Optimal Control

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0 Introduction

$$\dot{x} = f(t, x, u), \quad x(t_0) = x_0, \quad t \in [t_0, t_f]$$

$$f : [t_0, t_f] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$$

$$x = \text{state}, \quad u = \text{input}$$

Initial Value Problem (IVP)

Given $x_0, u(\cdot)$ we can compute $x(\cdot)$

↼ functions of time ⇀

When is this possible? It depends on f.

Lemma 0.1

(Sufficient conditions)

Existence & Uniqueness of solutions of ODEs.

Assume that

- f is piecewise continuous in t and u
- f is globally Lipschitz in x

$$\exists\, k(t, u) \text{ s.t. } \|f(t, x_1, u) - f(t, x_2, u)\| \leq k(t, u)\|x_1 - x_2\|, \ \forall x_1, x_2 \in \mathbb{R}^{n_x}$$

Then $x(\cdot)$ exists for all t and is unique.

Remarks

- Lipschitz continuous $\Rightarrow$ continuous, but not the converse
- $\sqrt{x}$ is continuous but not Lipschitz, $\dot{x} = \sqrt{x}$ does not have a unique solution
- Continously differentiable ($\mathcal{C}^1$) $\Rightarrow$ locally Lipschitz continous $\forall x_1, x_2 \in \mathcal{X} \subset \mathbb{R}^{n_x}$
- Locally Lipschitz continuous x guarantees existence & uniqueness for small enough times

In this course we will assume $f \in \mathcal{C}^1$ and implicitely assume that t_f is chosen such that $x(\cdot)$ exists in $[t_0, t_f]$.

We do not need to worry about existence & uniqueness!

Goal in Optimal Control:

Design u such that

1. $u(t) \in \underset{\uparrow}{\mathcal{U}}(t)$, $x(t) \in \underset{\uparrow}{\mathcal{X}}(t) \quad \forall t \in [t_0, t_f], \quad \mathcal{X} \subset \mathbb{R}^{n_x}$, $\mathcal{U} \subset \mathbb{R}^{n_u}$
   sets defining constraints on $u \& x$
   $\Rightarrow$ Admissible input/state trajectories

2. The system behaves optimally according to

$$\underset{\uparrow}{J}(u) = \int_{t_0}^{t_f} \underset{\uparrow}{l}(t, x(t), u(t))\, dt + \underset{\uparrow}{\varphi}(t_f, x(t_f))$$

   Cost function running cost terminal cost
   $\Rightarrow$ optimal behaviour

Formally, we can state the goal as follows:
Find an admissible input $u^\star$ which causes the dynamics to follow an admissible trajectory $x\star$ which minimizes J, that is

$$\int_{t_0}^{t_f} l(t, x^\star(t), u^\star(t))\, dt + \varphi(t_f, x^\star(t_f)) \leq \int_{t_0}^{t_f} l(t, x(t), u(t))\, dt + \varphi(t_f, x(t_f))$$

$\forall$ admissible x, u

Examples of cost functions

1) Minimum-time problem
   Goal: transfer the system from x_0 to a set $\mathcal{S}$ in the minimum time

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt \qquad (l = 1, \varphi = 0)$$

$$x(t_f) \in \mathcal{S}$$

   Note: t_f is also a decision variable! The unknowns are (u, t_f).

2) Minimum control-effort problem

$$J = \int_{t_0}^{t_f} \|u(t)\|^2\, dt$$

$$x(t_f) \in \mathcal{S}$$

3) Tracking problem

$$J = \int_{t_0}^{t_f} (x(t) - r(t))^T Q (x(t) - r(t)) dt$$

$Q > 0$ (positive definit matrix: symmetric & all eigenvalues positive)
$r(t)$ given signal

1 Nonlinear Programming

Nonlinear Programs (NLP) are general finite-dimensional optimization problems:

$$\min_x f(x)$$

$$\text{s.t. } g(x) \leq 0, \quad h(x) = 0$$

$f : \mathbb{R}^n \to \mathbb{R}$, objective function
$g : \mathbb{R}^n \to \mathbb{R}^{n_g}$, inequality constraints
$h : \mathbb{R}^n \to \mathbb{R}^{n_h}$, equality constraints
Feasible set:

$$D = \{x \in \mathbb{R}^n \mid g(x) \leq 0,\ h(x) = 0\}$$

$\overline{x} \in D$ feasible point

Definition 1.1

Global, local Minimizers
$x^\star \in \mathcal{D}$ Global Minimizer of the NLP if

$$f(x^\star) \leq f(x) \quad \forall x \in \mathcal{D}$$

$f(x^\star)$ is the Global Minimum (or Minimum)
Nomenclature: $x^\star$ is also called (optimal) solution, $F(x^\star)$ is optimal value
$x^\star$ is a strict global minimizer if $f(x^\star) < f(x) \quad \forall x \in \mathcal{D}$
$x^\star \in \mathcal{D}$ Local Minimizer if

$$\exists \varepsilon > 0, \text{ s.t. } f(x^\star) \leq f(x) \quad \forall x \in B_\varepsilon(x^\star) \cap \mathcal{D}$$

$$B_\varepsilon(x) := \{y \mid \|x - y\| \leq \varepsilon\} \qquad \|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \text{ any norm in } \mathbb{R}^n$$

Strict local Minimizer if inequality holds strictly
Global min $\underset{\not\leftarrow}{\rightarrow}$ local min

Solving an NLP boils down to finding global or local minimizers.
Does a solution always exist? No.