Consider a star with a density profile as a function of radius:

$$\rho(r) = \rho_0 \left(1 - \left(\frac{r}{R}\right)^2\right)$$

where is ρ_0 the central density (a constant) and R is the total radius of the star (a constant).

Part A

By integrating equations of stellar structure, show that the total mass of the star is given by the expression:

$$M = \frac{8\pi\rho_0 R^3}{15}$$

Solution:
For this part, we will need to apply the equation for the mass distribution inside a star. This equation relates the change in mass over the radius of the star to its density at r and r^2.

$$\frac{dM_r}{dr} = 4\pi\rho(r)r^2$$

Substitute the given definition of $\rho(r)$ into the above equation.

$$\frac{dM_r}{dr} = 4\pi\rho_0 \left(1 - \left(\frac{r}{R}\right)^2\right) r^2$$

After applying some algebraic simplification, we get

$$\frac{dM_r}{dr} = 4\pi\rho_0 \left(r^2 - \frac{r^4}{R^2}\right)$$

From here, we can integrate both sides and create a general equation for M_r.

$$M_r = \int 4\pi\rho_0 \left(r^2 - \frac{r^4}{R^2}\right) dr$$

$$M_r = 4\pi\rho_0 \left(\frac{r^3}{3} - \frac{r^5}{5R^2}\right) + C$$

However, for this part of the problem, we want the total mass of the star which is $M = M_R$. $C = 0$ because $M_0 = 0$. Substituting in R for r yields our desired result:

$$M = 4\pi\rho_0 \left(\frac{R^3}{3} - \frac{R^5}{5R^2}\right)$$

$$M = 4\pi\rho_0 \left(\frac{5R^3}{15} - \frac{3R^3}{15}\right)$$

$$M = \frac{8\pi\rho_0 R^3}{15}$$

Part B

Derive the following equation for central pressure:

$$P_{core} = \frac{15}{16\pi}\frac{GM^2}{R^4}$$

Solution:
To derive the equation for the pressure in the core of the star, we'll need to use the differential equation for hydrostatic equilirbium:

$$\frac{dP}{dr} = -G\frac{M_r(r)\rho(r)}{r^2}$$

We start by substituting values for $M_r(r)$ and $\rho(r)$ from the previous problem. Note that we substitute M_r here, NOT the final result of M because we need to integrate over r.

$$\frac{dP}{dr} = -G\frac{4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^5}{5R^2}\right)\rho_0\left(1 - \left(\frac{r}{R}\right)^2\right)}{r^2}$$

$$\frac{dP}{dr} = -4\pi\rho_0^2 G\frac{1}{r^2}\left(\frac{r^3}{3} - \frac{r^5}{3R^2} - \frac{r^5}{5R^2} + \frac{r^7}{5R^4}\right)$$

$$\frac{dP}{dr} = -4\pi\rho_0^2 G\frac{1}{15R^4}\left(5rR^4 - 5r^3R^2 - 3r^3R^2 + 3r^5\right)$$

Integrate with respect to r over 0 to R (over the whole star):

$$P = -4\pi\rho_0^2 G\frac{1}{15R^4}\int_0^R \left(5rR^4 - 5r^3R^2 - 3r^3R^2 + 3r^5\right) dr$$

$$P = -4\pi\rho_0^2 G\frac{1}{15R^4}\left(\frac{5}{2}r^2R^4 - \frac{5}{4}r^4R^2 - \frac{3}{4}r^4R^2 + \frac{1}{2}r^6\right)\Bigg|_{r=0}^{r=R}$$

Apply the definite integral bounds and simplify.

$$P = 4\pi\rho_0^2 G\frac{1}{15R^4}\left(\frac{5}{2}R^6 - \frac{5}{4}R^6 - \frac{3}{4}R^6 + \frac{1}{2}R^6\right)$$

$$P = -4\pi\rho_0^2 G\frac{R^2}{15}$$

Since P_{core} must be positive and is the magnitude of P at $r = 0$, we can take the absolute value of P to remove the negative sign.

$$P_{core} = 4\pi\rho_0^2 G\frac{R^2}{15}$$

All that we're missing here is to put ρ_0 in terms of M. From here, we'll solve for ρ_0^2 using M^2:

$$M^2 = \frac{64\pi\rho_0^2R^6}{225}$$

$$\rho_0^2 = \frac{225M^2}{64\pi^2 R^6}$$

Substituting this value into P results in our desired equation:

$$P_{core} = 4\pi\left(\frac{225M^2}{64\pi^2 R^6}\right)G\frac{R^2}{15} = \boxed{\frac{15}{16\pi}\frac{GM^2}{R^4}}$$