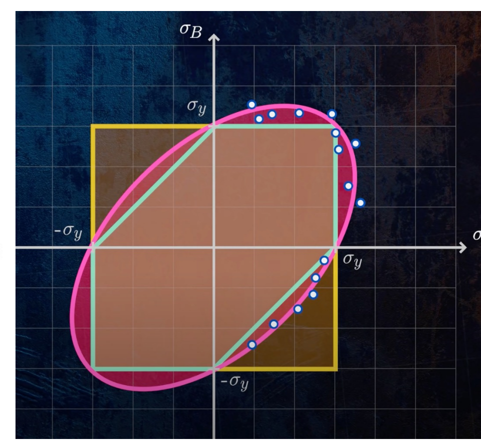
Von Mises Criterion



Tresca Criterion (more conservative)







R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

https://https://www.youtube.com/watch?v=xkbQnBAOFEg&t=9s

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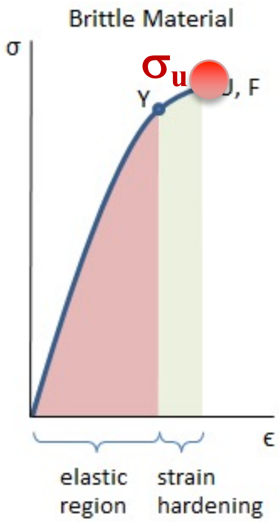
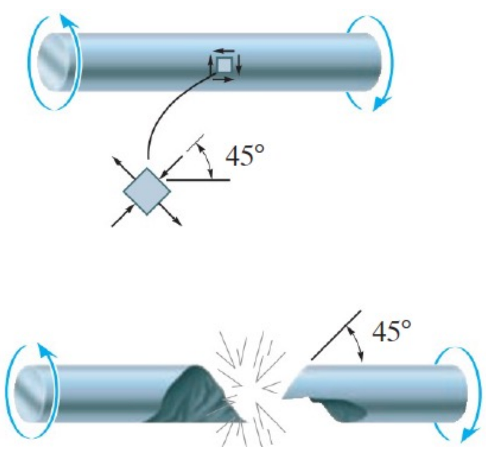
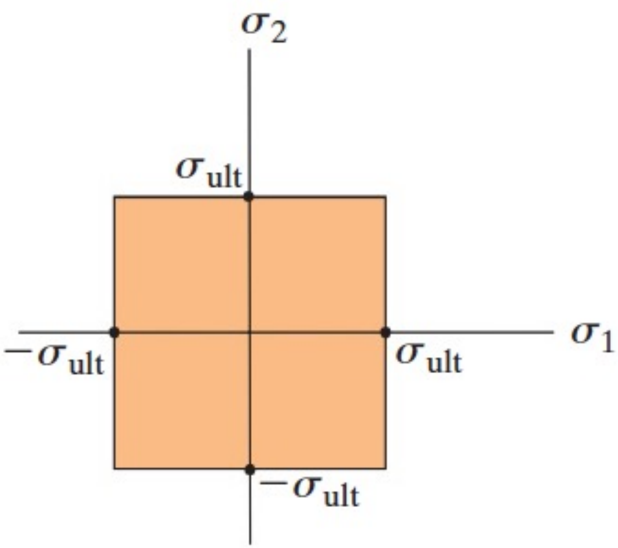


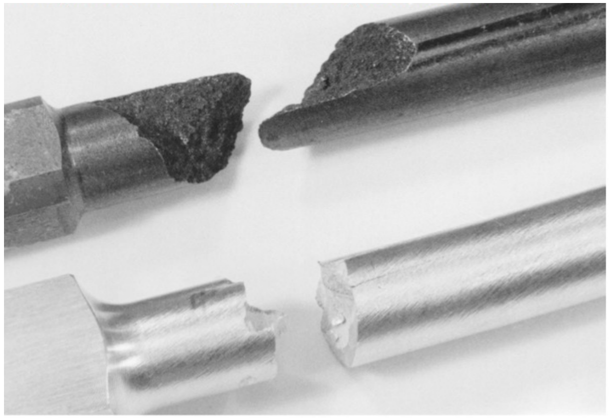
The maximum normal stress theory states that when a brittle material is subjected to a multiaxial state of stress, the material will fail when a principal tensile stress in the material reaches a value that is equal to the ultimate normal stress the material can sustain when it is subjected to simple tension.

Plane stress

$$|\sigma_1| = \sigma_{ult}$$
$$|\sigma_2| = \sigma_{ult}$$



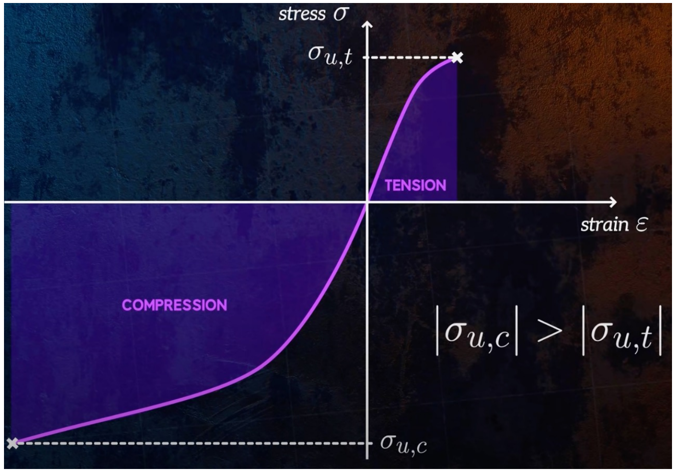


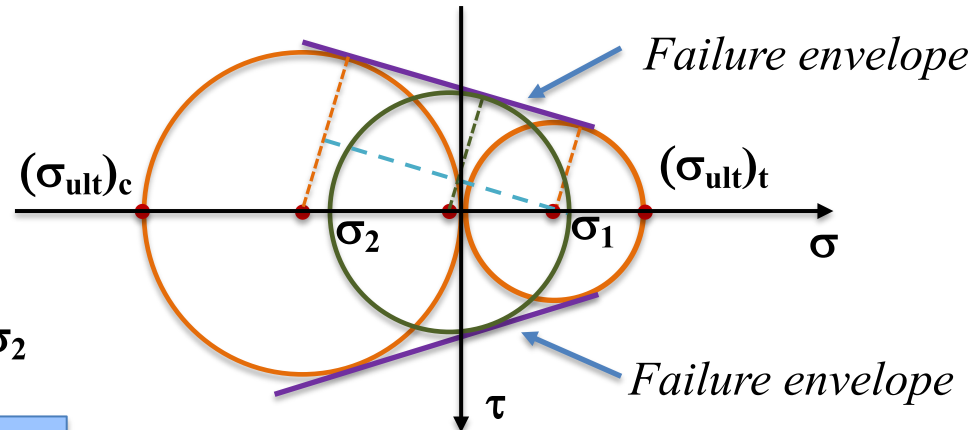


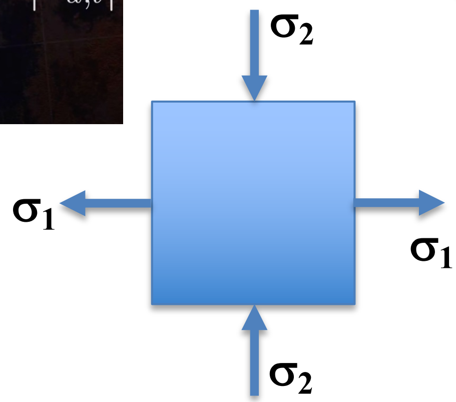
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

https://www.chegg.com/homework-help/questions-and-answers/following-figure-shows-results-torsion-tests-brittle-material-gray-cast-iron-top-ductile-m-q10030273

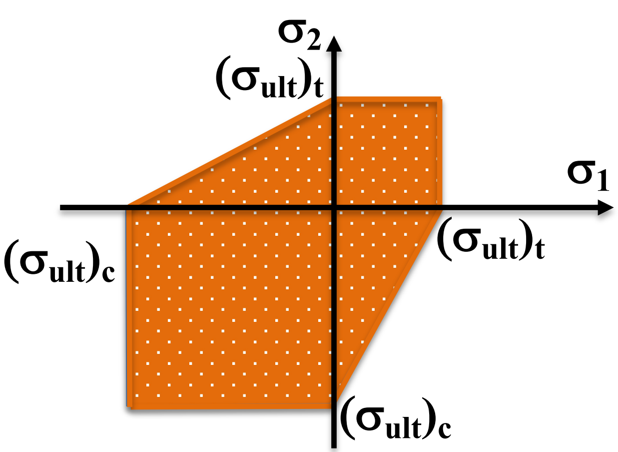
Brittle mater.: Mohr-Columb failure Criterion

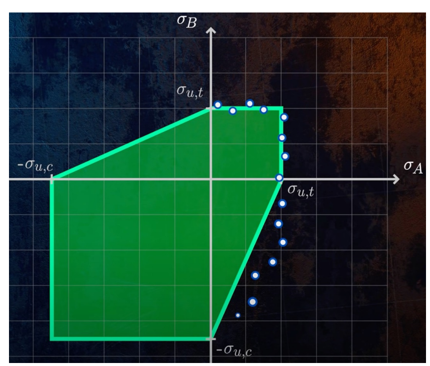


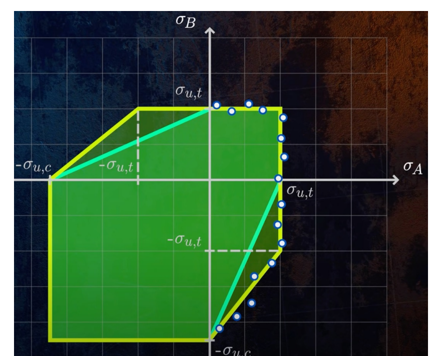




$$\frac{\sigma_1}{(\sigma_{ult})_t} - \frac{\sigma_2}{(\sigma_{ult})_c} = 1$$





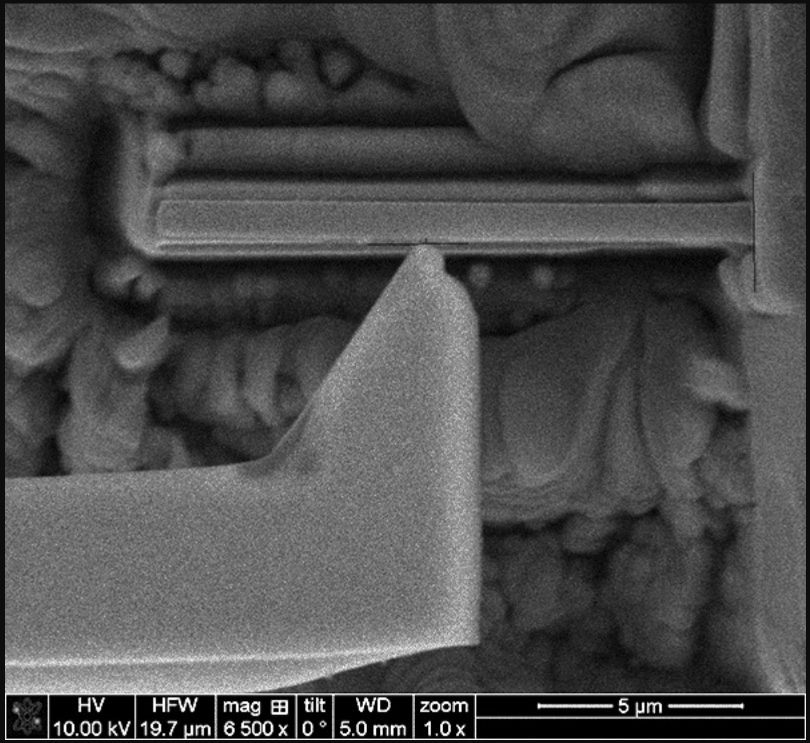


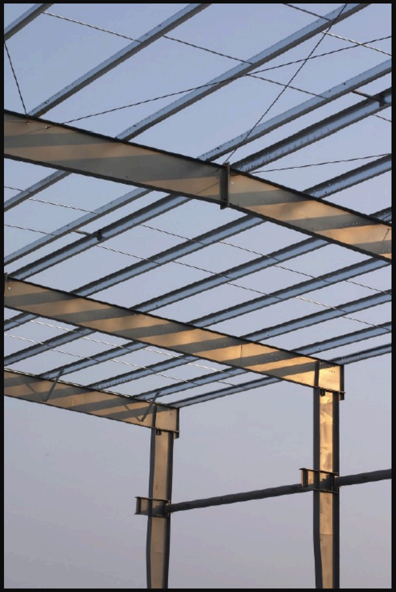
Modified Mohr

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

https://https://www.youtube.com/watch?v=xkbQnBAOFEg&t=9s

Deflection of beam





R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

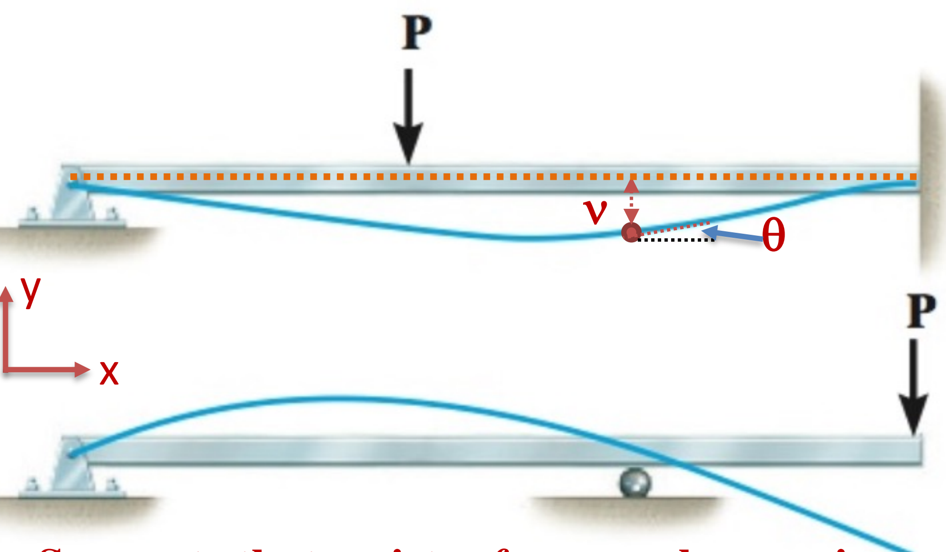
J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

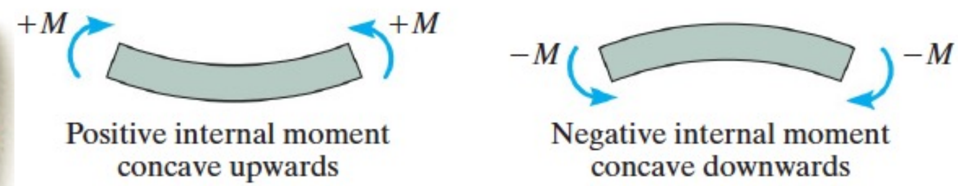
https://www.lboro.ac.uk/departments/materials/research/technique-development/in-situ-micro-bend-testing/

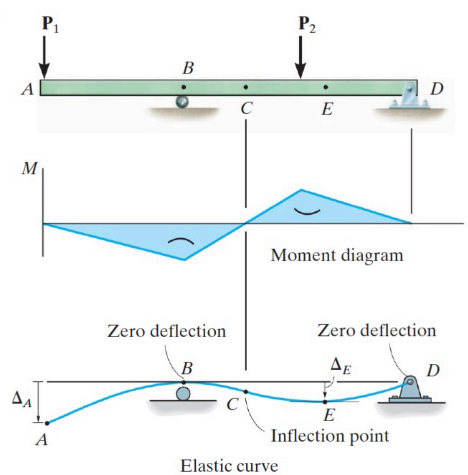
Deflection/Elastic Curve

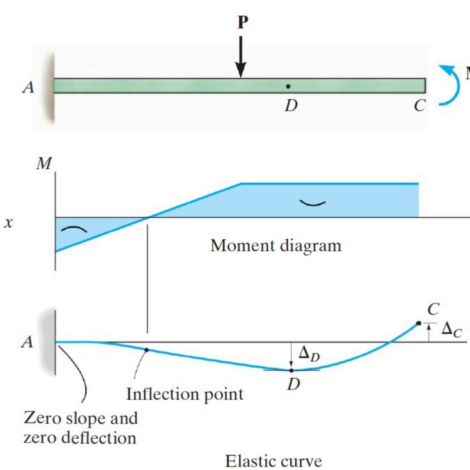
The deflection curve of the longitudinal axis that pass through the centroid of each cross-sectional area of a beam is called *the elastic curve or deflection curve*.

- ***Deflection (v):*** *the displacement in the y direction of any point on the axis of the beam*
- ***Angle of rotation/slope (θ) of the axis of the beam:*** *the angle between the x axis and the tangent to the deflection (elastic curve).*





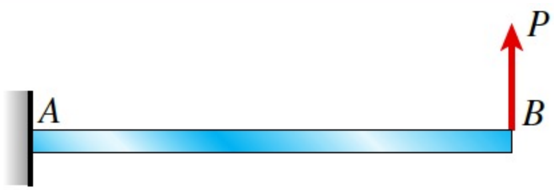




Supports that resist a *force*, such as a pin, restrict *displacement*, and those that resist a *moment*, such as a fixed wall, restrict *rotation* or *slope* as well as displacement.

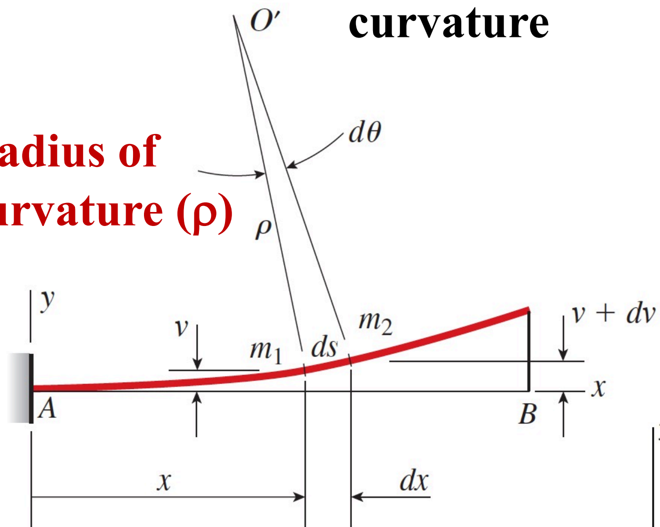
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Deflection/Elastic Curve



Center of curvature

Radius of curvature (ρ)



$\rho\, d\theta = ds$

By definition, the normal strain along Δs is

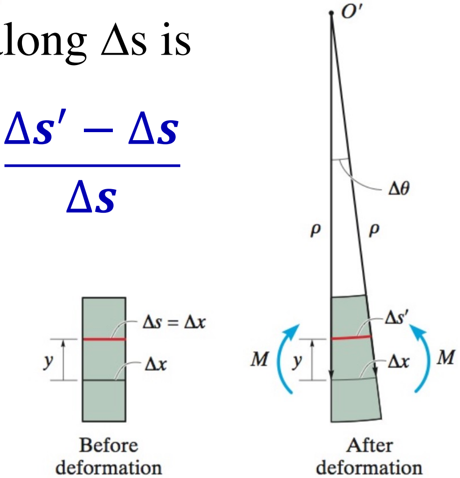
$$\Delta s = \Delta x = \rho \Delta\theta$$

$$\Delta s' = (\rho - y)\Delta\theta$$

$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

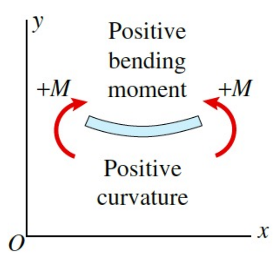
$$\epsilon = -\frac{y}{\rho} = -\kappa y$$

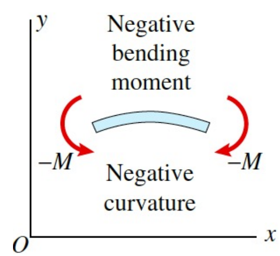
$$\sigma = E\epsilon = -\frac{Ey}{\rho} = -\frac{My}{I}$$

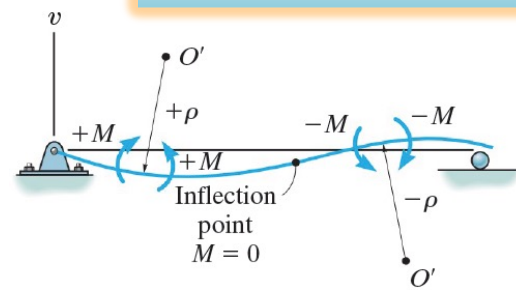


Moment-curvature equation

$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$







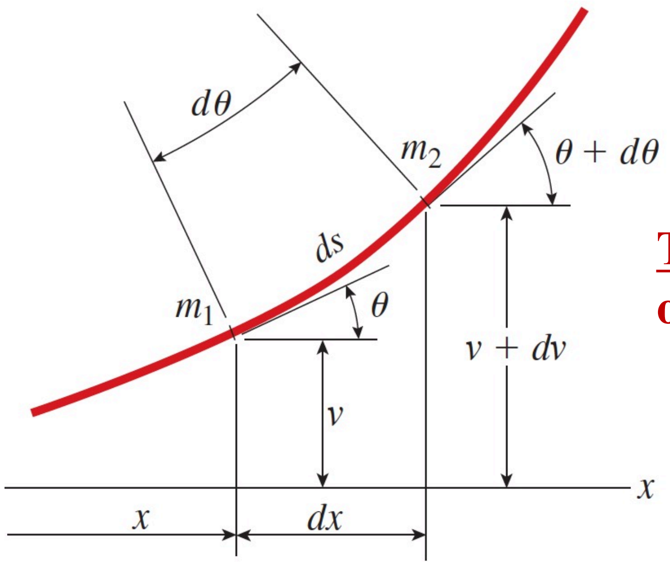
- **The curvature (κ):**
  $\kappa = 1/\rho = d\theta/ds$
- **A measure of how sharply a beam is bent**

- when M is positive, ρ extends above the beam, when M is negative, ρ extends below the beam
- **EI: flexural rigidity of the beam**, which is a measure of the resistance of a beam to bending

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Differential equation of the deflection curve



$$\frac{dv}{dx} = \tan\theta \qquad \frac{ds}{dx} = \cos\theta \qquad \frac{ds}{dv} = \sin\theta$$

$$(ds)^2 = (dx)^2 + (dv)^2 \qquad \frac{ds}{dx} = [1 + (dv/dx)^2]^{1/2}$$

The slope of the deflection curve is the first derivative dv/dx of the expression of the deflection v.

$$\theta = \arctan\frac{dv}{dx} \qquad \kappa = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d(\arctan\frac{dv}{dx})}{dx}\frac{dx}{ds}$$

$$\boxed{\kappa = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}}}$$

$$\frac{d(\arctan u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

$$ds \approx dx \qquad \kappa = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{dx}$$

$$\theta \approx \tan\theta = \frac{dv}{dx} \qquad \frac{d\theta}{dx} = \frac{d^2v}{dx^2}$$

$$\kappa = \frac{1}{\rho} = \frac{M}{EI} = \frac{d^2v}{dx^2}$$

Most engineering design codes will restrict the maximum deflection of a beam or shaft. The slope of the elastic curve, which is determined from dv/dx, will be very small, and its square will be negligible compared with unity.

$$\boxed{\kappa = \frac{1}{\rho} = \frac{d^2v}{dx^2} = \frac{M}{EI}}$$

Differential equation of the deflection curve

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Slope and deflection curve

Double Integration Method

$$\theta = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{1}{EI}\int M(x)dx + C_1$$

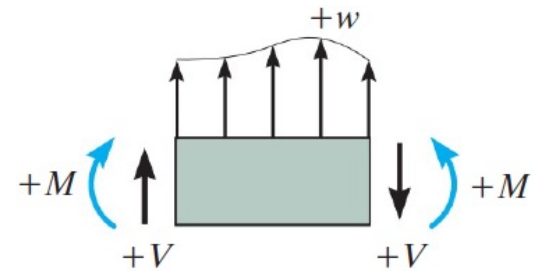
$$\frac{d^2v}{dx^2} = \frac{M(x)}{EI}$$

$$v = \frac{1}{EI}\iint M(x)dx + C_1x + C_2$$

$$\frac{d^3v}{dx^3} = \frac{V(x)}{EI}$$

$$\frac{d^4v}{dx^4} = \frac{\omega(x)}{EI}$$

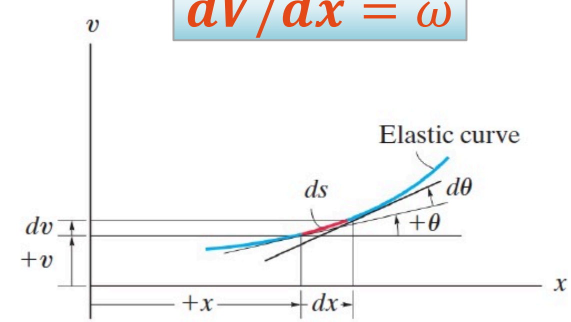
C_1 and C_2 are unknown constants that need to be determined.



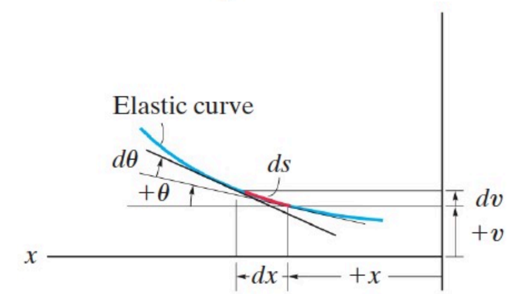
Positive sign convention

$$dM/dx = V$$

$$dV/dx = \omega$$



Positive sign convention



Positive sign convention

Positive deflection (v) is upwards, if x is positive to the right positive, slope θ is counterclockwise from the x axis; if positive x is directed to the left, θ is positive clockwise

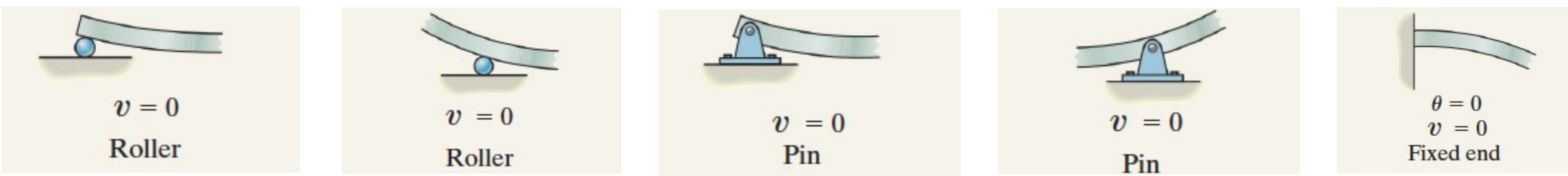
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Boundary and Continuity Conditions

$$\frac{dv}{dx} = \frac{1}{EI}\int M(x)dx + C_1$$

$$v = \frac{1}{EI}\iint M(x)dx + C_1x + C_2$$

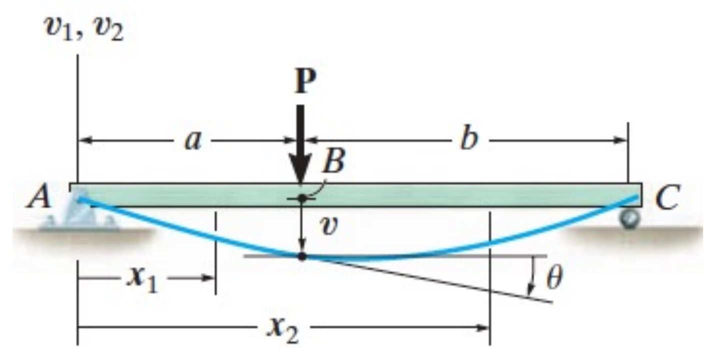
***Boundary conditions** pertain to the deflections and slopes at the supports of a beam.*



***Continuity conditions** occur at points where the regions of integration meet.*

Once the functions for the slope and deflection are obtained, they must give the ***same values*** for the slope and deflection at point B so the elastic curve is ***physically continuous***.

$$\theta_1(a) = \theta_2(a),\ v_1(a) = v_2(a)$$

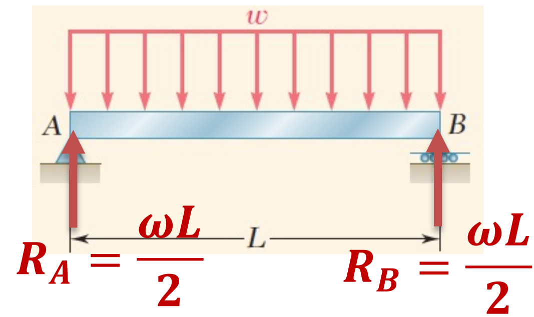


$0 \le x_1 \le a$ and $a \le x_2 \le (a + b)$

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

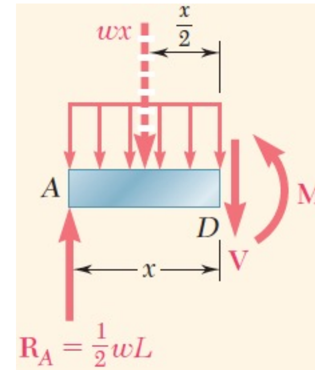
Concept application examples

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length. Determine the equation of the elastic curve and the maximum deflection of the beam.



Draw the FBD of the portion AD of the beam and take moments about D

$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2$$



Differential equation of the deflection curve

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

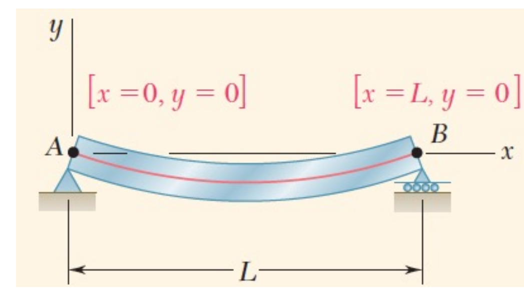
Integrating twice in x,

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_1$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_1x + C_2$$

- ❖ **x=0, y=0, C_2=0**
- ❖ **x=L, y=0**

$$C_1 = -\frac{\omega L^3}{24}$$



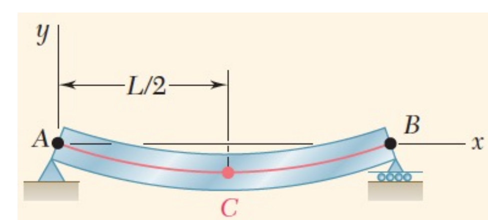
$$\theta = \frac{\omega}{24EI}(-4\omega x^3 + 6\omega Lx^2 - \omega L^3)$$

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x)$$

The slope of the beam is zero at the midpoint C for x = L/2

$$y_C = \frac{w}{24EI}\left(-\frac{L^4}{16} + 2L\frac{L^3}{8} - L^3\frac{L}{2}\right) = -\frac{5wL^4}{384EI}$$

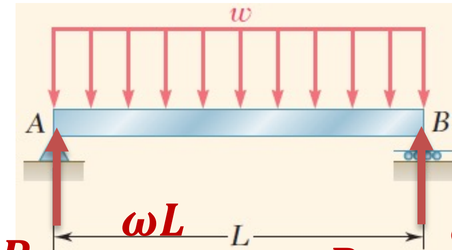
$$|y|_{max} = \frac{5wL^4}{384EI} \quad (\downarrow)$$



F.P. Beer et.al., Mechanics of Materials, 7th edition, McGrawHill Education.

Concept application examples

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length. Determine the equation of the elastic curve and the maximum deflection of the beam.



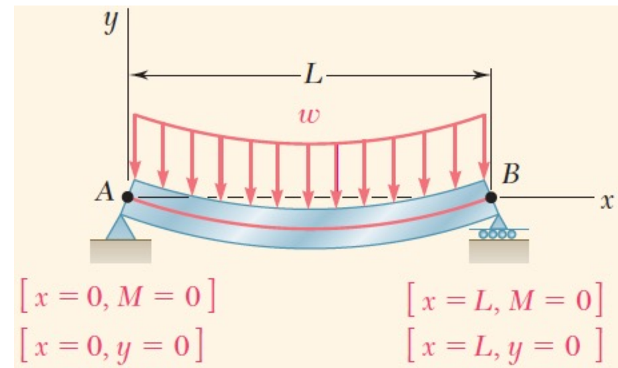
$R_A = \frac{\omega L}{2}$ $R_B = \frac{\omega L}{2}$

$$EI\frac{d^4y}{dx^4} = -w$$

$$EI\frac{d^3y}{dx^3} = V(x) = -wx + C_1$$

$$EI\frac{d^2y}{dx^2} = M(x) = -\frac{1}{2}wx^2 + C_1x + C_2$$

- **x=0, M=0, C_2=0**
- **x=L, M=0, $C_1 = \frac{\omega L}{2}$**



$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_3$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_3x + C_4$$

- **x=0, y=0, C_3=0**
- **x=L, y=0, $C_4 = -\frac{\omega L^3}{24}$**

$$\theta = \frac{\omega}{24EI}(-4\omega x^3 + 6\omega L x^2 - \omega L^3)$$

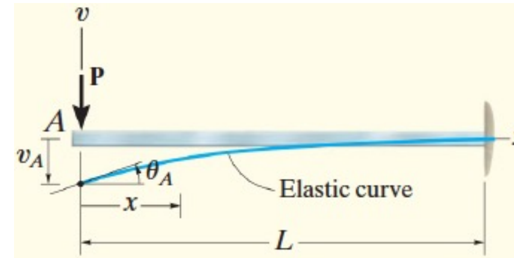
$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x)$$

$$|y|_{\max} = \frac{5wL^4}{384EI}\ (\downarrow)$$

F.P. Beer et.al., Mechanics of Materials, 7th edition, McGrawHill Education.

Concept application examples

The cantilevered beam shown in Fig. 12–11a is subjected to a vertical load **P** at its end. Determine the equation of the elastic curve. EI is constant.



Moment Function. From the free-body diagram, with **M** acting in the *positive direction*, Fig. 12–11b, we have

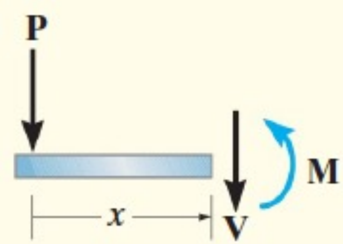
$$M = -Px$$

Slope and Elastic Curve.

$$EI\frac{d^2v}{dx^2} = -Px$$

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + C_1$$

$$EIv = -\frac{Px^3}{6} + C_1x + C_2$$



$$EI\frac{d^4v}{dx^4} = 0$$

$$EI\frac{d^3v}{dx^3} = C_1' = V \quad C_1' = -P.$$

$$EI\frac{d^3v}{dx^3} = -P$$

$$EI\frac{d^2v}{dx^2} = -Px + C_2' = M$$

$M = 0$ at $x = 0$, so $C_2' = 0$

Using the boundary conditions $dv/dx = 0$ at $x = L$ and $v = 0$ at $x = L$, Eqs. 2 and 3 become

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1L + C_2$$

$$C_1 = PL^2/2 \text{ and } C_2 = -PL^3/3.$$

$$\theta = \frac{P}{2EI}(L^2 - x^2)$$

$$v = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3)$$

Maximum slope and displacement occur at $A(x = 0)$, for which

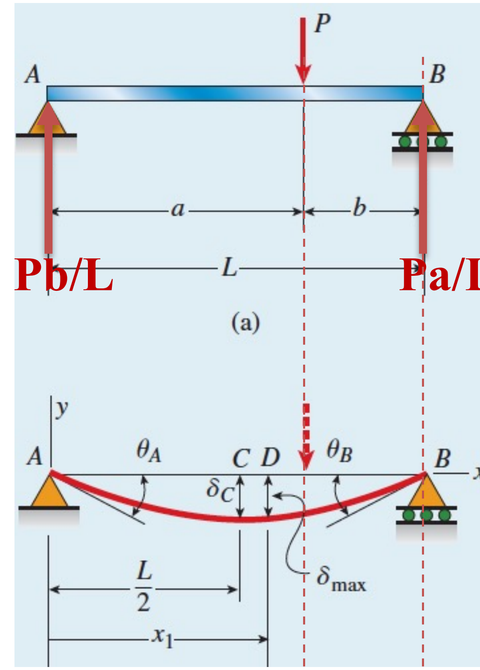
$$\theta_A = \frac{PL^2}{2EI}$$

$$v_A = -\frac{PL^3}{3EI}$$

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

A simple beam *AB* supports a concentrated load *P* acting at distances *a* and *b* from the left-hand and right-hand supports, respectively (Fig. 9-12a).

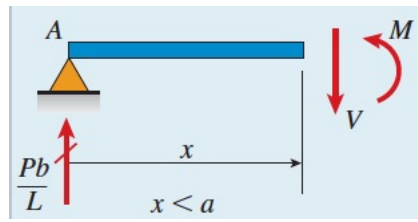
Determine the equations of the deflection curve, the angles of rotation θ_A and θ_B at the supports, the maximum deflection δ_{max}, and the deflection δ_C at the midpoint *C* of the beam (Fig. 9-12b). (*Note:* The beam has length *L* and constant flexural rigidity *EI*.)

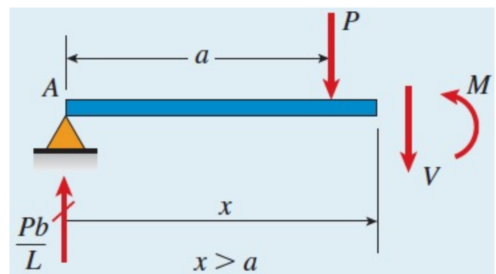


Bending moments in the beam

$$M = \frac{Pbx}{L} \qquad (0 \le x \le a)$$

$$M = \frac{Pbx}{L} - P(x - a) \qquad (a \le x \le L)$$

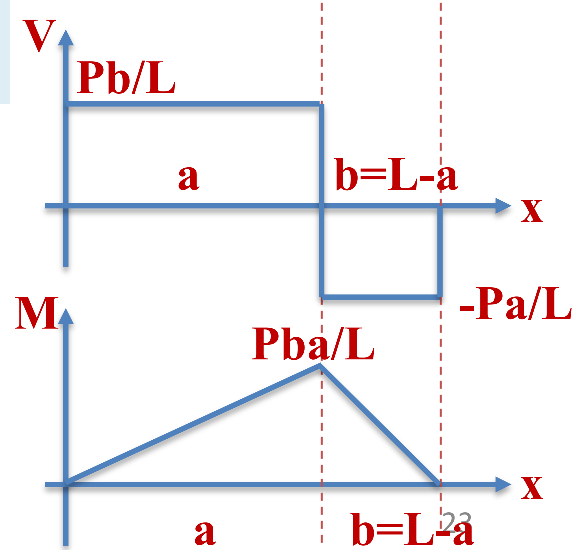




Differential equations of the deflection curve

$$EIv'' = \frac{Pbx}{L} \qquad (0 \le x \le a)$$

$$EIv'' = \frac{Pbx}{L} - P(x - a) \qquad (a \le x \le L)$$



Slopes and deflections of the beam

$$EIv' = \frac{Pbx^2}{2L} + C_1 \qquad (0 \le x \le a)$$

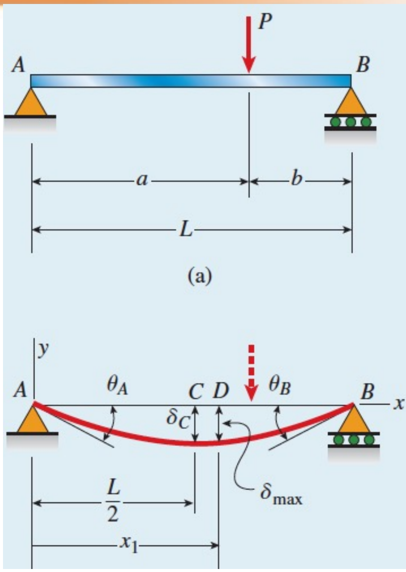
$$EIv' = \frac{Pbx^2}{2L} - \frac{P(x-a)^2}{2} + C_2 \qquad (a \le x \le L)$$

$$EIv = \frac{Pbx^3}{6L} + C_1 x + C_3 \qquad (0 \le x \le a)$$

$$EIv = \frac{Pbx^3}{6L} - \frac{P(x-a)^3}{6} + C_2 x + C_4 \qquad (a \le x \le L)$$

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Concept application examples



$$EIv' = \frac{Pbx^2}{2L} + C_1 \qquad (0 \le x \le a)$$

$$EIv' = \frac{Pbx^2}{2L} - \frac{P(x-a)^2}{2} + C_2 \qquad (a \le x \le L)$$

$$EIv = \frac{Pbx^3}{6L} + C_1 x + C_3 \qquad (0 \le x \le a)$$

$$EIv = \frac{Pbx^3}{6L} - \frac{P(x-a)^3}{6} + C_2 x + C_4 \qquad (a \le x \le L)$$

Boundary conditions

At $x = 0$, the deflection v is zero
At $x = L$, the deflection v is zero.

At $x = 0$ $\quad C_3 = 0$

Continuity conditions

At $x = a$, the slopes v′ for the two parts of the beam are the same.
At $x = a$, the deflections v for the two parts of the beam are the same

when $x = a$, $\quad \frac{Pba^2}{2L} + C_1 = \frac{Pba^2}{2L} + C_2$ or $C_1 = C_2$

$$\frac{Pba^3}{6L} + C_1 a + C_3 = \frac{Pba^3}{6L} + C_2 a + C_4$$

$C_3 = C_4$

$C_3 = C_4 = 0$

At $x = L$ $\quad \frac{PbL^2}{6} - \frac{Pb^3}{6} + C_2 L = 0$

$$C_1 = C_2 = -\frac{Pb(L^2 - b^2)}{6L}$$

$$v = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2) \qquad (0 \le x \le a)$$

$$v = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2) - \frac{P(x-a)^3}{6EI} \qquad (a \le x \le L)$$

$$\theta = v' = -\frac{Pb}{6LEI}(L^2 - b^2 - 3x^2) \qquad (0 \le x \le a)$$

$$\theta = v' = -\frac{Pb}{6LEI}(L^2 - b^2 - 3x^2) - \frac{P(x-a)^2}{2EI} \qquad (a \le x \le L)$$

$$\theta_A = v'(0) = \frac{-Pb(L^2 - b^2)}{6LEI} = \frac{-Pab(L+b)}{6LEI}$$

$$\theta_B = v'(L) = \frac{Pb(2L^2 - 3bL + b^2)}{6LEI} = \frac{Pab(L+a)}{6LEI}$$

Where the load is located, we can find the maximum θ_A?

$$\frac{d\theta_A}{db} = 0$$

$b = L/\sqrt{3} = 0.577L$ (or $a = 0.423L$).

$$(\theta_A)_{max} = \frac{PL^2\sqrt{3}}{27EI}$$

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Concept application examples

$$v = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2) \qquad (0 \le x \le a)$$

$$v = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2) - \frac{P(x-a)^3}{6EI} \qquad (a \le x \le L)$$

$$\theta = v' = -\frac{Pb}{6LEI}(L^2 - b^2 - 3x^2) \qquad (0 \le x \le a)$$

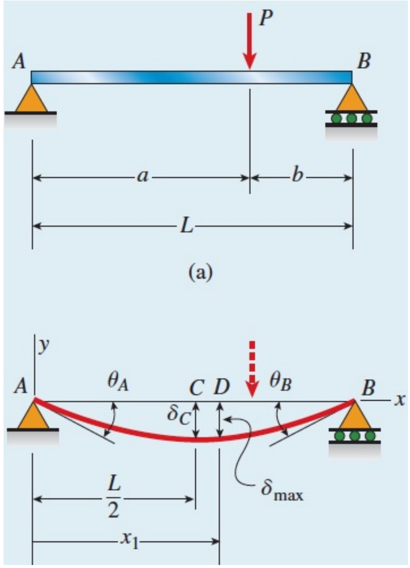
$$\theta = v' = -\frac{Pb}{6LEI}(L^2 - b^2 - 3x^2) - \frac{P(x-a)^2}{2EI} \qquad (a \le x \le L)$$

Maximum deflection of the beam Occurs at point D where the deflection curve has a horizontal tangent (θ=0)

If the load is to the right of the midpoint, that is, if a>b, point D is in the part of the beam to the left of the load.

$$v' = -\frac{Pb}{6LEI}(L^2 - b^2 - 3x^2) = 0 \qquad (0 \le x \le a)$$

$$x_1 = \sqrt{\frac{L^2 - b^2}{3}} \qquad (a \ge b)$$



$$v_{max} = -(v)_{x=x_1} = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}\ LEI} \qquad (a \ge b)$$

The maximum deflection of the beam depends on the position of the load P, that is, on the distance b. The maximum value of the maximum deflection (the "max-max" deflection) occurs when $b = L/2$ and the load is at the midpoint of the beam. This maximum deflection is equal to $PL^3/48EI$.

Deflection at the midpoint of the beam.

$$v_c = -v\left(\frac{L}{2}\right) = \frac{Pb(3L^2 - 4b^2)}{48EI} \qquad (a \ge b)$$

Special case (load at the midpoint of the beam).

$$v' = -\frac{P}{16EI}(L^2 - 4x^2) \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$v = -\frac{Px}{48EI}(3L^2 - 4x^2) \qquad \left(0 \le x \le \frac{L}{2}\right)$$

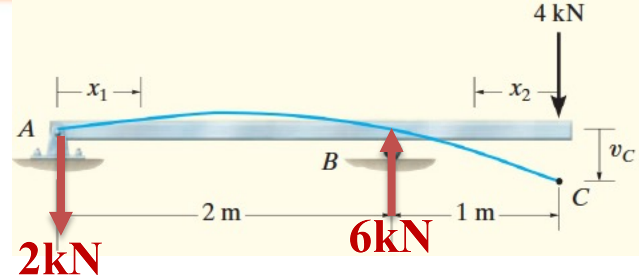
$$\theta_A = \theta_B = \frac{PL^2}{16EI}$$

$$\delta_{max} = \delta_C = \frac{PL^3}{48EI}$$

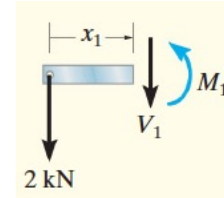
J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

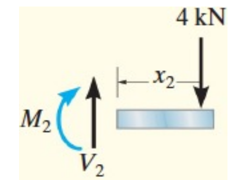
Examples

The beam in Fig. 12–13*a* is subjected to a load at its end. Determine the displacement at *C*. *EI* is constant.



Elastic Curve. The beam deflects into the shape shown in Fig. 12–13*a*. Due to the loading, two x coordinates will be considered, namely, $0 \le x_1 < 2$ m and $0 \le x_2 < 1$ m, where x_2 is directed to the left from C, since the internal moment is easy to formulate.





Moment Functions. $M_1 = -2x_1 \qquad M_2 = -4x_2$

Slope and Elastic Curve.

For $0 \le x_1 \le 2$:

$$EI\frac{d^2v_1}{dx_1^2} = -2x_1$$

$$EI\frac{dv_1}{dx_1} = -x_1^2 + C_1$$

$$EIv_1 = -\frac{1}{3}x_1^3 + C_1x_1 + C_2$$

For $0 \le x_2 \le 1$ m:

$$EI\frac{d^2v_2}{dx_2^2} = -4x_2$$

$$EI\frac{dv_2}{dx_2} = -2x_2^2 + C_3$$

$$EIv_2 = -\frac{2}{3}x_2^3 + C_3x_2 + C_4$$

$v_1 = 0$ at $x_1 = 0$; $\quad 0 = 0 + 0 + C_2$

$v_1 = 0$ at $x_1 = 2$ m;

$$0 = -\frac{1}{3}(2)^3 + C_1(2) + C_2$$

$$C_1 = \frac{4}{3}$$

The *four* constants of integration are determined using *three* boundary conditions, namely, $v_1 = 0$ at $x_1 = 0$, $v_1 = 0$ at $x_1 = 2$ m, and $v_2 = 0$ at $x_2 = 1$ m, and *one* continuity equation. Here the continuity of slope at the roller requires $dv_1/dx_1 = -dv_2/dx_2$ at $x_1 = 2$ m and $x_2 = 1$ m. There is a negative sign in this equation because the slope is measured positive counterclockwise from the right, and positive clockwise from the left, Fig. 12–9. (Continuity of displacement at B has been indirectly considered in the boundary conditions, since $v_1 = v_2 = 0$ at $x_1 = 2$ m and $x_2 = 1$ m.) Applying these four conditions yields

$v_2 = 0$ at $x_2 = 1$ m; $\quad 0 = -\frac{2}{3}(1)^3 + C_3(1) + C_4$

$$\left.\frac{dv_1}{dx_1}\right|_{x=2\,\text{m}} = -\left.\frac{dv_2}{dx_2}\right|_{x=1\,\text{m}}; \qquad -(2)^2 + C_1 = \boxed{-(-2(1)^2 + C_3)}$$

$$C_3 = \frac{14}{3} \qquad C_4 = -4$$

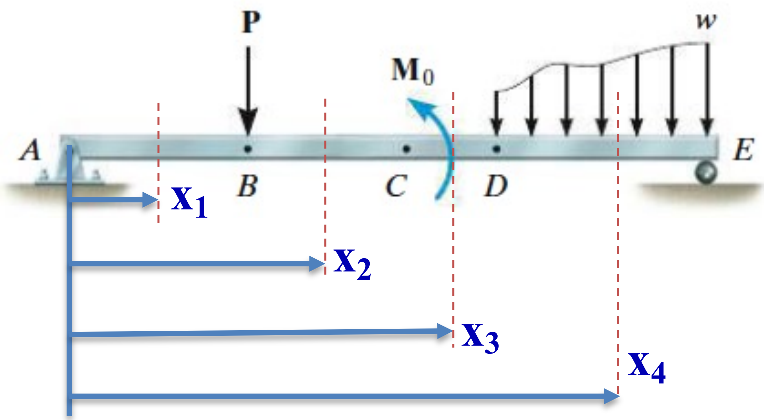
$$EIv_2 = -\frac{2}{3}x_2^3 + \frac{14}{3}x_2 - 4$$

The displacement at C is determined by setting $x_2 = 0$. We get

$$v_C = -\frac{4\,\text{kN}\cdot\text{m}^3}{EI}$$

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Macaulay's Method



Discontinuities areas:

- Begin/End of distributed loads
- Reactions
- Concentrated forces and moments

Macaulay Function

$$\langle x-a\rangle^n = \begin{cases} 0 & (x<a) \\ (x-a)^n & (x \geq a) \end{cases} \quad (n \geq 0)$$

"Off"

"On"

$$\langle x-a\rangle^0 = 1$$
$$\langle x-a\rangle^n = 0 \quad (n<0)$$

$$\frac{\partial \langle x-a\rangle^n}{\partial x} = \begin{cases} n\langle x-a\rangle^{n-1} & for\ n \geq 1 \\ \langle x-a\rangle^0 & for\ n = 1 \\ 0 & for\ n = 0 \end{cases}$$

$$\int \langle x-a\rangle^n dx$$
$$= \langle x-a\rangle^{n+1},\ n=-1,\ -2$$

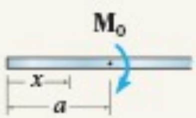
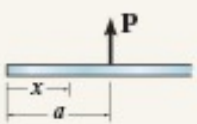
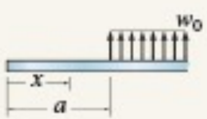
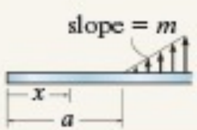
$$\int \langle x-a\rangle^n dx = \frac{\langle x-a\rangle^{n+1}}{n+1} + C \quad (n \geq 0)$$

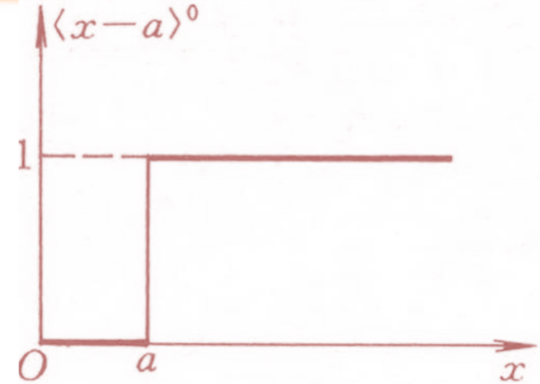
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

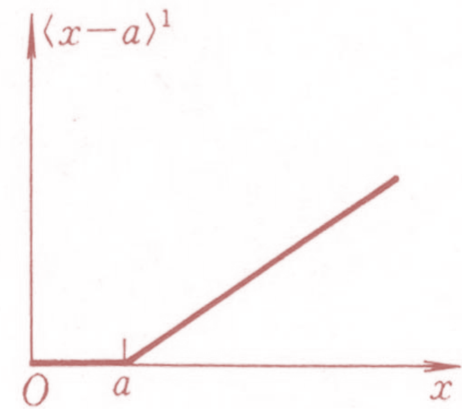
Macaulay's Notation

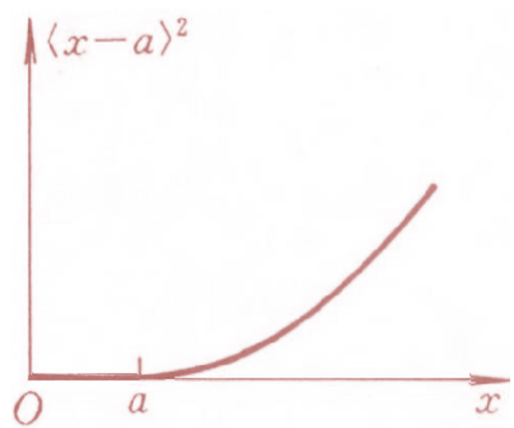
$$\langle x-a\rangle^n = \begin{cases} 0 & (x < a) \\ (x-a)^n & (x \geq a) \end{cases} \quad (n \geq 0)$$

$$\langle x-a\rangle^0 = 1$$
$$\langle x-a\rangle^n = 0 \quad (n<0)$$

Loading	Loading Function $w = w(x)$	Shear $V = \int w(x)dx$	Moment $M = \int V dx$
	$w = M_0\langle x-a\rangle^{-2}$	$V = M_0\langle x-a\rangle^{-1}$	$M = M_0\langle x-a\rangle^0$
	$w = P\langle x-a\rangle^{-1}$	$V = P\langle x-a\rangle^0$	$M = P\langle x-a\rangle^1$
	$w = w_0\langle x-a\rangle^0$	$V = w_0\langle x-a\rangle^1$	$M = \frac{w_0}{2}\langle x-a\rangle^2$
	$w = m\langle x-a\rangle^1$	$V = \frac{m}{2}\langle x-a\rangle^2$	$M = \frac{m}{6}\langle x-a\rangle^3$

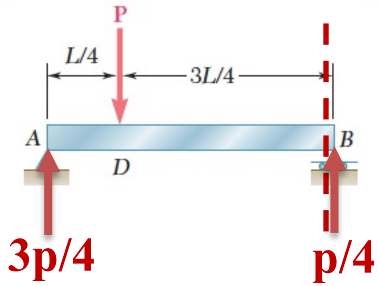






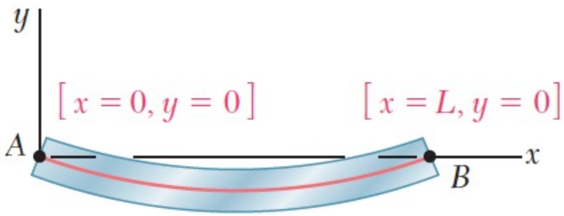
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Macaulay's Method



$$V(x) = \frac{3P}{4} - P\langle x - \tfrac{1}{4}L\rangle^0 \qquad M(x) = \frac{3P}{4}x - P\langle x - \tfrac{1}{4}L\rangle$$

$$EI\frac{d^2y}{dx^2} = \frac{3P}{4}x - P\langle x - \tfrac{1}{4}L\rangle$$



$$EI\,\theta = EI\frac{dy}{dx} = \frac{3}{8}Px^2 - \frac{1}{2}P\langle x - \tfrac{1}{4}L\rangle^2 + C_1$$

$$EI\,y = \frac{1}{8}Px^3 - \frac{1}{6}P\langle x - \tfrac{1}{4}L\rangle^3 + C_1x + C_2$$

$$0 = 0 - \frac{1}{6}P\langle 0 - \tfrac{1}{4}L\rangle^3 + 0 + C_2$$

$$C_2 = 0,$$

$x = L$, $y = 0$, and $C_2 = 0$

$$0 = \frac{1}{8}PL^3 - \frac{1}{6}P\langle\tfrac{3}{4}L\rangle^3 + C_1L$$

$$C_1 = -\frac{7PL^2}{128}$$

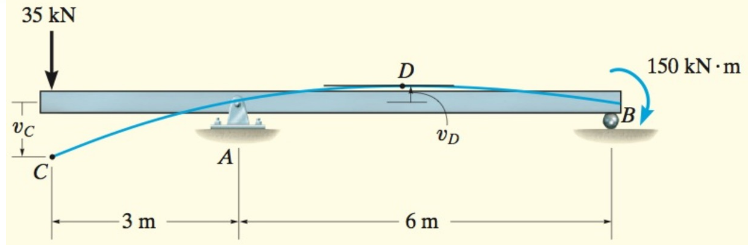
$$\theta = \frac{1}{EI}\left(\frac{3}{8}Px^2 - \frac{1}{2}P < x - \frac{1}{4}L >^2 - \frac{7}{128}PL^2\right)$$

$$y = \frac{1}{EI}\left(\frac{1}{8}Px^3 - \frac{1}{6}P < x - \frac{1}{4}L >^3 - \frac{7}{128}PL^2x\right)$$

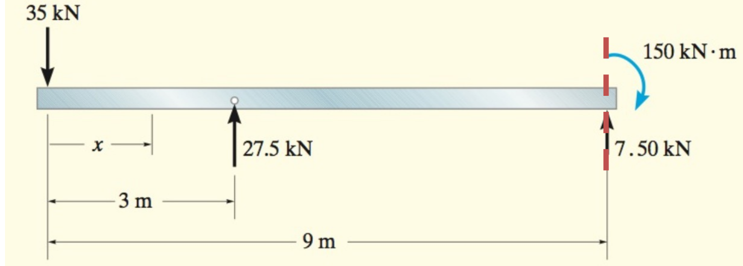
F.P. Beer et.al., Mechanics of Materials, 7th edition, McGrawHill Education.

Examples

Determine the maximum deflection of the beam shown in Fig. 12–18*a*. *EI* is constant.



$$M = -35\langle x - 0\rangle^1 + 27.5\langle x - 3\rangle^1$$
$$= \{-35x + 27.5\langle x - 3\rangle^1\}\ \text{kN}\cdot\text{m}.$$



Slope and Elastic Curve. Integrating twice yields

$$EI\frac{d^2v}{dx^2} = -35x + 27.5\langle x - 3\rangle^1$$

$$EI\frac{dv}{dx} = -17.5x^2 + 13.75\langle x - 3\rangle^2 + C_1$$

$$EIv = -5.8333x^3 + 4.5833\langle x - 3\rangle^3 + C_1x + C_2 \qquad (1)$$

$C_1 = 517.5, C_2 = -1395$

From Eq. 1, the boundary condition $v = 0$ at $x = 3$ m and $v = 0$ at $x = 9$ m gives

$$0 = -157.5 + 4.5833(3 - 3)^3 + C_1(3) + C_2$$

$$0 = -4252.5 + 4.5833(9 - 3)^3 + C_1(9) + C_2$$

$$EI\frac{dv}{dx} = -17.5x^2 + 13.75\langle x - 3\rangle^2 + 517.5 \qquad (2)$$

$$EIv = -5.8333x^3 + 4.5833\langle x - 3\rangle^3 + 517.5x - 1395 \qquad (3)$$

From Fig. 12–18*a*, maximum displacement can occur either at *C*, or at *D* where the slope $dv/dx = 0$. To obtain the displacement of *C*, set $x = 0$ in Eq. 3. We get

$$v_C = -\frac{1395\ \text{kN}\cdot\text{m}^3}{EI} \qquad \textit{Ans.}$$

The negative sign indicates that the displacement is downwards

To locate point *D*, use Eq. 2 with $x > 3$ m and $dv/dx = 0$.

$$0 = -17.5x_D^2 + 13.75(x_D - 3)^2 + 517.5$$
$$3.75x_D^2 + 82.5x - 641.25 = 0$$

$$x_D = 6.088\ \text{m}$$

$$EIv_D = -5.8333(6.088^3) + 4.5833(6.088 - 3)^3 + 517.5(6.088) - 1395$$

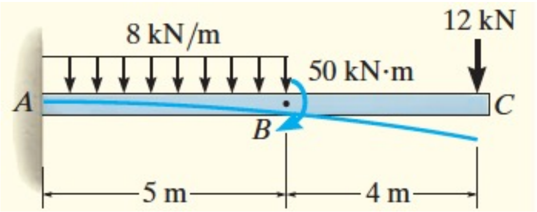
$$v_D = \frac{574\ \text{kN}\cdot\text{m}^3}{EI}$$

Comparing this value with v_C, we see that $v_{max} = v_C$.

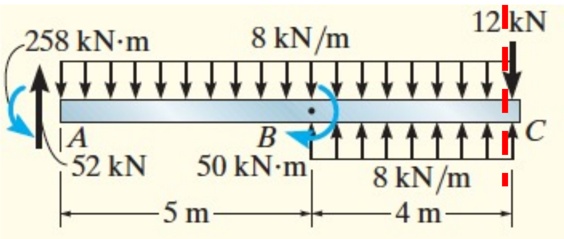
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Examples

Determine the equation of the elastic curve for the cantilevered beam shown in Fig. 12–17*a*. *EI* is constant.



$$w = 52\text{ kN}\langle x-0\rangle^{-1} - 258\text{ kN}\cdot\text{m}\langle x-0\rangle^{-2} - 8\text{ kN/m}\langle x-0\rangle^{0} + 50\text{ kN}\cdot\text{m}\langle x-5\text{ m}\rangle^{-2} + 8\text{ kN/m}\langle x-5\text{ m}\rangle^{0}$$



$$V = 52\langle x-0\rangle^{0} - 258\langle x-0\rangle^{-1} - 8\langle x-0\rangle^{1} + 50\langle x-5\rangle^{-1} + 8\langle x-5\rangle^{1}$$

$$M = -258\langle x-0\rangle^{0} + 52\langle x-0\rangle^{1} - \frac{1}{2}(8)\langle x-0\rangle^{2} + 50\langle x-5\rangle^{0} + \frac{1}{2}(8)\langle x-5\rangle^{2}$$
$$= (-258 + 52x - 4x^2 + 50\langle x-5\rangle^{0} + 4\langle x-5\rangle^{2})\text{ kN}\cdot\text{m}$$

$$EI\frac{d^2v}{dx^2} = -258 + 52x - 4x^2 + 50\langle x-5\rangle^{0} + 4\langle x-5\rangle^{2}$$

$$EI\frac{dv}{dx} = -258x + 26x^2 - \frac{4}{3}x^3 + 50\langle x-5\rangle^{1} + \frac{4}{3}\langle x-5\rangle^{3} + C_1$$

$$EIv = -129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^{2} + \frac{1}{3}\langle x-5\rangle^{4} + C_1x + C_2$$

$dv/dx = 0$ at $x = 0$, $C_1 = 0$;

$v = 0$ at $x = 0$, so $C_2 = 0$

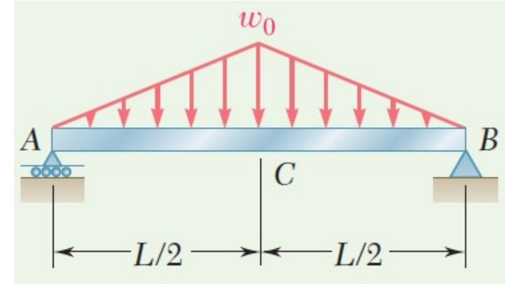
$$v = \frac{1}{EI}\left(-129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^{2} + \frac{1}{3}\langle x-5\rangle^{4}\right)\text{ m}$$

Examples

For the prismatic beam and loading shown, determine (a) the equation of the elastic curve and, (b) the maximum deflection

$$M(x) = -\frac{w_0}{3L}x^3 + \frac{2w_0}{3L}\langle x - \tfrac{1}{2}L\rangle^3 + \tfrac{1}{4}w_0 Lx$$

$$EI\frac{d^2y}{dx^2} = -\frac{w_0}{3L}x^3 + \frac{2w_0}{3L}\langle x - \tfrac{1}{2}L\rangle^3 + \tfrac{1}{4}w_0 Lx$$

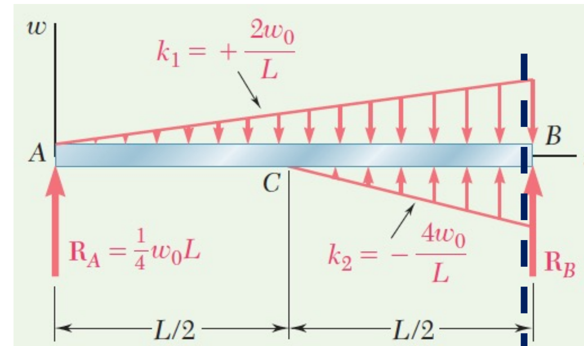


$$EI\,\theta = -\frac{w_0}{12L}x^4 + \frac{w_0}{6L}\langle x - \tfrac{1}{2}L\rangle^4 + \frac{w_0 L}{8}x^2 + C_1$$

$$EI\,y = -\frac{w_0}{60L}x^5 + \frac{w_0}{30L}\langle x - \tfrac{1}{2}L\rangle^5 + \frac{w_0 L}{24}x^3 + C_1 x + C_2$$

$x = 0,\ y = 0$

$C_2 = 0$

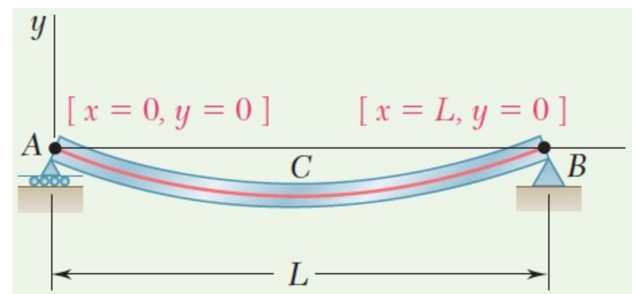


$x = L,\ y = 0$

$$0 = -\frac{w_0 L^4}{60} + \frac{w_0}{30L}\left(\frac{L}{2}\right)^5 + \frac{w_0 L^4}{24} + C_1 L \qquad C_1 = -\frac{5}{192}w_0 L^3$$

$$EI\;\theta = -\frac{w_0}{12L}x^4 + \frac{w_0}{6L}\langle x - \tfrac{1}{2}L\rangle^4 + \frac{w_0 L}{8}x^2 - \frac{5}{192}w_0 L^3$$

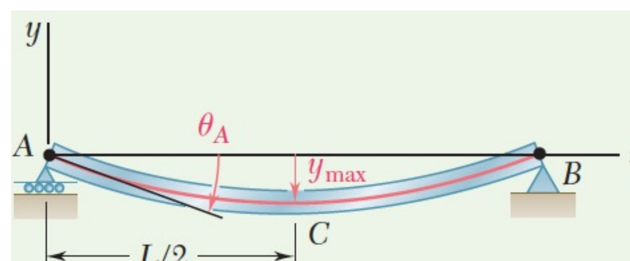
$$EI\,y = -\frac{w_0}{60L}x^5 + \frac{w_0}{30L}\langle x - \tfrac{1}{2}L\rangle^5 + \frac{w_0 L}{24}x^3 - \frac{5}{192}w_0 L^3 x$$



Maximum Deflection

$$EI\,y_{\max} = w_0 L^4\left[-\frac{1}{60(32)} + 0 + \frac{1}{24(8)} - \frac{5}{192(2)}\right] = -\frac{w_0 L^4}{120}$$

$$y_{\max} = \frac{w_0 L^4}{120EI}\downarrow$$



F.P. Beer et.al., Mechanics of Materials, 7th edition, McGrawHill Education.

Moment-Area Method

- **The moment-area method provides a <u>semigraphical technique</u>** for finding the slope and displacement at **<u>specific points</u>** on the elastic curve of a beam or shaft.
- This method requires calculating **<u>area segments of the beam's moment diagram</u>**
- If this diagram consists of simple shapes, the method is very convenient to use

Theorem 1

$$EI\frac{d^2v}{dx^2} = EI\frac{d}{dx}\left(\frac{dv}{dx}\right) = M(x)$$

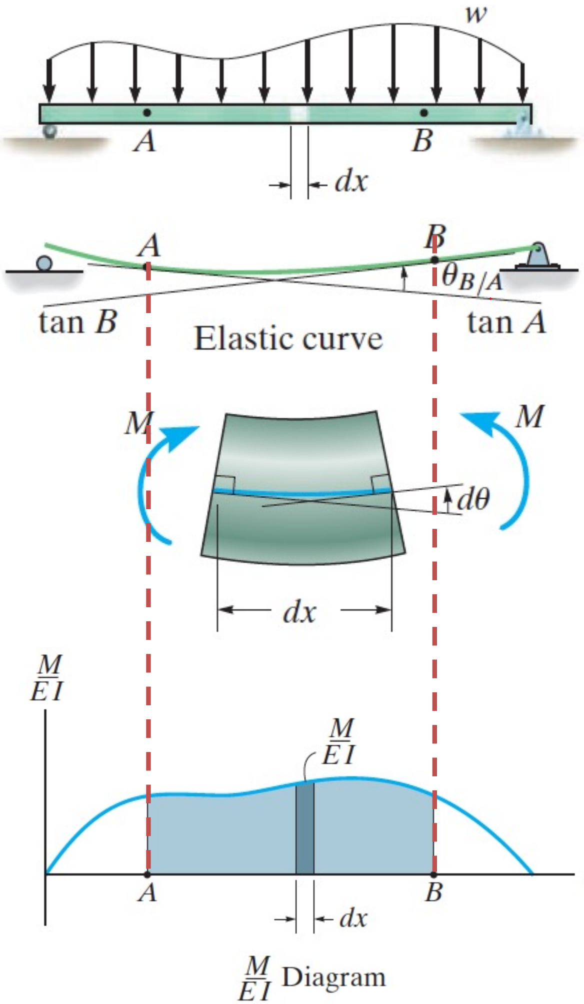
$$\theta = \frac{dv}{dx}$$

$$d\theta = \frac{M(x)}{EI}dx$$

$$\theta_{B/A} = \int_A^B \frac{M(x)}{EI}dx$$

The angle, measured in radians, between the tangents at any two points on the elastic curve equals the area under the <u>*M/EI*</u> diagram between these two points

<u>*$\theta_{B/A}$: the angle of the tangent at B measured with respect to the tangent at A*</u>. The $\theta_{B/A}$ is measured <u>*counterclockwise, from tangent A to tangent B*</u>, if the area under the M/EI diagram is <u>*positive*</u>. If the area <u>*is negative*</u>, the $\theta_{B/A}$ is measured <u>*clockwise*</u>.



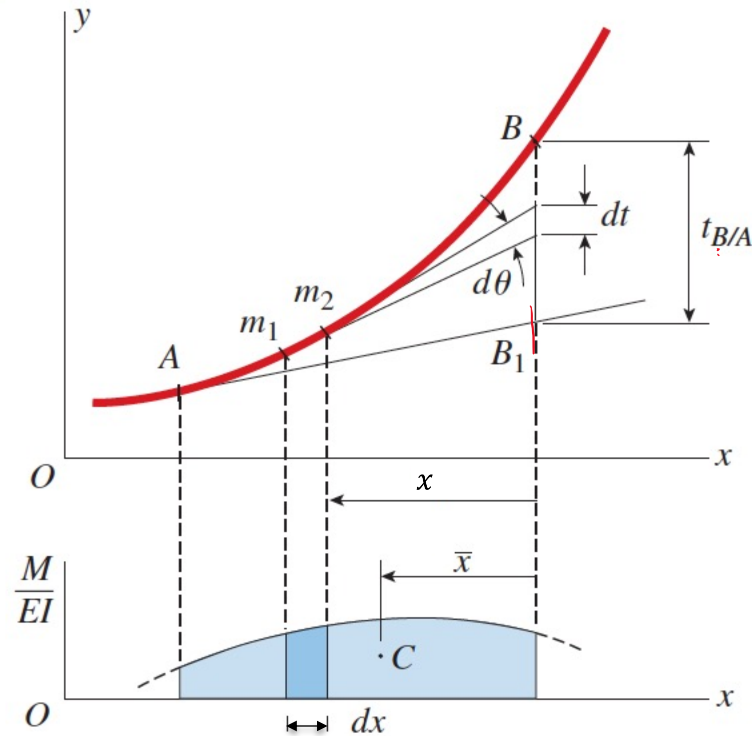
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Theorem 2

The vertical distance between points B and B_1 is referred to as the ***tangential deviation*** of B with respect to A (**$t_{B/A}$**).

$$dt = x\,d\theta \qquad d\theta = \frac{M(x)}{EI}dx$$

$$\boxed{dt = x\frac{M(x)}{EI}dx} \qquad \bar{x}\int dA = \int x\,dA$$



$$t_{B/A} = \int_A^B dt = \int_A^B x\frac{M(x)}{EI}dx = \bar{x}\int_A^B \frac{M(x)}{EI}dx$$

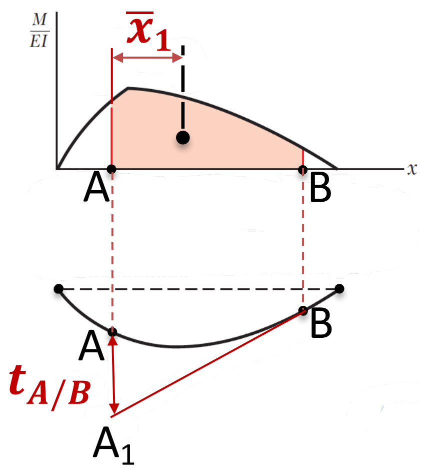
$$t_{B/A} = \bar{x}\int_A^B \frac{M(x)}{EI}dx$$

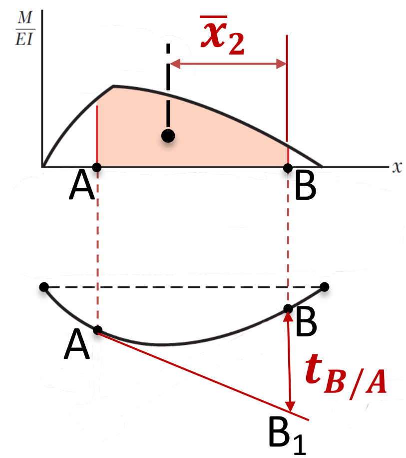
The tangential deviation of point B from the tangent at point A ($t_{B/A}$) is equal to the first moment of the area of the M/EI diagram between A and B, evaluated with respect to B.

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson. J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Moment-Area Method

The tangential deviation of A with respect to B ($t_{A/B}$)	The tangential deviation of B with respect to A ($t_{B/A}$)



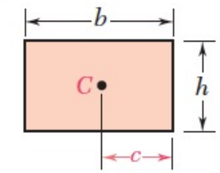
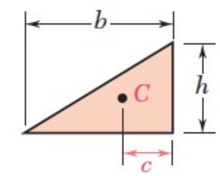
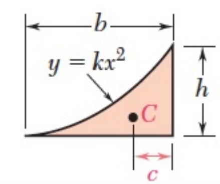
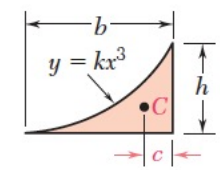
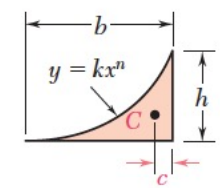


$$t_{A/B} = \bar{x}_1 \int_A^B \frac{M(x)}{EI}\,\mathrm{dx}$$

(Area between A and B) × (Distance between A and the centroid of the area)

$$t_{B/A} = \bar{x}_2 \int_A^B \frac{M(x)}{EI}\,\mathrm{dx}$$

(Area between A and B) × (Distance between B and the centroid of the area)

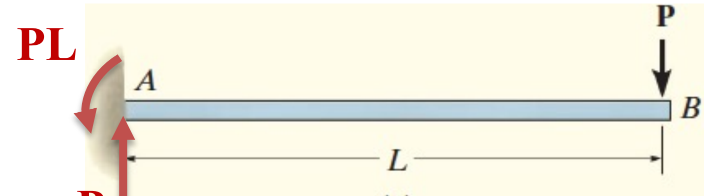
Shape		Area	c
Rectangle		bh	$\frac{b}{2}$
Triangle		$\frac{bh}{2}$	$\frac{b}{3}$
Parabolic spandrel		$\frac{bh}{3}$	$\frac{b}{4}$
Cubic spandrel		$\frac{bh}{4}$	$\frac{b}{5}$
General spandrel		$\frac{bh}{n+1}$	$\frac{b}{n+2}$

*If $t_{A/B}$ is calculated from the moment of a **positive M/EI area** between A and B, A is **above** the tangent extended from B. Similarly, **negative M/EI areas** indicate that A will be **below** the tangent extended from B. This same rule applies for $t_{B/A}$.*

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson. F.P. Beer et.al., Mechanics of Materials, 7th edition, McGrawHill Education.

Example

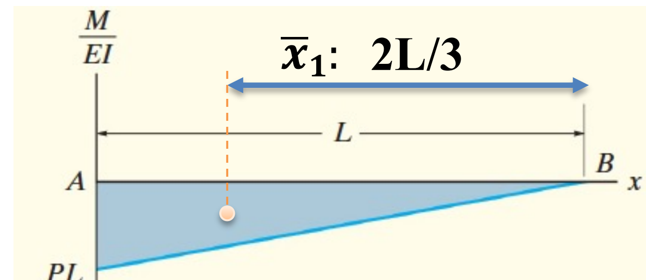
Determine the slope and deflection of the beam at point B. EI is constant



(a)

Elastic Curve. The force **P** causes the beam to deflect as shown in Fig. 12–21*c*. The tangent at *B* is indicated since we are required to find θ_B. Also, the tangent at the support (*A*) is shown. This tangent has a *known* zero slope. By the construction, the angle between tan *A* and tan *B* is equivalent to θ_B. Thus,

$$\theta_B = \theta_{B/A}$$



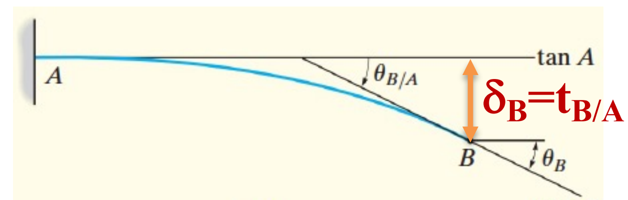
(b)

Moment-Area Theorem. Applying Theorem 1, $\theta_{B/A}$ is equal to the area under the M/EI diagram between *A* and *B*; that is,

$$\theta_B = \theta_{B/A} = \frac{1}{2}\left(-\frac{PL}{EI}\right)L$$

$$= -\frac{PL^2}{2EI} \qquad \textit{Ans.}$$

The *negative sign* indicates that the angle measured from the tangent at *A* to the tangent at *B* is *clockwise*. This checks, since the beam slopes downwards at *B*.



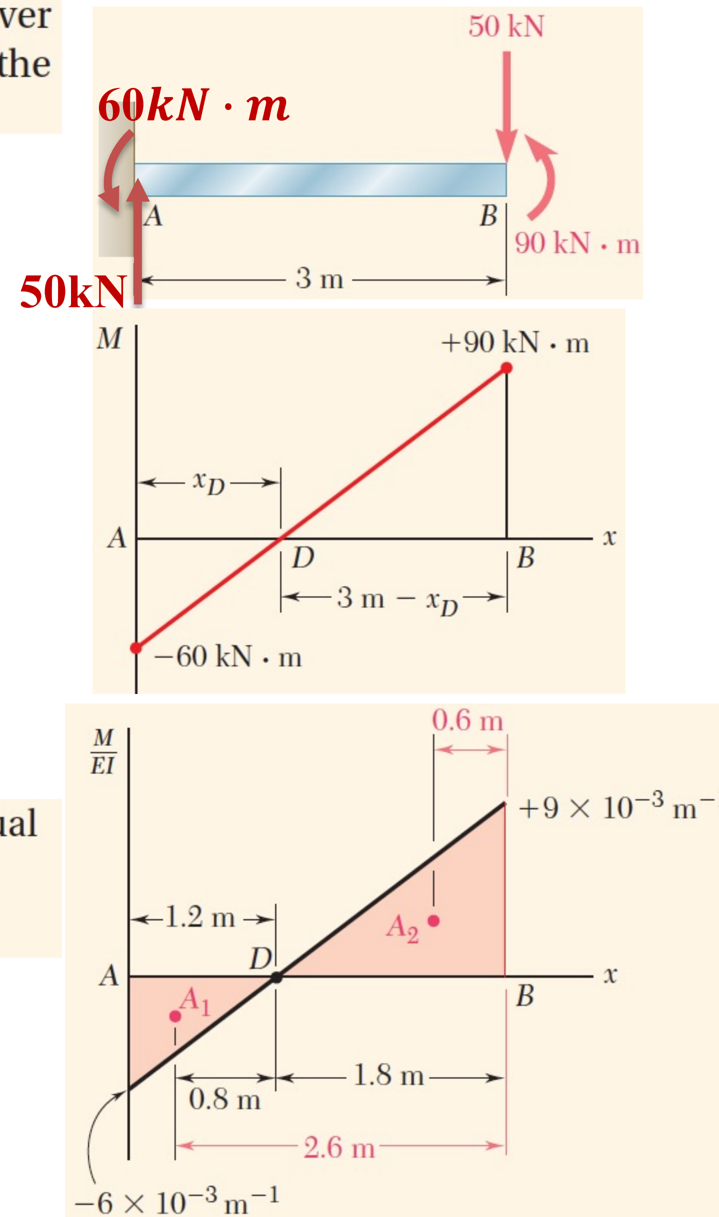
(c)

$$\delta_B = t_{B/A} = A\bar{x}_1 = \left(-\frac{PL^2}{2EI}\right)\left(\frac{2L}{3}\right) = -\frac{PL^3}{3EI}$$

Negative value indicates that point B lie below the tangent at A.

Example

Determine the slope and deflection at end B of the prismatic cantilever beam AB when it is loaded as shown (Fig. 9.30a), knowing that the flexural rigidity of the beam is $EI = 10 \text{ MN}\cdot\text{m}^2$.



$$\theta_{B/A} = \theta_B - \theta_A = \text{area from } A \text{ to } B = A_1 + A_2$$
$$= -\tfrac{1}{2}(1.2 \text{ m})(6 \times 10^{-3} \text{ m}^{-1}) + \tfrac{1}{2}(1.8 \text{ m})(9 \times 10^{-3} \text{ m}^{-1})$$
$$= -3.6 \times 10^{-3} + 8.1 \times 10^{-3}$$
$$= +4.5 \times 10^{-3} \text{ rad}$$

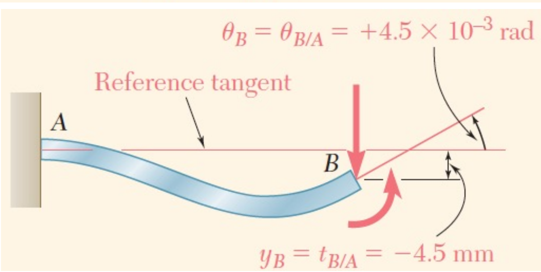
and, since $\theta_A = 0$,

$$\theta_B = +4.5 \times 10^{-3} \text{ rad}$$

$$t_{B/A} = A_1(2.6 \text{ m}) + A_2(0.6 \text{ m})$$
$$= (-3.6 \times 10^{-3})(2.6 \text{ m}) + (8.1 \times 10^{-3})(0.6 \text{ m})$$
$$= -9.36 \text{ mm} + 4.86 \text{ mm} = -4.50 \text{ mm}$$

Since the reference tangent at A is horizontal, the deflection at B is equal to $t_{B/A}$, so

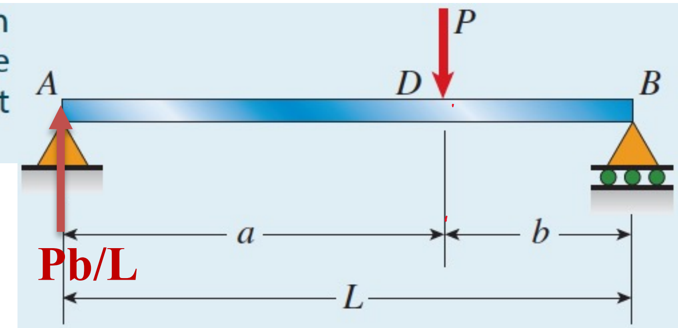
$$y_B = t_{B/A} = -4.50 \text{ mm}$$



F.P. Beer et.al., Mechanics of Materials, 7th edition, McGrawHill Education.

Example

A simple beam *ADB* supports a concentrated load *P* acting at the position shown in Fig. 9-26. Determine the angle of rotation θ_A at support *A* and the deflection δ_D under the load *P*. (*Note:* The beam has length *L* and constant flexural rigidity *EI*.)

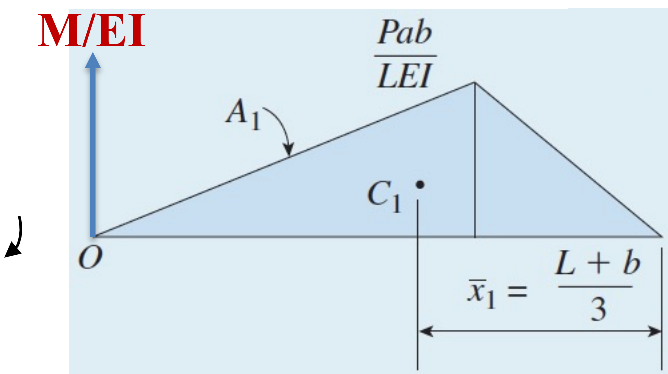


Angle of rotation at support A. *To find this angle, we construct the tangent AB_1 at support A. So, the distance BB_1 is the tangential deviation ($t_{B/A}$) of point B from the tangent at A*

The area of the entire *M/EI* diagram is

$$A_1 = \frac{1}{2}(L)\left(\frac{Pab}{LEI}\right) = \frac{Pab}{2EI}$$

$$\bar{x}_1 = \frac{L+b}{3}$$



$$t_{B/A} = A_1\bar{x}_1 = \frac{Pab}{2EI}\left(\frac{L+b}{3}\right) = \frac{Pab}{6EI}(L+b)$$

$$\boxed{\theta_A = \frac{t_{B/A}}{L} = \frac{Pab}{6LEI}(L+b)}$$

Deflection under the load

$$\delta_D = DD_1 - t_{D/A}$$

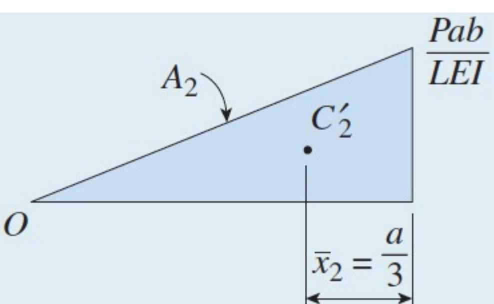
$$DD_1 = a\theta_A = \frac{Pa^2b}{6LEI}(L+b)$$

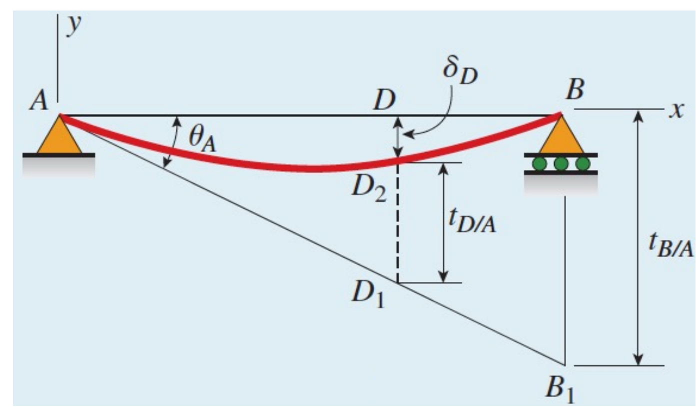
$$A_2 = \frac{1}{2}(a)\left(\frac{Pab}{LEI}\right) = \frac{Pa^2b}{2LEI}$$

$$\bar{x}_2 = \frac{a}{3}$$

$$t_{D/A} = A_2\bar{x}_2 = \left(\frac{Pa^2b}{2LEI}\right)\left(\frac{a}{3}\right) = \frac{Pa^3b}{6LEI}$$

$$\delta_D = \frac{Pa^2b}{6LEI}(L+b) - \frac{Pa^3b}{6LEI} = \frac{Pa^2b^2}{3LEI} \; (\downarrow)$$

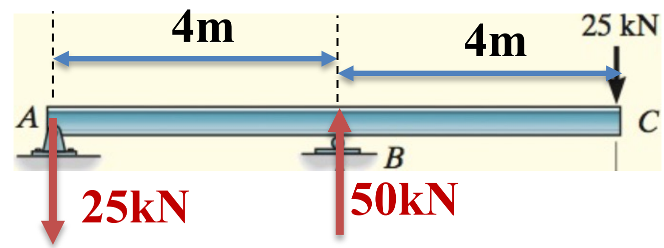




J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

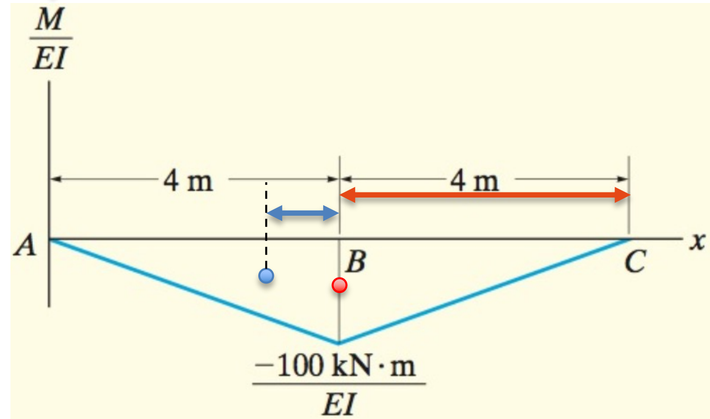
Example

Determine the displacement at point C for the steel overhanging beam shown in Fig. 12–26a. Take $E_{st} = 200\,\text{GPa}, I = 50(10^6)\,\text{mm}^4$.



Elastic Curve. The loading causes the beam to deflect as shown in Fig. 12–26c. We are required to find Δ_C. By constructing tangents at C and at the supports A and B, it is seen that $\Delta_C = |t_{C/A}| - \Delta'$. However, Δ' can be related to $t_{B/A}$ by proportional triangles; that is, $\Delta'/8 = |t_{B/A}|/4$ or $\Delta' = 2|t_{B/A}|$. Hence

$$\Delta_C = |t_{C/A}| - 2|t_{B/A}| \qquad (1)$$



Moment-Area Theorem. Applying Theorem 2 to determine $t_{C/A}$ and $t_{B/A}$, we have

$$t_{C/A} = (4\,\text{m})\left[\frac{1}{2}(8\,\text{m})\left(-\frac{100\,\text{kN}\cdot\text{m}}{EI}\right)\right]$$

$$= -\frac{1600\,\text{kN}\cdot\text{m}^3}{EI}$$

$$t_{B/A} = \left[\frac{1}{3}(4\,\text{m})\right]\left[\frac{1}{2}(4\,\text{m})\left(-\frac{100\,\text{kN}\cdot\text{m}}{EI}\right)\right] = -\frac{266.67\,\text{kN}\cdot\text{m}^3}{EI}$$

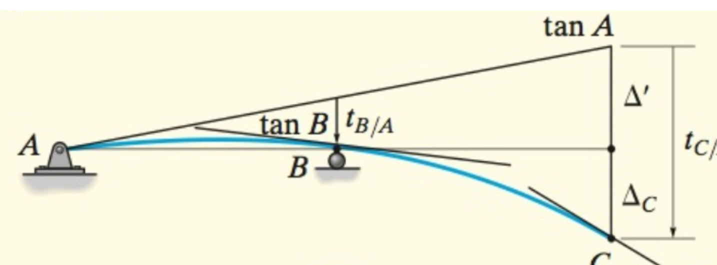
Why are these terms negative? Substituting the results into Eq. 1 yields

$$\Delta_C = \frac{1600\,\text{kN}\cdot\text{m}^3}{EI} - 2\left(\frac{266.67\,\text{kN}\cdot\text{m}^3}{EI}\right) = \frac{1066.67\,\text{kN}\cdot\text{m}^3}{EI}\downarrow$$

Realizing that the calculations were made in units of kN and m, we have

$$\Delta_C = \frac{1066.67(10^3)\,\text{N}\cdot\text{m}^3}{[200(10^9)\text{N/m}^2][50(10^{-6})\,\text{m}^4]}$$

$$= 0.1067\,\text{m} = 107\,\text{mm} \qquad \textit{Ans.}$$

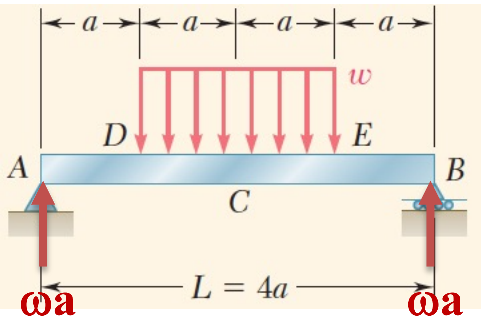


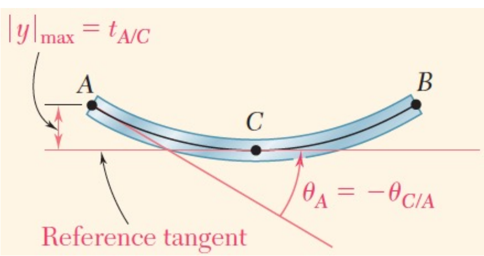
(c)

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Example

For the prismatic beam AB and the loading shown in Fig. 9.33a, determine the slope at a support and the maximum deflection.





$$\theta_{C/A} = \theta_C - \theta_A = -\theta_A \qquad \text{or} \qquad \theta_A = -\theta_{C/A}$$

$$A_1 = \frac{1}{2}(2a)\left(\frac{2wa^2}{EI}\right) = \frac{2wa^3}{EI}$$

$$A_2 = -\frac{1}{3}(a)\left(\frac{wa^2}{2EI}\right) = -\frac{wa^3}{6EI}$$

$$\theta_{C/A} = A_1 + A_2 = \frac{2wa^3}{EI} - \frac{wa^3}{6EI} = \frac{11wa^3}{6EI}$$

$$a = \tfrac{1}{4}L$$

$$\theta_A = -\theta_{C/A}$$

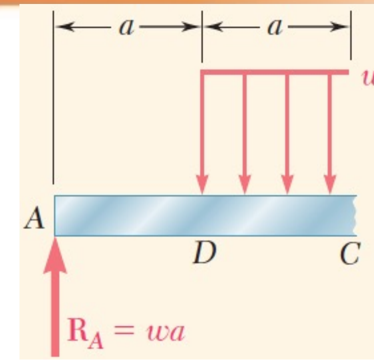
$$\theta_A = -\frac{11wa^3}{6EI} = -\frac{11wL^3}{384EI}$$

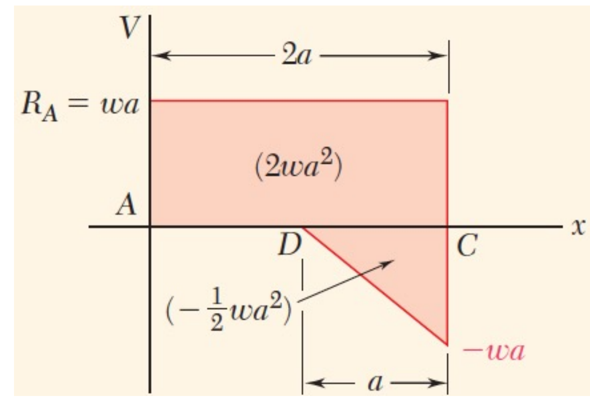
Since the tangent at the center C of the beam is horizontal, it is used as the reference tangent

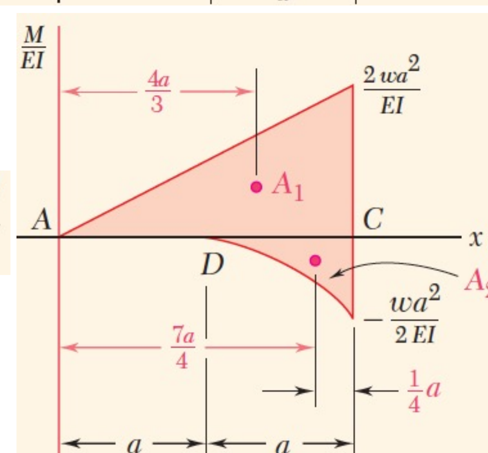
$$|y|_{max} = t_{A/C}$$

$$t_{A/C} = A_1\frac{4a}{3} + A_2\frac{7a}{4} = \left(\frac{2wa^3}{EI}\right)\frac{4a}{3} + \left(-\frac{wa^3}{6EI}\right)\frac{7a}{4} = \frac{19wa^4}{8EI}$$

$$|y|_{max} = t_{A/C} = \frac{19wa^4}{8EI} = \frac{19wL^4}{2048EI} \quad (\downarrow)$$







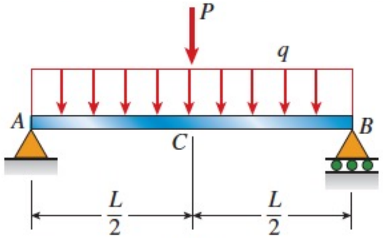
F.P. Beer et.al., Mechanics of Materials, 7th edition, McGrawHill Education.

Method of Superposition

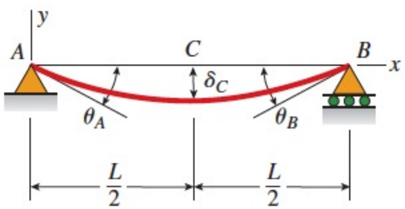
- Under suitable conditions, the deflection of a beam produced by several different loads acting simultaneously can be found by superposing the deflections produced by the same loads acting separately

- The justification for superposing deflections lies in the nature of the differential equations of the deflection curve. These equations are linear differential equations, because all terms containing the deflection v and its derivatives are raised to the first power. Therefore, the solutions of these equations for several loading conditions may be added algebraically, or superposed

- The principle of superposition is valid under the following conditions: (1) Hooke's law holds for the material, (2) the deflections and rotations are small, (3) the presence of the deflections does not alter the actions of the applied loads



(a)



For the uniform load $(\delta_c)_1 = -\dfrac{5qL^4}{384EI}$ $(\theta_A)_1 = -\dfrac{qL^3}{24EI}$

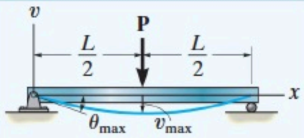
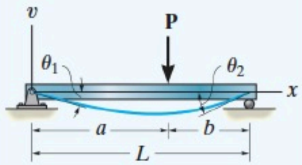
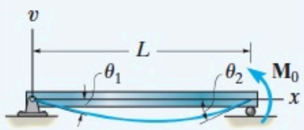
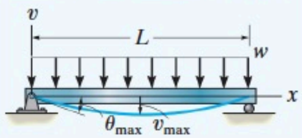
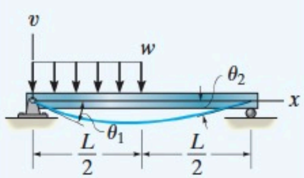
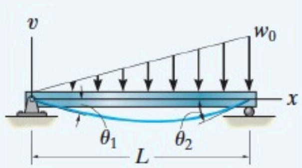
For the load P $(\delta_c)_2 = -\dfrac{PL^3}{48EI}$ $(\theta_A)_2 = -\dfrac{qL^3}{16EI}$

$\delta = (\delta_c)_1 + (\delta_c)_2$ $\theta = (\theta_A)_1 + (\theta_A)_2$

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Method of Superposition

Simply Supported Beam Slopes and Deflections

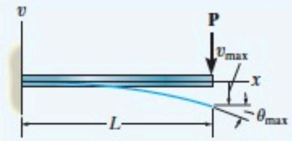
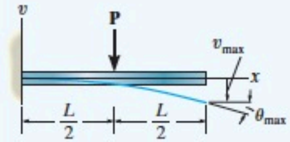
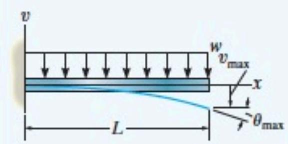
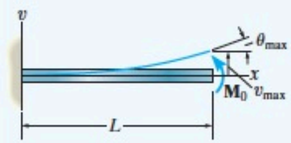
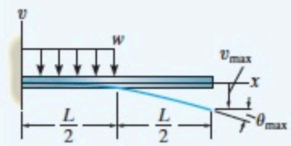
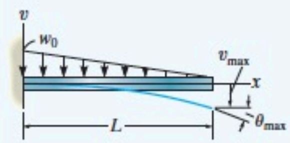
Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{16EI}$	$v_{max} = \frac{-PL^3}{48EI}$	$v = \frac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \frac{-Pab(L+b)}{6EIL}$ $\theta_2 = \frac{Pab(L+a)}{6EIL}$	$v\Big\rvert_{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \frac{-M_0L}{6EI}$ $\theta_2 = \frac{M_0L}{3EI}$	$v_{max} = \frac{-M_0L^2}{9\sqrt{3}EI}$ at $x = 0.5774L$	$v = \frac{-M_0x}{6EIL}(L^2 - x^2)$
	$\theta_{max} = \frac{-wL^3}{24EI}$	$v_{max} = \frac{-5wL^4}{384EI}$	$v = \frac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	$v\Big\rvert_{x=L/2} = \frac{-5wL^4}{768EI}$ $v_{max} = -0.006563\frac{wL^4}{EI}$ at $x = 0.4598L$	$v = \frac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \frac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \frac{-7w_0L^3}{360EI}$ $\theta_2 = \frac{w_0L^3}{45EI}$	$v_{max} = -0.00652\frac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \frac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

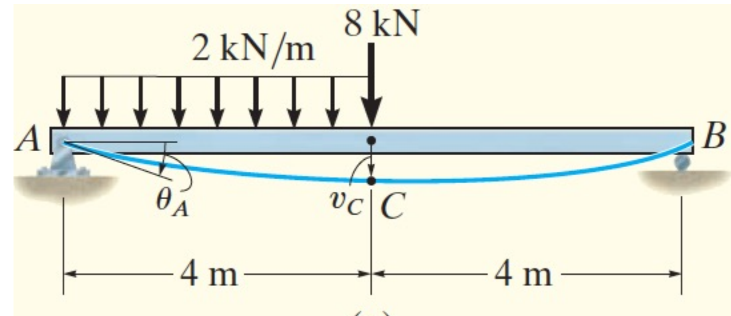
Method of Superposition

- *R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.*

Cantilevered Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-PL^2}{2EI}$	$v_{max} = \dfrac{-PL^3}{3EI}$	$v = \dfrac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \dfrac{-PL^2}{8EI}$	$v_{max} = \dfrac{-5PL^3}{48EI}$	$v = \dfrac{-Px^2}{12EI}(3L - 2x) \quad 0 \le x \le L/2$ $v = \dfrac{-PL^2}{48EI}(6x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \dfrac{-wL^3}{6EI}$	$v_{max} = \dfrac{-wL^4}{8EI}$	$v = \dfrac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \dfrac{M_0 L}{EI}$	$v_{max} = \dfrac{M_0 L^2}{2EI}$	$v = \dfrac{M_0 x^2}{2EI}$
	$\theta_{max} = \dfrac{-wL^3}{48EI}$	$v_{max} = \dfrac{-7wL^4}{384EI}$	$v = \dfrac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right) \quad 0 \le x \le L/2$ $v = \dfrac{-wL^3}{384EI}(8x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \dfrac{-w_0 L^3}{24EI}$	$v_{max} = \dfrac{-w_0 L^4}{30EI}$	$v = \dfrac{-w_0 x^2}{120EIL}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$

Determine the displacement at point C and the slope at the support A of the beam shown in Fig. 12–27*a*. EI is constant.

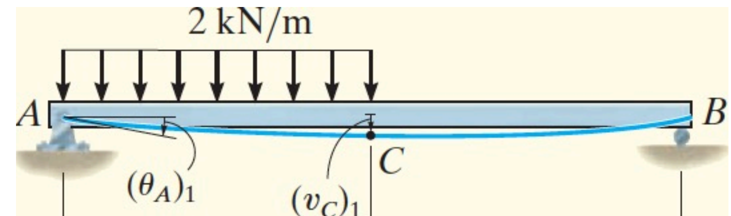


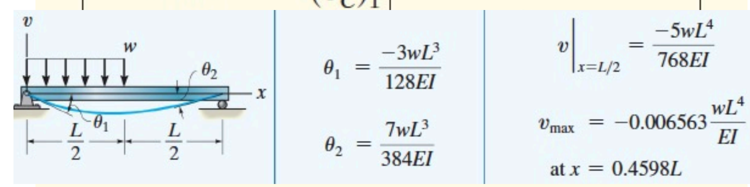
(a)

For the distributed loading,

$$(\theta_A)_1 = \frac{3wL^3}{128EI} = \frac{3(2\text{ kN/m})(8\text{ m})^3}{128EI} = \frac{24\text{ kN}\cdot\text{m}^2}{EI}$$

$$(v_C)_1 = \frac{5wL^4}{768EI} = \frac{5(2\text{ kN/m})(8\text{ m})^4}{768EI} = \frac{53.33\text{ kN}\cdot\text{m}^3}{EI}\downarrow$$

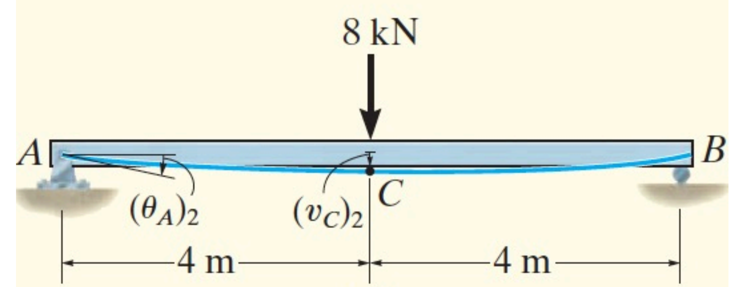




For the 8-kN concentrated force,

$$(\theta_A)_2 = \frac{PL^2}{16EI} = \frac{8\text{ kN}(8\text{ m})^2}{16EI} = \frac{32\text{ kN}\cdot\text{m}^2}{EI}$$

$$(v_C)_2 = \frac{PL^3}{48EI} = \frac{8\text{ kN}(8\text{ m})^3}{48EI} = \frac{85.33\text{ kN}\cdot\text{m}^3}{EI}\downarrow$$

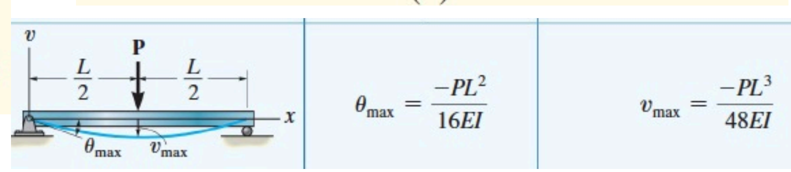


(c)

The displacement at C and the slope at A are the algebraic sums of these components. Hence,

$$\theta_A = (\theta_A)_1 + (\theta_A)_2 = \frac{56\text{ kN}\cdot\text{m}^2}{EI} \qquad \textit{Ans.}$$

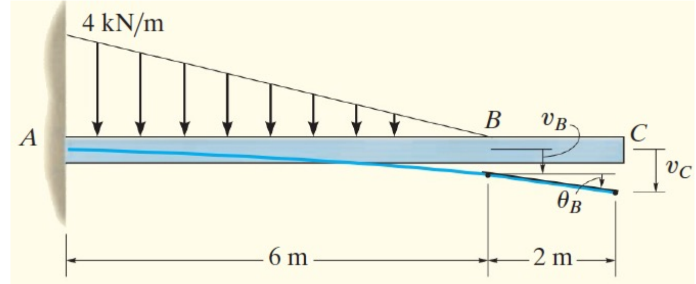
$$(+\downarrow) \qquad v_C = (v_C)_1 + (v_C)_2 = \frac{139\text{ kN}\cdot\text{m}^3}{EI}\downarrow \qquad \textit{Ans.}$$



R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

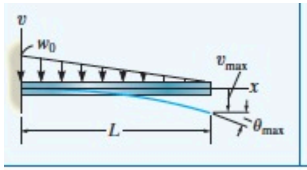
Example

Determine the displacement at the end C of the cantilever beam shown in Fig. 12–28. EI is constant.



$$\theta_B = \frac{w_0 L^3}{24EI} = \frac{4\ \mathrm{kN/m}(6\ \mathrm{m})^3}{24EI} = \frac{36\ \mathrm{kN \cdot m^2}}{EI}$$

$$v_B = \frac{w_0 L^4}{30EI} = \frac{4\ \mathrm{kN/m}(6\ \mathrm{m})^4}{30EI} = \frac{172.8\ \mathrm{kN \cdot m^3}}{EI} \downarrow$$



$$\theta_{max} = \frac{-w_0 L^3}{24EI} \qquad v_{max} = \frac{-w_0 L^4}{30EI}$$

The unloaded region BC of the beam remains straight, as shown in Fig. 12–28. Since θ_B is small, the displacement at C becomes

$$(+\downarrow) \qquad v_C = v_B + \theta_B(L_{BC})$$

$$= \frac{172.8\ \mathrm{kN \cdot m^3}}{EI} + \frac{36\ \mathrm{kN \cdot m^2}}{EI}(2\ \mathrm{m})$$

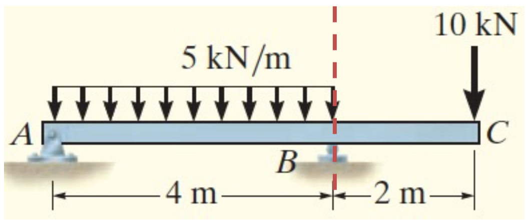
$$= \frac{244.8\ \mathrm{kN \cdot m^3}}{EI} \downarrow$$

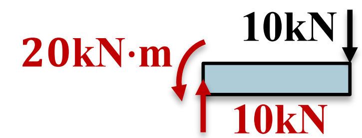
Ans.

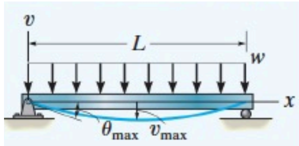
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Example

Determine the displacement at the end C of the overhanging beam shown in Fig. 12–29*a*. EI is constant.





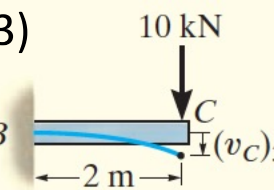


$$\theta_{max} = \frac{-wL^3}{24EI} \qquad v_{max} = \frac{-5wL^4}{384EI}$$

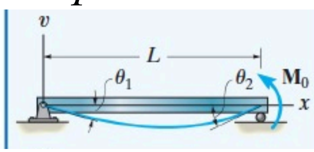
$$(\theta_B)_1 = \frac{wL^3}{24EI} = \frac{5\ \text{kN/m}(4\ \text{m})^3}{24EI} = \frac{13.33\ \text{kN}\cdot\text{m}^2}{EI} \circlearrowleft$$

$$(v_C)_1 = (2\ \text{m})\left(\frac{13.33\ \text{kN}\cdot\text{m}^2}{EI}\right) = \frac{26.67\ \text{kN}\cdot\text{m}^3}{EI}\uparrow$$

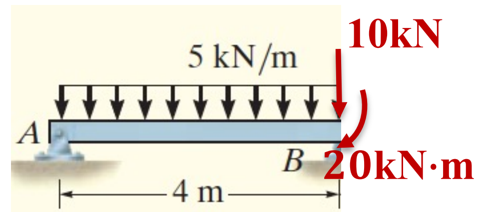
(3)



At B, the 20kN·m couple moment causes a slope, while the 10-kN force does not



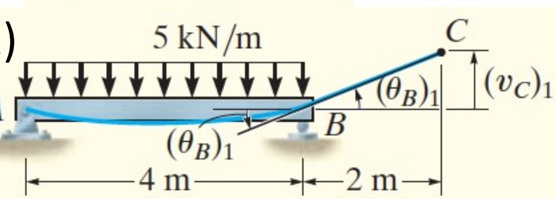
$$\theta_1 = \frac{-M_0L}{6EI} \qquad \theta_2 = \frac{M_0L}{3EI} \qquad v_{max} = \frac{-M_0L^2}{9\sqrt{3}\,EI} \text{ at } x = 0.5774L$$

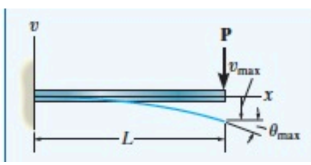


$$(\theta_B)_2 = \frac{M_0L}{3EI} = \frac{20\ \text{kN}\cdot\text{m}(4\ \text{m})}{3EI} = \frac{26.67\ \text{kN}\cdot\text{m}^2}{EI} \circlearrowright$$

$$(v_C)_2 = (2\ \text{m})\left(\frac{26.7\ \text{kN}\cdot\text{m}^2}{EI}\right) = \frac{53.33\ \text{kN}\cdot\text{m}^3}{EI}\downarrow$$

(1)





$$\theta_{max} = \frac{-PL^2}{2EI} \qquad v_{max} = \frac{-PL^3}{3EI}$$

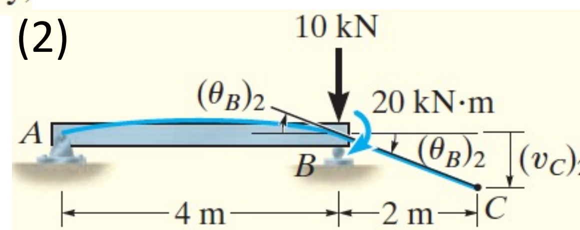
$$(v_C)_3 = \frac{PL^3}{3EI} = \frac{10\ \text{kN}(2\ \text{m})^3}{3EI} = \frac{26.67\ \text{kN}\cdot\text{m}^3}{EI}\downarrow$$

Summing these results algebraically,

$$v_C = -\frac{26.7}{EI} + \frac{53.3}{EI} + \frac{26.7}{EI}$$

$$= \frac{53.3\ \text{kN}\cdot\text{m}^3}{EI}\downarrow$$

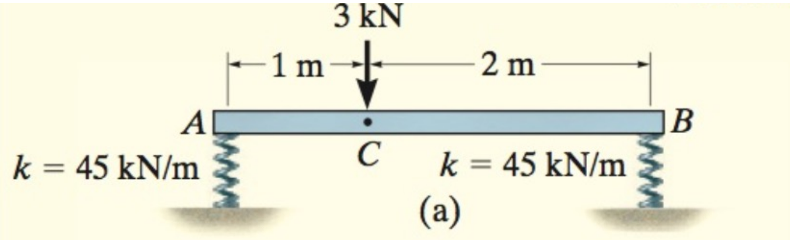
(2)



R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Example

The steel bar shown in Fig. 12–30*a* is supported by two springs at its ends *A* and *B*. Each spring has a stiffness of $k = 45\,\text{kN/m}$ and is originally unstretched. If the bar is loaded with a force of 3 kN at point *C*, determine the vertical displacement of the force. Neglect the weight of the bar and take $E_{st} = 200\,\text{GPa}, I = 4.6875(10^{-6})\,\text{m}^4$.

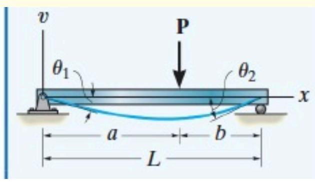


$$(v_A)_1 = \frac{2\,\text{kN}}{45\,\text{kN/m}} = 0.0444\,\text{m} \qquad (v_B)_1 = \frac{1\,\text{kN}}{45\,\text{kN/m}} = 0.0222\,\text{m}$$

If the bar is considered to be *rigid*, these displacements cause it to move into the position shown in Fig. 12–30*b*. For this case, the vertical displacement at *C* is

$$(v_C)_1 = (v_B)_1 + \left(\frac{2\,\text{m}}{3\,\text{m}}\right)[(v_A)_1 - (v_B)_1]$$

$$= 0.0222\,\text{m} + \frac{2}{3}(0.0444\,\text{m} - 0.0222\,\text{m}) = 0.037037\,\text{m}\downarrow$$



$$\theta_1 = \frac{-Pab(L+b)}{6EIL} \qquad \theta_2 = \frac{Pab(L+a)}{6EIL} \qquad v\Big|_{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$$

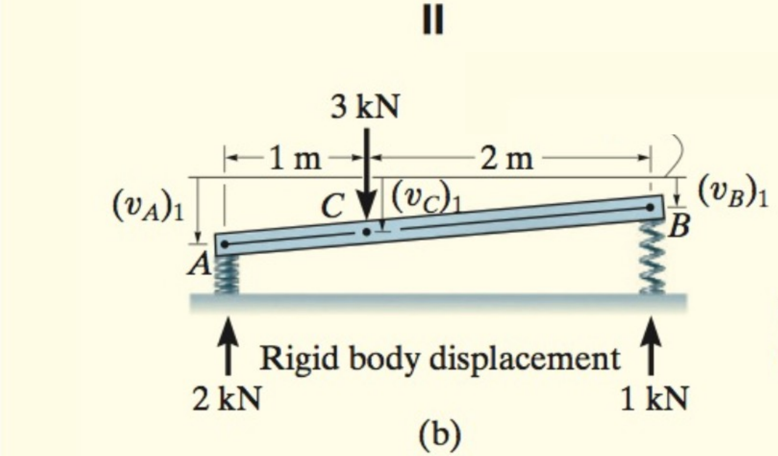
$$(v_C)_2 = \frac{Pab}{6EIL}(L^2 - b^2 - a^2)$$

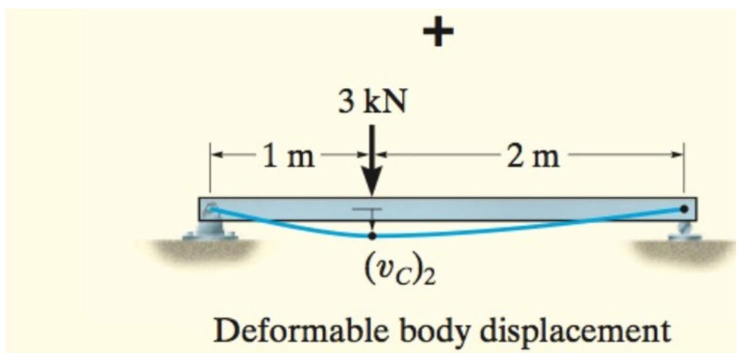
$$= \frac{[3(10^3)\text{N}](1\,\text{m})(2\,\text{m})[(3\,\text{m})^2 - (2\,\text{m})^2 - (1\,\text{m})^2]}{6[200(10^9)\text{N/m}^2][(4.6875(10^{-6})\,\text{m}^4](3\,\text{m})}$$

$$= 0.001422\,\text{m}$$

Adding the two displacement components, we get

$$(+\downarrow) \quad v_C = 0.037037\,\text{m} + 0.001422\,\text{m} = 0.038459\,\text{m} = 38.5\,\text{mm}\downarrow \quad \textit{Ans.}$$

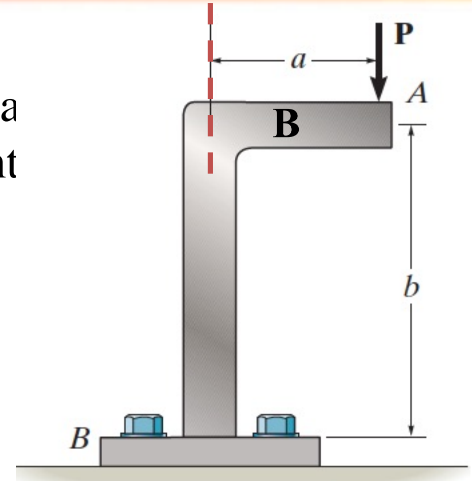


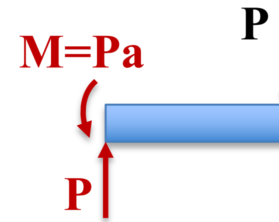


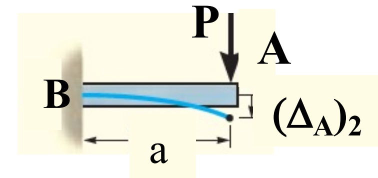
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Example

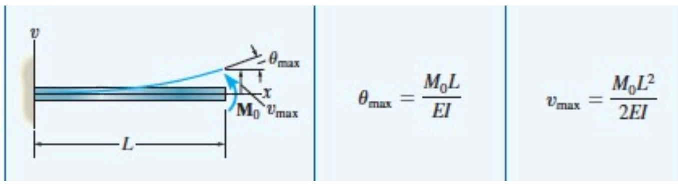
Determine the vertical deflection at the end A ofthe bracket. Assume that the bracket is fixed supported a its base B and neglect axial deflection. EI is constant



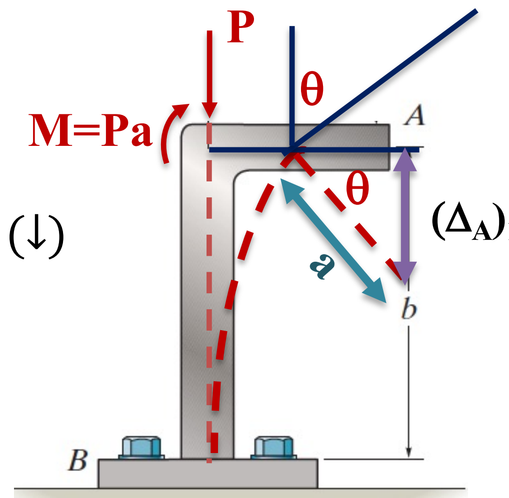


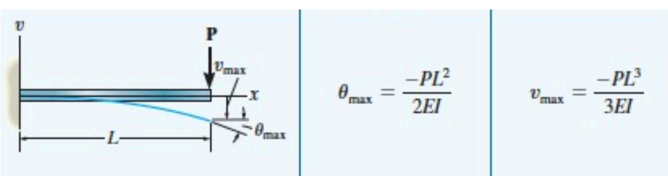


$$\Delta_A = (\Delta_A)_1 + (\Delta_A)_2$$



$$\theta = \frac{ML}{EI} = \frac{Pab}{EI} \quad (\Delta_A)_1 = \theta(a) = \frac{Pa^2 b}{EI} (\downarrow)$$





$$(\Delta_A)_2 = \frac{PL^3}{3EI} = \frac{Pa^3}{3EI} (\downarrow)$$

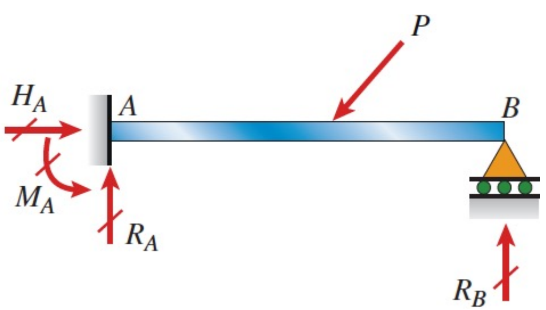
$$\Delta_A = (\Delta_A)_1 + (\Delta_A)_2$$

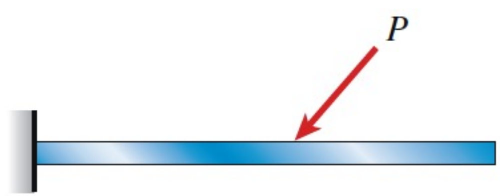
$$= \frac{Pa^2 b}{EI} + \frac{Pa^3}{3EI} = \frac{Pa^2(3b + a)}{3EI} \quad (\downarrow)$$

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

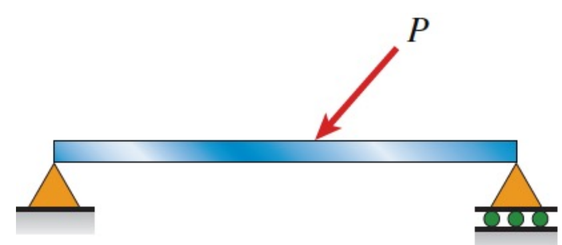
Statically indeterminate beams

- A member is **statically indeterminate** if the number of unknown reactions ***exceeds*** the available number of equilibrium equations.
- The number of reactions in excess of the number of equilibrium equations is called the ***degree of static indeterminacy (DSI)***.
- The excess reactions are called ***static redundants*** and must be selected in each case.
- The structure that remains when the redundants are released is called the ***released structure*** or the ***primary structure.***
- ***Method of Integration; Method of Superposition (compatibility condition)***

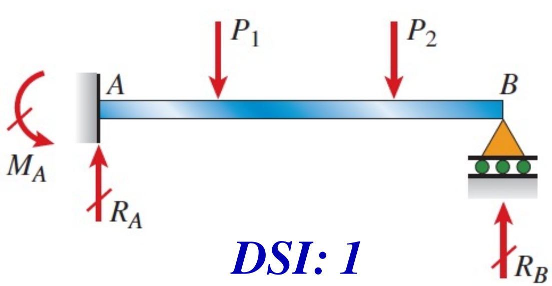




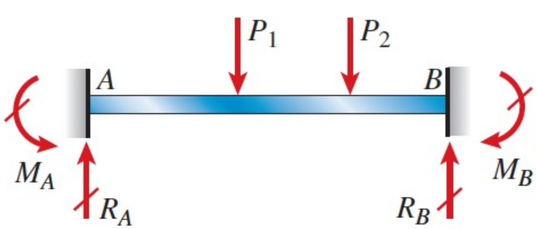
Select the Support B as the redundant



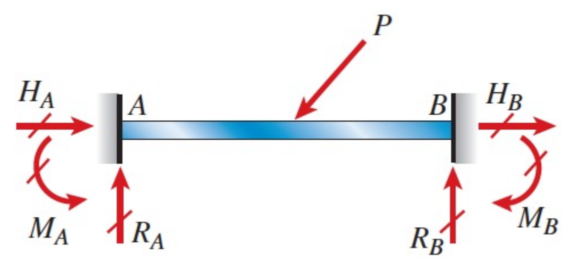
Select the reactive moment M_A *as the redundant*



DSI: 1



DSI: 2



DSI: 3

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson. J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

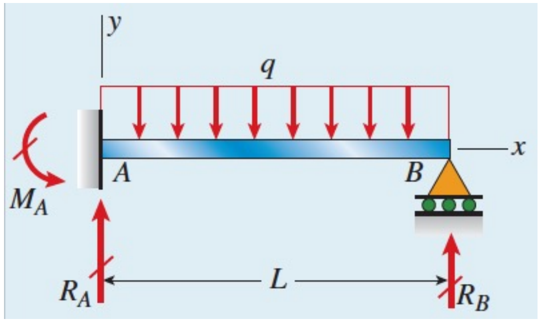
Statically indeterminate beams: Integration Method

- *The procedure is **essentially the same** as that for a statically determinate beam and consists of writing the differential equation, integrating to obtain its general solution, and then applying boundary and other conditions to evaluate the unknown quantities. The unknowns consist of the redundant reactions as well as the constants of integration.*
- *The differential equation for a beam may be solved in symbolic terms only when the beam and its loading are relatively simple and uncomplicated. However, in more complex situations the differential equations must be solved numerically.*

A beam AB of length L supports a uniform load of intensity q. Determine the reactions, shear forces, bending moments, slopes, and deflections of the beam.

Redundant reaction. Let us choose the reaction R_B at the simple support as the redundant. Then, by considering the equilibrium of the entire beam, we can express the other two reactions in terms of R_B:

$$R_A = qL - R_B \qquad M_A = \frac{qL^2}{2} - R_B L \qquad \textbf{(a,b)}$$



Bending moment. The bending moment M at distance x from the fixed support can be expressed in terms of the reactions as

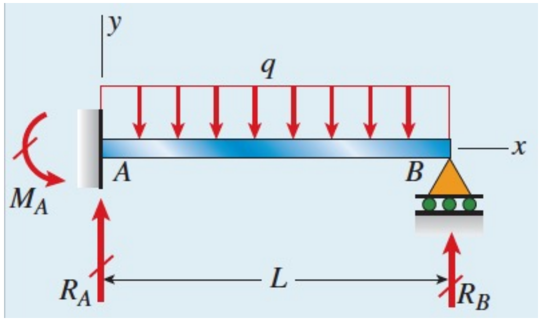
$$M = R_A x - M_A - \frac{qx^2}{2} \qquad \textbf{(c)}$$

$$M = qLx - R_B x - \frac{qL^2}{2} + R_B L - \frac{qx^2}{2} \qquad \textbf{(d)}$$

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Statically indeterminate beams: Integration Method

A beam AB of length L supports a uniform load of intensity q. Determine the reactions, shear forces, bending moments, slopes, and deflections of the beam.



$$M = qLx - R_B x - \frac{qL^2}{2} + R_B L - \frac{qx^2}{2} \qquad \text{(d)}$$

$$EIv'' = M = qLx - R_B x - \frac{qL^2}{2} + R_B L - \frac{qx^2}{2} \qquad \text{(e)}$$

$$EIv' = \frac{qLx^2}{2} - \frac{R_B x^2}{2} - \frac{qL^2 x}{2} + R_B L x - \frac{qx^3}{6} + C_1 \qquad \text{(f)}$$

$$EIv = \frac{qLx^3}{6} - \frac{R_B x^3}{6} - \frac{qL^2 x^2}{4} + \frac{R_B L x^2}{2} - \frac{qx^4}{24} + C_1 x + C_2 \qquad \text{(g)}$$

Boundary conditions.

$v(0) = 0 \qquad v'(0) = 0 \qquad v(L) = 0$

$C_1 = 0,\ C_2 = 0,$

$$R_B = \frac{3qL}{8}$$

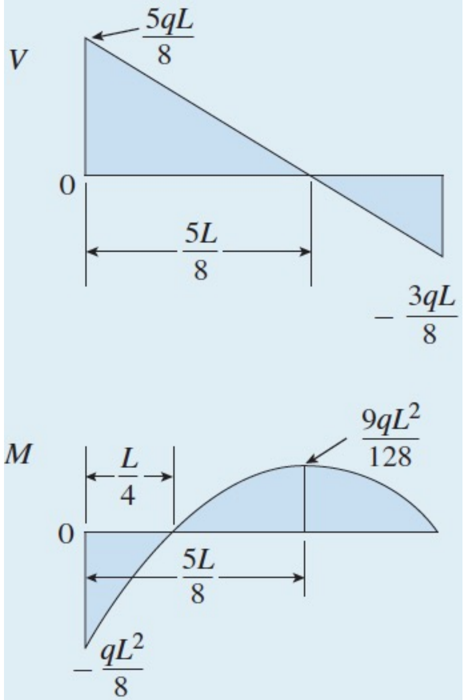
$$R_A = \frac{5qL}{8} \qquad M_A = \frac{qL^2}{8}$$

$$V = R_A - qx = \frac{5qL}{8} - qx$$

$$V_{max} = \frac{5qL}{8}$$

$$M = R_A x - M_A - \frac{qx^2}{2} = \frac{5qLx}{8} - \frac{qL^2}{8} - \frac{qx^2}{2}$$

$$M_{pos} = \frac{9qL^2}{128} \qquad M_{neg} = -\frac{qL^2}{8}$$

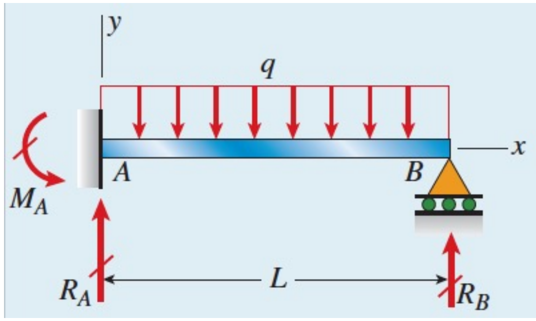


Statically indeterminate beams: Integration Method

Slopes and deflections of the beam

$$v' = \frac{qx}{48EI}(-6L^2 + 15Lx - 8x^2) \qquad (10\text{-}7)$$

$$v = -\frac{qx^2}{48EI}(3L^2 - 5Lx + 2x^2) \qquad (10\text{-}8)$$

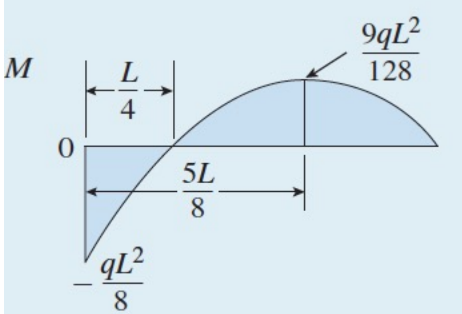


To determine the maximum deflection of the beam, we set the slope [Eq. (10-7)] equal to zero and solve for the distance x_1 to the point where this deflection occurs:

$$v' = 0 \quad \text{or} \quad -6L^2 + 15Lx - 8x^2 = 0$$

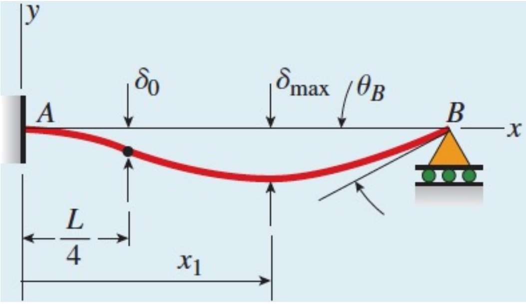
from which

$$x_1 = \frac{15 - \sqrt{33}}{16}L = 0.5785L \qquad (10\text{-}9)$$



$$\delta_{max} = -(v)_{x=x_1} = \frac{qL^4}{65{,}536\,EI}(39 + 55\sqrt{33})$$

$$= \frac{qL^4}{184.6\,EI} = 0.005416\frac{qL^4}{EI} \quad (\downarrow) \qquad (10\text{-}10)$$



To determine the angle of rotation θ_B at the simply supported end of the beam, we use Eq. (10-7), as

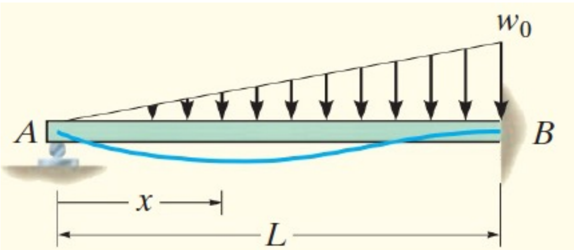
$$\theta_B = (v')_{x=L} = \frac{qL^3}{48EI} \qquad (10\text{-}12)$$

$$\delta_0 = -(v)_{x=L/4} = \frac{5qL^4}{2048\,EI} = 0.002441\frac{qL^4}{EI} \quad (\downarrow)$$

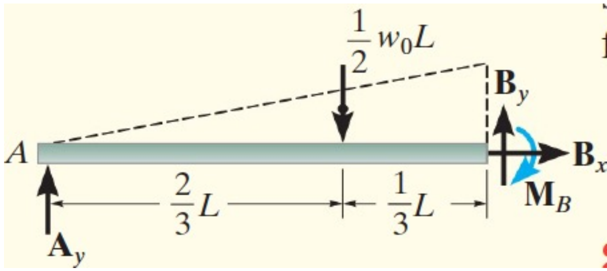
When x<L/4 , both the curvature and the bending moment are negative, and when x>L/4 , the curvature and bending moment are positive

Example: Integration Method

The beam is subjected to the distributed loading shown in Fig. 12–33*a*. Determine the reaction at A. EI is constant.



$$M = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

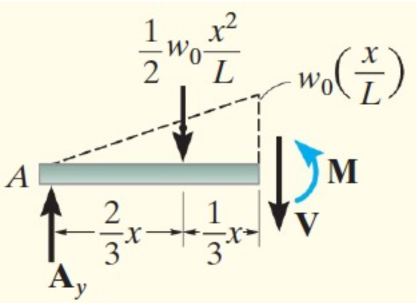


Slope and Elastic Curve

$$EI\frac{d^2v}{dx^2} = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

$$EI\frac{dv}{dx} = \frac{1}{2} A_y x^2 - \frac{1}{24} w_0 \frac{x^4}{L} + C_1$$

$$EIv = \frac{1}{6} A_y x^3 - \frac{1}{120} w_0 \frac{x^5}{L} + C_1 x + C_2$$



The *three* unknowns A_y, C_1, and C_2 are determined from the *three* boundary conditions $x = 0$, $v = 0$; $x = L$, $dv/dx = 0$; and $x = L$, $v = 0$. Applying these conditions we get

$$x = 0, v = 0; \qquad 0 = 0 - 0 + 0 + C_2$$

$$x = L, \frac{dv}{dx} = 0; \qquad 0 = \frac{1}{2} A_y L^2 - \frac{1}{24} w_0 L^3 + C_1$$

$$x = L, v = 0; \qquad 0 = \frac{1}{6} A_y L^3 - \frac{1}{120} w_0 L^4 + C_1 L + C_2$$

Solving,

$$A_y = \frac{1}{10} w_0 L \qquad \textit{Ans.}$$

$$C_1 = -\frac{1}{120} w_0 L^3 \qquad C_2 = 0$$

$$B_y = 2w_0 L/5$$

$$M_B = w_0 L^2/15$$

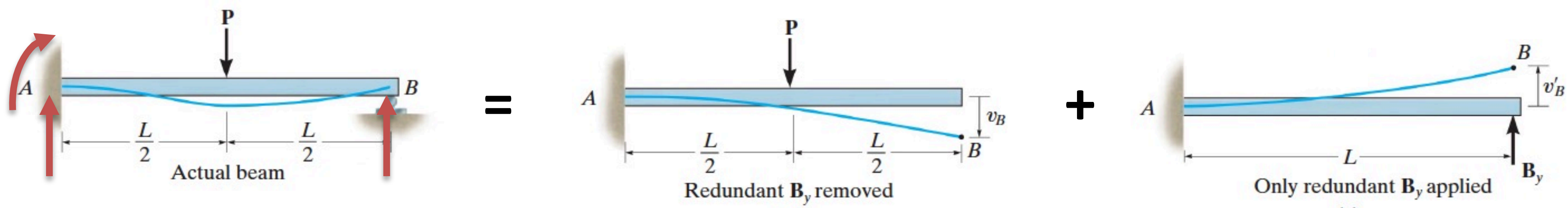
R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Statically indeterminate beams: Superposition Method

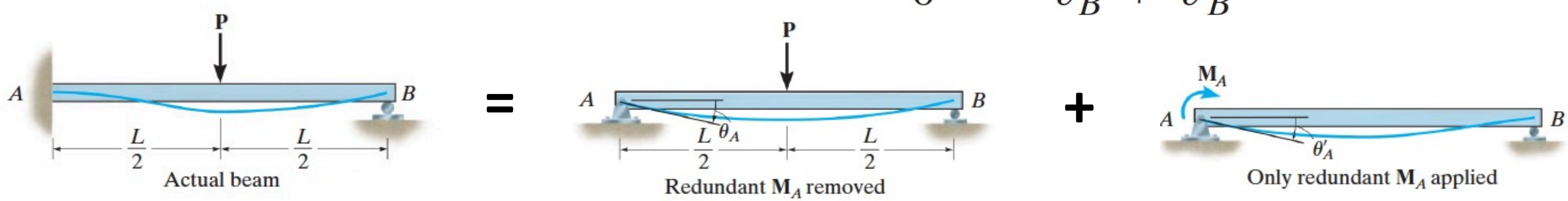
- Noting the degree of static indeterminacy, ***selecting the redundant reactions,*** and writing ***equations of equilibrium*** that relate other unknown reactions to redundants and loads.

- Imposing the original loads and the redundants upon the released structure to ***find the deflections by superposing the separate deflections*** due to the loads and the redundants.

- Writing the ***equations of compatibility*** to ensure the sum of these deflections to be the same as the deflections in the original beam (at those same points).



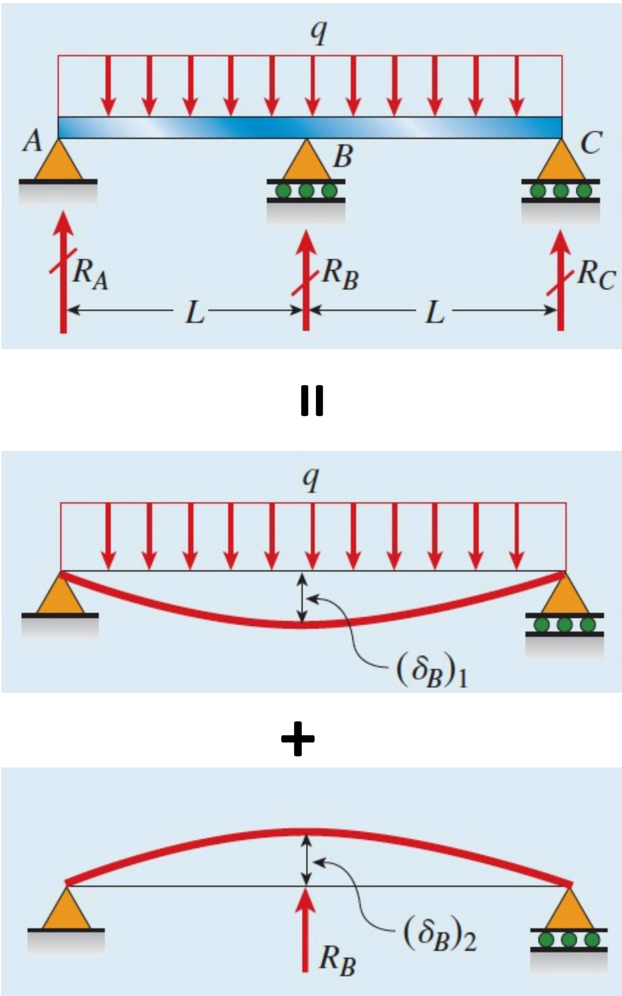
$$0 = -v_B + v'_B$$



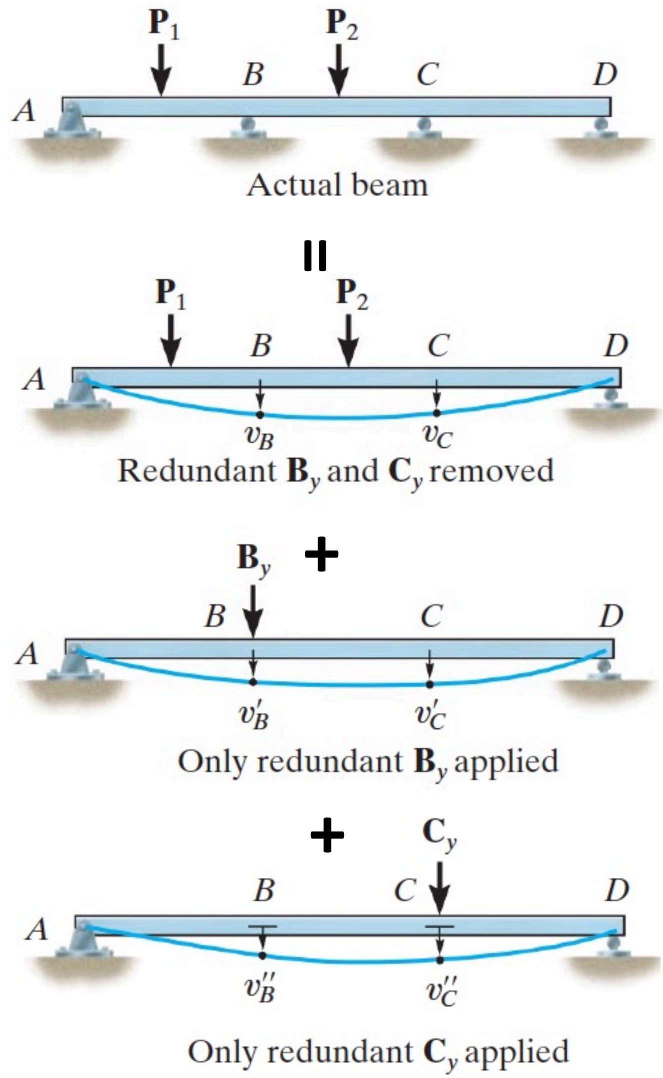
$$0 = \theta_A + \theta'_A$$

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition. R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Statically indeterminate beams: Superposition Method



$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$$

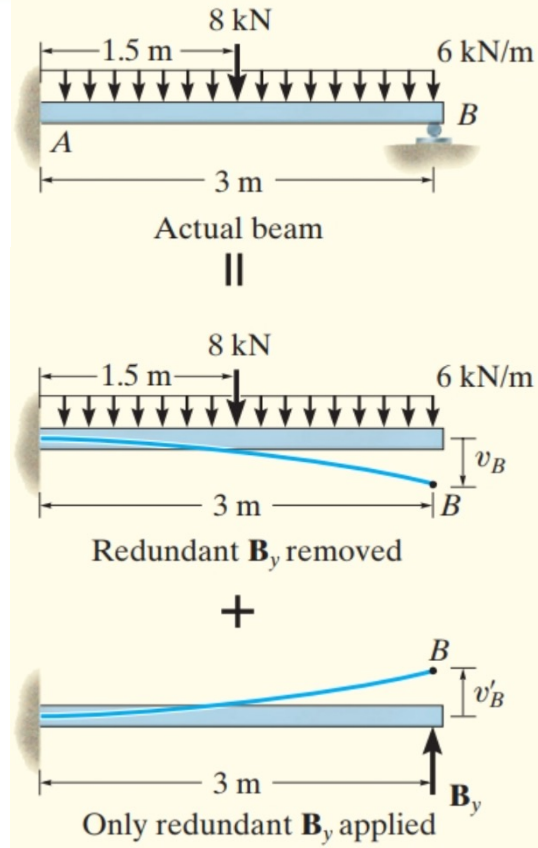


$$0 = v_B + v'_B + v''_B \qquad 0 = v_C + v'_C + v''_C$$

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition. R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Example: Superpostion Method

Determine the reactions at the roller support B of the beam shown in Fig. 12–45*a*, then draw the shear and moment diagrams. EI is constant.



Principle of Superposition. By inspection, the beam is statically indeterminate to the first degree. The roller support at B will be chosen as the redundant so that $\mathbf{B}_y$ will be determined *directly*. Figures 12–45*b* and 12–45*c* show application of the principle of superposition. Here we have assumed that $\mathbf{B}_y$ acts upward on the beam.

Compatibility Equation. $(+\downarrow)$ $\qquad 0 = v_B - v'_B$

$$v_B = \frac{wL^4}{8EI} + \frac{5PL^3}{48EI}$$

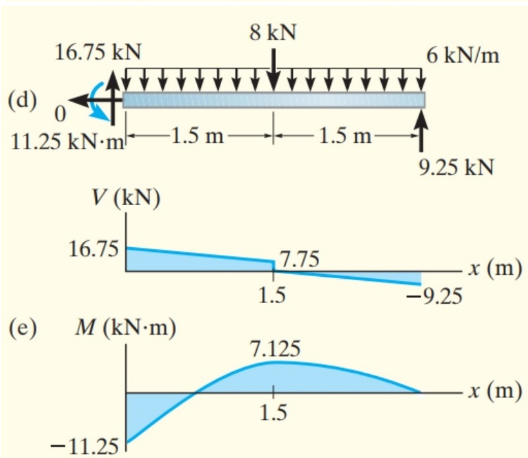
$$= \frac{(6\,\text{kN/m})(3\,\text{m})^4}{8EI} + \frac{5(8\,\text{kN})(3\,\text{m})^3}{48EI} = \frac{83.25\,\text{kN}\cdot\text{m}^3}{EI}\downarrow$$

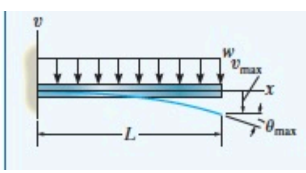
$$v'_B = \frac{PL^3}{3EI} = \frac{B_y(3\,\text{m})^3}{3EI} = \frac{9B_y}{EI}\uparrow$$

$$0 = \frac{83.25}{EI} - \frac{9B_y}{EI}$$

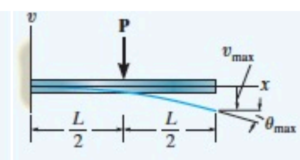
$$By = 9.25\,\text{kN}$$

Equilibrium Equations

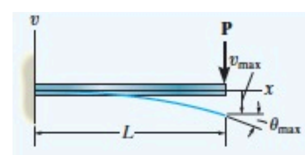




$$\theta_{max} = \frac{-wL^3}{6EI} \qquad v_{max} = \frac{-wL^4}{8EI}$$



$$\theta_{max} = \frac{-PL^2}{8EI} \qquad v_{max} = \frac{-5PL^3}{48EI}$$



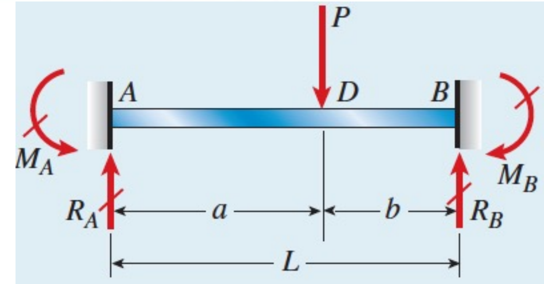
$$\theta_{max} = \frac{-PL^2}{2EI} \qquad v_{max} = \frac{-PL^3}{3EI}$$

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

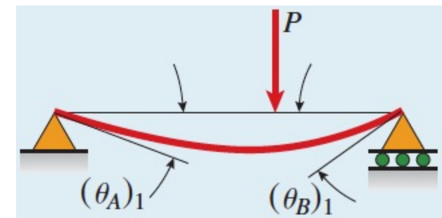
Example: Superpostion Method

A fixed-end beam AB (Fig. 10-15a) is loaded by a force P acting at an intermediate point D. Find the reactive forces and moments at the ends of the beam using the method of superposition. Also, determine the deflection at point D where the load is applied.

This beam has four unknown reactions (a force and a moment at each support), but only two independent equations of equilibrium are available. Therefore, the beam is statically indeterminate to the second degree. In this example, we will select the reactive moments M_A and M_B as the redundants.

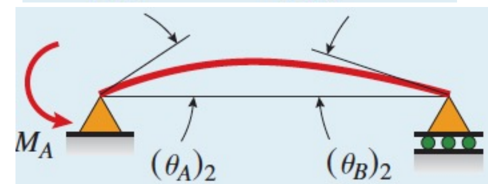


$$R_A = \frac{Pb}{L} + \frac{M_A}{L} - \frac{M_B}{L} \qquad R_B = \frac{Pa}{L} - \frac{M_A}{L} + \frac{M_B}{L}$$

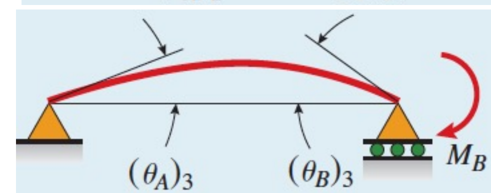


Equations of compatibility.

$$\theta_A = (\theta_A)_1 - (\theta_A)_2 - (\theta_A)_3 = 0$$
$$\theta_B = (\theta_B)_1 - (\theta_B)_2 - (\theta_B)_3 = 0$$



$$M_A = \frac{Pab^2}{L^2} \qquad M_B = \frac{Pa^2b}{L^2}$$

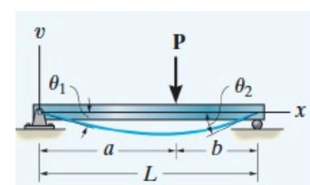


$$\frac{M_AL}{3EI} + \frac{M_BL}{6EI} = \frac{Pab(L+b)}{6LEI}$$

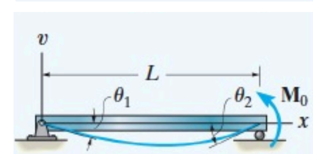
$$\frac{M_AL}{6EI} + \frac{M_BL}{3EI} = \frac{Pab(L+a)}{6LEI}$$

$$R_A = \frac{Pb^2}{L^3}(L + 2a)$$

$$R_B = \frac{Pa^2}{L^3}(L + 2b)$$



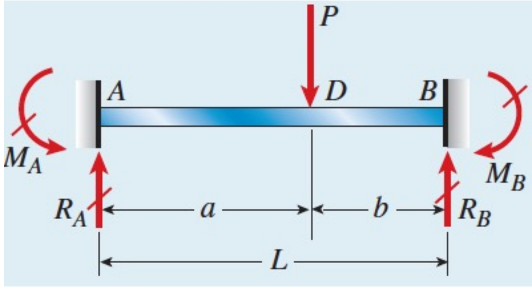
$$\theta_1 = \frac{-Pab(L+b)}{6EIL} \qquad \theta_2 = \frac{Pab(L+a)}{6EIL}$$



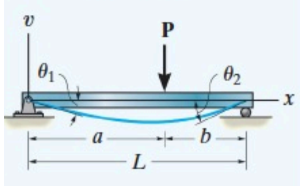
$$\theta_1 = \frac{-M_0L}{6EI} \qquad \theta_2 = \frac{M_0L}{3EI}$$

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Example: Superpostion Method

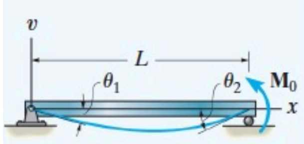


A fixed-end beam *AB* (Fig. 10-15a) is loaded by a force *P* acting at an intermediate point *D*. Find the reactive forces and moments at the ends of the beam using the method of superposition. Also, determine the deflection at point *D* where the load is applied.



$$\theta_1 = \frac{-Pab(L+b)}{6EIL} \qquad \theta_2 = \frac{Pab(L+a)}{6EIL}$$

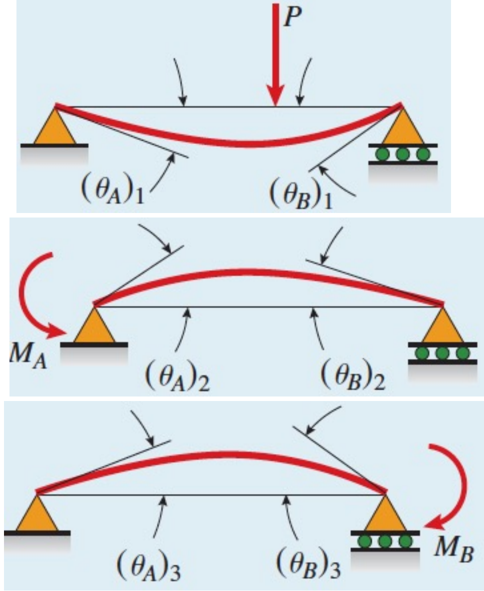
$$v\Big|_{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$$



$$\theta_1 = \frac{-M_0 L}{6EI} \qquad \theta_2 = \frac{M_0 L}{3EI}$$

$$v_{max} = \frac{-M_0 L^2}{9\sqrt{3}\,EI} \quad \text{at } x = 0.5774L$$

$$v = \frac{-M_0 x}{6EIL}(L^2 - x^2)$$



Deflection at point D

$$\delta_D = (\delta_D)_1 - (\delta_D)_2 - (\delta_D)_3$$

$$(\delta_D)_1 = \frac{Pa^2b^2}{3LEI} \, (\downarrow)$$

$$(\delta_D)_2 = \frac{M_A ab}{6LEI}(L+b) \qquad (\delta_D)_3 = \frac{M_B ab}{6LEI}(L+a)$$

$$(\delta_D)_2 = \frac{Pa^2b^3}{6L^3EI}(L+b)(\uparrow) \qquad (\delta_D)_3 = \frac{Pa^3b^2}{6L^3EI}(L+a)(\uparrow)$$

$$M_A = \frac{Pab^2}{L^2} \qquad M_B = \frac{Pa^2b}{L^2}$$

$$\delta_D = \frac{Pa^3b^3}{3L^3EI} \, (\downarrow)$$

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Example: Superpostion Method

The beam in Fig. 12–43*a* is fixed supported to the wall at *A* and pin connected to a 12-mm-diameter rod *BC*. If $E = 200$ GPa for both members, determine the force developed in the rod due to the loading. The moment of inertia of the beam about its neutral axis is $I = 186(10^6)\ \text{mm}^4$.

Principle of Superposition. By inspection, this problem is indeterminate to the first degree. Here *B* will undergo an unknown displacement v''_B, since the rod will stretch. The rod will be treated as the redundant and hence the force of the rod is removed from the beam at *B*, Fig. 12–43*b*, and then reapplied, Fig. 12–43*c*.

Compatibility Equation. At point *B* we require

$$(+\downarrow) \qquad v''_B = v_B - v'_B \qquad (1)$$

The displacements v_B and v'_B are determined from the table in Appendix C. v''_B is calculated from Eq. 4–2. Working in kilopounds and inches, we have

$$v''_B = \frac{PL}{AE} = \frac{F_{BC}(3\,\text{m})}{\left[(\pi/4)\,(0.012\,\text{m})^2\right]\left[200(10^9)\,\text{N/m}^2\right]} = 0.13263(10^{-6})\,F_{BC}$$

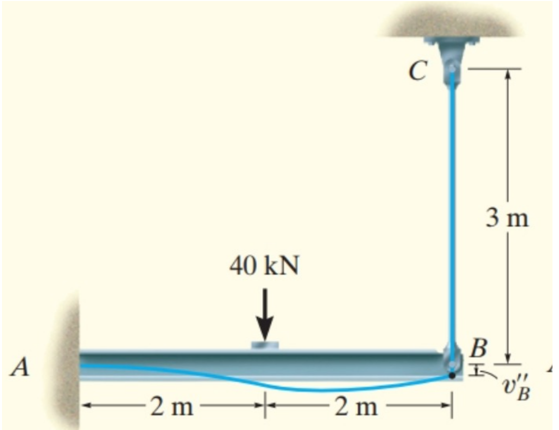
$$v_B = \frac{5PL^3}{48EI} = \frac{5\left[40(10^3)\,\text{N}\right](4\,\text{m})^3}{48\left[200(10^9)\,\text{N/m}^2\right]\left[186(10^{-6})\,\text{m}^4\right]} = 0.0071685\,\text{m}$$

$$v'_B = \frac{PL^3}{3EI} = \frac{F_{BC}(4\,\text{m})^3}{3\left[200(10^9)\,\text{N/m}^2\right]\left[186(10^{-6})\,\text{m}^4\right]} = 0.57348(10^{-6})\,F_{BC}$$

Thus, Eq. 1 becomes

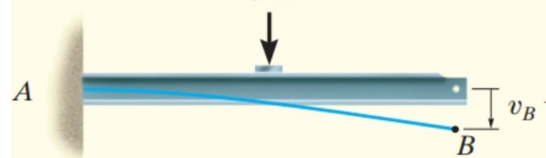
$$(+\downarrow) \qquad 0.13263(10^{-6})\,F_{BC} = 0.00716858 - 0.57348(10^{-6})\,F_{BC}$$

$$F_{BC} = 10.152(10^3)\,\text{N} = 10.2\,\text{kN} \qquad \textit{Ans.}$$



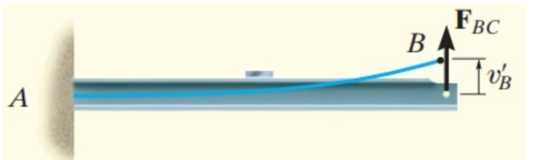
Actual beam and rod

(a)



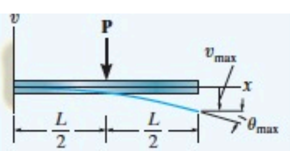
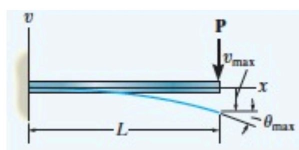
Redundant $\mathbf{F}_{BC}$ removed

(b)



Only redundant $\mathbf{F}_{BC}$ applied

(c)

	$\theta_{max} = \frac{-PL^2}{8EI}$	$v_{max} = \frac{-5PL^3}{48EI}$
	$\theta_{max} = \frac{-PL^2}{2EI}$	$v_{max} = \frac{-PL^3}{3EI}$

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Example: Superpostion Method

The beam in Fig. 12–43*a* is fixed supported to the wall at *A* and pin connected to a 12-mm-diameter rod *BC*. If $E = 200$ GPa for both members, determine the force developed in the rod due to the loading. The moment of inertia of the beam about its neutral axis is $I = 186(10^6)\,\text{mm}^4$.

Principle of Superposition. We can also solve this problem by removing the pin support at *C* and keeping the rod attached to the beam. In this case the 40-kN load will cause points *B* and *C* to be displaced downward the *same amount* v_C, Fig. 12–43*e*, since no force exists in rod *BC*. When the redundant force $\mathbf{F}_{BC}$ is applied at point *C*, it causes the end *C* of the rod to be displaced upward v'_C and the end *B* of the beam to be displaced upward v'_B, Fig. 12–43*f*. The difference in these two displacements, v_{BC}, represents the stretch of the rod due to $\mathbf{F}_{BC}$, so that $v'_C = v_{BC} + v'_B$. Hence, from Figs. 12–43*d*, 12–43*e*, and 12–43*f*, the compatibility of displacement at point *C* is

$$(+\downarrow) \qquad 0 = v_C - (v_{BC} + v'_B) \qquad (2)$$

From Solution I, we have

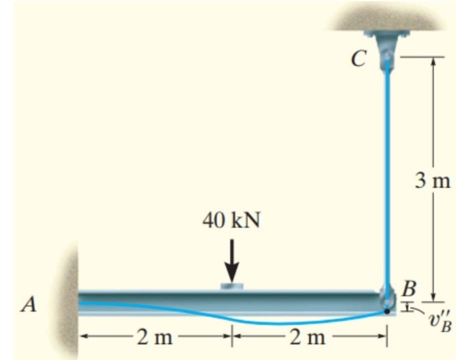
$$v_C = v_B = 0.0071685\,\text{m}\downarrow$$

$$v_{BC} = v''_B = 0.13263(10^{-6})\,F_{BC}\uparrow$$

$$v'_B = 0.57348(10^{-6})\,F_{BC}\uparrow$$

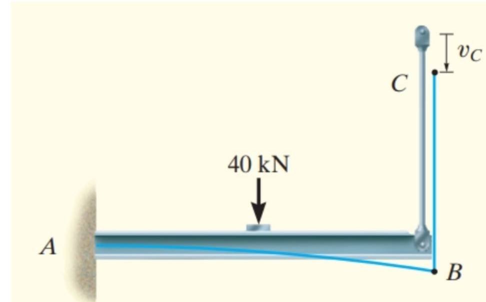
$$(+\downarrow) \qquad 0 = 0.0071685 - [0.13263(10^{-6})\,F_{BC} + 0.57348(10^{-6})\,F_{BC}]$$

$$F_{BC} = 10.152(10^3)\,\text{N} = 10.2\,\text{kN} \qquad \textit{Ans}$$

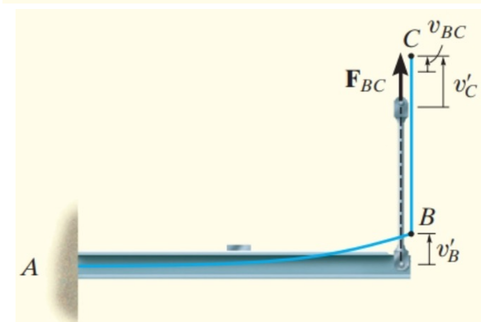


Actual beam and rod

(a)



Redundant $\mathbf{F}_{BC}$ removed

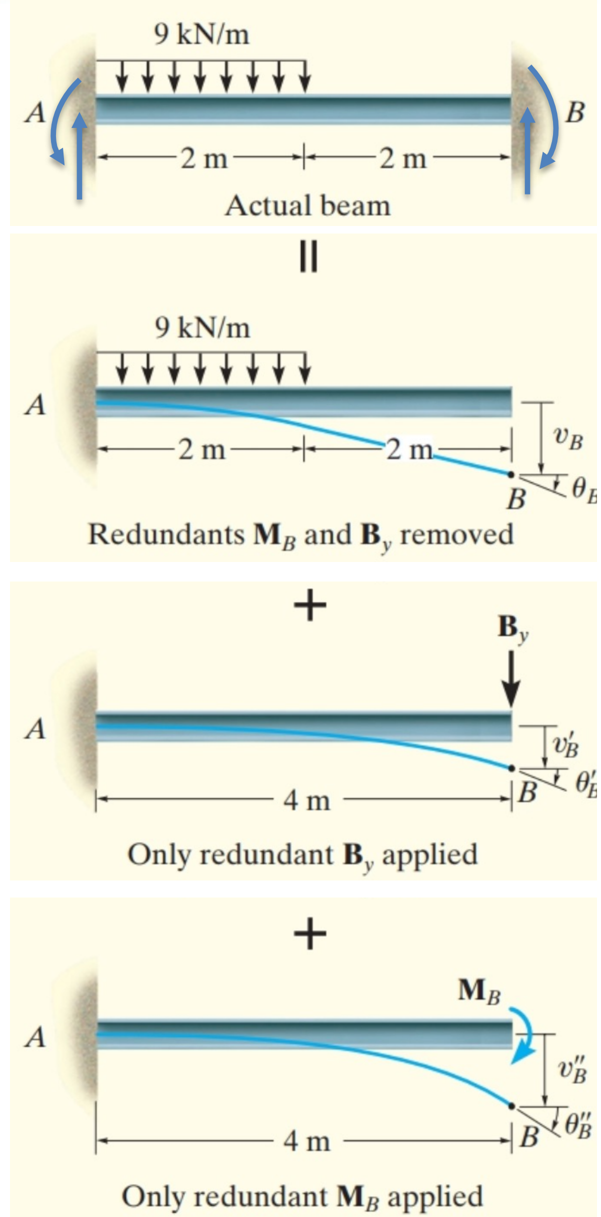


Only redundant $\mathbf{F}_{BC}$ applied

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Example: Superpostion Method

Determine the moment at *B* for the beam shown in Fig. 12–44*a*. *EI* is constant. Neglect the effects of axial load.



Principle of Superposition. Since the axial load on the beam is neglected, there will be a vertical force and moment at *A* and *B*. Here there are only two available equations of equilibrium ($\Sigma M = 0, \Sigma F_y = 0$) and so the problem is indeterminate to the second degree. We will assume that $\mathbf{B}_y$ and $\mathbf{M}_B$ are redundant, so that by the principle of superposition, the beam is represented as a cantilever, loaded *separately* by the distributed load and reactions $\mathbf{B}_y$ and $\mathbf{M}_B$, Figs. 12–44*b*, 12–44*c*, and 12–44*d*.

Compatibility Equations.

$$(\curvearrowleft +) \qquad 0 = \theta_B + \theta'_B + \theta''_B$$

$$(+\downarrow) \qquad 0 = v_B + v'_B + v''_B$$

$$\theta_B = \frac{wL^3}{48EI} = \frac{(9\,\text{kN/m})(4\,\text{m})^3}{48EI} = \frac{12\,\text{kN}\cdot\text{m}^2}{EI} \curvearrowright$$

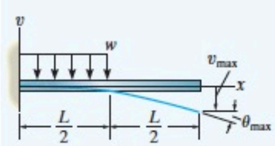
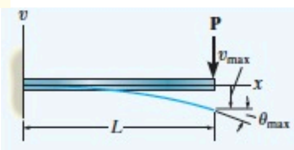
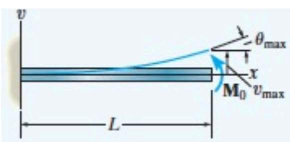
$$v_B = \frac{7wL^4}{384EI} = \frac{7(9\,\text{kN/m})(4\,\text{m})^4}{384EI} = \frac{42\,\text{kN}\cdot\text{m}^3}{EI} \downarrow$$

$$\theta'_B = \frac{PL^2}{2EI} = \frac{B_y(4\,\text{m})^2}{2EI} = \frac{8B_y}{EI} \curvearrowright$$

$$v'_B = \frac{PL^3}{3EI} = \frac{B_y(4\,\text{m})^3}{3EI} = \frac{21.33B_y}{EI} \downarrow$$

$$\theta''_B = \frac{ML}{EI} = \frac{M_B(4\,\text{m})}{EI} = \frac{4M_B}{EI} \curvearrowright$$

$$v''_B = \frac{ML^2}{2EI} = \frac{M_B(4\,\text{m})^2}{2EI} = \frac{8M_B}{EI} \downarrow$$

	$\theta_{max} = \frac{-wL^3}{48EI}$	$v_{max} = \frac{-7wL^4}{384EI}$
	$\theta_{max} = \frac{-PL^2}{2EI}$	$v_{max} = \frac{-PL^3}{3EI}$
	$\theta_{max} = \frac{M_0L}{EI}$	$v_{max} = \frac{M_0L^2}{2EI}$

$$(\curvearrowleft +) \qquad 0 = 12 + 8B_y + 4M_B$$

$$(+\downarrow) \qquad 0 = 42 + 21.33B_y + 8M_B$$

$$B_y = -3.375\,\text{kN}$$

$$M_B = 3.75\,\text{kN}\cdot\text{m}$$

R.C. Hibbeler, Mechanics of Materials, 9th edition, Pearson.

Strain Energy Method & Castigliano's Theorem

Additional, Not compulsory

https://www.youtube.com/watch?v=MvBqCeZllpQ

Strain Energy of Bending

A simple beam AB in pure bending under the action of two couples, each having a moment M. The deflection curve is a nearly flat circular arc of constant curvature. The angle θ subtended by this arc equals L/ρ, where L is the length of the beam and ρ is the radius of curvature.

$$\theta = \frac{L}{\rho} = \kappa L = \frac{ML}{EI}$$

Axial loading

$\delta = NL/EA$

Torsion

$\phi = TL/JG$

$$W = U = \frac{M\theta}{2}$$

$$U = \frac{M^2 L}{2EI}$$

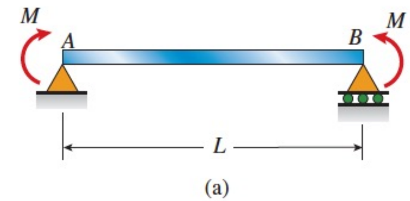
$$U = \frac{EI\theta^2}{2L}$$

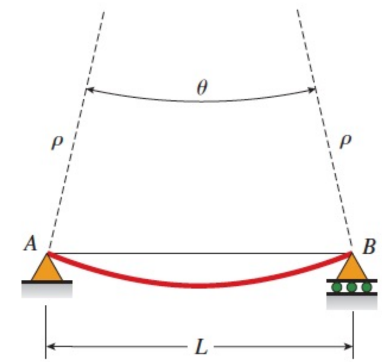
If the bending moment in a beam varies along its length, we may obtain the strain energy by taking an element and integrating along its length.

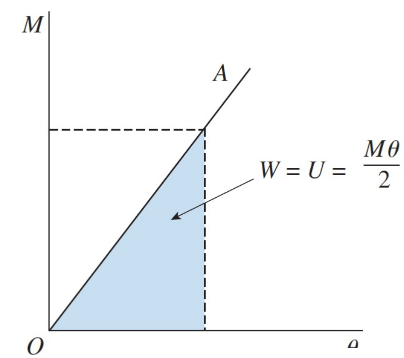
$$d\theta = \kappa dx = \frac{dx}{\rho} = \frac{d^2v}{dx^2}dx$$

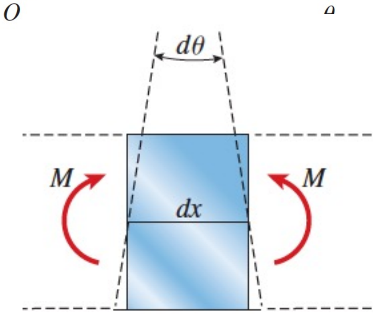
$$U = \int \frac{M^2 dx}{2EI}$$

$$U = \int \frac{EI(d\theta)^2}{2L} = \int \frac{EI}{2L}\left(\frac{d^2v}{dx^2}\right)^2 dx$$









J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Deflections Caused by a Single Load

$$U = \int \frac{M^2 dx}{2EI}$$

$$U = \int \frac{EI(d\theta)^2}{2L} = \int \frac{EI}{2L} \left(\frac{d^2v}{dx^2}\right)^2 dx$$

We considered only the effects of the M. If shear forces are present, additional strain energy will be stored in the beam. However, ***this strain energy is relatively small*** *(compared with the strain energy of bending) for beams in which the lengths are much greater than the depths (L/d>8). Therefore, in most beams the strain energy of shear may safely be disregarded.*

- In the case of a **single P**, the corresponding **deflection δ** is the deflection of the beam axis at the point *where the load is applied*, and must be measured *along the line of action of the load and is positive in the direction of the load*.
- In the case of a **single M_0**, the corresponding **angle of rotation θ** is the angle of rotation of the beam axis at the point where *the couple is applied*.

$$U = W = \frac{P\delta}{2}$$

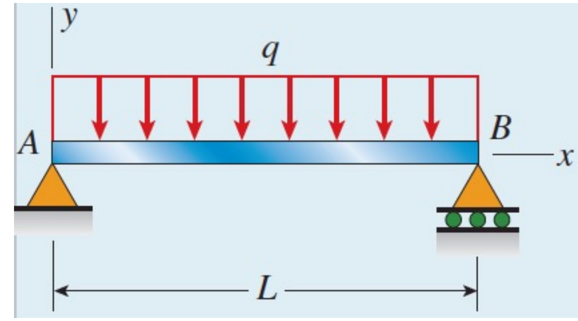
$$\delta = \frac{2U}{P}$$

$$U = W = \frac{M_0\theta}{2}$$

$$\theta = \frac{2U}{M_0}$$

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Example



A simple beam AB of length L supports a uniform load of intensity q (Fig. 9-33). (a) Evaluate the strain energy of the beam from the bending moment in the beam. (b) Evaluate the strain energy of the beam from the equation of the deflection curve. (*Note:* The beam has constant flexural rigidity EI.)

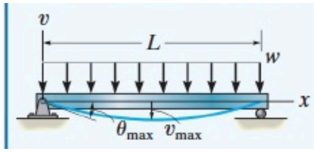
Strain energy from the bending moment.

$$M = \frac{qLx}{2} - \frac{qx^2}{2} = \frac{q}{2}(Lx - x^2)$$

$$U = \int_0^L \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L \left[\frac{q}{2}(Lx - x^2)\right]^2 dx = \frac{q^2}{8EI}\int_0^L (L^2x^2 - 2Lx^2 + x^4)dx$$

$$U = \frac{q^2L^5}{240EI}$$

Strain energy from the deflection curve.

	$\theta_{max} = \frac{-wL^3}{24EI}$	$v_{max} = \frac{-5wL^4}{384EI}$	$v = \frac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$

$$v = -\frac{qx}{24EI}(L^3 - 2Lx^2 + x^3)$$

$$\frac{dv}{dx} = -\frac{q}{24EI}(L^3 - 6Lx^2 + 4x^3) \qquad \frac{d^2v}{dx^2} = \frac{q}{2EI}(Lx - x^2)$$

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L \left[\frac{q}{2EI}(Lx - x^2)\right]^2 dx$$

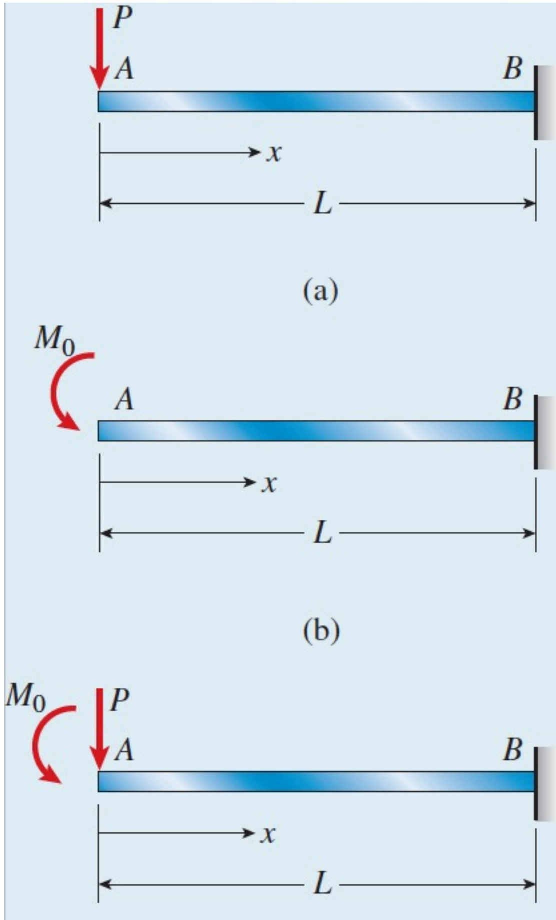
$$= \frac{q^2}{8EI}\int_0^L (L^2x^2 - 2Lx^3 + x^4)dx$$

$$U = \frac{q^2L^5}{240EI}$$

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Example

A cantilever beam AB (Fig. 9-34) is subjected to three different loading conditions: (a) a concentrated load P at its free end, (b) a couple M_0 at its free end, and (c) both loads acting simultaneously.



Beam with concentrated load P $M = -Px$

$$U = \int_0^L \frac{M^2 dx}{2EI} = \int_0^L \frac{(-Px)^2 dx}{2EI} = \frac{P^2L^3}{6EI} \qquad \frac{P\delta_A}{2} = \frac{P^2L^3}{6EI} \qquad \delta_A = \frac{PL^3}{3EI}$$

Beam with moment M_0 (Fig. 9-34b). In this case the bending moment is constant and equal to $-M_0$. Therefore, the strain energy

$$U = \int_0^L \frac{M^2 dx}{2EI} = \int_0^L \frac{(-M_0)^2 dx}{2EI} = \frac{M_0^2L}{2EI} \qquad \frac{M_0\theta_A}{2} = \frac{M_0^2L}{2EI} \qquad \theta_A = \frac{M_0L}{EI}$$

Beam with both loads acting simultaneously $M = -Px - M_0$

$$U = \int_0^L \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L (-Px - M_0)^2 dx$$

$$= \frac{P_2L^3}{6EI} + \frac{PM_0L^2}{2EI} + \frac{M_0^2L}{2EI}$$

we cannot calculate a deflection for a beam with two or more loads by equating the work done by the loads to the strain energy

$$\frac{P\delta_{A2}}{2} + \frac{M_0\theta_{A2}}{2} = \frac{P^2L^3}{6EI} + \frac{PM_0L^2}{2EI} + \frac{M_0^2L}{2EI}$$

two unknowns and only one equation

Strain energy is a quadratic function of the loads, not a linear function. Therefore, the principle of superposition does not apply to strain energy.

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Castigliano's Theorem

The partial derivative of the strain energy **of a structure with respect to any load is equal to the** ***displacement corresponding to that load.***

$$\delta_i = \frac{\partial U}{\partial P_i}$$

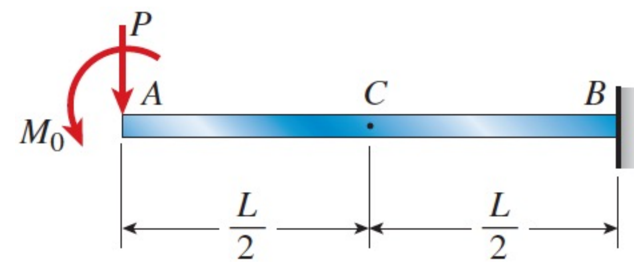
- When using the terms ***load and corresponding displacement*** in connection with Castigliano's theorem, it is understood that these terms are used **in a generalized sense**. ***(P vs δ, M vs θ, T vs ϕ)***
- The derivation requires that only conservative force be considered. These forces can be applied in any order and do work that is ***independent of the path and then create no energy loss (Linear-elastic behaviour)***
- ***The strain energy must be expressed as a function of the loads***

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Application of Catigliano’s Theorem

$$\delta_i = \frac{\partial U}{\partial P_i}$$

$$M = -Px - M_0$$



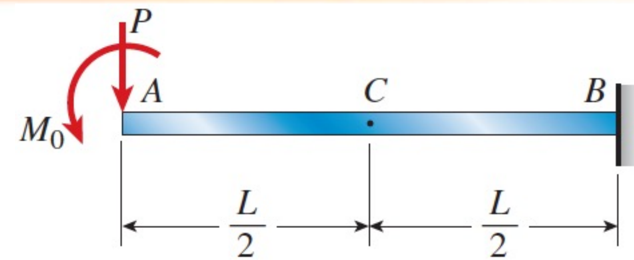
$$U = \int_0^L \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L (-Px - M_0)^2 dx = \frac{P^2L^3}{6EI} + \frac{PM_0L^2}{2EI} + \frac{M_0^2L}{2EI}$$

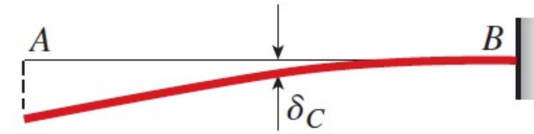
$$\delta_A = \frac{\partial U}{\partial P} = \frac{PL^3}{3EI} + \frac{M_0L^2}{2EI} \qquad \theta_A = \frac{\partial U}{\partial M_0} = \frac{PL^2}{2EI} + \frac{M_0L}{EI}$$

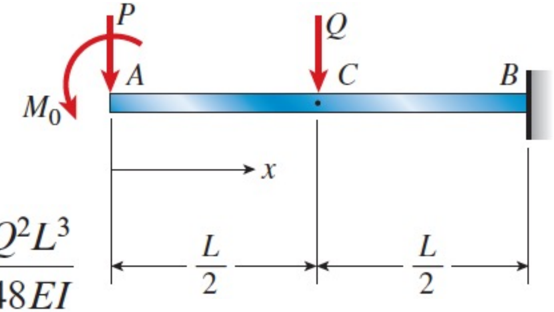
- ❖ **Specific numerical quantity: Treat external load P as a variable. Each internal M_i as a function of variable P (P=numerical quantity).**
- ❖ ***N*o load: apply a virtual load P. Each internal M_i as a function of variable P (P=0).**

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

❖ Application of a virtual (fictitious) load







$$M = -Px - M_0 \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$U_{AC} = \int_0^{L/2} \frac{M^2\,dx}{2EI} = \frac{P^2L^3}{48EI} + \frac{PM_0L^2}{8EI} + \frac{M_0^2L}{4EI}$$

$$M = -Px - M_0 - Q\left(x - \frac{L}{2}\right) \quad \left(\frac{L}{2} \le x \le L\right)$$

$$U_{CB} = \int_{L/2}^{L} \frac{M^2\,dx}{2EI} = \frac{7P^2L^3}{48EI} + \frac{3PM_0L^2}{8EI} + \frac{5PQL^3}{48EI} + \frac{M_0^2L}{4EI} + \frac{M_0Q^2}{8EI} + \frac{Q^2L^3}{48EI}$$

$$U = U_{AC} + U_{CB} = \frac{P^2L^3}{6EI} + \frac{PM_0L^2}{2EI} + \frac{5PQL^3}{48EI} + \frac{M_0^2L}{2EI} + \frac{M_0QL^2}{8EI} + \frac{Q^2L^3}{48EI}$$

$$(\delta_C)_0 = \frac{\partial U}{\partial Q} = \frac{5PL^3}{48EI} + \frac{M_0L^2}{8EI} + \frac{QL^3}{24EI} \qquad \delta_C = \frac{5PL^3}{48EI} + \frac{M_0L^2}{8EI}$$

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Application of Catigliano's Theorem

THE UNIVERSITY OF SYDNEY

- *The use of Castigliano's theorem for determining beam deflections may lead to* ***lengthy integrations****, especially when more than two loads act on the beam*
- *We can* ***bypass*** *the step of finding the U* ***by differentiating before integrating****.*

$$\delta_i = \frac{\partial U}{\partial P_i} = \frac{\partial}{\partial P_i}\int \frac{M^2 dx}{2EI} = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P_i}\right) dx$$

$$M = -Px - M_0$$

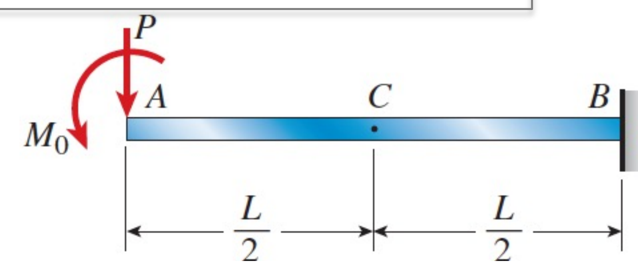
$$\frac{\partial M}{\partial P} = -x \qquad \frac{\partial M}{\partial M_0} = -1$$

$$\delta_A = \frac{1}{EI}\int_0^L (-Px - M_0)(-x)dx = \frac{PL^3}{3EI} + \frac{M_0L^2}{2EI}$$

$$\theta_A = \frac{1}{EI}\int_0^L (-Px - M_0)(-1)dx = \frac{PL^2}{2EI} + \frac{M_0L}{EI}$$

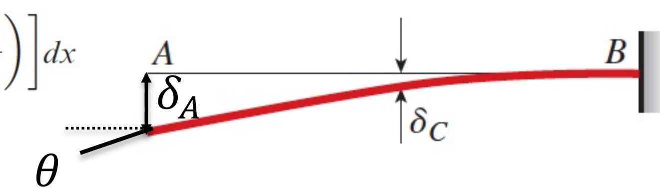
$$M = -Px - M_0 \qquad \frac{\partial M}{\partial Q} = 0 \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$M = -Px - M_0 - Q\left(x - \frac{L}{2}\right) \qquad \frac{\partial M}{\partial Q} = -\left(x - \frac{L}{2}\right) \qquad \left(\frac{L}{2} \le x \le L\right)$$



$$(\delta_C)_0 = \frac{1}{EI}\int_0^{L/2}(-Px - M_0)(0)dx + \frac{1}{EI}\int_{L/2}^{L}\left[-Px - M_0 - Q\left(x - \frac{L}{2}\right)\right]\left[-\left(x - \frac{L}{2}\right)\right]dx$$

$$\delta_C = \frac{1}{EI}\int_{L/2}^{L}[-Px - M_0]\left[-\left(x - \frac{L}{2}\right)\right]dx = \frac{5PL^3}{48EI} + \frac{M_0L^2}{8EI}$$

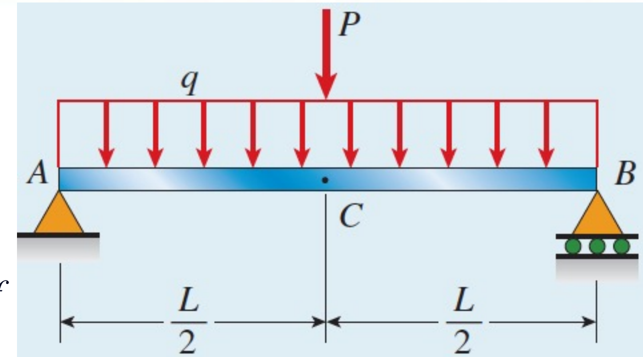


J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Example

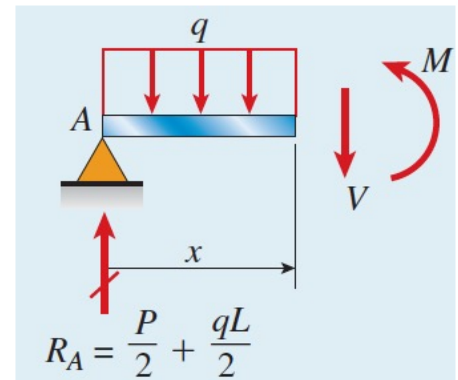
A simple beam AB supports a uniform load of intensity $q = 20$ kN/m and a concentrated load $P = 25$ kN (Fig. 9-40). The load P acts at the midpoint C of the beam. The beam has length $L = 2.5$ m, modulus of elasticity $E = 210$ GPa, and moment of inertia $I = 31.2 \times 10^2$ cm^4.

Determine the downward deflection δ_C at the midpoint of the beam by



Because the beam and its loading are symmetrical about the midpoint, the strain energy for the entire beam is equal to twice the strain energy for the left-hand half of the beam. Therefore, we need to analyze only the left-hand half of the beam.

$$R_A = \frac{P}{2} + \frac{qL}{2} \qquad M = R_A x - \frac{qx^2}{2} = \frac{Px}{2} + \frac{qLx}{2} - \frac{qx^2}{2} \qquad \frac{\partial M}{\partial P} = \frac{x}{2}$$



$$\delta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right)dx$$

$$= 2\int_0^{L/2} \frac{1}{EI}\left(\frac{Px}{2} + \frac{qLx}{2} - \frac{qx^2}{2}\right)\left(\frac{x}{2}\right)dx = \frac{PL^3}{48EI} + \frac{5qL^4}{384EI}$$

$$\delta_C = \frac{PL^3}{48EI} + \frac{5qL^4}{383EI}$$

$$= \frac{(25 \text{ kN})(2.5 \text{ m})^3}{48(210 \text{ GPa})(31.2 \times 10^{-6} \text{ m}^4)} + \frac{5(20 \text{ kN/m})(2.5 \text{ m})^4}{384(210 \text{ GPa})(31.2 \times 10^{-6} \text{ m}^4)}$$

$$= 1.24 \text{ mm} + 1.55 \text{ mm} = 2.79 \text{ mm}$$

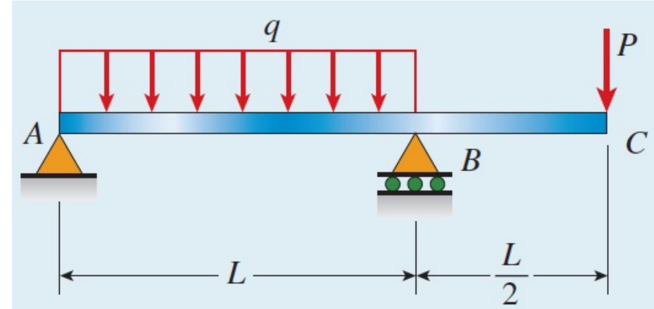
$$U = \int \frac{M^2 dx}{2EI} = 2\int_0^{L/2} \frac{1}{2EI}\left(\frac{Px}{2} + \frac{qLx}{2} - \frac{qx^2}{2}\right)^2 dx$$

Note: Numerical values cannot be substituted until *after* the partial derivative is obtained. If numerical values are substituted prematurely, either in the expression for the bending moment or the expression for the strain energy, it may be impossible to take the derivative.

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition

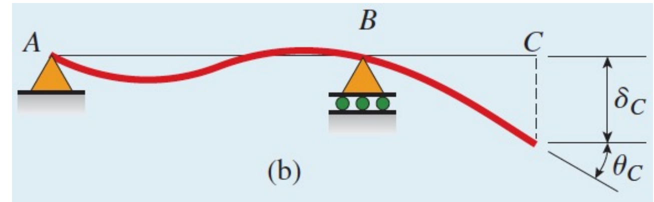
Example

A simple beam with an overhang supports a uniform load of intensity q on span AB and a concentrated load P at end C of the overhang (Fig. 9-42).
Determine the deflection δ_C and angle of rotation θ_C at point C. (Use



$$M_{AB} = R_A x_1 - \frac{qx_1^2}{2} = \frac{qLx_1}{2} - \frac{Px_1}{2} - \frac{qx_1^2}{2} \qquad (0 \le x_1 \le L)$$

where x_1 is measured from support A



$$M_{BC} = -Px_2 \quad \left(0 \le x_2 \le \frac{L}{2}\right)$$

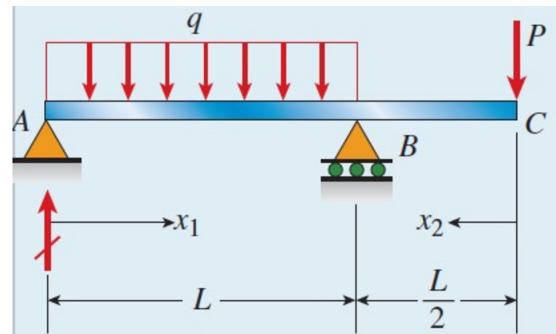
where x_2 is measured from point C

$$\frac{\partial M_{AB}}{\partial P} = -\frac{x_1}{2} \qquad (0 \le x_1 \le L)$$

$$\frac{\partial M_{BC}}{\partial P} = -x_2 \quad \left(0 \le x_2 \le \frac{L}{2}\right)$$

$$\delta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx$$

$$= \frac{1}{EI}\int_0^L M_{AB}\left(\frac{\partial M_{AB}}{\partial P}\right)dx + \frac{1}{EI}\int_0^{L/2} M_{BC}\left(\frac{\partial M_{BC}}{\partial P}\right)dx$$



$$R_A = \frac{qL}{2} - \frac{P}{2}$$

$$\delta_C = \frac{1}{EI}\int_0^L \left(\frac{qLx_1}{2} - \frac{Px_1}{2} - \frac{qx_1^2}{2}\right)\left(-\frac{x_1}{2}\right)dx_1 + \frac{1}{EI}\int_0^{L/2} (-Px_2)(-x_2)dx_2$$

$$\delta_C = \frac{PL^3}{8EI} - \frac{qL^4}{48EI}$$

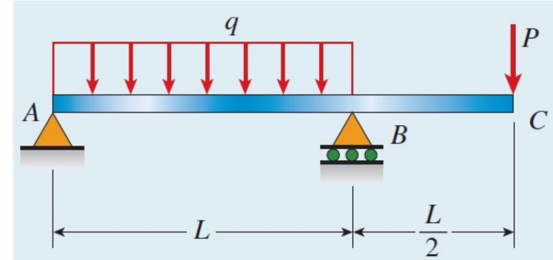
Since the load P acts downward, the deflection δ_C is also positive downward. In other words, if the preceding equation produces a positive result, the deflection is downward. If the result is negative, the deflection is upward.

The deflection at C is downward when P>qL/6, and upward when p<qL/6

J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition

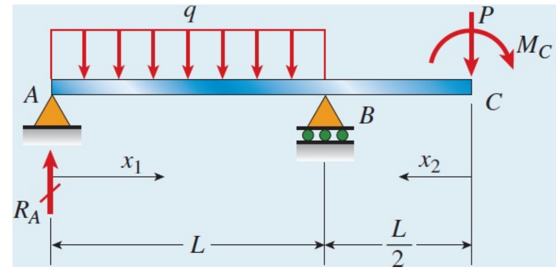
Example

A simple beam with an overhang supports a uniform load of intensity q on span AB and a concentrated load P at end C of the overhang (Fig. 9-42).
Determine the deflection δ_C and angle of rotation θ_C at point C. (Use



Angle of rotation θ_C at the end of the overhang

Since there is no load on the original beam corresponding to this θ, we must supply a fictitious load. Therefore, we place a couple of moment M_C at point C. Note that the couple M_C acts at the point on the beam where the θ is to be determined. Furthermore, it has the same clockwise direction as the θ.



$$R_A = \frac{qL}{2} - \frac{P}{2} - \frac{M_C}{L}$$

$$M_{AB} = R_A x_1 - \frac{qx_1^2}{2} = \frac{qLx_1}{2} - \frac{Px_1}{2} - \frac{M_C x_1}{L} - \frac{qx_1^2}{2} \quad (0 \le x_1 \le L)$$

$$M_{BC} = -Px_2 - M_C \quad \left(0 \le x_2 \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AB}}{\partial M_C} = -\frac{x_1}{L} \quad (0 \le x_1 \le L)$$

$$\frac{\partial M_{BC}}{\partial M_C} = -1 \quad \left(0 \le x_2 \le \frac{L}{2}\right)$$

$$\theta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_C}\right)dx$$

$$= \frac{1}{EI}\int_0^L M_{AB}\left(\frac{\partial M_{AB}}{\partial M_C}\right)dx + \frac{1}{EI}\int_0^{L/2} M_{BC}\left(\frac{\partial M_{BC}}{\partial M_C}\right)dx$$

The θ at C is clockwise when P>qL/7, and counterclockwise when p<qL/7

$$\theta_C = \frac{1}{EI}\int_0^L \left(\frac{qLx_1}{2} - \frac{Px_1}{2} - \frac{M_C x_1}{L} - \frac{qx_1^2}{2}\right)\left(-\frac{x_1}{L}\right)dx_1 + \frac{1}{EI}\int_0^{L/2} (-Px_2 - M_C)(-1)dx_2$$

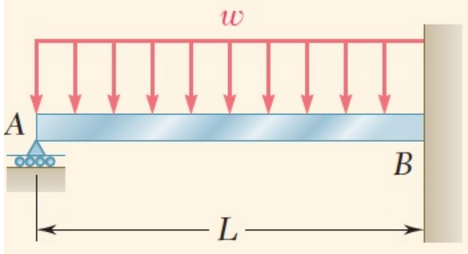
$$\theta_C = \frac{1}{EI}\int_0^L \left(\frac{qLx_1}{2} - \frac{Px_1}{2} - \frac{qx_1^2}{2}\right)\left(-\frac{x_1}{L}\right)dx_1 + \frac{1}{EI}\int_0^{L/2} (-Px_2)(-1)\,dx_2$$

$$\theta_C = \frac{7PL^2}{24EI} - \frac{qL^3}{24EI}$$

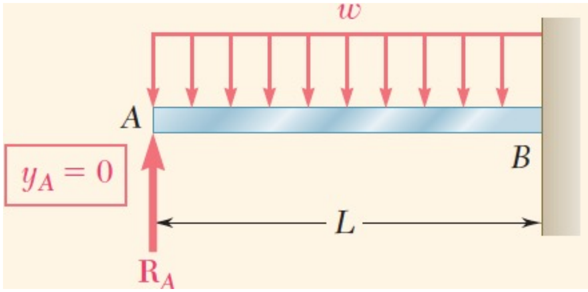
J.M. Gere and B.J. Goodno, Mechanics of Materials, 8th edition.

Example: statically indeterminate structures

Determine the reactions at the supports for the prismatic beam and load shown (Fig. 11.41*a*).



The beam is statically indeterminate to the first degree. The reaction at *A* is redundant and the beam is released from that support. The reaction $\mathbf{R}_A$ is considered to be an unknown load (Fig. 11.41*b*) and will be determined under the condition that the deflection y_A at *A* must be zero. By Castigliano's theorem, $y_A = \partial U/\partial R_A$, where *U* is the strain energy of the beam under the distributed load and the redundant reaction.



$$M = R_A x - \tfrac{1}{2} w x^2 \qquad \frac{\partial M}{\partial R_A} = x \qquad y_A = \frac{\partial U}{\partial R_A} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R_A}\,dx$$

$$y_A = \frac{1}{EI}\int_0^L \left(R_A x^2 - \frac{1}{2} w x^3\right) dx = \frac{1}{EI}\left(\frac{R_A L^3}{3} - \frac{w L^4}{8}\right)$$

Set $y_A = 0$ and solve for R_A:

$$R_A = \tfrac{3}{8} wL \qquad \mathbf{R}_A = \tfrac{3}{8} wL \uparrow$$

From the conditions of equilibrium for the beam, the reaction at *B* consists of the force and couple:

$$\mathbf{R}_B = \tfrac{5}{8} wL \uparrow \qquad \mathbf{M}_B = \tfrac{1}{8} wL^2 \circlearrowright$$

F.P. Beer et.al., Mechanics of Materials, 7th edition, McGrawHill Education.