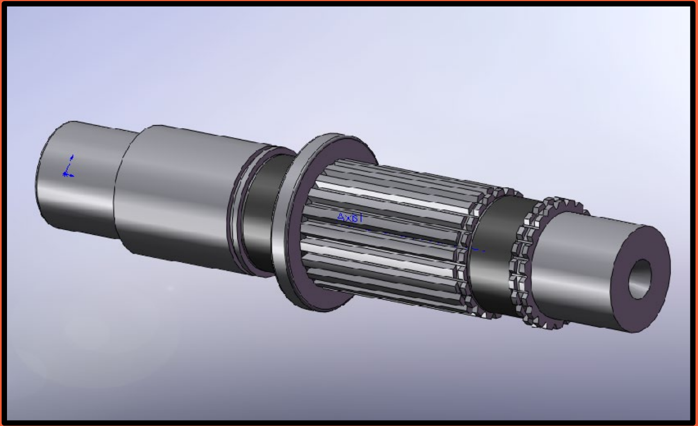
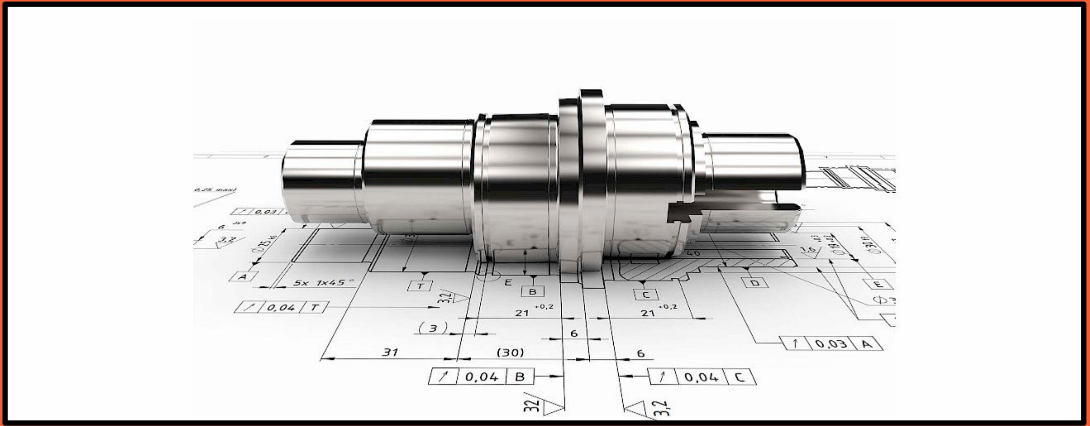
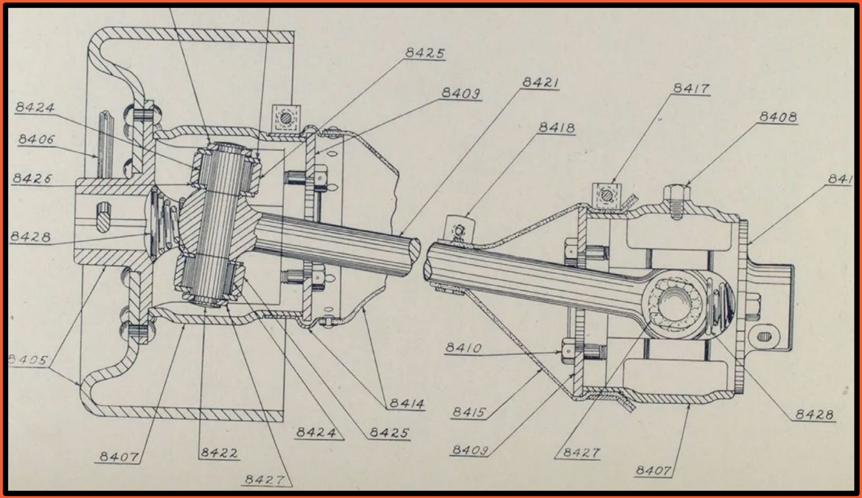
MECH3460
Shaft Design







THE UNIVERSITY OF SYDNEY

A short lecture on Design of Shafts as Part of MECH3460: Mechanical Design 2 By Paul Briozzo and Dr. Andrei Lozzi.

What is meant by Shaft Design?

Introduction

Shafts and axles have a very wide range of applications, consequently, although there are specific similarities, they can also vary widely in appearance.

Figure 1 gives an idea of what they can look like. There is a categorical difference between shafts and axles, it is that **shafts rotate about their long axis, whereas axle remain stationary and provide a pivot for other components to rotate on.** At times axles have been considered as just structural members and the methods of analysis that have been developed for shafts have not been applied to them. Yet the effective difference between typical examples of each is that axles may be just subjected to fewer load cycles than shafts.

The analysis is then the same for both, and in these notes, the name ‘shaft’ will stand for both items, unless for some particular reason the need arises to differentiate between them.

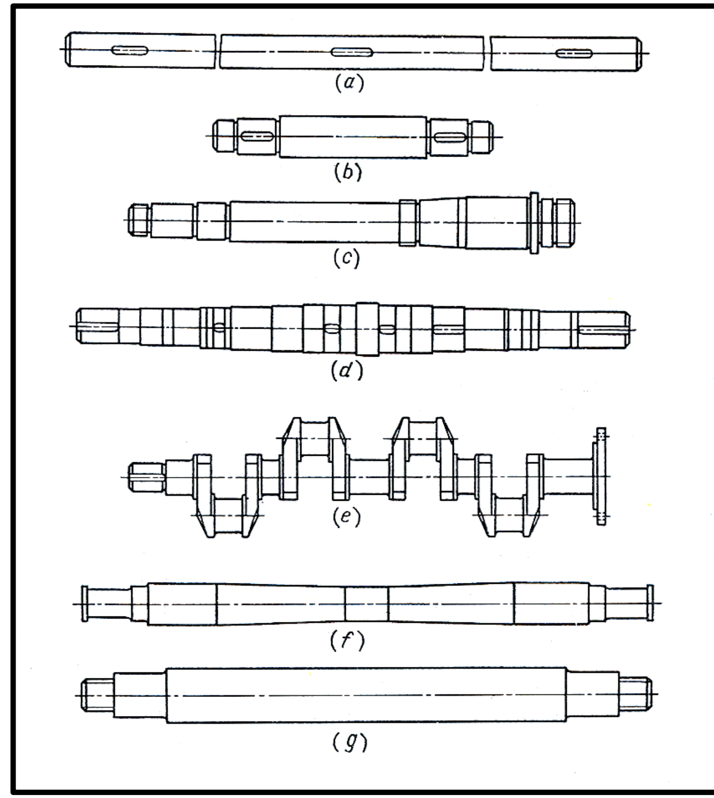


Figure 1.0

What is meant by Shaft Design? - Introduction

Figure 1a) Represents a plain constant diameter transmission shaft with or without keyways.

Figure 1b) & c) Describes machined stepped shafts for machine tools or gearboxes, with screw threads, keyways, and ground lands to possibly carry bearings or gears.

Figure 1d) & e) Represent a large steam turbine shaft and crankshaft respectively.

Figure 1f) Illustrates a rotating railway shaft, often referred incorrectly to as an 'axle'.

Figure 1g) Demonstrates a fixed axle as used on heavy trucks, trailers, and aircraft landing gear.

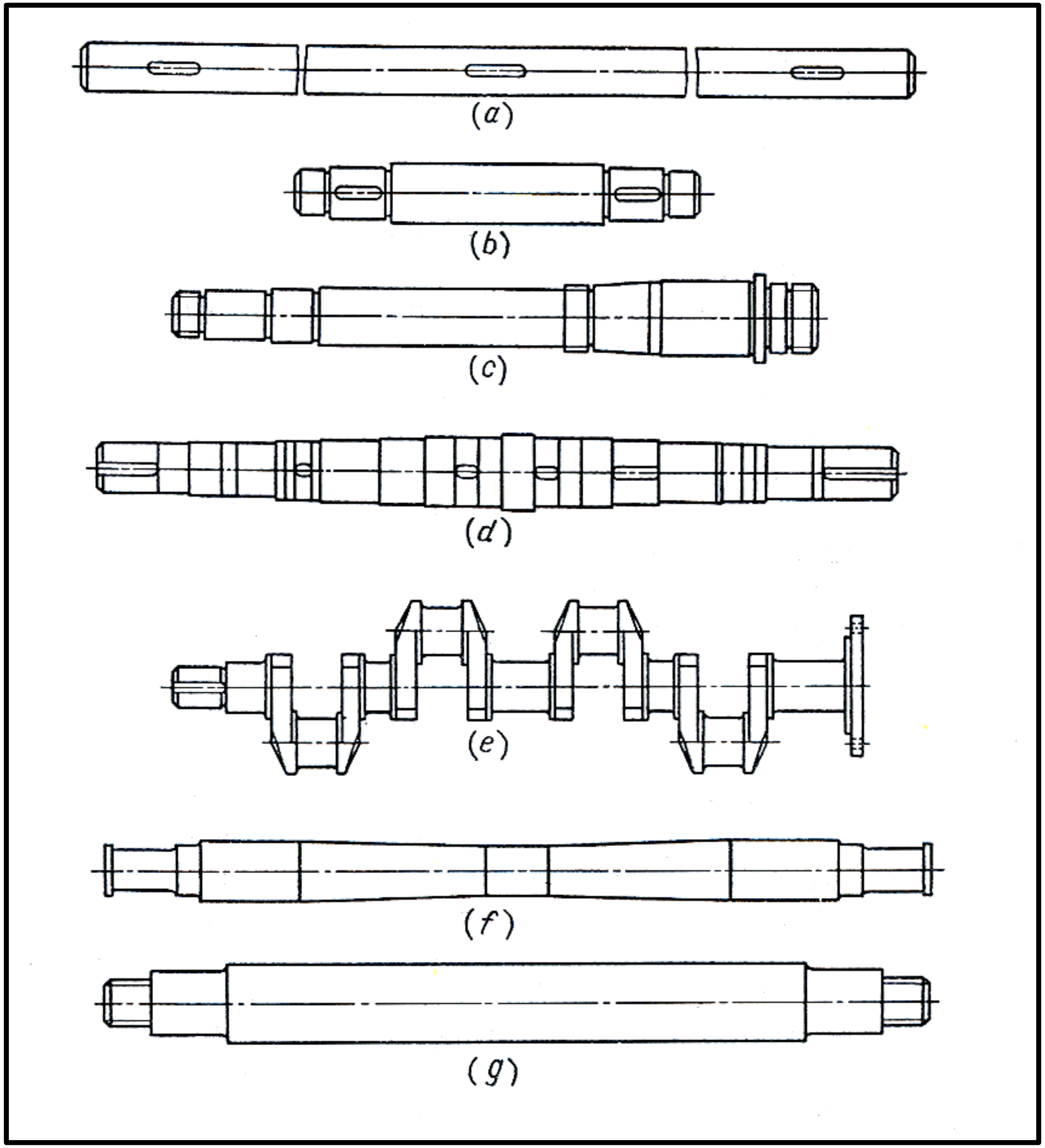
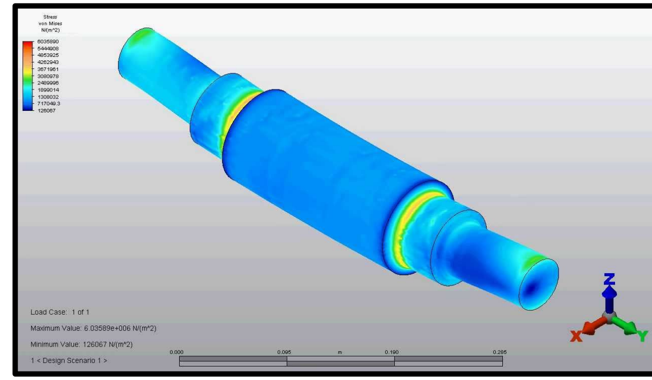


Figure 1.0 (a to g)

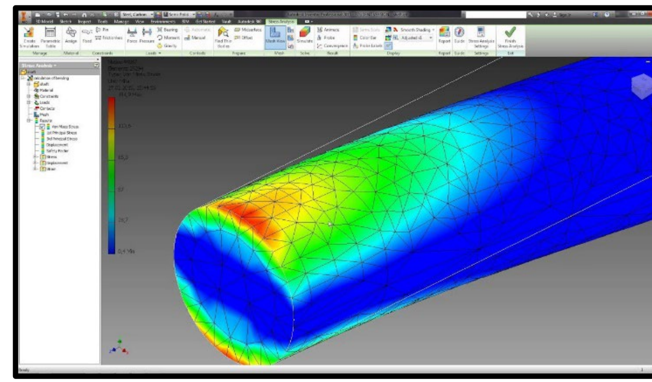
Shaft Design Analysis

There are two objectives to be kept in mind in the analysis of most shafts,



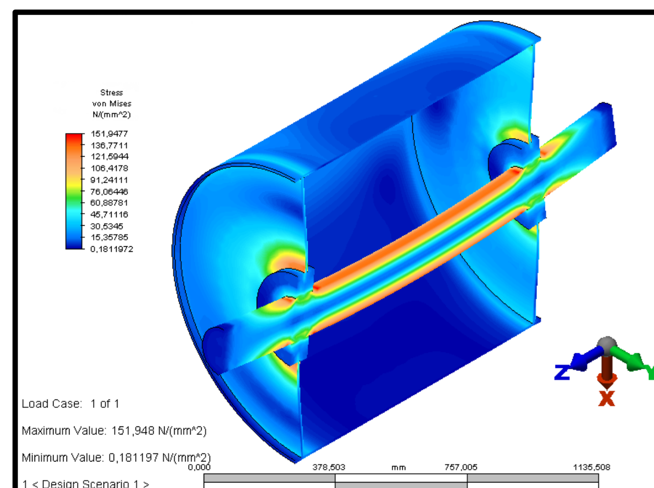
The stress levels have to be kept sufficiently low to provide the required service life, and the various deflections have to be controlled to ensure that the shaft can function as expected.

The loads that usually produce the greatest stresses and strains in shafts are moments, caused by transverse forces and reactions on the shaft, and the torque transmitted by the shaft.



Secondary loads may be longitudinal thrust forces down along the axis of a shaft and contact stresses caused possibly by interference fits.

Both the maximum allowable stresses and maximum allowable deflections are used separately and independently to determine shaft diameters.



It is common that deflection requirements dictated larger shaft diameters than the diameters required to give adequate fatigue life.

Shaft Design Analysis – Typical Problem Layout

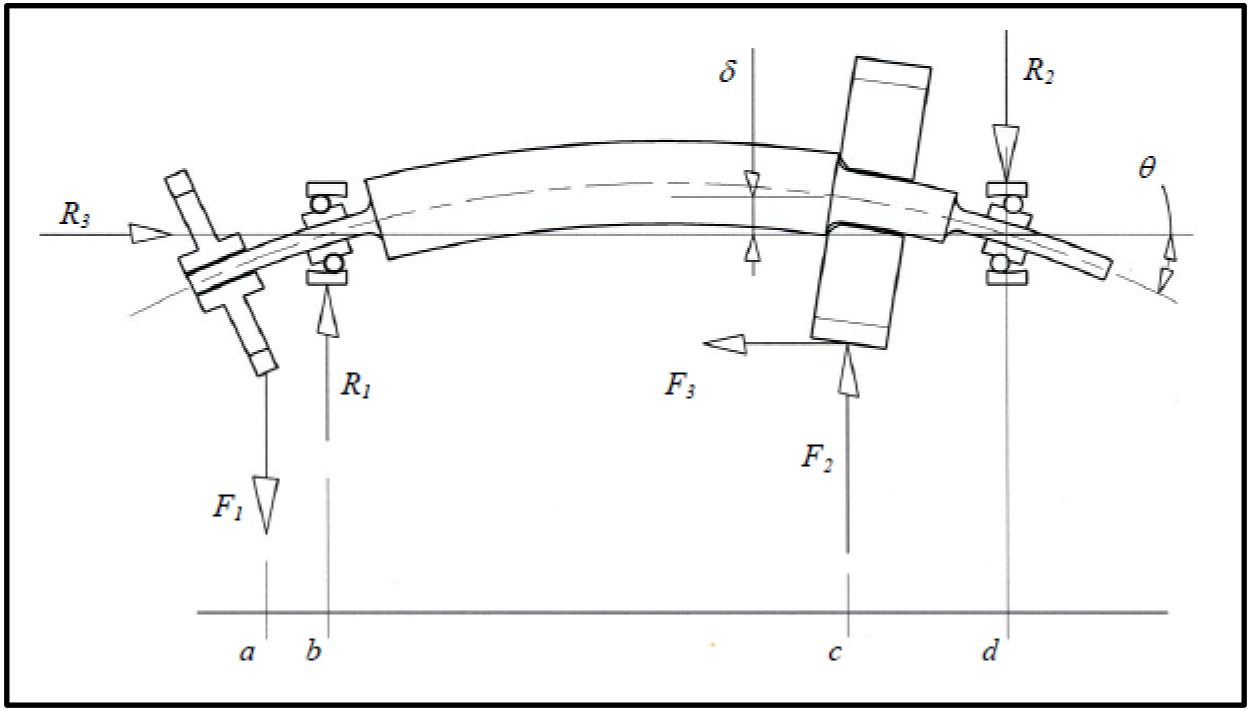


Figure 2.0

Figure 2.0 above is almost a caricature of a loaded shaft. At the station ***a*** there is a sprocket carrying the tension $\boldsymbol{F_1}$ from a chain. Bearings at ***b*** and ***d*** locate the shaft but they fix the shaft at only those stations.

The helical gear at ***c*** transmits a transverse force $\boldsymbol{F_2}$ and an axial force $\boldsymbol{F_3}$ to the shaft. The shaft is of greater diameter between bearings to reduce deflections there.

The bending moments nearly completely determine the lateral deflections δ at the gear and angular deflection θ at the bearing ***d***.

Note that single bearings should only be subjected to combinations of transverse and axial forces, **no bending moment**.

Shaft Design Analysis – Stress and Strain

Stress analysis determines shaft diameters at one location at the time. It uses moments, torques, surface finish, stress concentration, and other factors, all of which must apply to the location of interest, to determine the diameter of the shaft only at that location.

In contrast, **deflection analysis requires the overall shape and size of the shaft, as well as all the loads on it,** to be known before the deflections of the shaft can be calculated.

Hence the normal procedure is to use stress analysis to determine the minimum diameter of the shaft at critical locations, followed by some projections as to a reasonable shape everywhere else.

The deflections of the proposed shaft are then calculated, again at one location at a time. Finally, and obviously, the size of the shaft everywhere along its length has to be at least equal to **the larger of the two diameters dictated by the stress and deflection analysis**.

Shaft Design Analysis – Stress and Strain

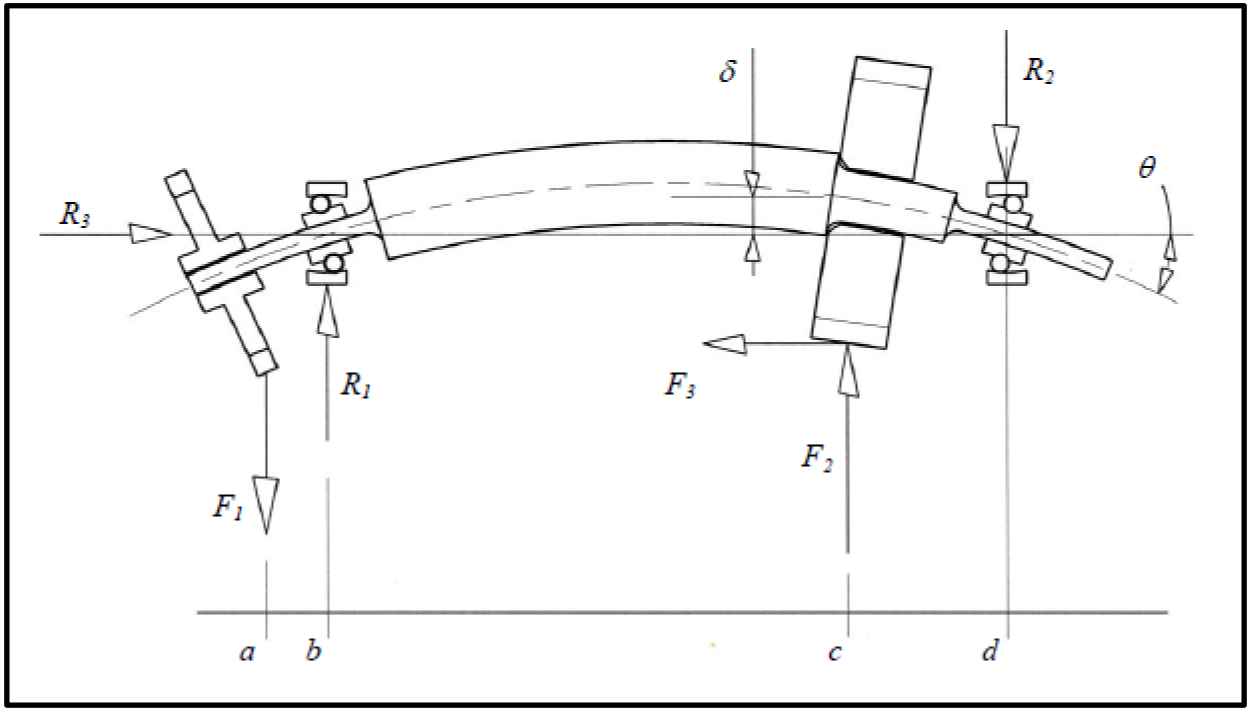


Figure 2.0

In Figure 2.0 above, **deflection analysis determined the minimum shaft diameter between *b* & *d*** to control the lateral δ and angular θ_c (shown but not labeled) deflection at the gear wheel, **c**.

Gear teeth have to make precise contact with each other or much of their capacity is wasted. **Stress analysis determined the minimum shaft diameters at the bearings**, since the extra cost of larger bearings can be significant, the shaft is shown machined down at the ends.

Angular deflection at the bearings θ has to be checked, because many bearings will tolerate very little misalignment before failing.

Shaft Design Analysis – Stress and Strain

Shown in Figure 3.0 (and 4.0 in the next slide) is a short section of a shaft where the stresses at position ***x* = *0*** are considered from a force ***F*** and a torque, ***T*** applied at ***x* = x_1**.

The force causes in part a moment about the ***Z*** axis, with its associated normal stress σ_x.

We know that σ_x varies linearly between maxima at ***y*= +/- *D*/2** and 0 at the ***Z*** axis.

The torque ***T*** causes a shear stress $\tau_{xy}(T)$ which is a maximum on the surface and points tangentially to the circumference.
We know that:

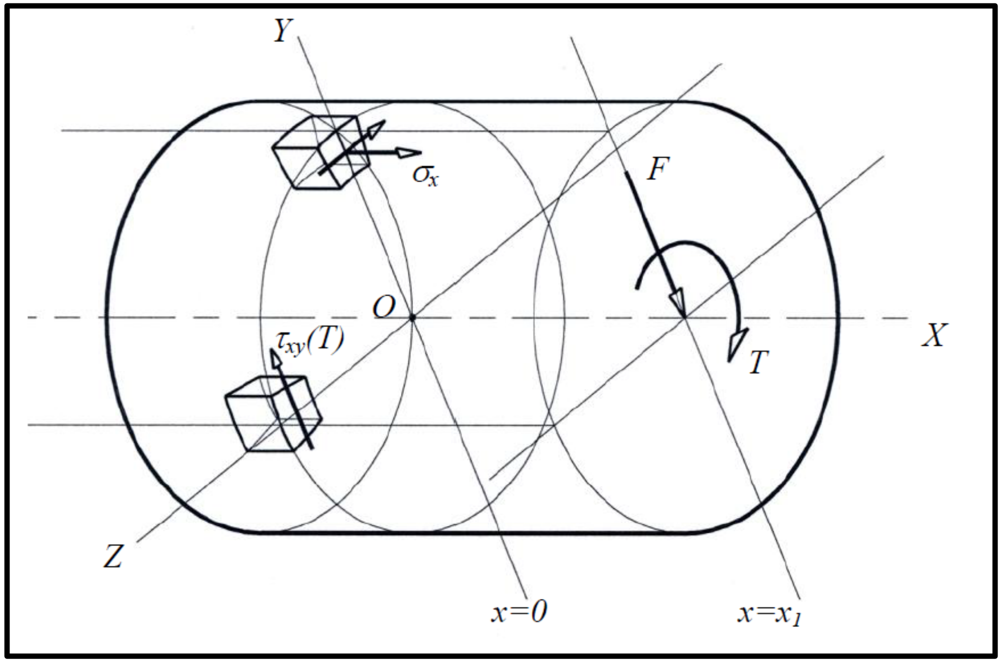


Figure 3.0

$$\sigma_x = \frac{F x_1 \, D/2}{I} \qquad \textbf{Eq 1}$$

$$\tau_{xy}(T) = \frac{T \, D/2}{J} \qquad \textbf{Eq 2}$$

Shaft Design Analysis – Stress and Strain

Figure 4.0 on the right is the same piece of the shaft as shown above, but here the transverse shear stress associated with force **F**, $\boldsymbol{\tau_{xy}}$**(F)** is shown.

This stress reaches a maximum on the neutral axis (***Z*** axis) and zero at ***y*** **=+/-** ***D*****/2**, in opposition to the variation in σ_{xy}.

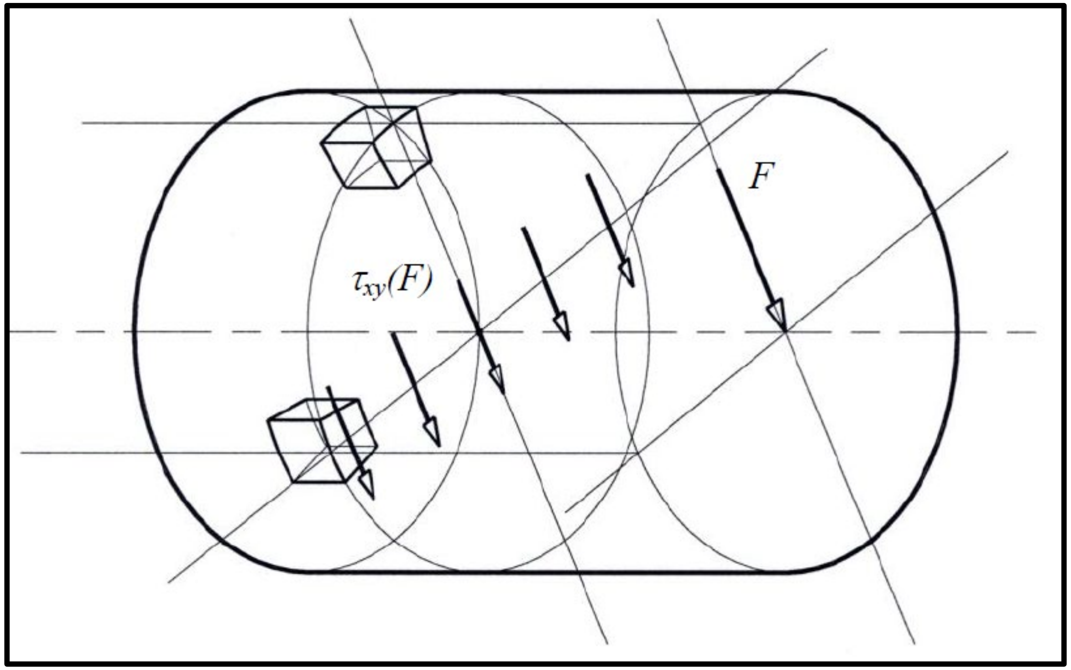


Figure 4.0

$$\sigma_x = \frac{F x_1 \, D/2}{I} \qquad \textbf{Eq 1}$$

$$\tau_{xy}(T) = \frac{T \, D/2}{J} \qquad \textbf{Eq 2}$$

Shaft Design Analysis – Transverse Shear Stress

This last stress $\tau_{xy}(F)$, is nearly always omitted from any consideration, but it is worthwhile to see why this is so, and perhaps when it may not be so. We can gauge the relative significance of this stress by comparing its magnitude to the magnitude of σ_x. The transverse shear stress is a maximum on the neutral axis. For a circular section this is given as:

$$\tau_{xy} = \frac{4F}{3A}, \qquad (\text{where } A = \pi D^2/4) \qquad \textbf{Eq 3}$$

Since the bending moment, and its associated normal stress increases with *x*, and the transverse shear stress remains constant, we can ask for what value of x_1 will the two stresses, $\tau_{xy}(F)$ and σ_x be of equal magnitude, i.e.:

$$\sigma_{xy} = \tau_{xy}(F) \qquad \textbf{Eq 4}$$

$$\frac{Fx_1 \, D/2}{\pi D^4/64} = \frac{4F}{3\pi D^2/4} \qquad \textbf{Eq5}$$

that is where: $x_1 = D/6$ i.e. x_1 is 0.17 of the shaft diameter! **Eq 6**

σ_x is 10 times larger than $\tau_{xy}(F)$ where $x_1 = 1.7\,D$, and 100 time where $x_1 = 17\,D$. Not only do transverse shear stresses go from their maximum to 0 exactly where normal stresses due to bending go from 0 to their maximum, but also these shear stresses are very small except for shafts with exceedingly short separation between loads and bearings. Proportionally transverse shear stress contribute more to total deflection than they do to total stress, but usually by less than a few % points.

Shaft Design Analysis – Stresses in Shafts

Stresses in Shafts. Given that the **highest coexisting biaxial stresses in Figure 3.0 are: σ_x and τ_{xy}*(T)*.** These stresses are shown on a Mohr's circle in Figure 5.

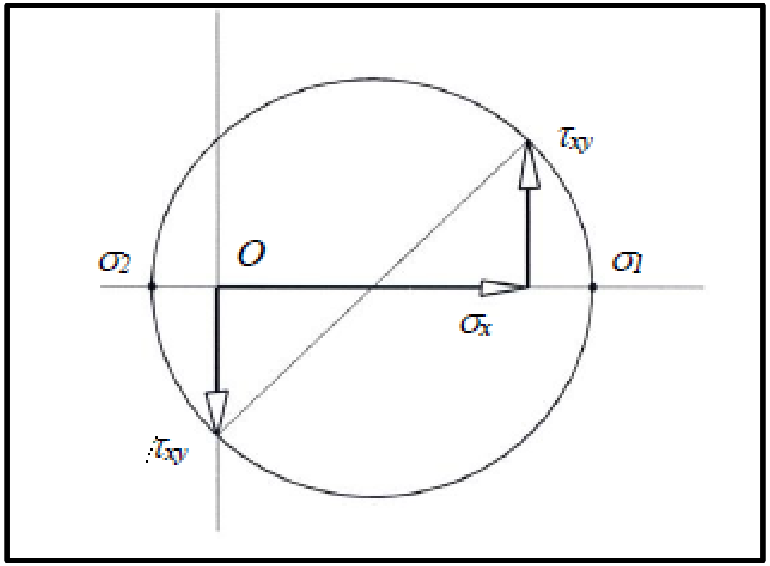


Figure 5.0

Mohr's circle of the significant stresses in a shaft caused by the shear force and torque shown on Figure 3.0

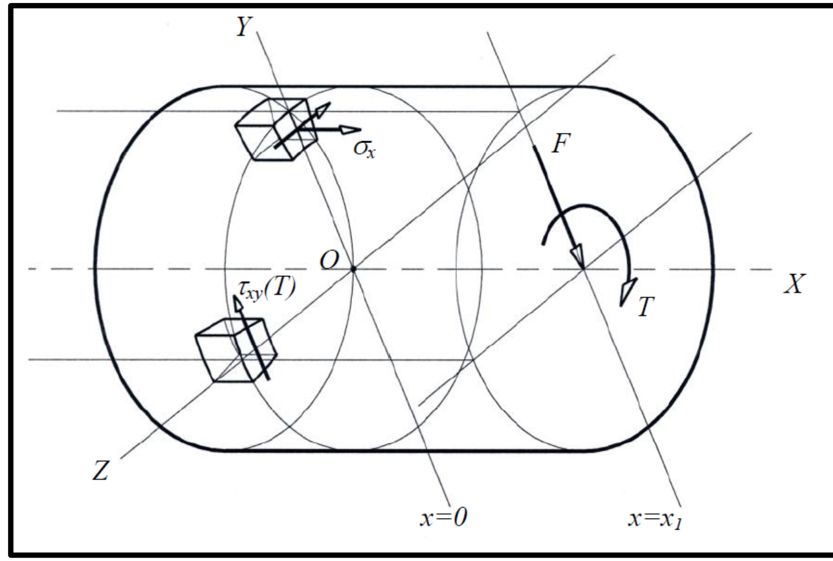


Figure 3.0

Giving the biaxial principal stresses:

$$\sigma_{1,2} = \frac{\sigma_x}{2} \pm \left(\left(\frac{\sigma_x}{2} \right)^2 + \tau_{xy}{}^2 \right)^{1/2} \quad \textbf{Eq 7}$$

Shaft Design Analysis – Stresses in Shafts

These are substituted in the Von Mises biaxial expression:

$$\sigma'^2 = \sigma_1^{\ 2} - \sigma_1\sigma_2 + \sigma_2^{\ 2} \qquad \textbf{Eq 8}$$

giving:

$$\sigma'^2 = \sigma_x^{\ 2} + 3\tau_{xy}^{\ 2} \qquad \textbf{Eq 9}$$

Since the force and torque may in general be made up of steady and alternating components, ***(F_m, F_a)*** and ***(T_m, T_a)***, we can calculate the mean and alternating distortion energy stresses starting with Equations 1 & 2 and get:

$$\sigma_x = \frac{Fx_1 \, D/2}{I} \qquad \textbf{Eq 1}$$

$$\tau_{xy}(T) = \frac{T \, D/2}{J} \qquad \textbf{Eq 2}$$

$$\sigma_m'^{\,2} = \sigma_{xm}^{\ 2} + 3\tau_{xym}^{\ 2} \qquad \textbf{Eq 10}$$

$$\sigma_a'^{\,2} = \sigma_{xa}^{\ 2} + 3\tau_{xya}^{\ 2} \qquad \textbf{Eq 11}$$

If there are mean and alternating axial forces on the shaft, the stresses resulting from those forces can be added to the normal stresses in Equation 10 & 11.

Shaft Design Analysis – The Goodman Option

Substituting equation 1 & 2, with the appropriate expansions for ***I*** & ***J*** and using the variable ***M* = *Fx*₁** for the moment, instead of ***F*** & ***x*₁** separately, into Equation 10 & 11. Then, substituting these expanded von Mises stresses into the Goodman equation:

$$\sigma_x = \frac{Fx_1 \, D/2}{I} \qquad \textbf{Eq 1}$$

$$\tau_{xy}(T) = \frac{T \, D/2}{J} \qquad \textbf{Eq 2}$$

$$\frac{1}{FS} = \frac{\sigma_a'}{S_f} + \frac{\sigma_m'}{S_u} \qquad \textbf{Eq 12}$$

Rearranging the resulting expanded Goodman relation, to give diameter ***D*** for a required factor of safety FS, after some time gives:

$$D = \left(\frac{32 \cdot FS}{\pi} \left(\frac{\left((k_f M_a)^2 + \frac{3}{4}(k_{fs} T_a)^2 \right)^{1/2}}{S_f} + \frac{\left((k_{fm} M_m)^2 + \frac{3}{4}(k_{fsm} T_m)^2 \right)^{1/2}}{S_{ut}} \right) \right)^{1/3} \qquad \textbf{Eq 13}$$

Where: $M_{a/m}$ - moment, alternating or mean
$T_{a,m}$ - torque, alternating or mean
$k_{f,fm}$ - stress concentration factor for alternating or mean moment
$k_{fs,sfm}$ - stress concentration factor for alternating or mean torque
S_f, S_u - fatigue and ultimate tensile strength

Shaft Design Analysis – The Goodman Option

If the average stress, or nominal stress, is below yield condition, then a stress concentration factor applies to the mean stress (see Norton p385)

$$\text{if: } \sigma_{av}\, k_f < S_y, \text{ then: } k_{fm} = k_f \qquad \textbf{Eq 14}$$

This is done on the grounds that in the stress concentration zone the stress cannot exceed the yield stress. Thus, a stress concentration factor can only apply if yielding does not occur. If yielding occurs in the whole load cycle:

$$k_{fm} = 1 \qquad \textbf{Eq 15}$$

The advantage of the Goodman condition is that substituting equations 1, 2 , 8, 9, 10, 11 into the linear equation 12, gives an expression that can be rearranged to give shaft diameters directly, i.e. Equation 13.

$$\boxed{D = \left(\frac{32 \cdot FS}{\pi} \left(\frac{\left((k_f M_a)^2 + \frac{3}{4}(k_{fs} T_a)^2 \right)^{1/2}}{S_f} + \frac{\left((k_{fm} M_m)^2 + \frac{3}{4}(k_{fsm} T_m)^2 \right)^{1/2}}{S_{ut}} \right) \right)^{1/3} \qquad \textbf{Eq 13}}$$

Shaft Design Analysis – The Gerber Option

The principal drawback to the use of the Goodman line is that it does not follow the experimental results well.

For fatigue failures under combined mean and alternating stresses. Some experimental data is shown in Figure 6.0. Note that this straight line is not a good fit to the distortion energy stresses at failure, i.e., the curved line above it.

The Goodman line understates the available strength by a variable amount, from about 20% to 50%. This additional mass may make the shaft (and additionally large ancillary equipment) uncompetitive in an industrial commercial arena.

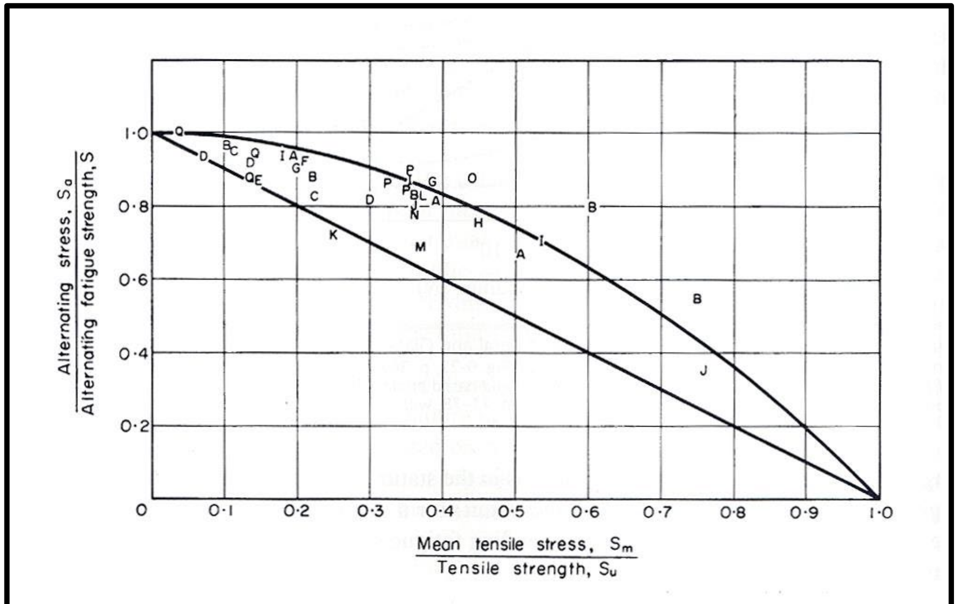


Figure 6.0

Experimental data for a steel specimen showing the Gerber parabola and the Goodman line, for tests between 10^6 to 10^8 cycles. The Gerber curve and the ASME ellipse both represent the average values reasonably well.

Shaft Design Analysis – The Gerber Option

An impediment to the use of the Gerber equation is that substituting equations 1, 2 , 8, 9, 10, 11 into the Gerber parabola Eq 16, raises the expressions of the right-hand sides of Eq 10 & 11 to the 2nd and 4th powers before the square root sign is lost. Extracting the shaft diameter as the dependent variable becomes a little messy but it can look like this:

$$\frac{FS\sigma_a'}{S_e} + \left(FS\frac{\sigma_m'}{S_u}\right)^2 = 1 \qquad \textbf{Eq 16}$$

letting: $a = \left((k_f M_a)^2 + \frac{3}{4}(k_{fs} T_a)^2\right) / S_e^2$ and $b = \left(M_m^2 + \frac{3}{4} T_m^2\right) / S_u^2$ **Eq 17a, b**

we get:

$$\boxed{D^6 = \frac{2(32FS/\pi)^2 b^2}{2b + a - (4ab + a^2)^{1/2}}} \qquad \textbf{Eq 18}$$

We can also use the numerical solver to determine the diameter that for the applied mean and alternating, moments and torques, gives a point that meets the condition defined by Equation 16. What is required are the equations mentioned above beginning with Equations 1 & 2, showing their relationship to functions 8, 9, **10, 11 and finally 16.**

Shaft Design Analysis – Shaft Material

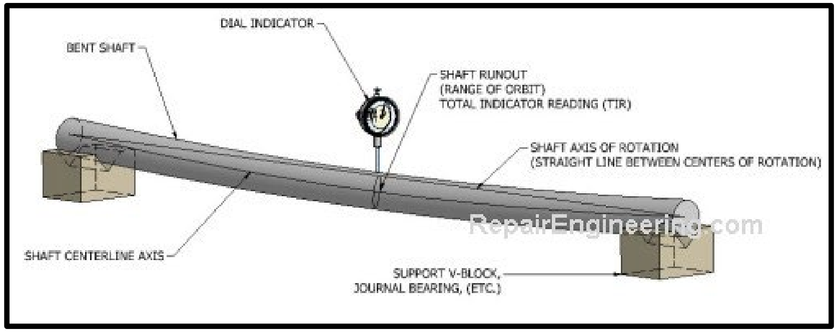
In the proportions that we usually make shafts, the requirements that they should run **adequately true (ie straight) under load, results in thicker and stiffer shafts over much of their lengths, than the need for an adequate fatigue life**.

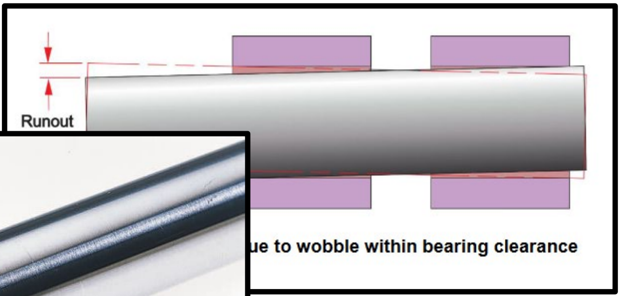
As a consequence, shafts are made from material of relatively high modulus of elasticity, that is steel. Since this modulus does not change much with alloying elements, **high quality low to medium carbon steel is preferred**.

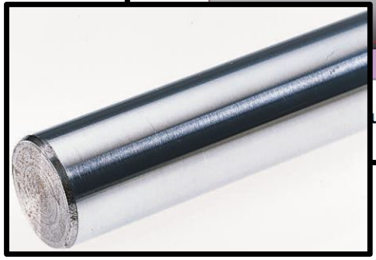
Stainless steel is employed where chemical action is significant.

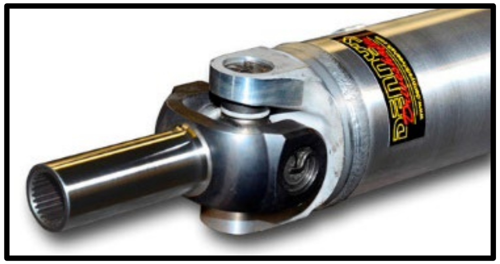
Cast iron is used where the shafts may be small or complex (crankshafts) and production costs may be significant.

Copper and Aluminum alloys are employed in rare circumstances where those materials have particular advantages that outweigh their low modulus.









Shaft Design Analysis – Shaft Material

Small mass-produced shafts are made from cold rolled stock, that comes with a smooth bright finish. These shafts are prone to warping or distorting when their surface is machined, as residual stresses are partly relieved, but this stock makes for cheap and rapid production.

Hot rolled or hot forged billets are more commonly used for larger industrial machines.

For precision, but usually limited production work, ground shaft stock is available.
This may be obtained in a range of standard diameters and in considerable lengths.
It can be in a normalized state, ready for extensive machining, or in a high-temper state, if no more than limited grinding is to be done.

Shaft Design Analysis – Deflection Calculations

Beam deflection formulae due to bending, which may be found in numerous texts on properties of solids, may be used to calculate the deflections of circular shafts.

These formulae deal with point and distributed loads, and a range of constraints at their reactions.

If a shaft has multiple loads at any location then the relevant formulae may be used to arrive at the deflections at that location, one load at the time. The final deflection is then the sum of the individual ones.

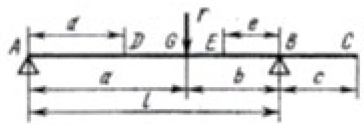
If the loads are not coplanar it will be necessary to find the vector sum of the deflections all points of interest along the shaft. Of course all this may be done provided that the elastic limit is not exceeded.

These formulae typically use Macaulay functions or brackets to describe the discontinuous loads and reactions very effectively.

But there is a significant limitation in their use as they are they apply to shafts of constant cross sections.

That is, if a shaft has significant steps the application of this method requires some careful considerations.

Formulas for Determining Elastic Angles of Inclination and Deflections of Shafts on Two Supports

Angles of inclination and deflections		
θ_A	$\frac{Fab\,(l+b)}{6EIl}$	$-\frac{F_1cl}{6EI}$
θ_B	$-\frac{Fab\,(l+a)}{6EIl}$	$\frac{F_1cl}{3EI}$
θ_C	θ_B	$\frac{F_1c\,(2l+3c)}{6EI}$
θ_D	$\frac{Fb\,(l^2-b^2-3d^2)}{6EIl}$	$\frac{F_1c\,(3d^2-l^2)}{6EIl}$
θ_E	$-\frac{Fa\,(l^2-a^2-3e^2)}{6EIl}$	—
θ_G	$\frac{Fab\,(b-a)}{3EIl}$	—
y_D	$\frac{Fbd\,(l^2-b^2-d^2)}{6EIl}$	$-\frac{F_1cd\,(l^2-d^2)}{6EIl}$
y_E	$\frac{Fae\,(l^2-a^2-e^2)}{6EIl}$	—
y_G	$\frac{Fa^2b^2}{3EIl}$	—
y_C	$\theta_B c$	$\frac{F_1c^2\,(l+c)}{3EI}$