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Lista 6 - GA

1.

a) $\vec{u} = (1,1,1)_E$

$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{(1,1,1)\cdot(1,1,1)}$

$= \sqrt{1^2+1^2+1^2} = \sqrt{3}$

b) $\vec{u} = 3\vec{i} + 4\vec{K}$

$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{(3,0,4)\cdot(3,0,4)}$

$= \sqrt{3^2+0^2+4^2} = \sqrt{25} = \pm 5$

c) $\vec{u} = -\vec{i} + \vec{j}$

$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{(-1,1,0)\cdot(-1,1,0)}$

$= \sqrt{(-1)^2+1^2+0^2} = \sqrt{2}$

d) $\vec{u} = 4\vec{i} + 3\vec{j} - \vec{K}$

$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{(4,3,-1)\cdot(4,3,-1)}$

$= \sqrt{4^2+3^2+(-1)^2} = \sqrt{16+9+1} = \sqrt{26}$

2.

a) Em $E = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ temos:

$$\begin{cases} \|\vec{e}_1\|^2 = \|\vec{e}_2\|^2 = \|\vec{e}_3\|^2 = 1 \\ \vec{e}_1\cdot\vec{e}_2 = \vec{e}_2\cdot\vec{e}_3 = \vec{e}_1\cdot\vec{e}_3 = 0, \end{cases}$$

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Logo, os vetores são unitários e ortogonais dois a dois, caracterizando E como base ortonormal.

b)

$\vec{u} = \vec{CD} + \vec{CB} = -\vec{e_2} + \vec{e_3}$

$\vec{v} = \vec{DC} + \vec{CB} = \vec{e_2} + \vec{e_3}$

$\vec{m} = \vec{GC} = -\vec{e_1}$

c) Em $F = (\vec{f_1}, \vec{f_2}, \vec{f_3})$ temos:

$$\begin{cases} \|\vec{f_1}\|^2 = \|\vec{f_2}\|^2 = \|\vec{f_3}\|^2 = 1 \\ \vec{f_1} \cdot \vec{f_2} = \vec{f_2} \cdot \vec{f_3} = \vec{f_1} \cdot \vec{f_3} = 0, \end{cases}$$

$$\|\vec{f_1}\|^2 = \|\vec{f_1}\| \, \|\vec{f_1}\| = \left\| \frac{\vec{u}}{\|\vec{u}\|} \right\| \left\| \frac{\vec{u}}{\|\vec{u}\|} \right\| =$$

$$\frac{\|\vec{u}\|}{\|\vec{u}\|} \cdot \frac{\|\vec{u}\|}{\|\vec{u}\|} = 1$$

$$\|\vec{f_2}\|^2 = \|\vec{f_2}\| \, \|\vec{f_2}\| = \left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| \left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| =$$

$$\frac{\|\vec{v}\|}{\|\vec{v}\|} \cdot \frac{\|\vec{v}\|}{\|\vec{v}\|} = 1$$

$$\|\vec{f_3}\|^2 = \|\vec{f_3}\| \, \|\vec{f_3}\| = \|\vec{m}\| \, \|\vec{m}\| = 1$$

$\vec{f_1} \cdot \vec{f_2} = 0 \iff \vec{f_1} \perp \vec{f_2}$ (ortogonalidade)

$\vec{f_1} \cdot \vec{f_3} = 0 \iff \vec{f_1} \perp \vec{f_3}$ (ortogonalidade)

FORON: $\vec{f_2} \cdot \vec{f_3} = 0 \iff \vec{f_2} \perp \vec{f_3}$ (ortogonalidade)

$$\begin{cases} \vec u = (0,-1,1)\vec e \\ \vec v = (0,1,1)\vec e \\ \vec w = (-1,0,0)\vec e \end{cases}$$

$$\vec f_1 = \frac{(0,-1,1)}{\sqrt{1+1}} = \left(0, -\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)\vec e$$

$$\vec f_2 = \frac{(0,1,1)}{\sqrt{1+1}} = \left(0, \frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)\vec e$$

$$\vec f_3 = \cancel{\frac{(-1,0,0)}{\sqrt{1+1}} = \left(-\frac{1}{\sqrt2}, 0, 0\right)\vec e} = \vec w = (-1,0,0)\vec e$$

d)

Matriz de F para E:

$$\begin{cases} \vec f_1 = -\frac{1}{\sqrt2}\vec e_2 + \frac{1}{\sqrt2}\vec e_3 \\ \vec f_2 = \frac{1}{\sqrt2}\vec e_2 + \frac{1}{\sqrt2}\vec e_3 \\ \vec f_3 = -1\,\vec e_1 \end{cases} \iff$$

$$\begin{pmatrix} \vec f_1 \\ \vec f_2 \\ \vec f_3 \end{pmatrix}^T = \left[\begin{pmatrix} 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \vec e_1 \\ \vec e_2 \\ \vec e_3 \end{pmatrix}\right]^T$$

$$\begin{pmatrix} \vec f_1 & \vec f_2 & \vec f_3 \end{pmatrix} = \begin{pmatrix} \vec e_1 & \vec e_2 & \vec e_3 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 0 & -1 \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \end{pmatrix}$$

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E F

Matriz de ~~$\vec{}$~~ para ~~$\vec{}$~~ :

$\{ \vec{e}_1 =$ É simplesmente a inversa de M:

$$[M \mid I_3] \rightarrow [I_3 \mid M^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 1 \end{array}\right] \quad l_1 \leftrightarrow l_3$$

$$\left[\begin{array}{ccc|ccc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} l_1 \leftarrow l_1 / \frac{1}{\sqrt{2}} \\ l_2 \leftarrow l_2 / + \frac{1}{\sqrt{2}} \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & \sqrt{2} \\ -1 & 1 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} l_1 \leftarrow l_1 - l_2 \\ l_2 \leftarrow l_2 + l_1 \\ l_3 \leftarrow + l_3 \cdot (-1) \end{array}$$

$$\left[\begin{array}{cccccc} 2 & 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} \\ 0 & 2 & 0 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} l_1 \leftarrow l_1 / 2 \\ l_2 \leftarrow l_2 / 2 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right] \quad \text{Logo:}$$

$$M^{-1} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -1 & 0 & 0 \end{pmatrix}$$

l)

FORON:

3

a)

$\vec{AB} = (5-2, 1-4, -3-3) = (3,-3,-6)$

$\vec{BC} = (0-5, -3-1, 1-(-3)) = (-5,-4,4)$

$\vec{CA} = (2-0, 4-(-3), 3-1) = (2,7,2)$

b)

$\|\vec{AB}\| = \sqrt{3^2+(-3)^2+(-6)^2} = \sqrt{9+9+36} = \sqrt{54}$

$\|\vec{BC}\| = \sqrt{(-5)^2+(-4)^2+4^2} = \sqrt{25+16+16} = \sqrt{57}$

$\|\vec{CA}\| = \sqrt{2^2+7^2+2^2} = \sqrt{4+49+4} = \sqrt{57}$

O triângulo isósceles possui dois lados com o mesmo comprimento, como é o caso com $\|BC\|$ sendo igual a $\|CA\|$.

c)

Ponto médio de $AB = \left(\frac{2+5}{2}, \frac{4+1}{2}, \frac{3+(-3)}{2}\right) = \left(\frac{7}{2}, \frac{5}{2}, 0\right)$

Ponto médio de $BC = \left(\frac{5+0}{2}, \frac{1+(-3)}{2}, \frac{-3+1}{2}\right) = \left(\frac{5}{2}, -1, -1\right)$

Ponto médio de $CA = \left(\frac{2+0}{2}, \frac{4+(-3)}{2}, \frac{3+1}{2}\right) = \left(1, \frac{1}{2}, 2\right)$

A mediatriz de $\vec{AB}$ é perpendicular a $\vec{AB}$:

$$(3,-3,-6) \cdot (0,0,1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -3 & -6 \\ 0 & 0 & 1 \end{vmatrix} = (3,-3,3)$$

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$x = \frac{7}{2} + 3t$

$y = \frac{5}{2} - 3t$

$z = 3t$

Para verificar se o ponto médio de AB pertence à mediatriz:

$\frac{7}{2} = \frac{7}{2} + 3t$

$\frac{5}{2} = \frac{5}{2} - 3t$

$0 = 0 + 3t$

Se observa que essas equações são verdadeiras para qualquer valor de t, significando que o ponto médio de AB pertence à mediatriz desse lado.

d)

$\vec{AB} + \vec{BC} + \vec{CA} = (3, -3, -6) + (-5, -4, 4) + (2, 7, 2)$

$= (3 - 5 + 2, -3 - 4 + 7, -6 + 4 + 2) = (0, 0, 0)$

Essa soma resulta em zero porque os vetores representam os lados de um triângulo, os quais somados em sequência voltam ao ponto de partida, sendo um vetor nulo.

4.

a)

① Prova da Desigualdade de Schwarz:

$\vec{u} = (u_1, u_2, u_3)$ $\vec{v} = (v_1, v_2, v_3)$

FORON: $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

(/ /)

$\| \vec{u} \| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ $\| \vec{v} \| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

$| \vec{u} \cdot \vec{v} | \leq \| \vec{u} \| \| \vec{v} \|$

$u_1 v_1 + u_2 v_2 + u_3 v_3 \leq \sqrt{u_1^2 + u_2^2 + u_3^2} \cdot \sqrt{v_1^2 + v_2^2 + v_3^2}$

② Prova da igualdade:

A igualdade ocorre somente se os vetores são proporcionais. Logo $\vec{u}$ é paralelo a $\vec{v}$.

$| \vec{u} \cdot \vec{v} | = \| \vec{u} \| \| \vec{v} \|$

b)

$\| \vec{u} + \vec{v} \| \leq \| \vec{u} \| + \| \vec{v} \|$

$\| \vec{u} + \vec{v} \|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + 2(\vec{u} \cdot \vec{v}) + \vec{v} \cdot \vec{v}$

Usando a Desigualdade de Schwarz:

$\vec{u} \cdot \vec{v} \leq \| \vec{u} \| \| \vec{v} \|$

$2(\vec{u} \cdot \vec{v}) \leq 2 \| \vec{u} \| \| \vec{v} \|$

Substituindo em $\| \vec{u} + \vec{v} \|^2$:

$\| \vec{u} + \vec{v} \|^2 \leq \| \vec{u} \|^2 + 2 \| \vec{u} \| \| \vec{v} \| + \| \vec{v} \|^2$

Aplicamos a raiz quadrada em ambos os lados:

FORON

$$\|\vec{u}+\vec{v}\| \leq \sqrt{\|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2}$$

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Provando assim a Desigualdade Triangular usando a Desigualdade de Schwarz.

c)

$$\|u+v\|^2 = (u+v)\cdot(u+v)$$
$$= u\cdot u + 2u\cdot v + v\cdot v$$
$$= \|u\|^2 + 2u\cdot v + \|v\|^2$$

$$\|u-v\|^2 = (u-v)\cdot(u-v)$$
$$= u\cdot v - 2u\cdot v + v\cdot v$$
$$= \|u\|^2 - 2u\cdot v + \|v\|^2$$

Subtraindo os dois resultados:

$$(\|u\|^2 + 2u\cdot v + \|v\|^2) - (\|u\|^2 - 2u\cdot v + \|v\|^2)$$
$$= \|\cancel{u}\|^2 + 2u\cdot v + \|\cancel{v}\|^2 - \|\cancel{u}\|^2 + 2u\cdot v - \|\cancel{v}\|^2$$
$$= 4u\cdot v$$

Logo: $4u\cdot v = \|u+v\|^2 - \|u-v\|^2$

5.

a) $\cos(\theta) = \dfrac{u\cdot v}{\|u\|\cdot\|v\|}$

$u\ v = (1\cdot(-2)) + (0\cdot 10) + (1\cdot 2) = -2+0+2=0$

FORON: $\|u\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$

$$\|v\| = \sqrt{(-2)^2 + 10^2 + 2^2} = \sqrt{4+100+4} = \sqrt{108} = 6\sqrt{3}$$

$$\cos(\theta) = \frac{0}{\sqrt{2}\cdot 6\sqrt{3}} = \frac{0}{6\sqrt{6}} = 0$$

Logo: $\theta = \frac{\pi}{2}$

b)

$$u\cdot v = (-1\cdot 1) + (1\cdot 1) + (1\cdot 1) = -1+1+1 = 1$$

$$\|u\| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$\|v\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$\cos(\theta) = \frac{1}{\sqrt{3}\cdot\sqrt{3}} = \frac{1}{3}$$

$\theta = \arccos\left(\frac{1}{3}\right) = 1,23$ radianos

c)

$$u\cdot v = (3\cdot 2) + (3\cdot 1) + (0\cdot -2)$$
$$= 6 + 3 + 0 = 9$$

$$\|u\| = \sqrt{3^2 + 3^2 + 0^2} = \sqrt{18} = 3\sqrt{2}$$

$$\|v\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\cos(\theta) = \frac{9}{3\sqrt{2}\cdot 3} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\theta = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

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d)

$$u \cdot v = (\sqrt{3} \cdot \sqrt{3}) + (1 \cdot 1) + (0 \cdot 2\sqrt{3})$$
$$= 3 + 1 + 0 = 4$$

$$\|u\| = \sqrt{(\sqrt{3})^2 + 1^2 + 0^2} = \sqrt{3 + 1 + 0} = \sqrt{4} = 2$$

$$\|v\| = \sqrt{(\sqrt{3})^2 + 1^2 + (2\sqrt{3})^2} = \sqrt{3 + 1 + 12} = \sqrt{16} = 4$$

$$\cos(\theta) = \frac{4}{2 \cdot 4} = \frac{1}{2}$$

Logo: $\theta = \frac{\pi}{3}$

6.

a)

$$u \cdot v = (x + 1)(x - 1) + 1 \cdot (-1) + 2 \cdot (-2) = 0$$

$(x^2 - 1) - 1 - 4 = 0 \qquad x^2 - 6 = 0$

$x^2 = 6 \qquad x = \pm\sqrt{6}$

Logo, para que u e v sejam ortogonais:

$x = \sqrt{6}$ ou $x = -\sqrt{6}$

b)

$$u \cdot v = (x \cdot 4) + (x \cdot x) + (4 \cdot 1) = 0$$

$4x + x^2 + 4 = 0 \qquad x^2 + 4x + 4 = 0$

FORON:

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1}$$

$$x = \frac{-4 \pm \sqrt{16-16}}{2} \qquad x = \frac{-4 \pm \sqrt{0}}{2}$$

$x = -2$

Logo, para que u e v sejam ortogonais:

$$x = -2$$

7.

a)

$$v \cdot w = \begin{pmatrix} i & j & K \\ 4 & -1 & 5 \\ 1 & -2 & 3 \end{pmatrix}$$

$= (5.(-2)-(-1).3)\,i - (4.3-1.1)\,j + (4.(-2)-1.1)\,K$

$= (-10+3)\,i - (12-1)\,j + (-8-1)\,K$

$= -7i - 11j - 9K$

Portanto: $v \cdot w = (-7, -11, -9)$

Para encontrar $\vec{u}$ ortogonal a v e w:

$u = (v \cdot w) \cdot (1,1,1)$

$$u = \begin{pmatrix} i & j & K \\ -7 & -11 & -9 \\ 1 & 1 & 1 \end{pmatrix} = (-11.1-(-9).1)\,i - ((-7).1-(-9).1)\,j + ((-7).1-(-11).1)\,K$$

$= (-11+9)\,i - (7+9)\,j + (-7+11)\,K$

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$= -2\,i - 16\,j + 4\,K$

Portanto, o vetor $\vec{u}$ é $(-2,-16,4)$ para que seja ortogonal a $v=(4,-1,5)$ e $w=(1,-2,3)$ e que tenha o produto interno $u\cdot(1,1,1)=-1$.

h)

$$v \cdot w = \begin{pmatrix} i & j & K \\ 2 & 3 & -1 \\ 2 & -4 & 6 \end{pmatrix}$$

$= (3.6-(-1).(-4))\,i - (2.6-2.(-1))\,j + (2.(-4)-2.3)\,K$

$= (18-4)\,i - (12+2)\,j + (-8-6)\,K$

$= 14\,i - 14\,j - 14\,K$

$$u = \frac{1}{\|v\cdot w\|}\cdot(14,-14,-14)$$

$\|v\cdot w\| = \sqrt{(14)^2+(-14)^2+(-14)^2}$

$= \sqrt{196+196+196} = \sqrt{588} = 14\sqrt{3}$

Então:

$$u = \frac{1}{14\sqrt{3}}\cdot(14,-14,-14) = \frac{1}{\sqrt{3}}\cdot(1,-1,-1)$$

$$u\cdot i = \frac{1}{\sqrt{3}}\cdot(1,-1,-1)\cdot(1,0,0) = \frac{1}{\sqrt{3}}\cdot 1 = \frac{1}{\sqrt{3}}$$

Então, $u=\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ forma um ângulo agudo com o vetor $i=(1,0,0)$

(/ /)

c)

$$u \cdot v = \| u \| \cdot \| v \| \cdot \cos\left(\frac{\pi}{4}\right) = \sqrt{5} \cdot 1 \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{2}}$$

$$= \frac{\sqrt{10}}{2}$$

$$u + v = (u_1 + v_1, u_2 + v_2) = (2+1, 1+1)$$

$$= (3, 2)$$

$$u - v = (u_1 - v_1, u_2 - v_2) = (2-1, 1-1)$$

$$= (1, 0)$$

$$(u+v) \cdot (u-v) = \| u + v \| \cdot \| u - v \| \cdot \cos(\theta)$$

$$(3, 2) \cdot (1, 0) = \sqrt{13} \cdot 1 \cdot \cos(\theta)$$

$$3 \cdot 1 + 2 \cdot 0 = \sqrt{13} \cdot 1 \cdot \cos(\theta)$$

$$3 = \sqrt{13} \cdot \cos(\theta)$$

$$\cos(\theta) = \frac{3}{\sqrt{13}} \qquad \theta = \arccos\left(\frac{3}{\sqrt{13}}\right)$$

8.

a)

$$v \cdot u = (1 \cdot 3) + (-1 \cdot -1) + (2 \cdot 1) = 3 + 1 + 2 = 6$$

$$\| \overset{u}{\cancel{v}} \|^2 = (3^2) + (-1^2) + (1^2) = 9 + 1 + 1 = 11$$

FORON:

$$\text{proj}_u(v) = \frac{6}{11} \cdot (3,-1,1)$$

$$\text{proj}_u(v) = \left(\frac{18}{11}, -\frac{6}{11}, \frac{6}{11}\right)$$

b)

$$v \cdot u = (1 \cdot -3) + (3 \cdot 1) + (5 \cdot 0)$$
$$= -3 + 3 + 0 = 0$$

$$\|u\|^2 = (-3^2) + (1^2) + (0^2) = 9 + 1 + 0 = 10$$

$$\text{proj}_u(v) = \frac{0}{10} \cdot (-3,1,0)$$

$$\text{proj}_u(v) = (0,0,0)$$

c)

$$v \cdot u = (-1 \cdot -2) + (1 \cdot 1) + (1 \cdot 2) = 2 + 1 + 2 = 5$$

$$\|u\|^2 = (-2)^2 + 1^2 + 2^2 = 4 + 1 + 4 = 9$$

$$\text{proj}_u(v) = \frac{5}{9} \cdot (-2,1,2$$

$$\text{proj}_u(v) = \left(-\frac{10}{9}, \frac{5}{9}, \frac{10}{9}\right)$$

d)

$$v \cdot u = (1 \cdot -2) + (2 \cdot (-4)) + (4 \cdot (-8)) =$$
$$= -2 - 8 - 32 = -42$$

FORON: $\|u\|^2 = (-2)^2 + (-4)^2 + (-8)^2 = 4 + 16 + 64 = 84$

$proj_u(v) = \frac{-42}{84} \cdot (-2, -4, -8)$

$proj_u(v) = (-\frac{1}{2}, -1, -2)$

9.

a)

$v \cdot u = (3 \cdot 2) + (-6 \cdot (-2)) + (0 \cdot 1)$
$= 6 + 12 + 0 = 18$

$\|u\|^2 = (2)^2 + (-2)^2 + 1^2 = 4 + 4 + 1 = 9$

$proj_u(v) = \frac{18}{9} \cdot (2i - 2j + K)$

$proj_u(v) = 4i - 4j + 2K$

$\|v\|^2 = (3)^2 + (-6)^2 + 0^2 = 9 + 36 + 0 = 45$

$proj_v(u) = \frac{18}{45} \cdot (3i - 6j)$

$proj_v(u) = \frac{2}{5} \cdot (3i - 6j)$

$proj_v(u) = \frac{6}{5} i - \frac{12}{5} j$

b) $y = v - proj_u(v)$

$y = (3i - 6j) - (4i - 4j + 2K)$

$y = -i - 2j - 2K$

FORON:

(/ /)

$p = (3i - 6j) - (-i - 2j - 2K)$

$p = 4i - 4j + 2K$

c)

$\text{área} = \| u \times v \|$

$$u \times v = \begin{vmatrix} i & j & K \\ 2 & -2 & 1 \\ 3 & -6 & 0 \end{vmatrix}$$

$u \times v = (i \cdot (-6) - j \cdot 0) - (i \cdot 3 - K \cdot 0) + (j \cdot 3 - K \cdot (-6))$

$u \times v = (-6i) - (3j) + (6K)$

$\| u \times v \| = \sqrt{(-6)^2 + (-3)^2 + 6^2} = \sqrt{36 + 9 + 36} = \sqrt{81} = 9$

$\text{área} = 9$

10.

a)

$$u \times v = \begin{vmatrix} i & j & K \\ 3 & 3 & 0 \\ 5 & 4 & 0 \end{vmatrix}$$

$u \times v = (3i \cdot 4 - 0) - (3j \cdot 5 - 0) + (9 - 12)K$

$u \times v = 12i - 15j - 3K$

FORON: $\| u \times v \| = \sqrt{12^2 + (-15)^2 + (-3)^2} = \sqrt{378}$

b)

$$u \times v = \begin{vmatrix} i & j & k \\ 7 & 0 & -5 \\ 1 & 2 & -1 \end{vmatrix}$$

$$u \times v = (0\cdot(-1) - (-5)\cdot 2)\,i - (7\cdot(-1) - (-5)\cdot 1)\,j + (7\cdot 2 - 0\cdot 1)\,k$$

$$u \times v = -10i + 2j + 14k$$

$$\|u \times v\| = \sqrt{(-10)^2 + 2^2 + 14^2} = \sqrt{300}$$

c)

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & -3 & 1 \\ 1 & 1 & 4 \end{vmatrix}$$

$$u \times v = (-3\cdot 4 - 1\cdot 1)\,i - (1\cdot 4 - 1\cdot 1)\,j + (1\cdot 1 - (-3)\cdot 1)\,k$$

$$u \times v = -13i - 3j + 4k$$

$$\|u \times v\| = \sqrt{(-13)^2 + (-3)^2 + 4^2} = \sqrt{194}$$

d)

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{vmatrix}$$

$$u \times v = (1\cdot 4 - 2\cdot 2)\,i - (2\cdot 4 - 2\cdot 4)\,j + (2\cdot 2 - 1\cdot 4)\,k$$

$$u \times v = 0i - 0j - 0k$$

$$\|u \times v\| = 0$$

11.

a)

$$\|u \times v\|^2 = (u \times v) \cdot (u \times v)$$

$$\|u \times v\|^2 = (u \cdot u)(v \cdot v) - (u \cdot v)^2$$

Sabendo que $u \cdot u = \|u\|^2$ e $v \cdot v = \|v\|^2$:

$$\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$$

b)

$$\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$$

$$\|u \times v\|^2 = 1^2 \cdot 5^2 - 3^2$$

$$\|u \times v\|^2 = 25 - 9$$

$$\|u \times v\|^2 = 16$$

$$\|u \times v\| = \sqrt{16} = 4$$

c)

Assumindo o vértice A como ponto de partida:

$$AB = l i$$

$$AC = \frac{l}{2}(-i + \sqrt{3} j)$$

$$AB \times AC = \begin{vmatrix} i & j & k \\ l & 0 & 0 \\ -\frac{l}{2} & \frac{\sqrt{3} l}{2} & 0 \end{vmatrix}$$

$$AB \times AC = (0-0) i - (0-0) j + \left(l \cdot \frac{\sqrt{3} l}{2} - \left(-\frac{l}{2} \cdot 0\right)\right) k$$

$$AB \times AC = \frac{\sqrt{3}\,l^2}{2} K$$

$$\|AB \times AC\| = \sqrt{\left(\frac{\sqrt{3}\,l^2}{2}\right)^2} \qquad \|AB \times AC\| = \sqrt{\frac{3l^4}{4}}$$

$$\|AB \times AC\| = \frac{l^2\sqrt{3}}{2}$$

12.

a)

$$a(2) + b(3) + c(4) = 9$$
$$2a + 3b + 4c = 9$$

$$\ast \begin{vmatrix} i & j & K \\ a & b & c \\ -1 & 1 & -1 \end{vmatrix} = -2i + 2K$$

$$(-b-c)i + (-a+c)j + (a+b)K = -2i + 2K$$

$$\left.\begin{array}{l} -b-c=-2 \\ c-a=0 \\ a+b=0 \end{array}\right\} \begin{array}{l} -b=-2+a \Rightarrow b=2-a \\ c=a \\ \end{array}$$

$$2a + 3(2-a) + 4a = 9$$
$$2a + \cancel{3a}\ 6 - 3a + 4a = 9$$
$$3a = 3$$
$$a = 1$$

Portanto: $\begin{cases} a=1 \\ b=1 \\ c=1 \end{cases}$

Logo: $X = 1i + 1j + 1K$

FORON:

b)

$$X \times (1,0,1) = 2(1,1,-1)$$

$$\begin{vmatrix} i & j & k \\ a & b & c \\ 1 & 0 & 1 \end{vmatrix} = 2i + 2j - 2k$$

$$(bc)i + (-c)j + (a-b)k = 2i + 2j - 2k$$

$$\left.\begin{aligned} bc &= 2 \\ -c &= 2 \\ a - b &= -2 \end{aligned}\right\} \quad \begin{aligned} c &= -2 \\ a &= b - 2 \\ b(b-2) &= 2 \\ b^2 - 2b - 2 &= 0 \end{aligned}$$

$b = \dfrac{2 + \sqrt{12}}{2}$ ou $b = \dfrac{2 - \sqrt{12}}{2}$

$b = 1 + \sqrt{3}$ ou $b = 1 - \sqrt{3}$

Se $b = 1 + \sqrt{3}$ então:

$a = (1 + \sqrt{3}) - 2 = -1 + \sqrt{3}$
$c = -2$

Se $b = 1 - \sqrt{3}$ então:

$a = (1 - \sqrt{3}) - 2 = -1 - \sqrt{3}$
$c = -2$

Portanto:

$X_1 = (-1 + \sqrt{3})i + (1 + \sqrt{3})j - 2k$
$X_2 = (-1 - \sqrt{3})i + (1 - \sqrt{3})j - 2k$

Checar se satisfaz a condição $\|X\| = \sqrt{6}$

Para X_1:

$$\|x_1\| = \sqrt{(-1+\sqrt{3})^2 + (1+\sqrt{3})^2 + (-2)^2}$$
$$\|x_1\| = \sqrt{1 - 2\sqrt{3} + 3 + 1 + 2\sqrt{3} + 3 + 4}$$
$$\|x_1\| = \sqrt{12} = 2\sqrt{3}$$

Para x_2:

$$\|x_2\| = \sqrt{(-1-\sqrt{3})^2 + (1-\sqrt{3})^2 + (-2)^2}$$
$$\|x_2\| = \sqrt{1 + 2\sqrt{3} + 3 + 1 - 2\sqrt{3} + 3 + 4}$$
$$\|x_2\| = \sqrt{12} = 2\sqrt{3}$$

Portanto, as soluções para x são:

$$x_1 = (-1+\sqrt{3})i + (1+\sqrt{3})j - 2k$$
$$x_2 = (-1-\sqrt{3})i + (1-\sqrt{3})j - 2k$$

c)

$$\|x\| = \sqrt{a^2 + b^2 + c^2} = \sqrt{3}$$

$$a^2 + b^2 + c^2 = 3$$

$$x \cdot u = a(-3) + b(0) + c(3) = -3a + 3c = 0$$
$$x \cdot v = a(2) + b(-2) + c(0) = 2a - 2b = 0$$

1. $-3a + 3c = 0 \rightarrow a = c$
2. $2a - 2b = 0 \rightarrow a = b$

Portanto, $a = b = c$.

$$x \cdot j = a(0) + b(1) + c(0) = b < 0$$

$$a^2 + a^2 + a^2 = 3 \qquad 3a^2 = 3 \qquad a^2 = 1 \qquad a = \pm 1$$

como $b < 0$, $a = b = c = -1$

FORON

/ /

Vetor x é:

$x = -i - j - K$

13.

a) $AB = B - A = (1, 1, -1) - (3, 2, -1) = (-2, -1, 0)$
$AD = D - A = (5, 3, 3) - (3, 2, -1) = (2, 1, 4)$

$$AB \times AD = \begin{vmatrix} i & j & K \\ -2 & -1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$$

$= (0-4)i - (0-8)j + (-2+2)K = -4i + 8j$

$\| AB \times AD \| = \sqrt{(-4)^2 + 8^2} = \sqrt{16 + 64} = \sqrt{80} = 4\sqrt{5}$

b)
$\text{Área} = \frac{1}{2} \| AB \times AC \|$

$$AB \times AC = \begin{vmatrix} i & j & K \\ -1 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$

$= (0-3)i - (0-0)j + (-1-0)K = -3i + 0j - K$
$= (-3, 0, -1)$

$\| AB \times AC \| = \sqrt{(-3)^2 + 0^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$

$\text{Área} = \frac{1}{2} \cdot \sqrt{10} = \frac{\sqrt{10}}{2}$

FORON: $\text{Área} = \frac{1}{2} \cdot \text{Base} \cdot \text{altura}$

(/ /)

$$\text{altura} = \frac{2 \cdot \text{área}}{\text{Base}}$$

$$\|AB\| = \sqrt{(-1)^2 + 1^2 + 0^2} = \sqrt{2} = \text{Base}$$

$$\text{altura} = \frac{2 \frac{\sqrt{10}}{2}}{\sqrt{2}} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$$

14.

a)

$$V \times W = \begin{vmatrix} i & j & K \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix}$$

$$V \times W = (Y_2 Z_3 - Z_2 Y_3) i - (X_2 Z_3 - Z_2 X_3) j + (X_2 Y_3 - Y_2 X_3) K$$

$$u \cdot (V \times W) = X_1 (Y_2 Z_3 - Z_2 Y_3) + Y_1 (X_2 Z_3 - Z_2 X_3) + Z_1 (X_2 Y_3 - Y_2 X_3)$$

$$X_1 Y_2 Z_3 - X_1 Z_2 Y_3 + Y_1 X_2 Z_3 - Y_1 Z_2 X_3 + Z_1 X_2 Y_3 - Z_1 Y_2 X_3$$

$$u \times V = \begin{vmatrix} i & j & k \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

$$u \times V = (Y_1 Z_2 - Z_1 Y_2) i - (X_1 Z_2 - Z_1 X_2) j + (X_1 Y_2 - Y_1 X_2) K$$

$$(u \times V) \cdot W = (Y_1 Z_2 - Z_1 Y_2) X_3 - (X_1 Z_2 - Z_1 X_2) Y_3 + (X_1 Y_2 - Y_1 X_2) Z_3$$

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$X_1 Y_2 Z_3 - X_1 Z_2 Y_3 + Y_1 X_2 Z_3 - Y_1 Z_2 X_3$
$+ Z_1 X_2 Y_3 - Z_1 Y_2 X_3$

Comparando as duas expressões:

$u \cdot (v \times w) = (u \times v) \cdot w$

b)

$$\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & -2 \\ -1 & 2 & 0 \end{vmatrix}$$

$= 1(1.0 - (-2).2) - 3(0.0 - (-1).2) + 2(0.2 - 1.(-1))$
$= 1(0+4) - 3(0+2) + 2(0+1)$
$= 1(4) - 3(2) + 2(1)$
$= 4 - 6 + 2$
$= 0$

Portanto, $[u, v, w] = 0$

1. $[u, w, v] = -[u, v, w]$
$[u, w, v] = -0 = 0$

2. $[v, 2w, u] = 2[v, w, u]$
$= 2.0 = 0$

3. $[u, 3v - 2u, w + 3u] = [u, v, w]$
$= 0$

15.

a) $AB = (1, 0, 1)$

FORON: $AF = (3, 5, 6) - (1, 0, 1) = (2, 5, 5)$

$$AB \times AF = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 2 & 5 & 5 \end{vmatrix}$$

$= (0-5)i - (1-2)j + (5-0)k$

$= -5i - j + 5k$

$\|AB \times AF\| = \sqrt{(-5)^2 + (-1)^2 + 5^2}$

$= \sqrt{25+1+25} = \sqrt{51}$

Logo: Área $= \sqrt{51}$

b) $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 5 & 6 \end{vmatrix}$

$= 1(2.6 - 2.5) - 0(2.3 - 2.6) + 1(2.5 - 2.3)$

$= 1(12-10) - 0(6-12) + 1(10-6)$

$= 1(2) - 0(-6) + 1(4)$

$= 2 + 0 + 4 = 6$

Logo: Volume $= 6$

c) a altura do paralelepípedo em relação à face ABCD é a componente z do vetor AF, sendo 6.

d) O volume do tetraedro EABD é igual a um terço do módulo [AB, BE, AF]. Sendo então $\frac{1}{3} \cdot 6 = 2$.

e) a altura do tetraedro em relação à face DEB é a z do vetor AB, que é 1.

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