Hermite interpolation with retractions on manifolds paper review

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Abstract

In the field of Riemannian optimization (optimization problems on Riemannian manifolds) our paper is focusing on an interpolation problem of data samples that are laying on some Riemannian manifold. In our specific setting, the data samples have a prescribed tangent vector at each data point. The demand is that the interpolant is a curve contained in the manifold that interpolates the data points and matches the prescribed derivatives at each point and in addition is globally continuously differentiable.
Such problem is called Hermite interpolation problem generalized to Riemannian manifolds.
In order to solve this problem one can think to use the exponential map (the geodesic endpoint curves induced by it) and it's inverse, the logarithmic map. An evident shortcoming of such an approach is that in many cases it is not obvious that there exist a computationally efficient way of computing the exponential map or its inverse. To address this drawback the paper suggests the usage of a broader family of manifold curves that are induced by a family of manifold valued functions called retractions. This family of curves is used by the authors in a generalization proposed by them to an algorithm which was originally designed for the Euclidean space. In the Euclidean setting, given two points, the original algorithm constructs a polynomial curve between them, the algorithm is called de Casteljau algorithm and the authors are generalizing it to manifolds and retraction endpoint curves in order to solve the Hermite interpolation problem on Riemannian manifolds.

Introduction

In This section I would want to give some background and formal definitions that I find crucial for understanding the main results and the general contribution of the paper.

I begin by introducing the concept of the Hermite interpolation problem.

The manifold Hermite interpolation problem is formulated as follows

Given $N+1$ tangent bundle data points $\{(p_i, v_i)\}_{i=0}^{N} \in T\mathcal{M}$ and scalar parameters $t_0 < t_1 < \cdots < t_N$, find a continuously differentiable curve $H : [t_0, t_N] \to \mathcal{M}$ such that

$$\begin{cases} H(t_i) = p_i, \\ \dot{H}(t_i) = v_i, \end{cases} \quad \forall i = 0, \ldots, N$$

In the Euclidean setup there is a famous algorithm named de Casteljau algorithm that produces polynomial endpoint curves with a certain degree of freedom in prescribing the derivatives of the produced curve at the end points. A formal description of the algorithm in the Euclidean setup is as follows

Given $x, y \in \mathbb{R}^D$ we let $\sigma_1(t; x, y) := (1-t)x + ty$ be the linear endpoint curve connecting x and y in $\mathbb{R}^D$. Now given $N+1$ control points $b_0, \ldots, b_N \in \mathbb{R}^D$ the following relation defines the recursive nature of the algorithm

$$\sigma_k(t, ; b_i, \ldots, b_{k+i}) := \sigma_1(t; \sigma_{k-1}(t; b_i, \ldots, b_{k+i-1}), \sigma_{k-1}(t; b_{i+1}, \ldots, b_{k+i})),$$

$$i = 0, \ldots, N-k$$

The polynomial curve σ_N constructed by recursively applying the above relation, satisfies the following properties

$$\sigma_N(0; b_0, \ldots, b_N) = b_0,$$

$$\sigma_N(1; b_0, \ldots, b_N) = b_N,$$

$$\dot{\sigma}(0; b_0, \ldots, b_N) = N(b_1 - b_0),$$

$$\dot{\sigma}(1; b_0, \ldots, b_N) = N(b_N - b_{N-1}),$$

As can be seen above, the property of the derivative of the σ_N curve at the endpoints enables us with allot of flexibility in the choice of a desired tangent vector at the endpoints since it only depends on the two control points b_1 and b_{N-1} and hence this kind of method of constructing endpoint curves is very useful for Hermite interpolation problems. The main result of this paper is the generalization of this algorithm into the general setting of Riemannian manifolds.

Before advancing into describing the main idea of the paper I would want to formally introduce the concept of retractions. A retraction R is a smooth map from the tangent bundle $T\mathcal{M}$ into the manifold $\mathcal{M}$ that is defined in a neighborhood of the origin of $T_x\mathcal{M}$ for all $x \in \mathcal{M}$. We denote R_x to be the restriction of R to $T_x\mathcal{M}$. a retraction R satisfies the following two properties for all $x \in \mathcal{M}$

$$R_x(0) = x,$$

$$DR_x(0) = I_{T_x\mathcal{M}}$$

We notice that the second property guarantees us for every $x \in \mathcal{M}$ a neighborhood of $\vec{0} \in T_x\mathcal{M}$ in which R_x is invertible. we denote the inverse map as R_x^{-1}.

The main idea of the paper

The main practical result developed in the paper is the generalization of the de Casteljau algorithm from the Euclidean space of $\mathbb{R}^D$ into an arbitrary Riemannian manifold $\mathcal{M}$ such that the linear endpoint curve defined by σ_1 in $\mathbb{R}^D$ is now replaced by a new family of endpoint curves called r-endpoint retraction curves and for $x, y \in \mathcal{M}$ [1] are given by the following formula

$$c_r(t; x, y) = R_{q(r)}((1-t)R^{-1}_{q(r)}(x) + tR^{-1}_{q(r)}(y))$$

Where $r, t \in [0,1]$ and $q(r) = R_x(rR_x^{-1}(y)$

This family of curves satisfies two very important properties in relation to our cause:

(i) $c_r(0; x, y) = x$ and $c_r(1; x, y) = y$ for every $r \in [0,1]$

(ii) $\dot{c}_0(0; , x, y) = R_x^{-1}(y)$ and $\dot{c}_0(1; , x, y) = -R_y^{-1}(x)$

Now by using the r-endpoint retraction curves as building blocks, given two points $(x, v_1), (y, v_2) \in T\mathcal{M}$ in the tangent bundle, we would want to construct a C^1 endpoint curve α connecting x and y s.t $\dot{\alpha}(0) = v_1$ and $\dot{\alpha}(1) = v_2$.

To help us do exactly that we are to use an important proposition given by the authors combined with the use of r-endpoint retraction curves and a careful tangent vectors selection, the proposition states the following

Given $b_0, b_1, b_2, b_3 \in \mathcal{M}$ and the following smooth curves

- $\beta_i : [0,1] \to \mathcal{M}$ joining b_i and b_{i+1} for $i = 0, 1, 2$

-$\beta_{01} : [0,1]^2 \to \mathcal{M}$ such that $\beta_{01}(\cdot, t)$ joins $\beta_0(t)$ and $\beta_1(t)$ for every $t \in [0,1]$

-$\beta_{12} : [0,1]^2 \to \mathcal{M}$ such that $\beta_{12}(\cdot, t)$ joins $\beta_1(t)$ and $\beta_2(t)$ for every $t \in [0,1]$

-$\beta_{012} : [0,1]^3 \to \mathcal{M}$ such that $\beta_{012}(\cdot, s, t)$ joins $\beta_{01}(s,t)$ and $\beta_{12}(s,t)$ for every $t, s \in [0,1]$

[1] Here I omit the the conversation about the neighborhood in which the retraction and it's inverse are well defined and just assume we operate in that domain where the maps are well-defined

And if in addition we have:

(i) $\beta_{01}(s,0)=\beta_0(s)$ and $\beta_{12}(s,1)=\beta_2(s)$

(ii) $\beta_{012}(s,0,0)=\beta_0(s)$ and $\beta_{012}(s,1,1)=\beta_2(s)$

Then the following curve

$$\beta(t)=\beta_{012}(t,t,t)$$

Is a smooth endpoint curve on $\mathcal{M}$ between the points b_0 and b_3 and also in addition we have $\dot{\beta}(0)=3\dot{\beta}_0(0)$ and $\dot{\beta}(1)=3\dot{\beta}_2(1)$

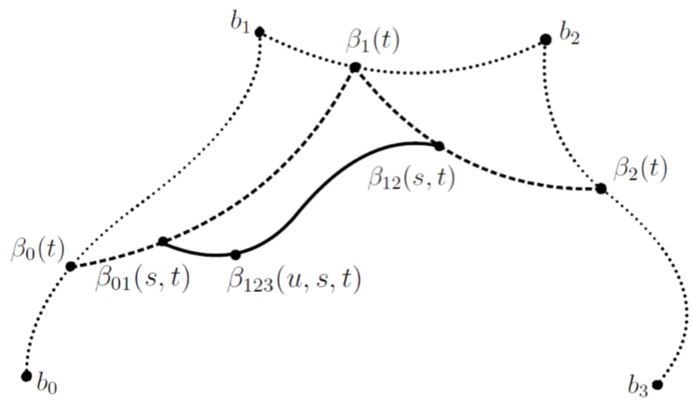

Figure 1: An illustration explaining the relation between $\beta_0, \beta_1, \beta_2, \beta_{01}, \beta_{02}, \beta_{012}$

Now we apply the proposition above and pick candidates from our r-endpoint retraction curves family for the roles of $\beta_0, \beta_1, \beta_2, \beta_{01}, \beta_{02}$ and β_{012} in the following manner[2]:

$$\beta_0(t)=c_0(t;b_0,b_1), \beta_1(t)=c_{r_1(t)}(t;,b_1,b_2), \beta_2(t)=c_1(t;b_2,b_3),$$

$$\beta_{01}(s,t)=c_{r_{01}(s,t)}(s,\beta_0(t),\beta_1(t), \beta_{12}(s,t)=c_{r_{12}(s,t)}(s,\beta_1(t),\beta_2(t)),$$

$$\beta_{012}(u,s,t)=c_{r_{012}(u,s,t)}(u;\beta_{01}(s,t),\beta_{12}(s,t))$$

Now by denoting

$$\beta(t)=\beta_{012}(t,t,t)$$

[2]Here it should be mentioned that the only restriction on $r_1, r_{01}, r_{12}, r_{012}$ is that they should be smooth functions and satisfy: $r_{01}(s,0)=0, r_{12}(s,1)=1, r_{012}(s,0,0)=0, r_{012}(s,1,1)=1$. for reasons of computational efficiency and general simplicity the authors chose the following permanent candidates:
$r_1(t)=\frac{1}{2}, r_{01}(s,t)=0, r_{12}(s,t)=1, r_{012}(u,s,t)=t$

We get our generalized de Casteljau curve β on 4 control points $b_0, b_1, b_2, b_3 \in \mathcal{M}$, which we denote altogether as $\beta(\cdot; b_0, b_1, b_2, b_3)$.

Next and almost finally we make a concrete choice of the two control points b_1 and b_2 and connect all previously shown pieces to form the desired solution of the Generalized Hermite interpolation problem presented.

Given $(b_0, v_0), (b_3, v_3) \in T\mathcal{M}$ we pick:

$$b_1 = R_{b_0}(\frac{1}{3}v_0), b_2 = R_{b_3}(-\frac{1}{3}v_3)$$

Now letting $\alpha(t) \equiv \alpha(t; p_0, v_0, p_3, v_3) := \beta(t; b_0, b_1, b_2, b_3)$, we get:

$$\alpha(0) = b_0, \alpha(1) = b_3, \dot{\alpha}(0) = v_0, \dot{\alpha}(1) = v_3$$

Now given $N + 1$ tangent bundle data points $\{(p_i, v_i)\}_{i=0}^{N} \in T\mathcal{M}$ and scalar parameters $t_0 < t_1 < \cdots < t_N$, by denoting $h_i = t_{i+1} - t_i$ for $i = 0, \ldots, N-1$, the manifold curve $H : [t_0, t_N] \to \mathcal{M}$ defined piecewise as

$$H(t)|_{[t_i, t_{i+1}]} = \alpha(\frac{t - t_i}{h_i}; p_i, h_i v_i, p_{i+1}, h_i v_{i+1})$$

Is our solution to the generalized Hermite Interpolation problem.

More details worth mentioning

A known result from differential geometry regarding Riemannian manifolds is the existence of so called geodesic convex neighborhoods around each point on the manifold. The meaning behind it is that for every two points in such a neighborhood of geodesic convexity, the endpoint geodesic curve connecting the two points and minimizing the distance between them is also contained in that neighborhood.
Since our paper generalizes the concept of geodesic endpoint curves to a endpoint retraction curves, the concept of retraction-convex sets is introduced and the existence of retraction-convex neighborhood around each point on the manifold in then proved.This theoretical labor is essential for the claim that for a given 4 sufficiently close control points, the retraction curves defined earlier, the retraction itself and it's inverse are actually well defined.

Another result established and proved in the paper is the bound on the interpolation error given by:

$$\max_{\tau \in [0,1]} d(\gamma(\tau), H(\tau)) < \kappa h^4$$

where $h = \max_{i=0,\ldots,N-1} t_{i+1} - t_i$ and the interpolated data samples are considered to be

$$\{\gamma(t_i), \dot{\gamma}(t_i)\}_{i=0}^{i=N} \in T\mathcal{M}$$

$N + 1$ samples of some smooth manifold curve γ and it's derivative $\dot{\gamma}$.