Optimal Control

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0 Introduction

$$\dot{x} = f(t, x, u), \quad x(t_0) = x_0, \quad t \in [t_0, t_f]$$

$$f : [t_0, t_f] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$$

$$x = \text{state}, \quad u = \text{input}$$

Initial Value Problem (IVP)

Given $x_0, u(\cdot)$ we can compute $x(\cdot)$

↼ functions of time ⇀

When is this possible? It depends on f.

Lemma 0.1 (Sufficient conditions)

Existence & Uniqueness of solutions of ODEs.
Assume that

- f is piecewise continuous in t and u
- f is globally Lipschitz in x

$$\exists\, k(t, u) \text{ s.t. } \|f(t, x_1, u) - f(t, x_2, u)\| \leq k(t, u)\|x_1 - x_2\|, \; \forall x_1, x_2 \in \mathbb{R}^{n_x}$$

Then $x(\cdot)$ exists for all t and is unique.

Remarks

- Lipschitz continuous $\Rightarrow$ continuous, but not the converse
- $\sqrt{x}$ is continuous but not Lipschitz, $\dot{x} = \sqrt{x}$ does not have a unique solution
- Continously differentiable $(\mathcal{C}^1) \Rightarrow$ locally Lipschitz continous $\forall x_1, x_2 \in \mathcal{X} \subset \mathbb{R}^{n_x}$
- Locally Lipschitz continuous x guarantees existence & uniqueness for small enough times

In this course we will assume $f \in \mathcal{C}^1$ and implicitely assume that t_f is chosen such that $x(\cdot)$ exists in $[t_0, t_f]$.

We do not need to worry about existence & uniqueness!

Goal in Optimal Control: Design u such that

1. $u(t) \in \mathcal{U}(t)$, $x(t) \in \mathcal{X}(t) \quad \forall t \in [t_0, t_f], \quad \mathcal{X} \subseteq \mathbb{R}^{n_x},\ \mathcal{U} \subseteq \mathbb{R}^{n_u}$
   sets defining constraints on $u \& x$
   $\Rightarrow$ Admissible input/state trajectories

2. The system behaves optimally according to

$$J(u) = \int_{t_0}^{t_f} l(t, x(t), u(t))\, dt + \varphi(t_f, x(t_f))$$

Cost function running cost terminal cost

$\Rightarrow$ optimal behaviour

Formally, we can state the goal as follows:
Find an admissible input $u^\star$ which causes the dynamics to follow an admissible trajectory $x\star$ which minimizes J, that is

$$\int_{t_0}^{t_f} l(t, x^\star(t), u^\star(t))\, dt + \varphi(t_f, x^\star(t_f)) \leq \int_{t_0}^{t_f} l(t, x(t), u(t))\, dt + \varphi(t_f, x(t_f))$$

$\forall$ admissible x, u

Examples of cost functions

1) Minimum-time problem
   Goal: transfer the system from x_0 to a set $\mathcal{S}$ in the minimum time

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt \qquad (l = 1, \varphi = 0)$$

$$x(t_f) \in \mathcal{S}$$

   Note: t_f is also a decision variable! The unknowns are (u, t_f).

2) Minimum control-effort problem

$$J = \int_{t_0}^{t_f} \|u(t)\|^2\, dt$$

$$x(t_f) \in \mathcal{S}$$

3) Tracking problem

$$J = \int_{t_0}^{t_f} (x(t) - r(t))^T Q (x(t) - r(t)) dt$$

$Q > 0$ (positive definit matrix: symmetric & all eigenvalues positive)
$r(t)$ given signal

1 Nonlinear Programming

Nonlinear Programs (NLP) are general <u>finite-dimensional</u> optimization problems:

$$\min_x f(x)$$

$$\text{s.t. } g(x) \leq 0, \quad h(x) = 0$$

$f : \mathbb{R}^n \to \mathbb{R}$, objective function
$g : \mathbb{R}^n \to \mathbb{R}^{n_g}$, inequality constraints
$h : \mathbb{R}^n \to \mathbb{R}^{n_h}$, equality constraints
Feasible set:

$$D = \{x \in \mathbb{R}^n \mid g(x) \leq 0,\ h(x) = 0\}$$

$\bar{x} \in D$ feasible point

Definition 1.1 (Global, local Minimizers)

$x^\star \in \mathcal{D}$ <u>Global Minimizer</u> of the NLP if

$$f(x^\star) \leq f(x) \quad \forall x \in \mathcal{D}$$

$f(x^\star)$ is the <u>Global Minimum</u> (or Minimum)
Nomenclature: $x^\star$ is also called (optimal) solution, $F(x^\star)$ is optimal value
$x^\star$ is a strict global minimizer if $f(x^\star) < f(x) \quad \forall x \in \mathcal{D}$
$x^\star \in \mathcal{D}$ <u>Local Minimizer</u> if

$$\exists \varepsilon > 0, \text{ s.t. } f(x^\star) \leq f(x) \quad \forall x \in B_\varepsilon(x^\star) \cap \mathcal{D}$$

$$B_\varepsilon(x) := \{y \mid \|x - y\| \leq \varepsilon\} \qquad \|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \text{ any norm in } \mathbb{R}^n$$

Strict local Minimizer if inequality holds strictly
Global min $\underset{\not\leftarrow}{\rightarrow}$ local min

Solving an NLP boils down to finding global or local minimizers.
Does a solution always exist? No.

Definition 1.2 (infimum)

Given $\mathcal{S} \subseteq \mathbb{R}$, $\inf(\mathcal{S})$ is the greatest lower bound of $\mathcal{S}$:

- $z \geq \inf(\mathcal{S}), \quad \forall z \in \mathcal{S} \quad$ (lower bound)
- $\forall \bar{\alpha} > \inf(\mathcal{S}) \quad \exists z \in \mathcal{S}$ s. t. $\bar{\alpha} > z \quad$ (greatest bound)

Example $\mathcal{S} = [-1, 1]$, $-50 = \inf(\mathcal{S})$? $\quad \rightarrow \quad$ No, $\inf(\mathcal{S}) = -1$

- Analogous: $\sup(\mathcal{S})$ is smallest upper bound.
- inf and sup always exist if $\mathcal{S} \neq \emptyset$
- $\inf(\mathcal{S})$ does not have to be an element of $\mathcal{S}$
- If $\mathcal{S}$ unbounded from below $\rightarrow \inf(\mathcal{S}) = -\infty$
- $\inf([a, b]) = \inf((a, b]) = a$

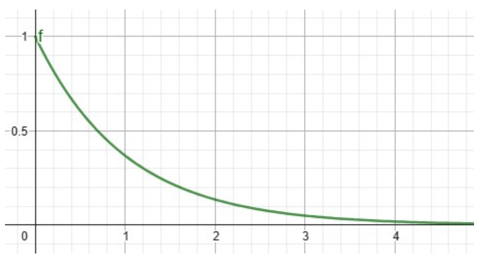
Connections with NLP?

$$f : \mathcal{D} \rightarrow \mathbb{R}$$

$$\inf(\underbrace{f(x) \,|\, x \in \mathcal{D}}_{\mathcal{S}}) := \bar{f} = \inf_{x \in \mathcal{D}} f(x) \quad \text{(similar to NLP)}$$

Whenever NLP has solution, then NLP ist equivalent to this, but $\nexists x^\star \in \mathcal{D}$ s. t. $f(x^\star) = \bar{f} \quad \rightarrow \quad$ infimum exists, but not minimum

Examples $f(x) = e^{-x}, \quad \mathcal{D} = [0, \infty), \quad \inf(\mathcal{S}) = 0$
$f(x) = x, \quad \mathcal{D} = \mathbb{R}, \quad \inf(\mathcal{S}) = -\infty$, min doesn't exist!



When does the infimum coincide with its optimal value?

Theorem 1.1 (Extreme value problem) (Weierstrass Theorem)

$f : D \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^n$
If:

- $f \in \mathcal{C}$ on $\mathcal{D}$
- $\mathcal{D}$ is compact
- $\mathcal{D} \neq \emptyset$

Then f attains a minimum on $\mathcal{D}$.

Definition 1.3 (Continuous function)

$f : \mathcal{D} \to \mathbb{R}$ is continuous at $x \in \mathcal{D}$ if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s. t. } \|x - x'\| < \delta \quad \Rightarrow \quad \|f(x) - f(x')\| < \epsilon$$

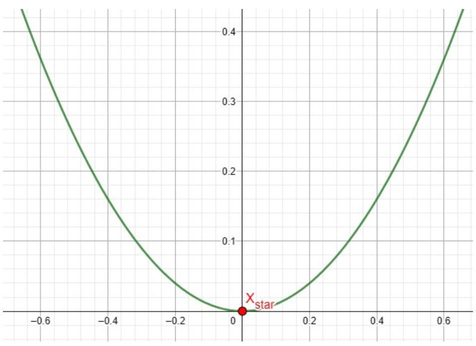
If f is continuous $\forall x \in \mathcal{D}$ then f is continuous on $\mathcal{D} \to f \in \mathcal{C}$

Implication for NLP: If f is $\mathcal{C}$ on $\mathcal{D}$ and $\mathcal{D}$ is compact and non-empty then [NLP] has a solution!

- $\mathcal{D} \subseteq \mathbb{R}^n$: in finite-dimensional spaces: compact = closed and bounded
  Not compact:
  - $(a, b]$ (not closed)
  - $(-\infty, b]$ (unbounded)

  Compact set:
  - $[a, b]$ $-\infty < a < b < \infty$

  Warning: $\mathcal{D}$ infinite dimensional (e. g. function space) then
  compact $\not\rightleftarrows$ bounded and closed



Theorem 1.1 is restrictive e. g. $f(x) = x^2, \mathcal{D} = (-\infty, \infty)$ has unique minimum

- Notation convention: Technically it is 'wrong' to write

$$\min_{x \in \mathcal{D}} f(x)$$

more compact is:

$$\underset{x \in \mathcal{D}}{\text{minimize}}\, f(x) \quad \text{or} \quad \inf_{x \in \mathcal{D}} f(x)$$

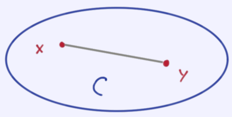
Goal of the Chapter: characterize necessary and sufficient conditions for $x^\star$ to be global minimizer of NLP.

Convexity

Definition 1.4 (Convex sets & functions)

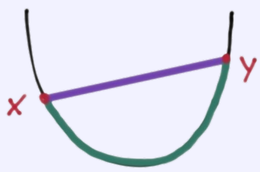
- A set $C \subseteq \mathbb{R}^n$ is <u>convex</u> (cvx) if $\forall x, y \in C$

$$\{z \mid z = \lambda x + (1-\lambda)\lambda,\ \lambda \in [0,1]\} \subseteq C$$



- Given a cvx set C, a function $f : C \to \mathbb{R}$ is cvx if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y \in C, \quad \lambda \in (0,1)$$



- f is strictly cvx if the inequality holds strictly.

Remarks

- The definition extends to vector functions $f : C \to \mathbb{R}^n$ for convex f_i
- $f : C_1 \times C_2 \to \mathbb{R}$
  $f(x, y)$ is jointly cvx, in x, y if $z := \begin{bmatrix} x \\ y \end{bmatrix}$, $f(z)$ is in cvx in z.

Example $f(x, y) = x^2 + y^2$, $z = \begin{bmatrix} x \\ y \end{bmatrix} \to f(z) = z_1^2 + z_2^2$

Definition 1.5

An NLP is a <u>convex program</u> if

- f is convex function,
- $\mathcal{D}$ is convex set.

Lemma 1.1

Let $x^\star$ be a local minimizer of cvx program. Then $x^\star$ is also global minimizer.

Proof: try as an erxercise

Minimizers of convex NLP form a convex set.
This set might be empty (Convex NLPs not guaranteed to have solution).
<u>However</u>: Unique solution for <u>strictly</u> convex NLPs, if a solution exists.

Lemma 1.2 (First/Second order conditions for convexity)

1. $f : C \to \mathbb{R}$ continuously differentiable on C. Then f is cvx iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \quad \forall x, y \in C$$

$$(\nabla f)_i = \frac{\partial f}{\partial x_i} \text{ is gradient (sometimes } f_{x_i})$$

2. f twice differentiable on C, then f convex iff

$$\nabla^2_{xx} f(x) \geq 0 \quad \forall x \in C$$

$$(\nabla^2_{xx})_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{(Hessian)}$$

- $A \geq 0$ means that: $A = A^T$ and pos semi-definite, i. e. all eigenvalues non-negative
- f strictly cvx if $\nabla^2_{xx} f(x) > 0 \quad \forall x \in C$ with $A > 0$ meaning pos definite and symmetric
- Interpretation: Curvature of function should be non-negative/positive
- For exercises to check convexity, the second condition is generelly useful. First condition is useful for proofs.

For $\mathcal{D} = \{x \mid g(x) \leq 0,\, h(x) = 0\}$ the following holds: If

$$\left.\begin{array}{l} \bullet\ g \text{ are convex functions,} \\ \bullet\ h \text{ are affine functions (i.e. } h(x) = 0 \Leftrightarrow Ax = b), \end{array}\right\} \text{sufficient}$$

then $\mathcal{D}$ is a convex set.

Example $a, b \in \mathbb{R}$

$$\begin{array}{ccc} \min\limits_x f & & \min\limits_x f \\ \text{s.t. } x^3 - 1 \leq 0 & \leftrightarrow & \text{s.t. } x - 1 \leq 0 \\ (ax + b)^2 = 0 & & ax + b = 0 \\ \text{non-convex} & & \text{convex} \\ \text{non-affine} & & \text{affine} \end{array}$$

<u>Moral to recognize convexity of NLP:</u>

1. Use definition of cvx NLP, cvx f, convex $\mathcal{D}$
2. If $\mathcal{D}$ written as equality/inequality-constraints, check g convex/h affine.
   If not, check further whether the feasible set is cvx or not (e.g. can be written equivalently with cvx g/affine h).

1.1 Unconstrained Problems

$$\mathcal{D} = \mathbb{R}^n$$

Assume throughout that $f \in \mathcal{C}^1$ (continuously differentiable).

Definition 1.6 (Descent Direction)

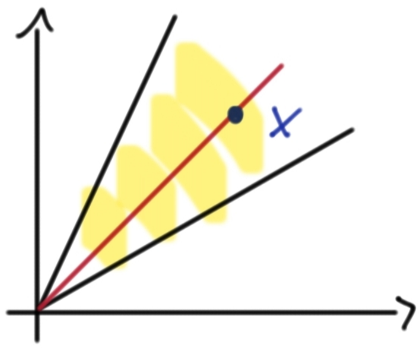
$d \in \mathbb{R}^n$ is a <u>descent direction</u> for f at $\bar{x} \in \mathbb{R}^n$ if

$$\exists \delta > 0 \quad \text{s.t.} \quad f(\bar{x} + \lambda d) < f(\bar{x}) \quad \forall \lambda \in (0, \delta).$$

$F(\bar{x})$: <u>Cone of decent directions</u>
Set of all descent directions of f at $\bar{x}$

A set $K \subseteq \mathbb{R}^n$ is a cone if it contains the full ray through any point in the set.

$$K \text{ cone if } \forall x \in K \text{ and } \rho \geq 0, \quad \rho x \in K$$



This is a geometric characterization of descent direction. It gives us a geometric condition for $x^\star$ to be a local minimizer.

Lemma 1.3 (Geometric Condition for local minimum)

$x^\star$ is a local minimizer iff

$$\mathcal{F}(\bar{x}) = \emptyset.$$

We want an algebraic condition to be able to compute or look for $x^\star$.

Lemma 1.4 (Algebraic first-order characterization of $\mathcal{F}$)

If $\nabla f(\bar{x}) \neq 0$, then

$$\mathcal{F}_0(\bar{x}) = \{d \mid \nabla f(\bar{x})^T d < 0\} = \mathcal{F}(\bar{x}).$$

Otherwise

$$\mathcal{F}_0(\bar{x}) \subseteq \mathcal{F}(\bar{x}).$$

Proof: try Taylor-series expansion of f at $\bar{x}$

Graphical interpretation:

∇f forms angles greater or equal than 90° with <u>all</u> descent directions.

Lemma 1.5 (First-order necessary condition for local minimum)

If $x^\star$ is a local minimizer, then

$$\underbrace{\nabla f(x^\star) = 0}_{\text{'stationary point'}}.$$

Proof: Contradiction

If $\nabla f(x^\star) \neq 0$, then $d = -\nabla f(x^*) \neq 0$. Therefore there exists a descent direction $d \in \mathcal{F}(x^\star)$ by Lemma 1.4. Thus $\exists \delta > 0$ s.t. $f(x^\star + \lambda d) < f(x^\star) \quad \forall \lambda \in (0, \delta)$.
This is a contradiction with the fact, that $x^\star$ is a minimizer. $\square$

Why only necessary?
It can't be a sufficient condition because in case where $\nabla f(x^\star) = 0$ we cannot use Lemma 1.4, e.g. $f_1(x) = -x^2$, $f_2(x) = x^3$, $\nabla f_1(0) = \nabla f_2(0) = 0$.