Guide to FYS1120 - Electrodynamics

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Contents

1 Electrostatics

1.1 Coulomb's Law

This equation describes the force that an electric source charge applies to a test charge.

$$\vec{F}\left(\vec{r}\right) = \frac{qQ}{4\pi\epsilon\mathfrak{r}^2}\hat{\mathfrak{r}} \tag{1}$$

$\vec{F}$ is the total force on the test charge from the source charge, measured in Newtons. q and Q are the test charge and source charge, respectively, measured in Coulombs. ϵ is the total electric field permittivity, and $\vec{\mathfrak{r}}$ is the vector pointing from the source charge to the test charge.

1.2 The Electric Field

Basics The electric field is a vector field that can be used to generalize Coulomb's Law for more complex situations. The electric field at an observation point is defined as:

$$\vec{E} \triangleq \frac{\vec{F}}{q} \tag{2}$$

Where $\vec{E}$ is the electric field at the observation point, $\vec{F}$ is the Coulomb Force applied to a test charge q located at the aforementioned observation point.

The electric field from a point charge (or source) can be written as:

$$\vec{E}\left(\vec{r}\right) = \frac{Q}{4\pi\epsilon\mathfrak{r}^2}\hat{\mathfrak{r}} \tag{3}$$

Superposition If we wish to find the electric field at an observation point for a series of N point charges, the principle of superposition states that the sum of the electric fields from these point charges gives the total electric field at our observation point:

$$\vec{E}_{\text{tot}}\left(\vec{r}\right) = \sum_{i=1}^{N}\vec{E}_i = \sum_{i=1}^{N}\frac{Q_i}{4\pi\epsilon\mathfrak{r}_i^2}\hat{\mathfrak{r}}_i \tag{4}$$

1.2.1 Continuous Charge Distributions

Line Charges We are still interested in finding the electric field at one *specific* observation point, but this time we have a charged line. If we look at this line charge as a collection of infinitessimal point charges, we can (per the superposition principle) find the total electric field at our observation point by summing over all these infinitessimal point charges.

Since the infinitessimal sum over a line is the same as a line integral, we can find our total electric field with a line integral - we must simply keep in mind that we can no longer use Q in our equation due to the nature of integration, and must instead use the line charge density (charge-per-unit-length) λ:

$$\vec{E}_{\text{tot}}\left(\vec{r}\right) = \int_C \vec{E}dl = \int_C \frac{\lambda\left(\vec{r}'\right)}{4\pi\epsilon\mathfrak{r}^2}\hat{\mathfrak{r}}dl \tag{5}$$

Surface Charges We can use the same principles from equation 5 to find the electric field emanating from a charged surface, only this time we need to integrate over the surface charge density (charge-per-unit-area) σ:

$$\vec{E}_{\text{tot}}\left(\vec{r}\right) = \iint_S \vec{E}da = \iint_S \frac{\sigma\left(\vec{r}'\right)}{4\pi\epsilon\mathfrak{r}^2}\hat{\mathfrak{r}}da \tag{6}$$

Volume Charges Now we wish to find the electric field from a volume charge; this time, we will integrate over the volumetric charge density (charge-per-unit-volume) ρ:

$$\vec{E}_{\text{tot}}\left(\vec{r}\right) = \iiint_V \vec{E}d\tau = \iiint_V \frac{\rho\left(\vec{r}'\right)}{4\pi\epsilon\mathfrak{r}^2}\hat{\mathfrak{r}}d\tau \tag{7}$$

1.3 Gauss' Law

It isn't always a simple task to find the electric field via direct integration; fortunately, the divergence theorem gives us Gauss' Law, which states that the charge of a closed surface is proportional to the total charge enclosed within it.

You may notice that there is a vector field $\vec{D}$ below - if you wish to know more about it, take a look at the electric displacement field section

Integral Form We can find the electric field at an observation point by creating a sphere, cylinder or box (around volume/point charges, line charges and surface charges respectively) such that our observation point is on this surface, and then using the total charge within this closed surface as follows:

$$\oiint_S \vec{D}\cdot d\vec{S} = \oiint_S \epsilon\vec{E}\cdot d\vec{S} = Q_{\text{free in S}} \tag{8}$$

Where $\vec{E}$ is the electric field inside the closed surface S, $Q_{\text{free in S}}$ is the sum of all the free charges in S, and ϵ is the permittivity of the medium.

Gradient Form This can also be written in gradient form:

$$\nabla\cdot\vec{D} = \nabla\cdot\left(\epsilon\vec{E}\right) = \rho \tag{9}$$

Where $\vec{E}$ is the electric field, ρ is the free particles' charge density distribution, and ϵ is the permittivity of the medium.

1.4 Symmetry

When trying to find the electric field around a charged object, integration can be frustrating; fortunately, we can simplify many of these integrals and expressions by taking the object's symmetry into account:

Spheres Assume a spherical coordinate system. For a charged perfectly-round sphere at the origin, the electric field will always point radially; as a result, we know the following is true:

$$\vec{E}(r,\phi,\theta) = E(r,\phi,\theta)\hat{r} \tag{10}$$

We also know that the aforementioned field cannot depend on anything other than the observation point's distance from the sphere (that is to say, it only depends on r, and not on θ or ϕ). We can thereby infer the following:

$$\vec{E}(r,\phi,\theta) = \vec{E}\left(r\right) = E\left(r\right)\hat{r} \tag{11}$$

Cylinders and Lines Assume a cylindrical coordinate system. If we have a charged, *infinitely-long* cylinder or line on the z axis, the electric field will always point radially such that:

$$\vec{E}(r,\phi,z) = E(r,\phi,z)\hat{r} \tag{12}$$

We also know that the field cannot depend on anything other than the observation point's distance from the cylinder (that is to say, it only depends on r, and not on θ or z). We then have that:

$$\vec{E}(r,\phi,z) = \vec{E}\left(r\right) = E\left(r\right)\hat{r} \tag{13}$$

Planes If we have a charged *infinite* plane, we know its electric field will always be normal to the plane (both above and below), which means that:

$$\vec{E}(x,y,z) = E(x,y,z)\hat{n} \tag{14}$$

Having an infinite plane implies that the emanated electric field lines will always point in the same direction, and have equal magnitudes – since the strength of an electric field depends on the density of its field lines, this means that the field is constant (that is to say, it is independent of x, y and z in a cartesian coordinate system); we then know that the following is true for all observation points *above or below* the plane:

$$\vec{E}(x,y,z) = E\hat{n} \tag{15}$$

1.5 The Electric Potential

According to Stokes' Theorem, we know that the curl of any electric field is zero:

$$\nabla \times \vec{E} = 0 \tag{16}$$

As a result, the field is conservative, and this implies that there exists a scalar potential (called the *electric potential*) V for $\vec{E}$. It is defined as:

$$\vec{E} = -\nabla V \tag{17}$$

Basics The electric potential can be written as follows:

$$V_{\mathrm{P}} \triangleq \frac{W}{q} = \int_{\mathrm{P}}^{\mathrm{ref}} \vec{E}\cdot d\vec{l} \tag{18}$$

W is the work it takes to move a test charge from P to *ref*, V_{P} is the electric potential at the point P, measured in volts, and $\vec{l}$ a path whose shape doesn't matter due to the field's conservative nature.

This integral represents a value which is *intuitively similar* to the effort needed to move something from P to a point *ref*; the key difference to note here is that this is not truly equivalent to *work* or *potential energy*, due to the fact that we are not taking the charge of the object to-be-moved into account. In short, the *electric potential*'s relation to the *electric field* is akin to the relation between *work* and *force*.

When using the electric potential to solve for other quantities (and our point of reference is not of particular importance), it is customary to set *ref* to infinity, since this simplifies our final expression. We then have the following:

$$V_{\text{P}} = \int_{\text{P}}^{\infty} \vec{E} \cdot d\vec{l} \tag{19}$$

For a point charge, we are then left with the following:

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon\mathfrak{r}} \tag{20}$$

Principle of Superposition To find the electric potential for N point charges, the principle of superposition states that:

$$V_{\text{tot}}(\vec{r}) = \sum_{i=1}^{N} V_i = \sum_{i=1}^{N} \frac{Q_i}{4\pi\epsilon\mathfrak{r}_i} \tag{21}$$

1.5.1 Continuous Charge Distributions

For a more in depth explanation, check out the electric fields section

Line Charges

$$V_{\text{tot}}(\vec{r}) = \int_C V\,dl = \int_C \frac{\lambda(\vec{r}')}{4\pi\epsilon\mathfrak{r}} dl \tag{22}$$

Surface Charges

$$V_{\text{tot}}(\vec{r}) = \iint_S V\,da = \iint_S \frac{\sigma(\vec{r}')}{4\pi\epsilon\mathfrak{r}} da \tag{23}$$

Volume Charges

$$V_{\text{tot}}(\vec{r}) = \iiint_V V\,d\tau = \iiint_V \frac{\rho(\vec{r}')}{4\pi\epsilon\mathfrak{r}} d\tau \tag{24}$$

1.5.2 Kirchhoff's Voltage Law

Kirchhoff's voltage law states that the directed sum of the electrical potential differences around any closed network is zero[1].

[1] https://en.wikipedia.org/wiki/Kirchhoff's_circuit_laws#Kirchhoff.27s_voltage_law_.28KVL.29

Since electric fields are always conservative, we have the following:

$$\oint_C \vec{E} \cdot d\vec{l} = 0 \tag{25}$$

As a result, we know that if we have a set of N points, and we find the difference of the electric potential between each point and end up looping back to the first point, the sum of potentials will always be zero:

$$V_{\text{tot}} = V_{1,2} + V_{2,3} + \cdots + V_{N-1,N} + V_{N,1} = 0 \tag{26}$$

1.6 Poisson's and Laplace's Equations

Assume that we have access to the electric potential, charge density distribution and permittivity in a given space, but lack the total charge of the object emitting the electric field - how can we use this information to our advantage and calculate the electric field and/or potential?

Fortunately, we have the following equations to help us out:

Poisson's Equation

$$\nabla^2 V = -\frac{\rho}{\epsilon} \tag{27}$$

Where V is the electric potential, ρ is the charge density distribution and ϵ is the permittivity of the medium.

Laplace's Equation In the case where there are no free charges (when $\rho = 0$), we can instead use Laplace's Equation:

$$\nabla^2 V = 0 \tag{28}$$

Using these equations, we can simply integrate over $\nabla^2 V$ and plug in the initial conditions (in this case, the known electric potentials) to get $-\vec{E}$ (which is the gradient of the electric potential) and/or V.

1.7 Boundary Conditions

Sometimes it's difficult to visualize what exactly is happening at the boundaries between different material. What if the electric field permittivities are different, for example?

To deal with this, we have the two following equations to help us out:

The Tangential Field

$$\vec{E}_{1t} = \vec{E}_{2t} \tag{29}$$

Where $\vec{E}_{1t}$ is the tangential component of the electric field on one side of the boundary, and $\vec{E}_{2t}$ is the tangential component of the electric field on the other.

The Normal Field

$$\vec{D}_{1n} - \vec{D}_{2n} = \sigma\hat{n} \iff \epsilon_1\vec{E}_{1n} - \epsilon_2\vec{E}_{2n} = \sigma\hat{n} \tag{30}$$

Where $\vec{D}_{1n}$ is the normal component of the electric displacement field on one side of the boundary, $\vec{D}_{2n}$ is the normal component of the electric displacement field on the other, and σ is the surface charge density at the boundary.

In essence, the electric fields on each side of the boundary point exactly in the same direction and magnitude tangentially, but are scaled differently in the normal direction proportionally to their respective media's electric field permittivities.

1.8 The Method of Images

The Uniqueness Theorem for Poisson's Equation Poisson's equation has a unique gradient of the solution for a large class of boundary conditions. In the case of electrostatics, this means that if an electric field satisfying the boundary conditions is found, then it is the complete electric field[2].

In practice, this means that if we've found one solution for Poisson's Equation with one particular set of boundary conditions, then that solution is universal for all fields containing the same boundary conditions.

The Method of Images The method of images (or method of mirror images) is a mathematical tool for solving differential equations, in which the domain of the sought function is extended by the addition of its mirror image with respect to a symmetry hyperplane. As a result, certain boundary conditions are satisfied automatically by the presence of a mirror image, greatly facilitating the solution of the original problem[3].

In other words, as a result of the aforementioned uniqueness theorem, it is possible to completely rewrite a problem into another simpler one; as long as the boundary conditions for the electric potential are still satisfied, the solution for the new problem will match that of the origin, more complex problem.

Example If there is a point charge at $z = h$ above an infinitely large grounded conductor at $z = 0$, then replacing the conductor with another point charge directly beneath it at $z = -h$ will allow the boundary conditions (those being that there is zero charge along all of the xy-plane) to remain unchanged, and thus give the same exact potential and electric field as the original problem.

1.9 Conductors

Materials that contain unbound electrons (as opposed to dielectrics/insulators) that can move about the material freely are known as *conductors*.

Properties of Conductors

[2] https://en.wikipedia.org/wiki/Uniqueness_theorem_for_Poisson's_equation
[3] https://en.wikipedia.org/wiki/Method_of_images

1. $\vec{E} = 0$ inside a conductor, since all the free charges *nearly* instantaneously move around the conductor and cancel each other out

2. $\rho = 0$ inside a conductor (this is a consequence of property 1. and Gauss' Law)

3. Any net charge resides on the surface

4. $V = \text{constant} \iff$ A conductor is an equipotential

5. $\vec{E}$ is strictly perpendicular to the surface just outside a conductor

6. $\vec{D}_n = \sigma\vec{n}$ right outside the conductor

1.9.1 Capacitors

Definition A capacitor consists of two conductors separated by a non-conductive region[4]. The non-conductive region can either be a vacuum or a dielectric.

A capacitor is assumed to be self-contained and isolated, with no net electric charge and no influence from any external electric field. The conductors thus hold equal and opposite charges on their facing surfaces[5], and the dielectric develops an electric field. In SI units, a capacitance of one farad means that one coulomb of charge on each conductor causes a voltage of one volt across the device[6].

An ideal capacitor is sufficiently characterized by a constant capacitance C, defined as the ratio of a positive or negative charge Q on each conductor to the voltage V between them[7]

$$C \triangleq \frac{Q}{V} \tag{31}$$

The potential V in equation 31 refers to the potential over the space between our conductors, and can be found through integration:

$$V = \int_a^b \vec{E} \cdot d\vec{l} \tag{32}$$

Symmetrical Conductors In the case of two conductors with symmetrical properties, such as a two parallel plates, a coaxial cable or a sphere within another:

$$C = \frac{Q}{V\left(a\right) - V\left(b\right)} \tag{33}$$

Where a is the position of the first conductor, and b is the position of the second.

Because the conductors (or plates) are close together, the opposing charges on the conductors attract one another due to the interactions between their own charges and the electric fields generated by the opposing conductor, allowing the capacitor to store more charge for a given voltage than when the conductors are separated; in short, moving two conductors closer together results in a larger capacitance.

[4]Ulaby, p.168
[5]Ulaby, p.157
[6]Ulaby, p.169
[7]Ulaby, p.168

Energy Storage in a Capacitor Now, we wish to determine exactly how much energy is stored in a capacitor; let us begin by rewriting equation 31 for an individual charged particle:

$$V(q) = \frac{q}{C} \tag{34}$$

The work required to charge a capacitor from 0 to Q is then:

$$W = \int_0^Q V(q)dq = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C} \tag{35}$$

We can rewrite equation 31 once more, this time in terms of our total charge:

$$Q = CV \tag{36}$$

So the energy stored in a capacitor is:

$$W_e \triangleq \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \tag{37}$$

1.10 The Energy Density of an Electric Field

Definition The energy density w_e of an electric field is defined as:

$$w_e \triangleq \frac{1}{2}\epsilon|\vec{E}|^2 = \frac{1}{2}\vec{D}\cdot\vec{E} \tag{38}$$

Usage This formula is very useful when it comes to calculating the total energy stored in a capacitor, and serves as an alternative to equation 37. To use the energy density formula, one should simply integrate as follows:

$$W_e = \iiint_V w_e d\tau = \frac{1}{2}\iiint \vec{D}\cdot\vec{E}d\tau \tag{39}$$

2 Electric Fields in Matter

2.1 Dielectrics

The Concept When dealing with fields going through vacuum, the electric field flows freely - when dealing with fields in materials, however, one must consider that there are far more variables to deal with. Dielectric materials (also known as *insulators*) are electrically neutral materials that are composed of a large number of microscopic dipoles (charged particles with opposing positive and negative sides) oriented randomly in all directions; this results in the sum of their charges adding up to approximately zero.

Resisting Change Ideal dielectric materials are not electrically charged by default, but what happens when they come into contact with an electric field? In short, all the dipoles *almost* immediately reorient themselves such that they are all aligned with the electric field - this means that the previously neutral material ends up being charged, with one end positive and the other negative.

2.2 Polarization

A useful quantification of the aforementioned orientation of dipoles is called the *polarization density* $\vec{P}$ of a dielectric medium. It can be understood as the vector field that expresses the density of permanent or induced electric dipole moments in a dielectric material[8]. One can also interpret the polarization as being a quantity that determines the strength of an electric displacement field.

$\vec{P}$ follows a few rules, depending on the properties of the medium it is in:

Linear Materials The polarization density $\vec{P}$ *depends* on $\vec{E}$ *linearly*

Isotropic Materials The polarization density $\vec{P}$ is *not dependent* on the *direction* of $\vec{E}$

Homogenous Materials The polarization density is *independent* of *position*, such that it can be defined as:

$$\vec{P} \triangleq \epsilon_0 \chi_e \vec{E} \tag{40}$$

Where ϵ_0 is the permittivity of free space, and χ_e is the electric susceptibility of the given material.

χ_e may be a tensor in some cases, but when a material is linear-isotropic-homogenous, its χ_e is simply a constant.

2.3 The Electric Displacement Field

Basics The electric displacement field $\vec{D}$ is a field much like $\vec{E}$, except that it accounts for the susceptibility of the dielectric materials it traverses. It consists of the electric field $\vec{E}$ multiplied by the vacuum permittivity ϵ_0 and an extra value called the *polarization density* $\vec{P}$:

$$\vec{D} \triangleq \epsilon_0 \vec{E} + \vec{P} \tag{41}$$

Where $\vec{P}$ is the *polarization density* of the medium in question. When a material is linear-isotropic-homogenous, $\vec{D}$ is proportional to the electric field $\vec{E}$, with a factor determined by the permittivity of the medium in question; this factor is is known as the medium's *relative permittivity* and is represented by $\epsilon_r = 1 + \chi_e$; this leaves us with the following simplified equation:

$$\vec{D} = \epsilon \vec{E} \tag{42}$$

Where $\epsilon \triangleq \epsilon_0 \epsilon_r$

3 Magnetostatics

When trying to model magnetic forces, it is vital to understand the mechanisms occuring in moving charges, since they are the basis from which the fundamentals of magnetostatics is built. Although we now have a grasp on the forces and fields emanating from electric charges

[8]Introduction to Electrodynamics (3rd Edition), D.J. Griffiths, Pearson Education, Dorling Kindersley, 2007, ISBN 81-7758-293-3

when focusing on one point in time, their movement is the process by which magnetic fields are generated, so we must begin by going through the following fundamental concepts:

3.1 Electric Current

Electric current, perhaps one of the most basic concepts in magnetodynamics, is also one of the most vital; later, it will be apparent why this is the case, but for now we will simply explain how to interpret electric currents in different circumstances.

An important consideration Before we begin, it is critical to note that all charge densities in magnetostatics refer *only* to the charge density distribution of particles **in motion** (also known as their *mobile charge density*).

Linear Electric Current Density

$$\vec{I} = NQ\vec{v} \tag{43}$$

Where $\vec{I}$ is the linear electric current density, Q is the charge of each particle, N is the density of charged particles (number of charges Q per-unit-length) and $\vec{v}$ is the velocity at which these charges are travelling. This can also be written as:

$$\vec{I} \triangleq \lambda\vec{v} \tag{44}$$

Where λ is the *mobile* linear charge density distribution.

Surface Electric Current Density

$$\vec{K} = NQ\vec{v} \tag{45}$$

Where $\vec{K}$ is the surface electric current density, Q is the charge of each particle, N is the density of charged particles (number of charges Q per-unit-area) and $\vec{v}$ is the velocity at which these charges are travelling. This can also be written as:

$$\vec{K} \triangleq \sigma\vec{v} \tag{46}$$

Where σ is the *mobile* surface charge density distribution.

Volumetric Electric Current Density

$$\vec{J} = NQ\vec{v} \tag{47}$$

Where $\vec{J}$ is the volumetric electric current density, Q is the charge of each particle, N is the density of charged particles (number of charges Q per-unit-volume) and $\vec{v}$ is the velocity at which these charges are travelling. This can also be written as:

$$\vec{J} \triangleq \rho\vec{v} \tag{48}$$

Where ρ is the *mobile* volume charge density distribution.

Now that we've grasped what the electric current density is, we can introduce at the concept of electric current:

Electric Current Defined as the change in charge over time:

$$I = \frac{dQ}{dt} \tag{49}$$

When we are working with a cable, equation 49 can also be written as:

$$I \triangleq \iint_S \vec{J} \cdot d\vec{a} \tag{50}$$

Where S is the cross-section of a the cable through which charges are flowing.

3.2 Ohm's Law, Conductivity and Resistivity

In reality, electric currents will *almost* never flow freely through materials – it is therefore necessary to take a material's *conductivity* into account to determine how efficiently it will carry such a current. For the sake of perspective, a *perfect conductor* is said to have a conductivity of $\sigma = \infty$, while a *perfect insulator* has a conductivity of $\sigma = 0$.

Ohm's Law To find the electric current of a material, we can use a material's conductivity in the following equation:

$$\vec{J} = \sigma \vec{E} \tag{51}$$

Where σ is defined as the conductivity of the particular material we are investigating, and **not** the mobile surface charge density introduced in equation 46.

Resistivity If conductivity is a measure of the ease with which electrons flow through a material, then resistivity (ρ) is the measure of how difficult it is for a current to flow. It is related to conductivity as follows:

$$\rho = \frac{1}{\sigma} \tag{52}$$

Resistance Not to be confused with *resistivity*, the resistance (R) of an object is – in a sense – a measure of its total resistivity:

$$R = \rho \frac{L}{A} = \frac{L}{\sigma A} \tag{53}$$

Where R is the resistance, L is the length of the object and A is its cross-sectional area

3.2.1 Electric Power

Power is defined as work done per-unit-time:

$$P \triangleq \frac{W}{t} = \frac{VQ}{t} = VI \tag{54}$$

Power Loss We now wish to understand how much energy (or power) we are losing due to a material's inefficienc – let us begin with the following; since work is defined as force over distance, in the case of electric fields it can be written as:

$$W = q\vec{E} \cdot \vec{v}dt \tag{55}$$

Let us rewrite the above:

$$Nd\tau q\vec{E} \cdot \vec{v}dt = \vec{J} \cdot \vec{E}d\tau dt \tag{56}$$

Power loss per-unit-volume can then be defined as:

$$p_j = \vec{J} \cdot \vec{E} \tag{57}$$

And the total power loss as:

$$P_j \triangleq \iiint_V p_j d\tau = \iiint_V \vec{J} \cdot \vec{E}d\tau \tag{58}$$

Special Case – Even Charge Distributions Assuming we have a cable in which the current is spread out evenly over its cross-section, we can simplify equation 58 such that:

$$P_j = VI = \frac{V^2}{R} = RI^2 \tag{59}$$

Where R is the cable's resistance measured in Ohms

Charge Conservation In all fields of physics, it is vital to take laws of conservation into account - it is also important to realize that this is not a bad thing! These rules simply give us more tools with which we can solve problems. In the case of electrodynamics, the conservation of charge law states that if the net current leaving a junction, point or source is positive, then the given area will eventually be left with no charged particles (and vice-versa):

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \tag{60}$$

Where ρ is a charge density distribution.

3.2.2 Kirchhoff's Current Law

At any node (or junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node; in other words, the algebraic sum of currents in a network of conductors meeting at a point is zero[9]

$$\sum_i I_i = 0 \tag{61}$$

Where I is the electric current

$$\oiint_S \vec{J} \cdot d\vec{a} = 0 \tag{62}$$

$$\nabla \cdot \vec{J} = 0 \tag{63}$$

[9] https://en.wikipedia.org/wiki/Kirchhoff's_circuit_laws#Kirchhoff.27s_current_law_.28KCL.29

3.3 Biot-Savart's Law

Biot–Savart's law is used for computing the resultant magnetic field $\vec{B}$ at position $\vec{r}$ generated by a steady current I: a continual flow of charges which is constant in time and the charge neither accumulates nor depletes at any point. The law is a physical example of a line integral, being evaluated over the path C in which the electric currents flow[10].

$$\vec{B}(\vec{r}) \triangleq \frac{\mu_0}{4\pi} \oint_C \frac{I d\vec{l} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} \tag{64}$$

Where $\vec{B}$ is the magnetic flux density measured in Teslas, μ_0 is the permeability and I is the electric current measured in Amperes.

Magnetic Force Interestingly enough, due to the nature of a charged particle's movement, magnetic forces act orthogonally both to a charged particle's velocity and the magnetic field which surrounds it:

$$\vec{F} = Q\vec{v} \times \vec{B} \tag{65}$$

Where $\vec{F}$ is the resulting force in Newtons, Q is a test charge, $\vec{v}$ is the velocity of the test charge and $\vec{B}$ is a magnetic field.

Usage Solving the integral of a cross product may seem daunting at first glance (and it certainly can be in some cases), but it is often possible to use an argument to simplify the equation such that we are left with something more friendly.

Try to visualize what the cross product would result in on its own, and determine if its answer is constant along the whole curve, surface or volume; if this is the case, simply replace the cross product expression with this simpler one, and then integrate over it.

3.4 Differential Elements of Electric Current

The main source of a magnetic field is the electric current differential, which can be written in one of three ways:

Line Element

$$I d\vec{l} \tag{66}$$

Surface Element

$$\vec{K} da \tag{67}$$

Volume Element

$$\vec{J} d\tau \tag{68}$$

These will be very useful, so take note!

[10] Electromagnetism (2nd Edition), I.S. Grant, W.R. Phillips, Manchester Physics, John Wiley & Sons, 2008, ISBN 978-0-471-92712-9

3.5 Lorentz' Force

This equation represents the sum of all electromagnetic forces, and is composed of the earlier Coulomb force equation as well as the aforementioned Magnetic Force:

$$\vec{F} = Q\vec{E} + Q\vec{v} \times \vec{B} \tag{69}$$

3.6 The Hall Effect

If an electric current flows through a conductor in a magnetic field, the magnetic field exerts a transverse force on the moving charge carriers which tends to push them to one side of the conductor. A buildup of charge at the sides of the conductors will balance this magnetic influence, producing a measurable voltage between the two sides of the conductor[11].

3.7 Magnetic Forces

Here are a few useful equations that can be used in practical situations:

Magnetic Force on a Cable

$$\vec{F} = \oint I d\vec{l} \times \vec{B} \tag{70}$$

Magnetic Moment This is a value that is useful when determining the torque from a magnetic field on a charged object:

$$\vec{m} \triangleq I\vec{a} \tag{71}$$

Where $\vec{m}$ is the magnetic moment, I is the electric current and $\vec{a}$ is the surface normal

It can be used directly in the *torque equation*:

$$\vec{T} = \vec{m} \times \vec{B} \tag{72}$$

3.8 Magnetic Flux

The magnetic flux Φ of a surface S is defined as the sum of the normal components of the magnetic field lines passing through S. It is represented by the following equation:

$$\Phi \triangleq \iint_S \vec{B} \cdot d\vec{a} \tag{73}$$

Special Case – Closed Surfaces When integrating over a closed surface, we find that:

$$\Phi = \oiint_S \vec{B} \cdot d\vec{a} = 0 \tag{74}$$

The reason our flux in equation 74 is zero for *any* closed surface is that all magnetic field lines circulate, and therefore must reconnect at some point; this therefore implies that all lines entering a closed surface must exit. Once again, this will always leave us with a net zero magnetic flux.

[11] http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/Hall.html

3.9 Ampère's Law

Recall our introduction to Gauss' Law earlier in this guide – has it not made our lives easier in many situations? Fortunately, we have an analogous formula in magnetostatics called **Ampère's Law**, which we will introduce shortly.

Integral Form We can use Ampère's Law to relate the amount of current going through a closed curve C to the magnetic field at that point with:

$$\oint_{\text{C}} \vec{H} \cdot d\vec{l} = I_{\text{total through C}} \tag{75}$$

In a vacuum, we also have that:

$$\oint_{\text{C}} \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{total through C}} \tag{76}$$

In a linear and isotropic material, we have that:

$$\oint_{\text{C}} \vec{B} \cdot d\vec{l} = \mu_0 \mu_r I_{\text{total through C}} \tag{77}$$

Where μ_r is a material's relative permeability – this will be defined later in equation 119.

Differential Form Using a logical process similar to that in Gauss' Law, it is possible to represent this equation with a gradient instead of an integral:

$$\nabla \times \vec{H} = \vec{J} \tag{78}$$

3.10 EMF

We define the **electromotive force** or **emf** in statics as:

$$\varepsilon \triangleq \oint_{\text{circuit}} \vec{f} \cdot d\vec{l} \tag{79}$$

Where $\vec{f}$ is the sum of external forces.

In dynamics, it is defined as:

$$\varepsilon \triangleq \oint_{\text{circuit}} \left(\vec{f} + \vec{E}\right) \cdot d\vec{l} \tag{80}$$

The dynamic definition of the electromotive force is essentially the same as the static form – the main difference is that it takes non-conservative electric fields into account.

Terminology The term electromotive *force* is a misnomer – it is more intuitive (and correct) to imagine it being a sort of electromotive *potential*, since it represents the sum of the external work per-unit-charge from a source in a circuit. Oftentimes when dealing with electric circuits, it is represented as a battery voltage.

3.11 Faraday's Law

Although we are dealing with magnetostatics, it is important to recall that we are still using time-based variables (such as $\vec{J}$) to describe a phenomenon we describe as static. As a result, we can still use a change over time to describe the *static emf*:

$$\varepsilon = \frac{d}{dt} \iint_S \vec{B} \cdot d\vec{a} \tag{81}$$

This can also be written as:

$$\varepsilon = -\frac{\partial \Phi}{\partial t} \tag{82}$$

Circuits As mentioned previously, we can view the *emf* as the total external voltage when dealing with electric circuits. This means that:

$$\sum_i \varepsilon_i = RI \tag{83}$$

In addition to this, we have that the *emf* in a circuit is:

$$\varepsilon = \oint\limits_{\text{circuit}} \left(\vec{f}_b + \vec{f}_m + \vec{E} \right) \cdot d\vec{l} = \int\limits_{\text{over}R} \vec{E} \cdot d\vec{l} \tag{84}$$

Where $\vec{f}_m = \vec{v} \times \vec{B}$ and $\vec{f}_b$ is the force-per-unit-charge from an external power source (usually interpreted as a *battery* voltage).

So we can describe our *total resistance* in terms of the electrical field:

$$R = \frac{1}{I} \int \vec{E} \cdot d\vec{l} \tag{85}$$

Multicoiled Inductors For *one* coil, we've already seen in equation 82 that:

$$\varepsilon = -\frac{\partial \Phi}{\partial t} \tag{86}$$

For a solenoid with N coils, we have that:

$$\varepsilon = -\frac{\partial \Phi_{\text{tot}}}{\partial t} \tag{87}$$

Where:

$$\Phi_{\text{tot}} = \sum_{i=1}^{N} \Phi_i \tag{88}$$

Constant Φ Assuming that we have a conductor with N coils, each with an equal magnetic flux Φ_1, we can state that:

$$\Phi = N\Phi_1 \implies \varepsilon = -N \frac{\partial \Phi_1}{\partial t} \tag{89}$$

3.11.1 Differential Form

Faraday's Law in differential form is:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{90}$$

3.12 The Circuit Equation

Now that equations 81 – 90 have given us an understanding with regards to the different ways electromotive forces can affect a circuit, we can conclude that all circuits are governed by the following rule:

$$\sum \varepsilon = RI \implies V_b - \frac{d\Phi}{dt} = RI \tag{91}$$

Where V_b is our external voltage (or battery voltage)

3.13 Lenz' Law

> "The direction of current induced in a conductor by a changing magnetic field due to Faraday's law of induction will be such that it will create a magnetic field that opposes the change that produced it." – Heinrich Friedrich Lenz, 1834

Lenz's law is shown by the negative sign in Faraday's law of induction (equation 82), which indicates that the induced EMF and the change in magnetic flux have opposing signs. It is a qualitative law that specifies the direction of induced current but says nothing about its magnitude.[12]

3.14 Inductance

Inductance is the property of an electrical conductor by which a change in current through it induces an electromotive force in both the conductor itself and in any nearby conductors by mutual inductance[13]. We will introduce the concept of mutual inductance later in section 5.5, but for now here is a short introduction explaining the basic concepts behind inductance in general. To begin, we must understand that:

$$\Phi \propto B \propto I \tag{92}$$

Where Φ is the *total flux* in a circuit or solenoid.

We then define **inductance** as:

$$L \triangleq \frac{\Phi}{I} \tag{93}$$

Using *Faraday's Law*, we can state that:

$$\varepsilon = -L\frac{\partial I}{\partial t} \tag{94}$$

[12] Giancoli, Douglas C. (1998). Physics: principles with applications (5th ed.). p. 624.
[13] Sears and Zemansky 1964

4 Circuit Analysis

4.1 Direct Current

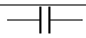
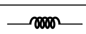
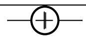
4.1.1 Kirchhoff's Voltage Law

> "The algebraic sum of the products of the resistances of the conductors and the currents in them in a closed loop is equal to the total *emf* available in that loop."[14]

This can also be written as:

$$\sum_{k=1}^{n} V_k = 0 \tag{95}$$

4.1.2 Linear Circuit Components

Value	Symbol	Circuit Symbol	Circuit Equation
Resistance	R		$V = RI$
Capacitor	C		$I = C\frac{dV}{dt}$
Inductor	L		$V = L\frac{dI}{dt}$
Perfect Conductor	I		$V = 0$
Ground		$\perp$	$V = 0$
Voltage Source			$V_0 = \text{constant}$

4.1.3 Mechanical Analogy

Electrical Name	Electrical Symbol	Mechanical Name	Mechanical Symbol
Charge	Q	Displacement	$\vec{x}$
Current	$I = \frac{dQ}{dt}$	Velocity	$\vec{v} = \frac{d\vec{x}}{dt}$
Voltage	V	Force	$\vec{F}$
Capacitance	$C\frac{dV}{dt} = I$	Spring Force	$\frac{d\vec{F}}{dt} = k\vec{v}$
Inductance	$V = L\frac{dI}{dt}$	Inertia	$\vec{F} = m\vec{a}$
Resistance	$V = RI$	Friction	$\vec{F} = k\vec{v}$
Capacitor Energy	$W = \frac{Q^2}{2C}$	Spring Energy	$W = \frac{1}{2}kx^2$

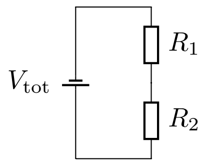
4.1.4 Circuits in Series and Parallel Configurations

Series When a circuit is in *series*, we have that a set of resistances are all lined up such that the current flows through them one at a time in sequence.

Mathematically speaking, we have that the *current* over all resistances is constant when they are in series:

$$I_{\text{tot}} = I_1 = I_2 = \cdots = I_n \tag{96}$$

[14] https://en.wikipedia.org/wiki/Kirchhoff%27s_circuit_laws#Kirchhoff.27s_voltage_law_.28KVL.29



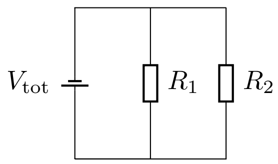
In addition to this, the resistances can all be summed up and considered as *one* large resistance:

$$R_{\text{tot}} = R_1 + R_2 + \cdots + R_n \tag{97}$$

Finally, we can also find the total voltage over the entire circuit by summing up the individual voltages over each resistance:

$$V_{\text{tot}} = V_1 + V_2 + \cdots + V_n \tag{98}$$

Parallel When a circuit is in a *parallel* configuration, we have that a set of resistances are separated such that the current flows through each of them simultaneously:



Instead of the current being constant (as in a series), we now have that the *voltage* is constant over each resistance that is in parallel:

$$V_{\text{tot}} = V_1 = V_2 = \cdots = V_n \tag{99}$$

The rule for the sum of resistances is different than that for series configurations as well, and follows an inverse summation pattern:

$$\frac{1}{R_{\text{tot}}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n} \tag{100}$$

Since current has to do with a physical amount of particles, we have that the total current in the circuit is equal to the sum of the currents over each branch of a parallel configuration:

$$I_{\text{tot}} = I_1 + I_2 + \cdots + I_n \tag{101}$$

4.1.5 Loss of Energy

If we wish to determine the amount of power lost due to the resistance in a cable, we mentioned previously in equation 59 that:

$$P = RI^2 = \frac{V^2}{R} \tag{102}$$

If we wish to find out how much energy is lost over a resistance, we can integrate over equation 102 to find the total energy loss:

$$E = \int_0^\infty P dt = \int_0^\infty \frac{V^2}{R} dt \tag{103}$$

4.2 Alternating Current

When dealing with alternating current, we can no longer assume that the voltage remains constant over time as it does with direct current; we must instead assume that the voltage follows an oscillatory pattern:

$$V(t) = V_0 \cos(\omega t + \varphi) \tag{104}$$

Where V_0 is the *peak voltage*, ω is the *angular frequency* and φ is the *phase*.

In this section, complex numbers will be used to represent different quantities in our circuits. To begin with, we can rewrite the above as:

$$V(t) = \Re\left\{\hat{V}e^{i\omega t}\right\} \tag{105}$$

Where $\hat{V}$ is defined as:

$$\hat{V} \triangleq |\hat{V}|e^{i\varphi} = V_0 e^{i\varphi} \tag{106}$$

We can move between our previous equation and our first one via the following steps:

$$V(t) = \Re\left\{\hat{V}e^{i\omega t}\right\} = \Re\left\{|\hat{V}|e^{i\varphi}e^{i\omega t}\right\} = |\hat{V}|\Re\left\{e^{i(\omega t+\varphi)}\right\} = |\hat{V}|\cos(\omega t + \varphi) \tag{107}$$

4.2.1 Circuit Components

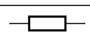
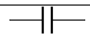
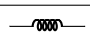
When dealing with alternating currents, we interpret each separate component as an *impedance* with different properties each defined by a particular equation. The reason for this is that we wish to be able to represent the effects of all components with a single equation:

$$\hat{V} = \hat{Z}\hat{I} \tag{108}$$

Where $\hat{Z}$ is the impedance of the component in question, a value that is similar to R in direct current circuits. It is also important to note that:

$$\hat{I} = |\hat{I}|e^{i\theta} \tag{109}$$

Below is a chart with an overview of the impedances of different components (in other words, what should be plugged into $\hat{Z}$ for each of the components listed):

Value	Symbol	Circuit Symbol	$\hat{Z}$ Equation
Resistance	R		$\hat{Z} = R$
Capacitor	C		$\hat{Z} = \frac{1}{i\omega C}$
Inductor	L		$\hat{Z} = i\omega L$

Useful Equations Here are some formulae relating different values in alternating currents:

$$I = C\frac{dV}{dt} \implies \Re\left\{\hat{I}e^{i\omega t}\right\} = C\frac{d}{dt}\Re\left\{\hat{V}e^{i\omega t}\right\} = \Re\left\{Ci\omega\hat{V}e^{i\omega t}\right\} \tag{110}$$

And:

$$V = L\frac{dI}{dt} \implies \Re\left\{\hat{V}e^{i\omega t}\right\} = L\frac{dI}{dt}\Re\left\{\hat{I}e^{i\omega t}\right\} = \Re\left\{i\omega L\hat{I}e^{i\omega t}\right\} \tag{111}$$

5 The Magnetic Field in Matter

When dealing with magnetic fields in matter, it is necessary to understand the fundamental processes that occur on a microscopic scale; just as we previously discussed electric dipoles/bound charges in section 2.1, we will now examine the behavior of magnetic dipoles in magnetic fields. A good place to begin is to look at the *magnetic dipole moment* $\vec{m} = I\vec{S}$. This equation represents the magnetic dipole moment of a single infinitely small *closed* curve circulating about a surface vector $\vec{S}$.

An important quantity we've previously mentioned in equation 72 is *torque* $\vec{T} = \vec{m} \times \vec{B}$. When exposed to a magnetic field, a material's magnetic dipoles will want to reorient themselves such that their torque end up being zero.

In reality, the aforementioned model is not entirely representative of what truly occurs on a microscopic level – it is simply accurate enough for our purposes. In practice, we will be using the **magnetic dipole density**, which takes all the dipole moments in a volume V into account:

$$\vec{M} = \frac{\sum\limits_{\text{in } dV} \vec{m}}{dV} \tag{112}$$

If we have N magnetic dipoles per-unit-volume, we have that:

$$\vec{M} = N\vec{m} \tag{113}$$

We can then state with some certainty that the total bound current flowing through a closed loop C is:

$$I_{\text{bound through } C} = \oint_C \vec{M} \cdot d\vec{l} \tag{114}$$

In the end, we can extend **Ampère's Law** (originally mentioned in equation 75) such that we are left with a more complete form:

$$\frac{1}{\mu_0} \oint_{\text{C}} \vec{B} \cdot d\vec{l} = \oint_{\text{C}} \vec{M} \cdot d\vec{l} + I_{\text{free through } C} \tag{115}$$

This gives us our $\vec{H}$-field:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \tag{116}$$

5.1 Linear and Isotropic Materials

Recall that when dealing with electric fields in matter, it is often possible to simplify some of the mathematics behind the definition of the electric displacement field $\vec{D}$ based on material properties. Fortunately, we can also classify materials in a similar way relative to magnetic fields:

Linear Materials The magnetic polarization density $\vec{M}$ *depends* on $\vec{B}$ *linearly*

Isotropic Materials The magnetic polarization density $\vec{M}$ is *not dependent* on the *direction* of $\vec{B}$

Using the above definitions, magnetically *linear* and *isotropic* materials have the following properties:

$$\vec{M} \propto \vec{B} \implies \vec{H} \propto \vec{B} \tag{117}$$

This means that we can define a new factor χ_m called the **magnetic susceptibility** to simplify equation 116:

$$\vec{M} = \chi_m \vec{H} \implies \vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right) = \mu_0 \left(1 + \chi_m\right) \vec{H} \tag{118}$$

Then we define a new value μ_r called the relative permeability as:

$$\mu_r \triangleq 1 + \chi_m \tag{119}$$

We then have that:

$$\vec{B} = \mu_0 \mu_r \vec{H} \tag{120}$$

We also have one more new factor μ called the **absolute permeability**:

$$\mu = \mu_0 \mu_r \tag{121}$$

Finally, we have that for linear and isotropic materials, the $\vec{H}$ and $\vec{B}$ fields are related thusly:

$$\vec{H} = \mu \vec{B} \tag{122}$$

5.2 Three Types of Magnetic Materials

We classify different materials into three different categories, depending on their *relative permeabilities*:

Paramagnetic Materials

- $\mu_r > 1$ yet $\mu_r \approx 1$
- Aluminium is Paramagnetic ($\mu_r = 1.00002$)

Diamagnetic Materials

- $\mu_r < 1$ yet $\mu_r \approx 1$
- Water is Diamagnetic ($\mu_r = 0.99991$)

Ferromagnetic Materials In this case, we are no longer dealing with linear media; instead, we have a situation where we approximate that $\mu_r \in [10^2, 10^5]$. In addition to this, we also approximate the following:

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right) \approx \mu_0 \vec{M} \tag{123}$$

As mentioned previously, these materials are not linear – instead, their *magnetic dipole density* is related to the $\vec{H}$-field with a **Hysteresis Curve** – a mathematical tool that models the remanent magnetization of a material.

5.3 Boundary Conditions

When dealing with magnetic fields crossing over several materials, it is important to understand the relationship between the field on each side of these boundaries. We have two main equations to deal with this:

$$\vec{H}_{1,\text{tangent}} - \vec{H}_{2,\text{tangent}} = \vec{K} \times \hat{n} \tag{124}$$

And:

$$\vec{B}_{1,\text{normal}} = \vec{B}_{2,\text{normal}} \tag{125}$$

5.4 Magnetic Circuits

In this section, we will begin by comparing the equations of magnetic circuits and electric circuits:

Electric Circuits	Magnetic Circuits
$\vec{J}$	$\vec{B}$
$I = \int_{\text{cr-sec}} \vec{J} \cdot d\vec{a}$	$\Phi = \int_{\text{cr-sec}} \vec{B} \cdot d\vec{a}$
$\vec{J} = \sigma \vec{E}$	$\vec{B} = \mu \vec{H}$
σ	μ
ε	$N \cdot I$
$R = \frac{l}{\sigma S}$	$R_m = \frac{l}{\mu S}$

We've discussed the majority of the quantities shown above, however there is one which has not been mentioned; this is the **reluctance** of a material.

Magnetic Reluctance A concept used in the analysis of magnetic circuits, reluctance is analogous to resistance in an electrical circuit, but rather than dissipating electric energy it stores magnetic energy. In likeness to the way an electric field causes an electric current to follow the path of least resistance, a magnetic field causes magnetic flux to follow the path of least magnetic reluctance.[15]

5.5 Inductance

Inductance (mentioned briefly in section 3.14)is the property of an electrical conductor by which a change in current through it induces an electromotive force in both the conductor itself and in any nearby conductors by mutual inductance[16]. The two types of inductance – **mutual inductance** and **self inductance** – are defined as follows:

Mutual Inductance This occurs when the change in current in one inductor induces a voltage in another nearby inductor[17] and can be represented by the equation below:

$$L_{1,2} = \frac{\Phi_{1,2}}{I_1} = \frac{\Phi_{2,1}}{I_2} = L_{2,1} \tag{126}$$

[15] https://en.wikipedia.org/wiki/Magnetic_reluctance
[16] Sears and Zemansky 1964:743
[17] https://en.wikipedia.org/wiki/Inductance#Mutual_inductance_of_two_wire_loops

As one can observe, mutual inductance is symmetrical in the sense that it is equal from both sides of the occurence.

Self Inductance We must also consider the effects of an inductor on itself, which is represented by the following:

$$L_{1,1} = \frac{\Phi_{1,1}}{I_1} \tag{127}$$

Note - Total Flux It is very important to keep the definition of *total flux* in mind when calculating the inductance in a solenoid with multiple coils – always be sure to take the sum of *all* the individual fluxes through each individual coil, and not simply the flux of a single coil.

Inductance Matrix It is interesting to note that this leaves us with a symmetrical matrix:

$$L = \begin{bmatrix} L_{1,1} & L_{1,2} \\ L_{2,1} & L_{2,2} \end{bmatrix} = \begin{bmatrix} \frac{\Phi_{1,1}}{I_1} & \frac{\Phi_{1,2}}{I_1} \\ \frac{\Phi_{2,1}}{I_2} & \frac{\Phi_{2,2}}{I_2} \end{bmatrix} = \begin{bmatrix} \frac{\Phi_{1,1}}{I_1} & \frac{\Phi_{1,2}}{I_1} \\ \frac{\Phi_{1,2}}{I_1} & \frac{\Phi_{2,2}}{I_2} \end{bmatrix} \tag{128}$$

5.6 Perfect Transformers

Using the principle of inductance, we can create a mechanism by which we can change the voltage of an electric cable via a magnetic circuit and a pair of inductors; the idea here is to have the cable with the initial voltage in a solenoid around one side of the magnetic circuit, and then have another solenoid with a different number of coils on the opposite side of the magnetic circuit, such that a current with a different voltage is induced in the new solenoid.

The number of coils is the main factor to take into account when dealing with a perfect transformer; below, we have the relationship between the initial and new voltages:

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \tag{129}$$

We also have a similar expression for our initial and new currents:

$$\frac{I_2}{I_1} = \frac{N_1}{N_2} \tag{130}$$

5.7 Energy in an Inductor

The work needed to change a magnetic flux from Φ to $\Phi + \Delta\Phi$ is:

$$dA_m = Id\Phi \tag{131}$$

If we now assume that we have a linear medium, we get that $d\Phi = LdI$, which we can use to find the energy stored in an inductor:

$$W_m = \frac{1}{2}LI^2 \tag{132}$$

For a system of inductors with several coils, we can use the following:

$$W_m = \sum_{i,j} \frac{1}{2} L_{ij} I_i I_j \tag{133}$$

5.8 Force from Energy

If we have a field with the energy W_m, we can determine the force it applies on an object with the following equation:

$$\vec{F} = -\nabla W_m \tag{134}$$

If we have a case where there is a *constant current*, then we can simplify this:

$$\vec{F} = \nabla W_m \tag{135}$$

5.9 Displacement Current

In some cases, there is the possibility of a magnetic field existing without there strictly being a current through a closed loop – normally we are used to the following formulation of *Ampères Law*:

$$\nabla \times \vec{H} = \vec{J} \tag{136}$$

In reality, due to the aforementioned possibility it is necessary to extend equation 136, giving us the **Ampère-Maxwell Equation**:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{137}$$

The expression $\frac{\partial \vec{D}}{\partial t}$ is known as the **displacement density**, and is of great importance when dealing with many situations – for instance, when finding the magnetic field in the current-free space between two parallel plates in a capacitor.

In many cases, the displacement density can be difficult to calculate – however, if we know that we are looking at a *vacuum*, then we have that:

$$\frac{\partial \vec{D}}{dt} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{138}$$

5.10 Conservation of Charge

As we mentioned earlier, conservation laws are ubiquitous in the realm of physics, and charge is absolutely no exception; to model the conservation of charge, we can either represent it with an integral formula:

$$\oiint_S \vec{J} \cdot d\vec{a} = -\frac{d}{dt} \iiint_V \varrho d\tau \tag{139}$$

Or in the differential form shown previously in equation 60:

$$\nabla \cdot \vec{J} = -\frac{\partial \varrho}{\partial t} \tag{140}$$

Intuitively, one can simply imagine that the current exits a closed surface S at the cost of a decreasing charge in V (where V is the volume enclosed by S)

6 Reference

6.1 Notation

6.1.1 Vectors

There are three vectors in this text that appear similar, but have three distinct definitions; they are:

$$\vec{r}, \text{ The location of the observation point or test charge} \tag{1}$$

$$\vec{r}', \text{ The location of the source charge} \tag{2}$$

$$\vec{\mathfrak{r}} = \vec{r} - \vec{r}', \text{ The vector from the source charge to the observation point} \tag{3}$$

6.1.2 Infinitessimals

In this guide, we use the following notation to represent differential quantities:

Curves The *line-element* of a curve is written as dl or $d\vec{l}$

Surfaces The *surface-element* of a surface is written as da or $d\vec{a}$

Volumes The *volume-element* of a volume is written as $d\tau$ or $d\vec{\tau}$

6.1.3 Densities

When describing electric field densities as a function of $\vec{r}'$, the following is used:

Curves The linear electric field density is written as $\lambda\left(\vec{r}'\right)$

Surfaces The surface electric field density is written as $\sigma\left(\vec{r}'\right)$

Volumes The volumetric electric field density is written as $\varrho\left(\vec{r}'\right)$

6.1.4 Integrals

This guide uses the following integral notation:

Curves The scalar-valued sum of a scalar function F integrated over the curve C:

$$I = \int_C F dl \tag{4}$$

The vector-valued sum of a vector function $\vec{F}$ integrated over the curve C:

$$\vec{I} = \int_C \vec{F} dl \tag{5}$$

The scalar-valued sum of a scalar function $\vec{F}$ integrated over the line elements of the curve C:

$$I = \int_C \vec{F} \cdot d\vec{l} \tag{6}$$

Surfaces The scalar-valued sum of a scalar function F integrated over the surface S:

$$I = \iint_S F da \tag{7}$$

The vector-valued sum of a vector function $\vec{F}$ integrated over the surface S:

$$\vec{I} = \iint_S \vec{F} da \tag{8}$$

The scalar-valued sum of a scalar function $\vec{F}$ integrated over the surface elements of the surface S:

$$I = \iint_S \vec{F} \cdot d\vec{a} \tag{9}$$

Volumes The scalar-valued sum of a scalar function F integrated over the volume S:

$$I = \iiint_V F d\tau \tag{10}$$

The vector-valued sum of a vector function $\vec{F}$ integrated over the volume S:

$$\vec{I} = \iiint_V \vec{F} d\tau \tag{11}$$

The scalar-valued sum of a scalar function $\vec{F}$ integrated over the volume elements of the volume S:

$$I = \iiint_V \vec{F} \cdot d\vec{\tau} \tag{12}$$

Loops A one-dimensional loop, which consists of an enclosed curve that creates a surface:

$$\oint_C dl \tag{13}$$

Closed Surfaces A two-dimensional closed surface, which contains a volume:

$$\oiint_S da \tag{14}$$

6.2 Permeability, Permittivity and Susceptibility

Electric Field Permittivity The total electric field permittivity is a combination of the permittivity of free space and the permittivity of the specific medium:

$$\epsilon = \epsilon_0 \epsilon_r \tag{15}$$

Where $\epsilon_0 \approx 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N \cdot m^2}}$ and ϵ_r depends on the medium

Magnetic Field Permeability The total magnetic field permeability is a combination of the permeability of free space and the permeability of the specific medium:

$$\mu = \mu_0 \mu_r \tag{16}$$

Where $\mu_0 \approx 4\pi \times 10^{-7} \frac{\mathrm{N}}{\mathrm{A}^2}$ and μ_r depends on the medium

6.3 Maxwell's Equations

$$\nabla \cdot \vec{D} = \varrho_{\mathrm{V}} \tag{17}$$

$$\nabla \cdot \vec{B} = 0 \tag{18}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{19}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \tag{20}$$

6.4 Quantities and SI Units

Quantity	Quantity Name	SI-Unit	SI-Unit Name
$\vec{B}$	Magnetic Field	T	Teslas
C	Capacitance	F	Farads
$\vec{D}$	Electric Displacement Field	$\mathrm{C/m^2}$	Coulombs/Area
$\vec{E}$	Electric Field	N/C or V/m	Volts/Meter
$\vec{F}$	Force	N	Newtons
$\vec{f}$	Sum of External Forces	N	Newtons
G	Conductance	S	Siemens
$\vec{H}$	Magnetic Field	A/m	Amperes/Meter
I	Electric Current	A	Amperes
$\vec{J}$	Volumetric Current Density	$\mathrm{A/m^2}$	Amperes/Area
$\vec{K}$	Surface Current Density	A/m	Amperes/Meter
L	Inductance	H	Henries
$\vec{M}$	Magnetic Polarization	A/m	Amperes/Meter
$\vec{m}$	Magnetic Moment	$\mathrm{A/m^2}$ or N·m/T	Newton-Meters/Tesla
$\vec{P}$	Polarization Density	$\mathrm{C/m^2}$	Coulombs/Area
P	Power	W	Watts
P_j	Power Loss	W	Watts
p_j	Power Loss Density	$\mathrm{W/m^3}$	Watts/Volume
Q or q	Charge	C	Coulombs
R	Resistance	Ω	Ohms
R_m	Reluctance	$\mathrm{H^{-1}}$	Inverse Henries
$\vec{r}$	Distance	m	Meters
$\vec{T}$	Torque	N·m	Newton-Meters
t	Time	s	Seconds
V	Electric Potential	J/C or V	Volts
$\vec{v}$	Velocity	m/s	Meters/Second
W	Work	J	Joules
W_e	Energy in a Capacitor	J	Joules
w_e	Electric Field Energy Density	$\mathrm{J/m^3}$	Joules/Volume
ε	Electromotive Force	V	Volts
ϵ	Permittivity	F/m	Farads/Meter
ϵ_0	Permittivity of Free Space	F/m	Farads/Meter
λ	Linear Free Charge Density	C/m	Coulombs/Meter
μ	Permeability	H/m	Henries/Meter
μ_0	Permeability of Free Space	H/m	Henries/Meter
ϱ	Volumetric Free Charge Density	$\mathrm{C/m^3}$	Coulombs/Volume
ϱ	Resistivity	Ω·m	Ohm-Meters
σ	Surface Free Charge Density	$\mathrm{C/m^2}$	Coulombs/Area
σ	Conductivity	S/m	Siemens/Meter
Φ	Magnetic Flux	V·s or Wb	Webers
χ_e	Electric Susceptibility	Dimensionless	N/A