Consider the structural model

$$\begin{aligned} y_{i1} &= -y_{i2}'\gamma + u_{i1} \\ y_{i2} &= \boldsymbol{B}_2' x_i + u_{i1} \end{aligned}$$

Or in matricial form

$$\boldsymbol{Y\Gamma} = \boldsymbol{XB} + \boldsymbol{U}$$

$$\boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0_p' \\ \gamma & \boldsymbol{I}_p \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0_k & \boldsymbol{B}_2 \end{bmatrix}$$

Using $q = 1 + p$,

$\boldsymbol{Y}:$	$n \times q$	$\boldsymbol{\Gamma}:$	$q \times q$	$\gamma:$	$p \times 1$
$\boldsymbol{X}:$	$n \times k$	$\boldsymbol{B}:$	$k \times q$	$\boldsymbol{B}_2:$	$k \times p$
$\boldsymbol{U}:$	$q \times q$				

Using $\boldsymbol{V} = \boldsymbol{U\Gamma}^{-1}$, I want to solve

$$\min_{\gamma\,,\,\boldsymbol{B}_2} \left|\boldsymbol{V}'\boldsymbol{V}\right|$$

To do so, I define

$$\boldsymbol{R}_1 = \underset{q^2\times q^2}{\begin{bmatrix} 1 & & \\ & \mathcal{O}_p & \\ & & \boldsymbol{I}_{pq} \end{bmatrix}} \qquad \boldsymbol{R}_2 = \underset{kq\times kq}{\begin{bmatrix} \boldsymbol{I}_k & \\ & \mathcal{O}_{kp} \end{bmatrix}} \qquad r_1 = \begin{bmatrix} 1 \\ 0_p \\ \operatorname{vec}\begin{bmatrix} 0_p' \\ \boldsymbol{I}_p \end{bmatrix} \end{bmatrix} \qquad r_2 = 0_{kp}$$

And write the problem using linear restrictions

$$\min_{\boldsymbol{\Gamma}\,,\,\boldsymbol{B}} \left|\boldsymbol{U}'\boldsymbol{U}\right| \quad \text{s.t.} \quad \begin{cases} \boldsymbol{R}_1 \operatorname{vec}(\boldsymbol{\Gamma}) = r_1 \\ \boldsymbol{R}_2 \operatorname{vec}(\boldsymbol{B}) = r_2 \end{cases}$$

Where I used the fact that $\left|\boldsymbol{\Gamma}\right| = 1$.

1 Unrestricted Problem

Form the Lagrangean

$$\mathcal{L} = \log \left| \boldsymbol{V}'\boldsymbol{V} \right| + \lambda_1' \left[\boldsymbol{R}_1 \mathrm{vec}\left(\boldsymbol{\Gamma} \right) - r_1 \right] - \lambda_2' \left[\boldsymbol{R}_2 \mathrm{vec}\left(\boldsymbol{B} \right) - r_2 \right]$$

Using some algebra, we obtain the first order conditions

$$\frac{\partial}{\partial \boldsymbol{\Gamma}} \mathcal{L} = \left[(\boldsymbol{U}'\boldsymbol{U})^{-1} \otimes \boldsymbol{Y}' \right] \mathrm{vec}\left(\boldsymbol{U} \right) + \boldsymbol{R}_1 \lambda_1 = 0$$

$$-\frac{\partial}{\partial \boldsymbol{B}} \mathcal{L} = \left[(\boldsymbol{U}'\boldsymbol{U})^{-1} \otimes \boldsymbol{X}' \right] \mathrm{vec}\left(\boldsymbol{U} \right) + \boldsymbol{R}_2 \lambda_2 = 0$$

As we want the system to be polynomial, we multiply the first equation by $(\boldsymbol{U}'\boldsymbol{U}) \otimes \boldsymbol{I}_q$ and the second one by $(\boldsymbol{U}'\boldsymbol{U}) \otimes \boldsymbol{I}_k$ to obtain

$$\mathrm{vec}\left(\boldsymbol{Y}'\boldsymbol{U} \right) + \left[(\boldsymbol{U}'\boldsymbol{U}) \otimes \boldsymbol{I}_q \right] \boldsymbol{R}_1 \lambda_1 = 0$$

$$\mathrm{vec}\left(\boldsymbol{X}'\boldsymbol{U} \right) + \left[(\boldsymbol{U}'\boldsymbol{U}) \otimes \boldsymbol{I}_k \right] \boldsymbol{R}_2 \lambda_2 = 0$$

We must impose the structures of the matrices and

$$\lambda_1 = \begin{bmatrix} \mu_{1(1)} \\ 0_p \\ \mu_{1(2:q^2-p)} \end{bmatrix} \qquad \lambda_2 = \begin{bmatrix} \mu_{2(1:k)} \\ 0_{kp} \end{bmatrix}$$

2 LR Test (Kuhn-Tucker)

Let $g : \mathbb{R}^p \times \mathbb{R}^{kp} \to \mathbb{R}^m$ be a polynomial and $b = \operatorname{vec} \boldsymbol{B}_2$. Let's say we would like to test

$$\mathbb{H}_0 : g(\gamma, b) \leq 0_m$$

using a LR statistic. I use the fact that

$$\gamma = \begin{bmatrix} 0_p & \boldsymbol{I}_p & \mathcal{O}_{p\times pq} \end{bmatrix} \operatorname{vec}(\boldsymbol{\Gamma}) := \boldsymbol{R}_3' \operatorname{vec}(\boldsymbol{\Gamma})$$

$$b = \begin{bmatrix} \mathcal{O}_{kp\times k} & \boldsymbol{I}_{kp} \end{bmatrix} \operatorname{vec}(\boldsymbol{B}) := \boldsymbol{R}_4' \operatorname{vec}(\boldsymbol{B})$$

($\boldsymbol{R}_3$ is $q^2 \times p$ and $\boldsymbol{R}_4$ is $kq \times kp$) to frame the problem as

$$\min_{\boldsymbol{\Gamma}, \boldsymbol{B}} \left|\boldsymbol{U}'\boldsymbol{U}\right| \quad \text{s.t.} \quad \begin{cases} \boldsymbol{R}_1 \operatorname{vec}(\boldsymbol{\Gamma}) = r_1 \\ \boldsymbol{R}_2 \operatorname{vec}(\boldsymbol{B}) = r_2 \\ g\big(\boldsymbol{R}_3' \operatorname{vec} \boldsymbol{\Gamma}, \boldsymbol{R}_4' \operatorname{vec} \boldsymbol{B}\big) \leq 0_m \end{cases}$$

The restriction is still polynomial. The Lagrangean is now

$$\mathcal{L} = \log\left|\boldsymbol{U}'\boldsymbol{U}\right| + \lambda_1' \left[\boldsymbol{R}_1 \operatorname{vec}(\boldsymbol{\Gamma}) - r_1\right] - \lambda_2' \left[\boldsymbol{R}_2 \operatorname{vec}(\boldsymbol{B}) - r_2\right] + \lambda_3' \left[g\big(\boldsymbol{R}_3' \operatorname{vec} \boldsymbol{\Gamma}, \boldsymbol{R}_4' \operatorname{vec} \boldsymbol{B}\big) + h(y)\right]$$

Where $\lambda_3 \in \mathbb{R}^m$ and $h(y) = (y_1^2, \cdots, y_m^2)$. That is, the y_i's are slack variables. FOC:

$$\frac{\partial}{\partial \boldsymbol{\Gamma}} \mathcal{L} = \left[(\boldsymbol{U}'\boldsymbol{U})^{-1} \otimes \boldsymbol{Y}'\right] \operatorname{vec}(\boldsymbol{U}) + \boldsymbol{R}_1 \lambda_1 + \boldsymbol{R}_3 \frac{\partial}{\partial \gamma} g\big(\boldsymbol{R}_3' \operatorname{vec} \boldsymbol{\Gamma}, \boldsymbol{R}_4' \operatorname{vec} \boldsymbol{B}\big) \lambda_3 = 0$$

$$-\frac{\partial}{\partial \boldsymbol{B}} \mathcal{L} = \left[(\boldsymbol{U}'\boldsymbol{U})^{-1} \otimes \boldsymbol{X}'\right] \operatorname{vec}(\boldsymbol{U}) + \boldsymbol{R}_2 \lambda_2 + \boldsymbol{R}_4 \frac{\partial}{\partial b} g\big(\boldsymbol{R}_3' \operatorname{vec} \boldsymbol{\Gamma}, \boldsymbol{R}_4' \operatorname{vec} \boldsymbol{B}\big) \lambda_3 = 0$$

Eliminating the inverses and adding the remaining KT conditions,

$$\begin{aligned} \operatorname{vec}(\boldsymbol{Y}'\boldsymbol{U}) + \left[(\boldsymbol{U}'\boldsymbol{U}) \otimes \boldsymbol{I}_q\right] \boldsymbol{R}_1 \lambda_1 + \left[(\boldsymbol{U}'\boldsymbol{U}) \otimes \boldsymbol{I}_q\right] \boldsymbol{R}_3 \frac{\partial}{\partial \gamma} g(\gamma, b) \lambda_3 &= 0 \\ \operatorname{vec}(\boldsymbol{X}'\boldsymbol{U}) + \left[(\boldsymbol{U}'\boldsymbol{U}) \otimes \boldsymbol{I}_k\right] \boldsymbol{R}_2 \lambda_2 + \left[(\boldsymbol{U}'\boldsymbol{U}) \otimes \boldsymbol{I}_k\right] \boldsymbol{R}_4 \frac{\partial}{\partial b} g(\gamma, b) \lambda_3 &= 0 \\ g(\gamma, b) + h(y) &= 0 \\ f(\lambda_3, y) &= 0 \end{aligned}$$

Where $f(\lambda_3, y) = (\lambda_{3i}\, y_i)_{i=1}^m$.

3 Simulation

Consider $p = k = 1$ and the true parameters

$$\theta_0 = \begin{bmatrix} \gamma_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

I generated a sample of $n = 10^5$ observations and obtained

$$\widehat{\theta}_{FIML} = \begin{bmatrix} 1.9977 \\ 3.0122 \end{bmatrix}$$

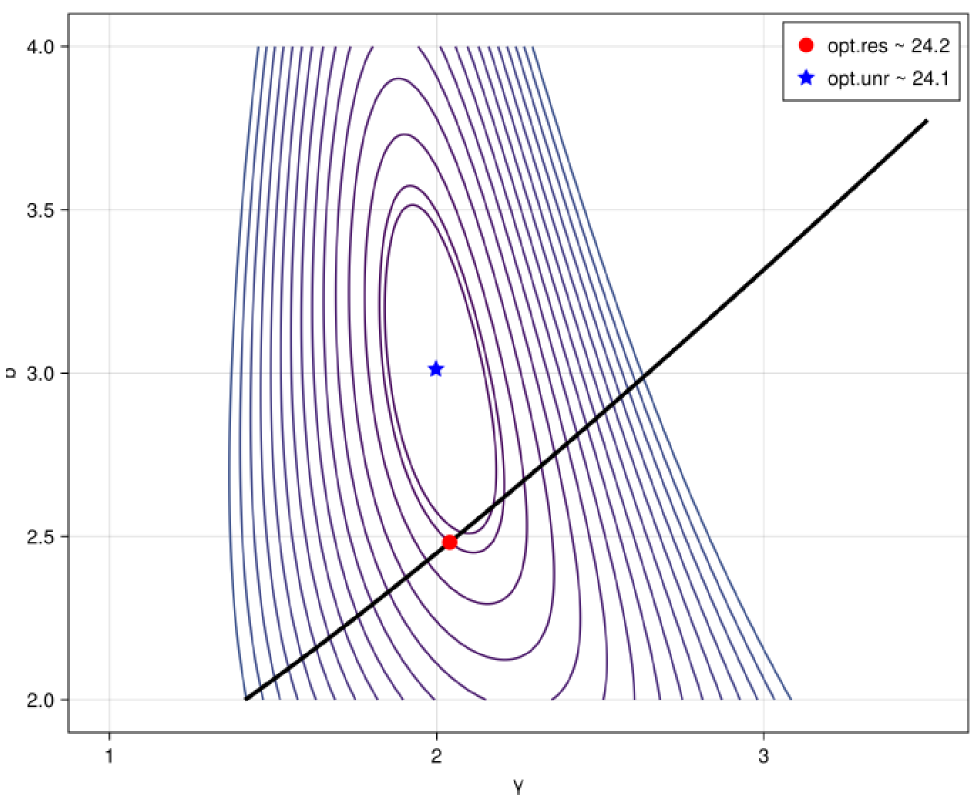
Now, consider

$$\mathbb{H}_0 : b^2 - \gamma^2 \leq 2$$

Using the Kuhn-Tucker method, I obtain

$$\widetilde{\theta}_{\mathbb{H}_0} \approx \begin{bmatrix} 2.04 \\ 2.48 \end{bmatrix}$$

This runs in around 15sec. But it looks like we have a numerical problem:



The LR statistic obtained is $\lambda_n \approx \frac{n}{2}(24.2 - 24.1) = 5000$. Loosely applying Theorem 4.3 in Drton, we reject the null.