$\delta_2 = 0{\cdot}001$ and $n_p = 3$ or 6, respectively. Obviously, for fixed $\delta_1$ and $\delta_2$ and fixed degree $2n$, only fixed pairs of cutoff frequencies $\Omega_p$ and $\Omega_s$ can be obtained choosing different values $n_p$. Since only $n$ different values for $n_p$ can be chosen, there exist only $n$ different nonrecursive filters for fixed $n$, $\delta_1$ and $\delta_2$. A lowpass–lowpass transformation usable in the design of recursive filters to obtain any cutoff frequency is not applicable here, since it transforms the nonrecursive filter into a recursive one with nonlinear phase. However, the transformation into highpass, bandpass or bandstop with fixed cutoff frequencies is easily effected.

Using this method, roughly 400 filters have been calculated with $2n$ between 6 and 42, $\delta_1 = 10^{-2} \ldots 10^{-1}$, $\delta_2 = 5 \times 10^{-5} \ldots 10^{-1}$ and different values of $n_p$, and the parameters of these filters are available.

As in References 1 and 5, a figure of merit can be defined by

$$D = 2n(\Omega_s - \Omega_p) = D(n, n_p, \delta_1, \delta_2)$$

The smaller is $D$, the better is the filter.

As indicated, $D$ depends in general on $n$, $n_p$ and the tolerated deviation in the passband and stopband. An examination of the results shows this expression to become independent of $n$ and nearly independent of $n_p$ for $2n > 30$. We define

$$D_0 = 2n(\Omega_s - \Omega_p) = D_0(\delta_1, \delta_2)$$

This limit, achieved for the filters calculated so far, has been drawn in Fig. 3 as a function of $\delta$ with $\delta_1$ as parameter. The corresponding values published in References 1 and 5 are indicated. A comparison with a recursive filter may be of interest. A frequency response, similar to the two examples of Fig. 2, which requires $2n = 20$ unit delays in the nonrecursive case takes only five delays in the recursive case if an elliptic approximation is chosen. The number of multiplications for each output value is 10 for the nonrecursive structure and seven for the recursive one. This comparison indicates the price to be paid for the linear phase we obtain with the nonrecursive filters treated here.

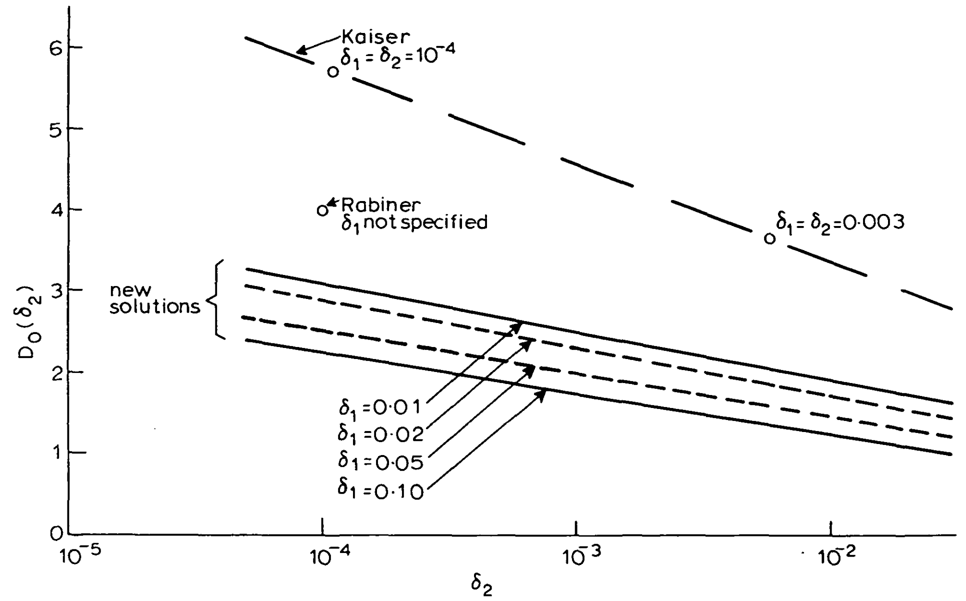


**Fig. 3** *Figure of merit $D_0$ as a function of tolerated deviation $\delta_2$ in stopband*

The author is very grateful to Prof. H. W. Schuessler for his suggestions and encouragement.

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*27th April 1970*

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# DESIGN OF NONRECURSIVE DIGITAL FILTERS WITH MINIMUM PHASE

*Indexing term: Digital filters*

A method is described of transforming nonrecursive filters with equal-ripple attenuation in the passband, stopband and linear phase into those with minimum phase and half the degree, but again with equal-ripple attenuation in the passband and stopband.

In another letter the design of nonrecursive digital filters with linear phase and equal-ripple attenuation in the passband and the stopband has been considered.[1] This letter presents a method of transforming these filters to those with the same type of attenuation behaviour, but with minimum phase and half the degree. The filter with linear phase can, to begin with, be described by

$$H(z) = \frac{1}{z^{2n}} \frac{1}{2} \sum_{\mu=0}^{n} d_\mu (z^{n+\mu} + z^{n-\mu}) = \frac{1}{z^{2n}} H_0(z)$$

With $z = \exp(j2\pi\Omega)$, we obtain

$$H(\Omega) = \frac{1}{\exp(jn2\pi\Omega)} \sum_{\mu=0}^{n} d_\mu \cos \mu 2\pi\Omega$$

$$= \frac{1}{\exp(jn2\pi\Omega)} H_0(\Omega)$$

Let the coefficients $d_\mu$ be chosen in such a way that $H_0(\Omega)$ has the equal-ripple behaviour shown in Fig. 1 with tolerated deviations $\delta_1$ in the passband and $\delta_2$ in the stopband. The Figure indicates additionally the poles and zeros of $H(z)$ in the $z$ plane.

We now define a transfer function $H_1(z)$ by

$$H_1(z) = H(z) + \delta_2 \frac{1}{z^n}$$

which has a frequency response

$$|H_1(\Omega)| = H_0(\Omega) + \delta_2$$

The frequency response and the pole-zero pattern are shown in Fig. 2. As indicated, the filter has zeros of second order on the unit circle. Expressed with its poles and zeros, the transfer function $H_1(z)$ becomes

$$H_1(z) = \frac{d_n}{2z^{2n}} \prod_{\mu=1}^{n} (z-z_{0\mu})(z-z_{0\mu}^{-1}) \quad \text{with} \quad |z_{0\mu}| \leq 1$$

which can be written as

$$H_1(z) = \frac{(-1)^n d_n}{2z^n \prod_{\mu=1}^{n} z_{0\mu}} \prod_{\mu=1}^{n} (z-z_{0\mu})(z^{-1}-z_{0\mu})$$

For $z = \exp(j2\pi\Omega)$, the magnitude of $H_1$ turns out to be a square:

$$|H_1(\Omega)| = \frac{|d_n|}{2\left|\prod_{\mu=1}^{n} z_{0\mu}\right|} \prod_{\mu=1}^{n} |\{\exp(j2\pi\Omega)-z_{0\mu}\}|^2$$

We now define a transfer function $H_2(z)$ with frequency response

$$|H_2(\Omega)| = \sqrt{|H_1(\Omega)|}$$

To obtain a transfer function with minimum phase we choose those zeros of $H_1(z)$ which are inside the unit circle and a simple zero at those points of the unit circle where $H_1(z)$ has a pair of zeros. In addition, we pick half the poles located at the origin (see Fig. 3). We obtain

$$H_2(z) = \sqrt{\left(\frac{d_n}{2\prod_{\mu=1}^{n} z_{0\mu}}\right)} \prod_{\mu=1}^{n} (z-z_{0\mu}) \frac{1}{z^n}$$

Since $|H_1(\Omega)|$ approximates to $1+\delta_2$ in the passband, we have to divide by $\sqrt{(1+\delta_2)}$ to obtain the wanted approximation of 1. Finally, we obtain

$$H_3(z) = \sqrt{\left\{\frac{d_n}{(1+\delta_2)\, 2\prod_{\mu=1}^{n} z_{0\mu}}\right\}} \prod_{\mu=1}^{n} (z-z_{0\mu}) \frac{1}{z^n}$$

whose frequency response approximates to 1 in the passband with a deviation

$$\delta_1' = \sqrt{\left(1+\frac{\delta_1}{1+\delta_2}\right)} - 1$$

and to zero in the stopband with a deviation

$$\delta_2' = \sqrt{\left(\frac{2\delta_2}{1+\delta_2}\right)}$$

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**Reference**

1 HERRMANN, O.: 'Design of nonrecursive digital filters with linear phase', see p. 328

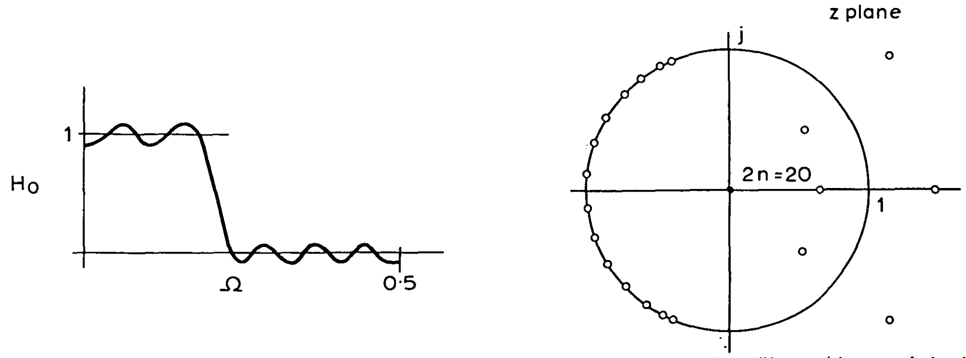


**Fig. 1** *Frequency response and pole-zero pattern of a nonrecursive filter with equal-ripple attenuation and linear phase*

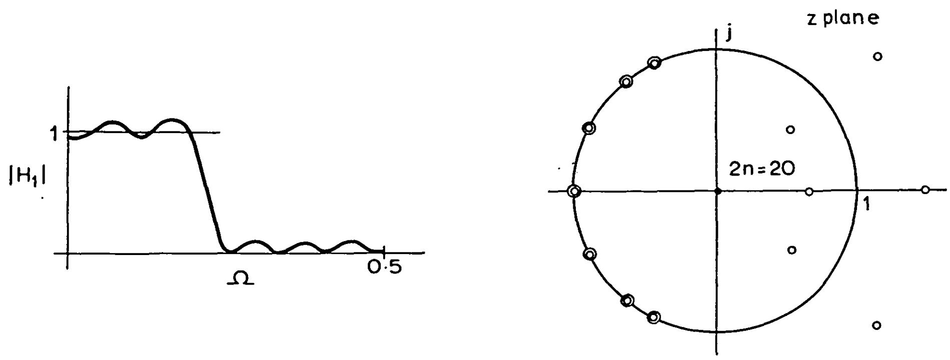


**Fig. 2** *Frequency response and pole-zero pattern of an auxiliary transfer function $H_1(z)$*

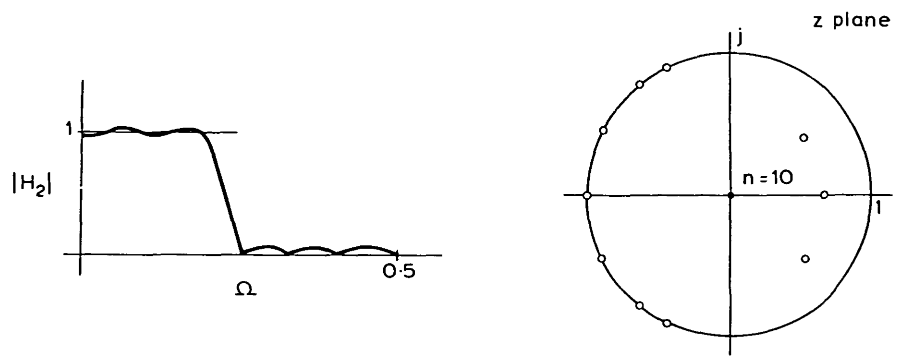


**Fig. 3** *Frequency response and pole-zero pattern of a nonrecursive filter with equal-ripple attenuation and minimum phase*