Generalized Rational Sampling Rate Conversion Polyphase FIR Filter

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***Abstract*—In this letter, we derived a mathematical expression that can help in developing a generalized rational sampling rate conversion polyphase finite impulse response filter, for all relatively prime values of the upsampling (L) and downsampling (M) factors. In contrast to the existing approaches, the proposed structure efficiently exploits the involved noncausality to eliminate input delay requirements. In addition, the minimization of output delay requirement is presented. A numerical example is also studied to validate the proposed structure. We further evaluate the performance of the proposed structure in terms of total delay requirements ($\mathcal{D}$), multiplication complexity ($\mathcal{M}$), and addition complexity ($\mathcal{A}$). Compared to a similar recent approach, the proposed structure is found to be more efficient in terms of $\mathcal{D}$ for $L < M$ and $L > M$. However, it has reduced computational complexity in terms of $\mathcal{A}$ than the existing approach for $L < M$ and same for $L > M$, whereas both approaches have same $\mathcal{M}$ for $L < M$ and $L > M$.**

***Index Terms*—Multirate system, polyphase finite impulse response (FIR) filter, sampling rate conversion.**

I. Introduction

The development of new paradigms such as Internet of Things, cognitive cooperative communication, has setup a strong demand to research community for designing efficient multirate signal processing algorithms. The block diagram of a conventional multirate system is shown in Fig. 1, where $x(n)$ is the input signal with sampling frequency f_{in}, L denotes an upsampler, M is the downsampler, and $h(n)$ is the impulse response of finite impulse response (FIR) filter which removes the image spectra generated by L as well as the aliased spectra produced by M. $y(m)$ is the output signal with sampling frequency $f_{\text{out}} = \frac{L}{M} f_{\text{in}}$ which can be given as $y(m) = \sum_{n=0}^{N-1} x(n)h(mM - nL)$. The multirate system of Fig. 1 operates at maximum sampling frequency Lf_{in}, thus it has high computational complexity. Whereas, the polyphase structures operate at intermediate sampling frequency (Type-1 at Lf_{in}/M and Type-2 at f_{in}) have relatively lower computational complexity [1]–[4].

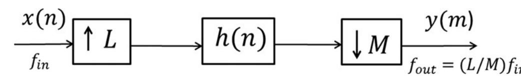

Fig. 1. Schematic diagram of a rational sampling rate conversion filter.

Hsiao in [5] has used network transformation principles and developed two simple polyphase structures for the cases $L > M$ and $L < M$. Both the structures are generalized in the respective range of L/M and perform filtering at minimum sampling frequency f_{in}/M, but they required more delay units. Using real valued fast cyclic convolution algorithms, single generalized structure for rational rate sampling was presented in [6]. The configuration of such structure is independent of M and L, however, the input and output are required to be arranged in specific polyphase decomposition. The optimization of polyphase filter coefficient matrix is another approach for designing the multirate systems as described in [7]–[11], however, none of these papers have presented the general structure. Bi *et al.* in [12] have addressed the problem of higher delay requirements of [5], and presented a design approach with reduced delay requirements. Moreover, noncausality is a crucial issue which is invariably involved in designing the computationally efficient multirate systems. This feature enhances the delay requirement of the existing approaches as explicitly mentioned in [13] for its structure. Therefore, to the best of author's knowledge, a generalized polyphase structure [for arbitrary conversion rate (L/M)] which can efficiently handle the noncausality with reduced computational complexity, lower delay requirements, and easy implementation is not yet available in the literature.

In the light of preceding discussions, we have developed a polyphase rational sampling rate conversion filter that operates at sampling frequency f_{in}/M in achieving $f_{\text{out}} = \frac{L}{M} f_{\text{in}}$ for arbitrary values of L and M. The simplicity of the proposed structure lies in the fact that it can be built with the aid of single mathematical relationship by incorporating L and M at predefined locations. The proposed system exploits the involved noncausality to eliminate the input delay requirements such that M consecutive samples followed by respective sequential samples are the desired inputs of M parallel filtering branches.

II. Proposed Approach

We develop a mathematical expression as per Proposition 1 of Section II-A, which facilitates the design of a generalized polyphase rate conversion filter that performs arithmetic operations at f_{in}/M, as shown in Fig. 2. The proposed generalized structure arranges the input samples as per the requirements of

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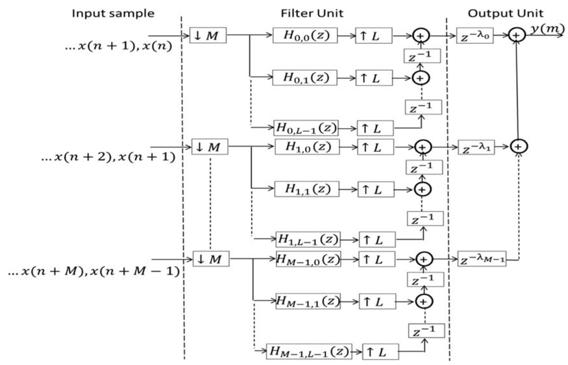

Fig. 2. Proposed structure of generalized polyphase FIR filter having M subfilter. Each subfilter consists of L cosubfilter.

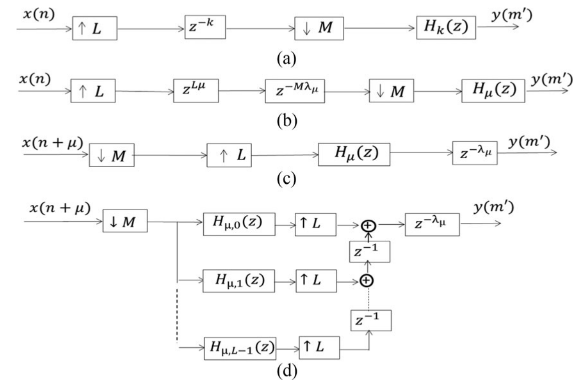

Fig. 3. (a) kth branch of Type-1 polyphase rate converter. (b) After substitution $k = M\lambda_\mu - L\mu$. (c) μth branch after rearranging the input and output delays. (d) μth branch of the proposed structure. The output signal $y(m')$ is same for all representation of Fig. 3.

M subfilters of the filtering unit. Its output unit adds the outputs of these subfilters after introducing suitable delay in each subfilter. The output delay requirements can be further reduced, as described in Section II-B. The performance measures are presented in Section II-C.

A. Derivation of the Mathematical Expression for Designing the Proposed System

The recent similar studies did not efficiently exploit the noncausality with the rearrangement of subfilters in designing the polyphase FIR filter, and hence lead to the separate structures for $L < M$ and $L > M$. Motivated by this, we restructure the polyphase FIR filter and develop a generalized structure which is applicable for $L < M$ and $L > M$, whose desired mathematical expression is given as follows.

Proposition 1: A Type-1 polyphase rational sampling rate converter can be reorganized using network transformation concepts, and by changing the input sequence from $x(n)$ to $x(n + \mu)$, the modified structure can be mathematically represented as

$$H_e(z) = \sum_{\mu=0}^{M-1}\left[\sum_{\gamma=0}^{L-1}\underbrace{\left[\sum_{j=0}^{\lceil \frac{N}{ML}\rceil -1} h_{(jML+M\gamma+M\lambda_\mu-L\mu)} z^{-j}\right]}_{\triangleq H_{\mu,\gamma}(z)} z^{-\gamma}\right] z^{-\lambda_\mu} \tag{1}$$

where μ is the branch index, N denotes the length of the FIR filter, $\lceil \frac{N}{ML} \rceil$ denotes the largest integer that is greater than or equal to $\frac{N}{ML}$, $h_{(jML+M\gamma+M\lambda_\mu-L\mu)}$ denotes the jth coefficient of the γth cosubfilter for the μth branch, and λ_μ is the delay of μth branch. This expression is only valid for the predefined locations of L and M, as shown[1] in Fig. 2.

Proof. The filter of Fig. 1 is realized using polyphase decomposition principles, and L and M moved into each subfilter branch. Furthermore, M is shifted before the coefficient multipliers, therefore, the equivalent diagram of the kth branch with input $x(n)$ and output $y(m')$ is shown in Fig. 3 (a). It can be mathematically represented as

$$H(z) = \sum_{k=0}^{M-1} z^{-k} H_k(z), \tag{2}$$

$$\text{where } H_k(z) = \sum_{i=0}^{\lceil \frac{N}{M}\rceil -1} h_{(iM+k)} z^{-i}. \tag{3}$$

Considering that L and M are relatively prime[2], the substitution $k = M\lambda_\mu - L\mu$ yields two changes in Fig. 3(a) as follows: 1) The delay z^{-k} is replaced by two isolated delay blocks, $z^{L\mu}$ and $z^{-M\lambda_\mu}$. 2) This kth branch is mapped[3] into the μth branch of the proposed structure, however, the filter coefficients remain unchanged. Such changes are shown in Fig. 3(b). To proceed further, $z^{L\mu}$ is moved to the extreme left side and $z^{-M\lambda_\mu}$ is shifted to the extreme right side of the μth branch. Then, L and M can be interchanged. Moreover, z^{μ} can be eliminated by accommodating its effect in changing the input sample from $x(n)$ to $x(n + \mu)$. Such input samples can be obtained using the tapped shift register or commutator. Also, as described in [3], [7], and [17], this step eliminates the need of accounting the z^{μ} in delay computation. Consequently, the modified structure of the μth branch can be represented, as shown in Fig. 3(c). Recalling (2), by combining all M branches, as represented in Fig. 3(c), the desired mathematical expression can now be expressed as

$$H_e(z) = \sum_{\mu=0}^{M-1} H_\mu(z) z^{-\lambda_\mu}, \tag{4}$$

$$\text{where } H_\mu(z) = \sum_{i=0}^{\lceil \frac{N}{M}\rceil -1} h_{(iM+M\lambda_\mu-L\mu)} z^{-i}. \tag{5}$$

[1] After finding the coefficients of all cosubfilters and the delay of each subfilter using (1), L must be inserted after coefficient multipliers in each cosubfilter, and each subfilter should be connected to input samples through M.

[2] Note that z^{-k} can be replaced with $z^{-(\lambda_\mu M - L\mu)}$ (if the integers λ_μ and μ satisfying $k = \lambda_\mu M - L\mu$ for integer value of k), and L and M in cascade can also be interchanged.

[3] For example, we consider a scenario when $M = 3$ and $L = 4$, so that $k = 3\lambda_\mu - 4\mu$, therefore, $k = 0, 1, 2$ branches of Type-1 polyphase filter are mapped as $\mu = 0, 2, 1$ branches of the proposed system by introducing $\lambda_0 = 0$, $\lambda_2 = 3$, $\lambda_1 = 2$.

Furthermore, polyphase decomposition of the subfilter $H_\mu(z)$ in (5) into L parallel cosubfilters (where all polyphase branches have same sample at input side and the delays are present at the output side) is done. Then, L is moved into each cosubfilter branch at its extreme right, therefore, Fig. 3(c) is modified into Fig. 3(d). Consequently, $H_\mu(z)$ can be mathematically expressed as

$$H_\mu(z) = \sum_{\gamma=0}^{L-1} z^{-\gamma} H_{\mu,\gamma}(z) \tag{6}$$

where $H_{\mu,\gamma}(z)$ represents the γth cosubfilter branch of μth subfilter which can be given as

$$H_{\mu,\gamma}(z) = \sum_{j=0}^{\lceil \frac{N}{ML} \rceil - 1} h_{(M(\gamma+jL)+M\lambda_\mu - L\mu)} z^{-j}. \tag{7}$$

By invoking (7) into (6) and the result into (4), the mathematical expression as shown in (1) is obtained. ■

B. Minimization of the Delay Requirements of the Output Unit

The output unit in Fig. 2 has total number of delay equals to $\sum_{\mu=0}^{M-1} \lambda_\mu$. However, such delays can be further minimized by selecting the minimum value of λ_μ which can map the kth branch into the μ branch, for $k, \mu \in [0, M-1]$. Moreover, such mapping helps in finding the lower and upper limits of λ_μ, which can be explained as follows. When $k = \mu = 0$, we have $\lambda_\mu = \frac{k+L\mu}{M} = 0$, therefore, the lower limit of λ_μ is 0. For $k = \mu = M-1$, we have $\lambda_\mu = \frac{M-1}{M}(L+1)$; since the ratio $\frac{M-1}{M}$ is always less than one for any value of M, it leads to $\lambda_\mu = \lceil \frac{(M-1)}{M}(L+1) \rceil = L+1$. Thus, the upper limit[4] of λ_μ is $L+1$. Hence, to minimize the output delays, $\bar{\lambda} = \max(\lambda_0, \lambda_1, \ldots, \lambda_{M-1})$ should be considered for designing a ladder of delays with the aim of replacing all output delay elements of all branches by suitably rearranging the adders (without using additional adders), as illustrated with an example in Section III.

C. Performance Measures of the Proposed Structure

The performance of the proposed system can be evaluated in terms of total delay requirements ($\mathcal{D}$), multiplication complexity ($\mathcal{M}$), and addition complexity ($\mathcal{A}$) as explained hereunder.[5]

1) Total delay requirements ($\mathcal{D}$): The total delay required in the proposed system can be evaluated as

$$\begin{aligned} \mathcal{D} = {} & \text{total delays in output unit } (\bar{\lambda}) \\ & + \text{total delays in filter unit } (N - M). \end{aligned} \tag{8}$$

2) Multiplication complexity ($\mathcal{M}$): It is defined as the total number of multiplications per output sample. Therefore, the $\mathcal{M}$ can be mathematically expressed as

$$\mathcal{M} = \text{filter length } (N)/\text{upsampler } (L). \tag{9}$$

3) Addition complexity ($\mathcal{A}$): It is defined as the total number of additions per output sample, and can be mathematically expressed as

$$\begin{aligned} \mathcal{A} = {} & (\text{total adders in filter unit } (N - ML) + \text{total} \\ & \text{adders in output unit}(M-1))/(\text{upsampler}(L)). \end{aligned} \tag{10}$$

In deriving the expression of $\mathcal{A}$, under standard assumptions, we only consider the two-input adders and did not consider the addition elements representing the time domain interlace operations.

Remarks: The proposed structure has following differences than that of [5].

1) The proposed structure has better arrangement of the polyphase branches which leads to simpler implementation with reduced delay, as described in Section II-A. The further minimization of delay requirement is described in Section II-B. Such arrangements cannot be made for the arbitrary values of L and M in the structures of [5].
2) The proposed structure has robust mathematical formulation which leads to a simple design procedure, as illustrated in Section III. Whereas, in [5], the design guidelines are not clearly mentioned.
3) In [5], two separate structures were presented for the cases $L > M$ and $L < M$, while we propose a single structure for the arbitrary values of L and M. Furthermore, the performance comparisons of the proposed structure with [5] in terms of $\mathcal{D}$, $\mathcal{M}$, and $\mathcal{A}$ are discussed in Section IV.

III. Example of the Proposed Structure

In this section, a rational sampling rate converter with a factor of L/M is studied. For this, we consider two cases of interest; Case 1: when $L = 2$, $M = 3$, and $N = 12$ and Case 2: when $L = 3$, $M = 2$, and $N = 12$. The designing of a rate conversion filter under Case 1 is explained as follows.

Step-1: The range of variables μ, γ, and j as indicated in (1) is evaluated as $0 \le \mu \le 2$, $0 \le \gamma \le 1$, and $0 \le j \le 1$, respectively.

Step-2: The value of λ_μ can be solved by using the relation $\lambda_\mu = \frac{\mu L + k}{M}$ and appropriate values of k, for $\mu = 0, 1, 2$. Consequently, we obtained $\lambda_\mu = 0, 1, 2$.

Step-3: Using the values of μ, γ, j, and λ_μ, as obtained above, the filter coefficients of each cosubfilter with the aid of (1) can be obtained as: $H_{0,0}(z) = h_0 + h_6 z^{-1}$; $H_{0,1}(z) = h_3 + h_9 z^{-1}$; $H_{1,0}(z) = h_1 + h_7 z^{-1}$; $H_{1,1}(z) = h_4 + h_{10} z^{-1}$; $H_{2,0}(z) = h_2 + h_8 z^{-1}$; $H_{2,1}(z) = h_5 + h_{11} z^{-1}$.

[4] As suggested, the designer should choose the minimum possible value of λ_μ, for $\mu \in [0, M-1]$, therefore, it may be possible that none among $M-1$ values of λ_μ attains the upper limit of $L+1$.

[5] From [5], $\mathcal{M}$ is same as (9), whereas $\mathcal{D}$ and $\mathcal{A}$ are evaluated as

$$\mathcal{D} = \begin{cases} p_i + q_i + N - M, \text{for } L > M \\ p_i + q_i + N - L, \text{for } L < M \end{cases},$$

$$\mathcal{A} = \begin{cases} (N - ML + M - 1)/L, \text{for } L > M \\ (N - L)/L, \text{for } L < M \end{cases}$$

where the variables p_i and q_i are as defined in [5].

TABLE I
PERFORMANCE COMPARISON OF THE PROPOSED STRUCTURE WITH [5]

Parameter	$\mathcal{D}$		$\mathcal{M}$		$\mathcal{A}$		Parameter	$\mathcal{D}$		$\mathcal{M}$		$\mathcal{A}$	
M, L, N	$\mathcal{D}^\dagger$	$\mathcal{D}^\S$	$\mathcal{M}^\dagger$	$\mathcal{M}^\S$	$\mathcal{A}^\dagger$	$\mathcal{A}^\S$	M, L, N	$\mathcal{D}^\dagger$	$\mathcal{D}^\S$	$\mathcal{M}^\dagger$	$\mathcal{M}^\S$	$\mathcal{A}^\dagger$	$\mathcal{A}^\S$
$3, 2, 30$	30	29	15	15	14	13	$2, 3, 30$	30	30	10	10	8.3	8.3
$5, 3, 60$	64	58	20	20	19	16.3	$3, 5, 60$	64	61	12	12	9.4	9.4
$7, 3, 42$	48	38	14	14	13	9	$3, 7, 42$	48	44	6	6	3.2	3.2
$11, 3, 66$	76	58	22	22	21	14.3	$3, 11, 66$	76	71	6	6	3.1	3.1

†From the derived results in [5]. §From the proposed derived results.

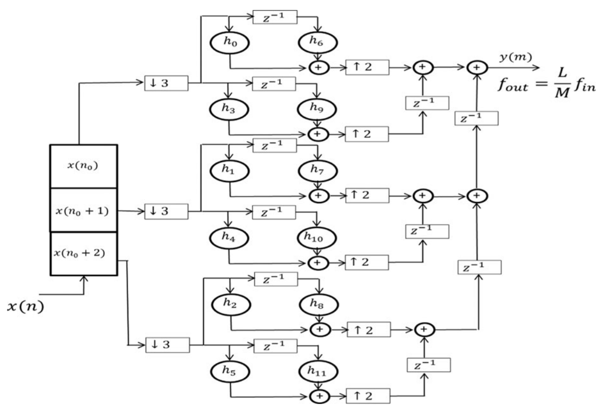

Fig. 4. Polyphase implementation of rational sampling rate for $L = 2$ and $M = 3$ at $n = n_0$ time instant.

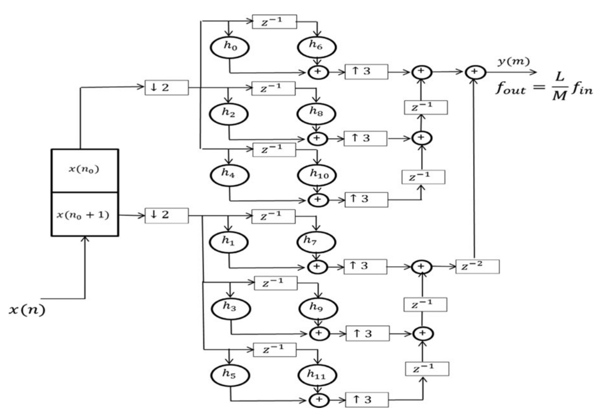

Fig. 5. Polyphase implementation of rational sampling rate for $L = 3$ and $M = 2$ at $n = n_0$ time instant.

Step-4: Recalling Fig. 2, using the coefficients calculated in Step-3 and inserting the L and M at their appropriate places (as discussed in Section II-A), the filter unit and input unit can be built.

Step-5: According to Section II-B, we have $\bar{\lambda} = 2$, whereas the upper limit of $\lambda_\mu = 3$. Therefore, the output unit with $\bar{\lambda} = 2$ can be built. Finally, the input unit, the filter unit, and the output unit can be integrated to obtain the desired system, as shown in Fig. 4.

Under Case-2, the desired system (as shown in Fig. 5) can be obtained following similar steps as mentioned above. Hence, it can readily be concluded that the proposed structure is applicable under both scenarios, i.e., $L > M$ and $L < M$.

IV. RESULTS AND DISCUSSIONS

Here, we present the performance of the proposed generalized polyphase FIR structure in terms of delay requirement $(\mathcal{D})$, multiplication complexity $(\mathcal{M})$, and addition complexity $(\mathcal{A})$ for various values of M, L, and N. For a fair comparison, we have also shown the results, as presented[6] in [5].

The various performance measures are shown in Table I. It can be observed from the Table I that when $L < M$ (for given value of N), the proposed structure has the lower $\mathcal{D}$ [obtained using (8)] and lower $\mathcal{A}$ [found using (10)] than the structure presented in [5], which results into the lower computational complexity. However, both the structures have same $\mathcal{M}$. For instance, when $L = 3, M = 5$ with $N = 60$, the number of $\mathcal{D}$ units required by the proposed structure are 58, whereas such requirement is 64 in case of [5]. The $\mathcal{A}$ of proposed structure is 16.3, while in [5], it is 19. Moreover, it can also be seen from Table I that when the value of M is very large in comparison to the value of L (i.e., $L = 3 < M = 11$), the addition complexity of our proposed structure is much lower than that of the polyphase structure given by [5]. Furthermore, when $L > M$, the proposed structure requires lower $\mathcal{D}$ units compared to [5], however, it has same performance in terms of $\mathcal{M}$ and $\mathcal{A}$. For example, when $L = 5, M = 3, N = 60$, we have $\mathcal{M} = 12$, $\mathcal{A} = 9.4$ for both structures, however, the proposed structure requires only 61 delays compared to the 64 require in [5].

V. CONCLUSION

In this letter, we have presented a novel structure for rational sampling rate applications along with relevant mathematical analysis. It is simple to implement through software and hardware tools. By taking the advantage of noncausality, a simpler implementation of the input samples of the proposed structure is presented. We also present a technique to reduce the delay requirement of the output unit. Compared to [5], the proposed structure has better performance in terms of $\mathcal{D}$ for both $L < M$ and $L > M$ scenarios. However, its addition complexity is lower than [5] for $L < M$ and same in the case of $L > M$. Being generalized structure, it is suitable for the applications where dynamic changes in the sampling frequency are required.

[6] Note that [14]–[16] are based on different philosophy, thus their comparisons with the proposed structure are totally out of the scope of this letter.

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