1 Basics

The multiplication $\boldsymbol{AB}$ stacks

$$\boldsymbol{AB} = \begin{bmatrix} \boldsymbol{A}b_1 & \cdots & \boldsymbol{A}b_m \end{bmatrix}$$

Example 1.1. Let $\boldsymbol{X}_{n\times k}$ and $\boldsymbol{Y}_{n\times m}$. Then

$$\boldsymbol{X}'\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{X}' \begin{bmatrix} y_{11} \\ \vdots \\ y_{n1} \end{bmatrix} & \cdots & \boldsymbol{X}' \begin{bmatrix} y_{1m} \\ \vdots \\ y_{nm} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix} \begin{bmatrix} y_{11} \\ \vdots \\ y_{n1} \end{bmatrix} & \cdots & \begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix} \begin{bmatrix} y_{1m} \\ \vdots \\ y_{nm} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^n x_i y_{i1} & \cdots & \sum_{i=1}^n x_i y_{im} \end{bmatrix} = \sum_{i=1}^n x_i \begin{bmatrix} y_{i1} & \cdots & y_{im} \end{bmatrix} = \sum_{i=1}^n x_i y_i'$$

■

2 Kronecker Product

Given $\boldsymbol{A}_{I_A\times J_A}$ and $\boldsymbol{B}_{I_B\times J_B}$,

$$\boldsymbol{A}\otimes\boldsymbol{B} = \begin{bmatrix} a_{ij}\boldsymbol{B} \end{bmatrix}_{ij} \qquad \sim \quad I_A I_B \times J_A J_B$$

Lema 2.1

(i) $(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D}) = \boldsymbol{AC}\otimes\boldsymbol{BD}$

(ii) $(\boldsymbol{A}\otimes\boldsymbol{B})' = \boldsymbol{A}'\otimes\boldsymbol{B}'$

(iii) $(\boldsymbol{A}\otimes\boldsymbol{B})^{-1} = \boldsymbol{A}^{-1}\otimes\boldsymbol{B}^{-1}$

(iv) $\operatorname{tr}(\boldsymbol{A}\otimes\boldsymbol{B}) = \operatorname{tr}(\boldsymbol{A})\operatorname{tr}(\boldsymbol{B})$

The following result implies that $\boldsymbol{A}\otimes\boldsymbol{B}$ is nonsingular iff both $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonsingular.

Theorem 2.2. Let $\boldsymbol{A}_{N_A\times N_A}$ and $\boldsymbol{B}_{N_B\times N_B}$. Then

(i) $|\boldsymbol{A}\otimes\boldsymbol{B}| = |\boldsymbol{A}|^{N_B}|\boldsymbol{B}|^{N_A}$

(ii) $\operatorname{rank}(\boldsymbol{A}\otimes\boldsymbol{B}) = \operatorname{rank}(\boldsymbol{A})\operatorname{rank}(\boldsymbol{B})$

■

3 The vec operator

Let $\boldsymbol{A}_{I\times J} = \begin{bmatrix} a_1 & \cdots & a_j \end{bmatrix}$.

$$\mathrm{vec}\,(\boldsymbol{A}) = \begin{bmatrix} a_1 \\ \vdots \\ a_J \end{bmatrix} \qquad \sim \quad IJ \times 1$$

Lemma 3.1

(i) $\mathrm{vec}\,(\boldsymbol{ABC}) = (\boldsymbol{C}' \otimes \boldsymbol{A})\mathrm{vec}\,(\boldsymbol{B})$

(ii) $\mathrm{vec}\,(\boldsymbol{A} + \boldsymbol{B}) = \mathrm{vec}\,(\boldsymbol{A}) + \mathrm{vec}\,(\boldsymbol{B})$

(iii) $\mathrm{tr}\,(\boldsymbol{A}'\boldsymbol{B}) = \mathrm{vec}\,(\boldsymbol{A})'\mathrm{vec}\,(\boldsymbol{B})$[1]

Corollary 3.2. Let $\boldsymbol{A}_{I_A \times J_A}$ and $\boldsymbol{B}_{I_B \times J_B}$.

$$\begin{aligned} \mathrm{vec}\,(AB) &= (\boldsymbol{B}' \otimes \boldsymbol{I}_{I_A})\mathrm{vec}\,(\boldsymbol{A}) \\ &= (\boldsymbol{B}' \otimes \boldsymbol{A})\mathrm{vec}\,(\boldsymbol{I}_N) \\ &= (\boldsymbol{I}_{J_B} \otimes \boldsymbol{A})\mathrm{vec}\,(\boldsymbol{B}) \end{aligned}$$

■

[1]Here $\boldsymbol{A}$ and $\boldsymbol{B}$ must have the same dimensions.

4 Standard Calculus

Let $f : \mathbb{R}^n \to \mathbb{R}^m$.

$$Df(x) = \frac{\partial}{\partial x'} f(x) = \big[\nabla' f_i(x)\big]_i \quad \sim m \times n$$

$$D'f(x) = \frac{\partial}{\partial x} f(x) = \big[\nabla f_j(x)\big]_j \quad \sim n \times m$$

So these objects are just transposes of each other. The most common cases are

$f(x)$	$Df(x)$	$D'f(x)$
$\boldsymbol{A}x$	$\boldsymbol{A}$	$\boldsymbol{A}'$
$x'\boldsymbol{A}\,x$	$x'(\boldsymbol{A}+\boldsymbol{A}')$	$(\boldsymbol{A}+\boldsymbol{A}')x$
$g(x)'\boldsymbol{A}\,g(x)$	$g(x)'(\boldsymbol{A}+\boldsymbol{A}')Dg(x)$	$D'g(x)(\boldsymbol{A}+\boldsymbol{A}')g(x)$
$f(x)'g(x)$[2]	$f(x)'Dg(x) + g(x)'Df(x)$	$D'f(x)g(x) + D'g(x)f(x)$
$\frac{p(x)}{q(x)}$	$\times$	$\frac{1}{q(x)}\nabla p(x) - \frac{p(x)}{q(x)^2}\nabla q(x)$
$m\big(h(x)\big)$[3]	$m'\big(h(x)\big)Dh(x)$	$\times$

[2] $f, g : \mathbb{R}^n \to \mathbb{R}^m$

[3] $h : \mathbb{R}^n \to \mathbb{R}$ and $m : \mathbb{R} \to \mathbb{R}$

5 Matrix Calculus

Let $F : \mathbb{R}^{I_1 \times J_1} \to \mathbb{R}^{I_2 \times J_2}$. We identify the domain of F as $\mathbb{R}^{I_1 \times J_1} \sim \mathbb{R}^{I_1 J_1}$ and define

$$DF(\boldsymbol{X}) = \frac{\partial}{\partial \vec{\boldsymbol{X}}'} \operatorname{vec} F(\vec{\boldsymbol{X}}) \qquad \sim \quad I_2 J_2 \times I_1 J_1$$

Where $\vec{\boldsymbol{X}} := \operatorname{vec}(\boldsymbol{X})$.

Example 5.1. Let $F(\boldsymbol{X}) = \boldsymbol{A}\boldsymbol{X}$. Then

$$\operatorname{vec} F(\boldsymbol{X}) = \operatorname{vec}(\boldsymbol{A}\boldsymbol{X}) = \big(\boldsymbol{I}_{J_X} \otimes \boldsymbol{A}\big) \operatorname{vec}(\boldsymbol{X})$$

So we can apply the rule from standard calculus

$$DF(\boldsymbol{X}) \;=\; \frac{\partial}{\partial \vec{\boldsymbol{X}}'} \big(\boldsymbol{I}_{J_X} \otimes \boldsymbol{A}\big)\vec{\boldsymbol{X}} \;=\; \boldsymbol{I}_{J_X} \otimes \boldsymbol{A}$$

■

$F(\boldsymbol{X})$	$DF(\boldsymbol{X})$
$\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}$	$\boldsymbol{B}' \otimes \boldsymbol{A}$
$\boldsymbol{X}^{-1}$	$-(\boldsymbol{X}')^{-1} \otimes \boldsymbol{X}^{-1}$
$G(\boldsymbol{X})H(\boldsymbol{X})$	$\big(\boldsymbol{I}_{J_H} \otimes G(\boldsymbol{X})\big) DH(\boldsymbol{X}) + \big(H(\boldsymbol{X})' \otimes \boldsymbol{I}_{I_G}\big) DG(\boldsymbol{X})$

$f(\boldsymbol{X})$	$Df(\boldsymbol{X})$
$\operatorname{tr}(\boldsymbol{A}\boldsymbol{X})$	$\operatorname{vec}'(\boldsymbol{A}')$
$\operatorname{tr}(\boldsymbol{X}\boldsymbol{A}\boldsymbol{X}')$	$\operatorname{vec}'\big[\boldsymbol{X}(\boldsymbol{A} + \boldsymbol{A}')\big]$
$\lvert\boldsymbol{X}\rvert$	$\lvert\boldsymbol{X}\rvert \operatorname{vec}'\big[(\boldsymbol{X}')^{-1}\big]$
$\log\lvert\boldsymbol{X}\rvert$	$\operatorname{vec}'\big[(\boldsymbol{X}')^{-1}\big]$
$\lvert\boldsymbol{X}'\boldsymbol{X}\rvert$	$\lvert\boldsymbol{X}'\boldsymbol{X}\rvert \operatorname{vec}'\big[\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\big]$
$\log\lvert\boldsymbol{X}'\boldsymbol{X}\rvert$	$\operatorname{vec}'\big[\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\big]$
$\lvert\boldsymbol{X}\boldsymbol{X}'\rvert$	$\lvert\boldsymbol{X}\boldsymbol{X}'\rvert \operatorname{vec}'\big[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}\big]$

Theorem 5.2 (Chain rule)**.** If $F : \mathbb{R}^{I_1 \times J_1} \to \mathbb{R}^{I_2 \times J_2}$ and $F : \mathbb{R}^{I_2 \times J_2} \to \mathbb{R}^{I_3 \times J_3}$,

$$D(G \circ F)(\boldsymbol{X}) = DG\big(F(\boldsymbol{X})\big) \times DF(\boldsymbol{X})$$

■