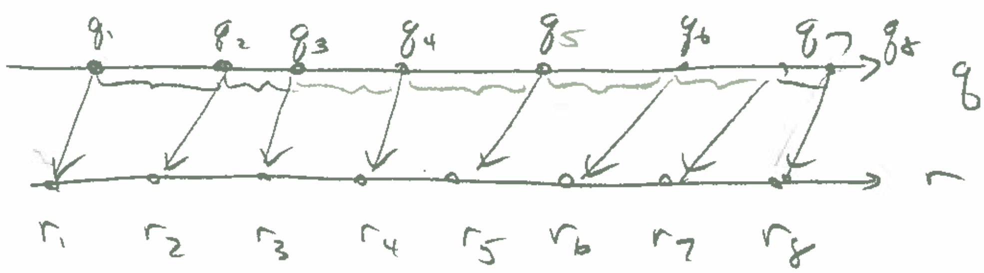
We're given two ordered sequences representing points on a time line.

$$\underset{\sim}{r} = r_1, r_2, \ldots, r_N \qquad \text{reference.}$$

$$\underset{\sim}{q} = q_1, q_2, \ldots, q_N \qquad \text{another sequence.}$$

We seek a sequence $\underset{\sim}{s}$ that close to $\underset{\sim}{r}$ but with spacings like $\underset{\sim}{q}$.



One candidate is a linear interpolate

$$\underline{s} = \varphi \underset{\sim}{r} + (1-\varphi)\underline{q}$$

$$\varphi = 0 \Rightarrow \underline{s} = \underline{q}$$

$$\varphi = 1 \Rightarrow \underline{s} = \underline{r}$$

How to measure performance?

$\|\underline{s} - \underline{r}\|$ ← distance of $\underline{s}$ from $\underline{r}$

$\|\Delta s - \Delta q\|$ ← distance of differences in s from differences in q.

If $\underline{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix}$ then $\Delta s = \begin{bmatrix} s_2 - s_1 \\ s_3 - s_2 \\ \vdots \\ s_N - s_{N-1} \end{bmatrix}$ } $N-1$ elements.

or

$$\underline{\Delta s} = \underbrace{\underbrace{N-1\left\{\begin{bmatrix} -1 & 1 & & & 0 \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & 0 & & & \\ & & & -1 & 1 \end{bmatrix}\right.}_{N}}_{\underline{\underline{\Delta}}} \underbrace{\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix}}_{s.}$$

Alessandro's note suggested various weightings

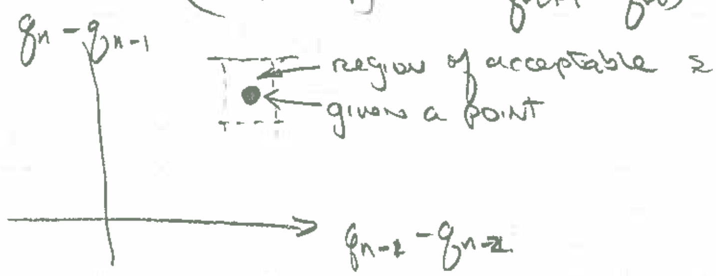
$$\cancel{\| \underline{\Delta s} - \underline{\Delta g} \| \leq}$$

$$b \leq \frac{s_{n+1} - s_n}{g_{n+1} - g_n} \leq a$$

$s_{n+1} - s_n$ positive, $g_{n+1} - g_n$ positive, $b \leq 1$, $a \geq 1$

$$\Rightarrow \quad b\,\underbrace{(g_{n+1} - g_n)} \leq \underline{s_{n+1} - s_n} \leq a\,\underline{(g_{n+1} - g_n)}$$

$$(s_{n+1} - s_n) - b\,(g_{n+1} - g_n) \geq 0$$
$$(s_{n+1} - s_n) - a\,(g_{n+1} - g_n) \leq 0$$



This is like an $\|\cdot\|_\infty$ norm though not exactly the same because of the differences in a & b.

Suppose we just say that we want

(a) $\quad \| \Delta s - \Delta g \|$ small

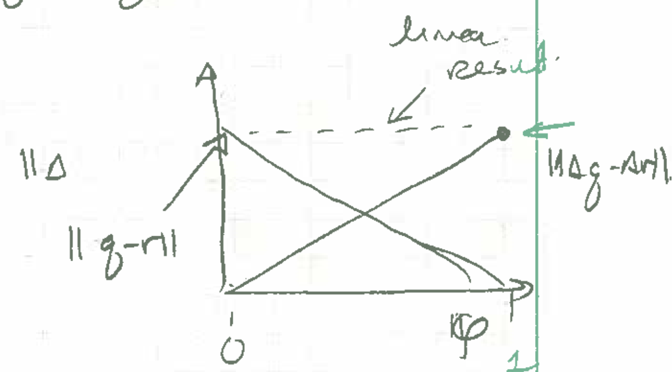
or $\quad \| \Delta s - \Delta g \| \leq \varepsilon \| \Delta g \|$ ← relatively small.

Linear Interpolation

Suppose. $\underline{s} = \varphi \underline{r} + (1-\varphi)\underline{q}$

$$\|s-r\| = \|(\varphi-1)r + (1-\varphi)q\|$$
$$= (1-\varphi)\|q-r\|$$

$$\|\Delta s - \Delta q\| = \|\varphi \Delta r + (1-\varphi)\Delta q - \Delta q\|$$
$$= \|\varphi \Delta r - \varphi \Delta q\|$$
$$= \varphi \, \|\Delta q - \Delta r\|$$



Probably seek a ~~better~~ a quadratic weighting to achieve a "balanced" tradeoff.

Least Squares

Find $\underline{s}$ which minimizes $J = \|\underline{s}-\underline{r}\|_2^2 + \alpha \|\Delta s - \Delta q\|_2^2$

$$J = (s-r)^T(s-r) + \alpha(\Delta s - \Delta q)^T W (\Delta s - \Delta q)$$

↑ optional weighting

$$= (s-r)^T(s-r) + \alpha(s-q)^T \Delta^T W \Delta (s-q)$$

$$\frac{\partial J}{\partial s^T} = \cancel{2} s^T - \cancel{2} r^T + \cancel{2}\alpha\, s^T \Delta^T W \Delta - \cancel{2}\alpha\, q^T \Delta^T W \Delta^T = 0$$

$$(I + \alpha \Delta^T W \Delta)s = r + \alpha \Delta^T W \Delta q$$

Solve for s.

Note that for $W = I$ we have

$$\Delta^T \Delta = \underbrace{\begin{bmatrix} 1 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & 2 & \\ 0 & & & & 2 \; -1 \; 1 \end{bmatrix}}_{N} \Bigg\} N$$

So a second order difference operator. We'll call this D_2. Interpolate is the solution to $(W = I)$

For
$$\boxed{(\underbrace{I + \alpha D_2}_{\text{tridiagonal}}) s = r + \alpha D_2 q}$$

Iterative solution $\Rightarrow$

$$s_{k+1} = (r + \alpha D_2 q) - \alpha D_2 s_k$$

$$\boxed{s_{k+1} = r + \alpha D_2 (q - s_k)}$$

$s_0 = r$

this will not be stable for large α. So forget it.

So need a tridiagonal solver for

$$\boxed{(I + \alpha D_2) s = r + \alpha D_2 q}$$

Given α,