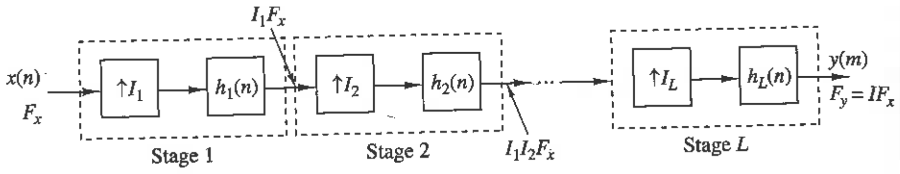

Figure 11.6.1 Multistage implementation of interpolation by a factor I.

$I/D = 130/63$. Although, in theory, this rate alteration can be achieved exactly, the implementation would require a bank of 130 polyphase filters and may be computationally inefficient. In this section we consider methods for performing sampling rate conversion for either $D \gg 1$ and/or $I \gg 1$ in multiple stages.

First, let us consider interpolation by a factor $I \gg 1$ and let us assume that I can be factored into a product of positive integers as

$$I = \prod_{i=1}^{L} I_i \tag{11.6.1}$$

Then, interpolation by a factor I can be accomplished by cascading L stages of interpolation and filtering, as shown in Fig. 11.6.1. Note that the filter in each of the interpolators eliminates the images introduced by the upsampling process in the corresponding interpolator.

In a similar manner, decimation by a factor D, where D may be factored into a product of positive integers as

$$D = \prod_{i=1}^{J} D_i \tag{11.6.2}$$

can be implemented as a cascade of J stages of filtering and decimation as illustrated in Fig. 11.6.2. Thus the sampling rate at the output of the ith stage is

$$F_i = \frac{F_{i-1}}{D_i}, \qquad i = 1, 2, \ldots, J \tag{11.6.3}$$

where the input rate for the sequence $\{x(n)\}$ is $F_0 = F_x$.

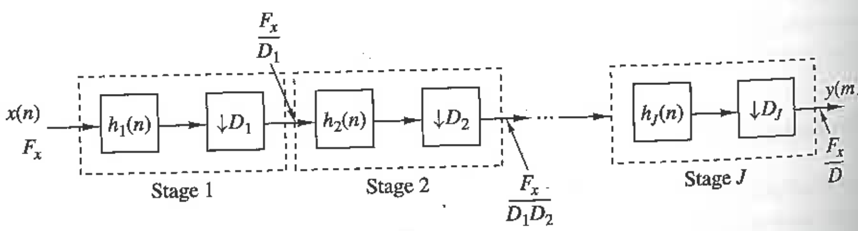

Figure 11.6.2 Multistage implementation of decimation by a factor D.

To ensure that no aliasing occurs in the overall decimation process, we can design each filter stage to avoid aliasing within the frequency band of interest. To elaborate, let us define the desired passband and the transition band in the overall decimator as

$$\begin{aligned} &\text{Passband: } 0 \le F \le F_{pc} \\ &\text{Transition band: } F_{pc} \le F \le F_{sc} \end{aligned} \tag{11.6.4}$$

where $F_{sc} \le F_x/2D$. Then, aliasing in the band $0 \le F \le F_{sc}$ is avoided by selecting the frequency bands of each filter stage as follows:

$$\begin{aligned} &\text{Passband: } 0 \le F \le F_{pc} \\ &\text{Transition band: } F_{pc} \le F \le F_i - F_{sc} \\ &\text{Stopband: } F_i - F_{sc} \le F \le \frac{F_{i-1}}{2} \end{aligned} \tag{11.6.5}$$

For example, in the first filter stage we have $F_1 = F_x/D_1$, and the filter is designed to have the following frequency bands:

$$\begin{aligned} &\text{Passband: } 0 \le F \le F_{pc} \\ &\text{Transition band: } F_{pc} \le F \le F_1 - F_{sc} \\ &\text{Stopband: } F_1 - F_{sc} \le F \le \frac{F_0}{2} \end{aligned} \tag{11.6.6}$$

After decimation by D_1, there is aliasing from the signal components that fall in the filter transition band, but the aliasing occurs at frequencies above F_{sc}. Thus there is no aliasing in the frequency band $0 \le F \le F_{sc}$. By designing the filters in the subsequent stages to satisfy the specifications given in (11.6.5), we ensure that no aliasing occurs in the primary frequency band $0 \le F \le F_{sc}$.

EXAMPLE 11.6.1

Consider an audio-band signal with a nominal bandwidth of 4 kHz that has been sampled at a rate of 8 kHz. Suppose that we wish to isolate the frequency components below 80 Hz with a filter that has a passband $0 \le F \le 75$ and a transition band $75 \le F \le 80$. Hence $F_{pc} = 75$ Hz and $F_{sc} = 80$. The signal in the band $0 \le F \le 80$ may be decimated by the factor $D = F_x/2F_{sc} = 50$. We also specify that the filter have a passband ripple $\delta_1 = 10^{-2}$ and a stopband ripple of $\delta_2 = 10^{-4}$.

The length of the linear phase FIR filter required to satisfy these specifications can be estimated from one of the well-known formulas given in the Section 10.2.7. Recall that a particularly simple formula for approximating the length M, attributed to Kaiser, is

$$\hat{M} = \frac{-10\log_{10}\delta_1\delta_2 - 13}{14.6\Delta f} + 1 \tag{11.6.7}$$

where Δf is the normalized (by the sampling rate) width of the transition region [i.e., $\Delta f = (F_{sc} - F_{pc})/F_s$]. A more accurate formula proposed by Herrmann et al. (1973) is

$$\hat{M} = \frac{D_\infty(\delta_1, \delta_2) - f(\delta_1, \delta_2)(\Delta f)^2}{\Delta f} + 1 \tag{11.6.8}$$

where $D_\infty(\delta_1, \delta_2)$ and $f(\delta_1, \delta_2)$ are defined as

$$D_\infty(\delta_1, \delta_2) = [0.005309(\log_{10}\delta_1)^2 + 0.07114(\log_{10}\delta_1) - 0.4761]\log_{10}\delta_2 - [0.00266(\log_{10}\delta_1)^2 + 0.5941\log_{10}\delta_1 + 0.4278] \tag{11.6.9}$$

$$f(\delta_1, \delta_2) = 11.012 + 0.51244[\log_{10}\delta_1 - \log_{10}\delta_2] \tag{11.6.10}$$

Now a single FIR filter followed by a decimator would require (using the Kaiser formula) a filter of (approximate) length

$$\hat{M} = \frac{-10\log_{10}10^{-6} - 13}{14.6(5/8000)} + 1 \approx 5152$$

As an alternative, let us consider a two-stage decimation process with $D_1 = 25$ and $D_2 = 2$. In the first stage we have the specifications $F_1 = 320$ Hz and

$$\text{Passband: } 0 \le F \le 75$$

$$\text{Transition band: } 75 < F \le 240$$

$$\Delta f = \frac{165}{8000}$$

$$\delta_{11} = \frac{\delta_1}{2}, \qquad \delta_{21} = \delta_2$$

Note that we have reduced the passband ripple δ_1 by a factor of 2, so that the total passband ripple in the cascade of the two filters does not exceed δ_1. On the other hand, the stopband ripple is maintained at δ_2 in both stages. Now the Kaiser formula yields an estimate of M_1 as

$$\hat{M}_1 = \frac{-10\log_{10}\delta_{11}\delta_{21} - 13}{14.6\Delta f} + 1 \approx 167$$

For the second stage, we have $F_2 = F_1/2 = 160$ and the specifications

$$\text{Passband: } 0 \le F \le 75$$

$$\text{Transition band: } 75 < F \le 80$$

$$\Delta f = \frac{5}{320}$$

$$\delta_{12} = \frac{\delta_1}{2}, \qquad \delta_{22} = \delta_2$$

Hence the estimate of the length M_2 of the second filter is

$$\hat{M}_2 \approx 220$$

Therefore, the total length of the two FIR filters is approximately $\hat{M}_1 + \hat{M}_2 = 387$. This represents a reduction in the filter length by a factor of more than 13.

The reader is encouraged to repeat the computation above with $D_1 = 10$ and $D_2 = 5$.

It is apparent from the computations in Example 11.6.1 that the reduction in the filter length results from increasing the factor Δf, which appears in the denominator in (11.6.7) and (11.6.8). By decimating in multiple stages, we are able to increase the width of the transition region through a reduction in the sampling rate.

In the case of a multistage interpolator, the sampling rate at the output of the ith stage is

$$F_{i-1} = I_i F_i, \qquad i = J, J-1, \ldots, 1$$

and the output rate is $F_0 = IF_J$ when the input sampling rate is F_J. The corresponding frequency band specifications are

$$\text{Passband: } 0 \le F \le F_p$$

$$\text{Transition band: } F_p < F \le F_i - F_{sc}$$

The following example illustrates the advantages of multistage interpolation.

EXAMPLE 11.6.2

Let us reverse the filtering problem described in Example 11.6.1 by beginning with a signal having a passband $0 \le F \le 75$ and a transition band of $75 \le F \le 80$. We wish to interpolate by a factor of 50. By selecting $I_1 = 2$ and $I_2 = 25$, we have basically a transposed form of the decimation problem considered in Example 11.6.1. Thus we can simply transpose the two-stage decimator to achieve the two-stage interpolator with $I_1 = 2$, $I_2 = 25$, $\hat{M}_1 \approx 220$, and $\hat{M}_2 \approx 167$.

11.7 Sampling Rate Conversion of Bandpass Signals

In this section we consider the decimation and interpolation of bandpass signals. We begin by noting that any bandpass signal can be converted into an equivalent lowpass signal (see Section 6.5.2) whose sampling rate can be changed using the already developed techniques. However, a simpler and more widely used approach concerns bandpass discrete-time signals with *integer-band positioning*. The concept is similar to the one discussed for continuous-time bandpass signals in Section 6.4.

To be specific, suppose that we wish to decimate by a factor D an integer-positioned bandpass signal with spectrum confined to the bands

$$(k-1)\frac{\pi}{D} < |\omega| < k\frac{\pi}{D} \tag{11.7.1}$$

where k is a positive integer. A bandpass filter defined by

$$H_{BP}(\omega) = \begin{cases} 1, & (k-1)\frac{\pi}{D} < |\omega| < k\frac{\pi}{D} \\ 0, & \text{otherwise} \end{cases} \tag{11.7.2}$$

would normally be used to eliminate signal frequency components outside the desired frequency range. Then direct decimation of the filtered signal $v(n)$ by the factor D