A Group-Delay Interpretation of Polyphase Filters*

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Previously, polyphase interpolation and decimation were derived from the Noble identities and motivated for reasons of computational efficiency. Here we present a different interpretation of the (ideal) polyphase filter.

Assume that $H(z)$ is an ideal lowpass filter with gain L, cutoff $\frac{\pi}{L}$, and constant group delay of d:

$$H\left(e^{i\omega}\right)=\begin{cases} Le^{-(id\omega)} \text{ if } \omega\in\left[-\frac{\pi}{L},\frac{\pi}{L}\right) \\ 0 \text{ if } \omega\in\left[-\pi,-\frac{\pi}{L}\right) \wedge \left[\frac{\pi}{L},\pi\right) \end{cases}$$

Recall that the polyphase filters are defined as

$$\forall p, p\in\{0,\dots,L-1\} : \left(h_p[k]=h[kL+p]\right)$$

In other words, $h_p[k]$ is an advanced (by p samples) and downsampled (by factor L) version of $h[n]$ (see Figure 1).

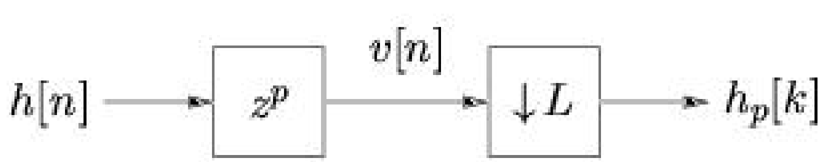

Figure 1

The DTFT of the p^{th} polyphase filter impulse response is then

$$H_p(z)=\frac{1}{L}\sum_{l=0}^{L-1}V\left(e^{(-i)\frac{2\pi}{L}l}z^{\frac{1}{L}}\right) \quad (1)$$

*Version 2.12: Sep 16, 2005 2:30 pm -0500 where $V(z) = H(z) z^p$

$$H_p(z) = \frac{1}{L} \sum_{l=0}^{L-1} e^{(-i)\frac{2\pi}{L} lp} z^{\frac{p}{L}} H\left(e^{(-i)\frac{2\pi}{L} l} z^{\frac{1}{L}}\right) \quad (1)$$

$$\begin{aligned} H_p\left(e^{i\omega}\right) &= \frac{1}{L} \sum_{l=0}^{L-1} e^{i\frac{\omega - 2\pi l}{L} p} H\left(e^{i\frac{\omega-2\pi l}{L}}\right) \\ &= \forall \omega, |\omega| \leq \pi : \left(\frac{1}{L}\left(e^{i\frac{\omega}{L} p} H\left(e^{i\frac{\omega}{L}}\right)\right)\right) \\ &= \forall \omega, |\omega| \leq \pi : \left(e^{(-i)\frac{d-p}{L}\omega}\right) \end{aligned} \quad (1)$$

Thus, the ideal p^{th} polyphase filter has a constant magnitude response of one and a constant group delay of $\frac{d-p}{L}$ samples. The implication is that if the input to the p^{th} polyphase filter is the unaliased T-sampled representation $x[n] = x_c(nT)$, then the output of the filter would be the unaliased T-sampled representation $y_p[n] = x_c\left(\left(n - \frac{d-p}{L}\right) T\right)$ (see Figure 2).

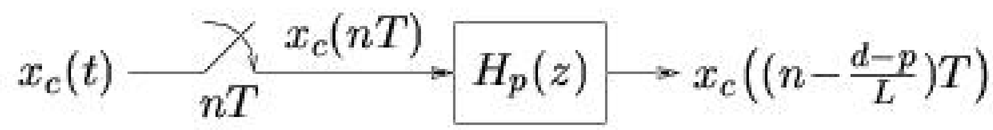

Figure 2

Figure 3 shows the role of polyphase interpolation filters assume zero-delay ($d = 0$) processing. Essentially, the p^{th} filter interpolates the waveform $\frac{p}{L}$-way between consecutive input samples. The L polyphase outputs are then interleaved to create the output stream. Assuming that $x_c(t)$ is bandlimited to $\frac{1}{2T}$Hz, perfect polyphase filtering yields a perfectly interpolated output. In practice, we use the casual FIR approximations of the polyphase filters $h_p[k]$ (which which correspond to some casual FIR approximation of the master filter $h[n]$).

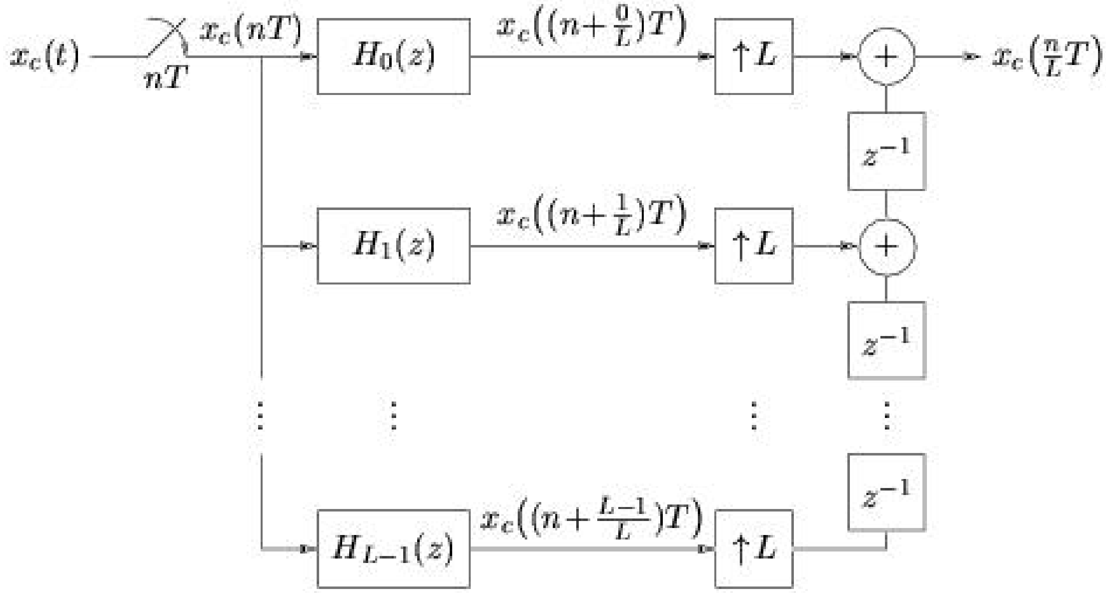
$x_c(t)$
nT
$x_c(nT)$
$H_0(z)$
$x_c\left((n+\frac{0}{L})T\right)$
$\uparrow L$
$+$
$x_c\left(\frac{n}{L}T\right)$
z^{-1}
$H_1(z)$
$x_c\left((n+\frac{1}{L})T\right)$
$\uparrow L$
$+$
z^{-1}
z^{-1}
$H_{L-1}(z)$
$x_c\left((n+\frac{L-1}{L})T\right)$
$\uparrow L$

Figure 3