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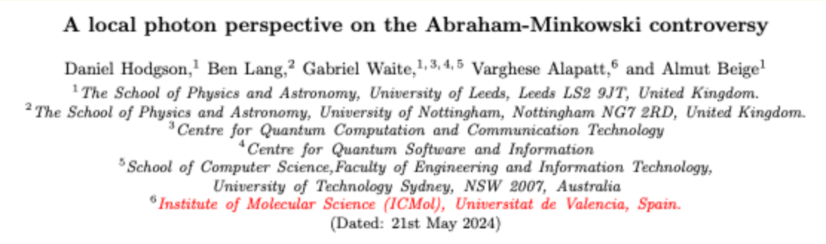
(Kind of) Masters Thesis



A local photon perspective on the Abraham-Minkowski controversy

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UTS
PhD – Quantum Algorithms and Complexity

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Efficient classical algorithms for calculating the free energy of quantum spin systems on trees

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Motivation

Given a local Hamiltonian defined over a spin system, what is the computational complexity of computing the free energy of the system?

Does there exist an efficient approximation algorithm for calculating the free energy of a quantum spin system on a tree?

(A surface overview of the technical details.)
(I will keep it light.)

Consider a graph $G = (\mathcal{V}, \mathcal{E})$. For a finite subset $\Lambda \subset \mathcal{V}(G)$ then for any $X \subset \Lambda$, define an 'interaction' $\Phi(X)$ as a function over the set X. The Hamiltonian is then defined as

$$H_\Lambda := \sum_{X \subset \Lambda} \Phi(X).$$

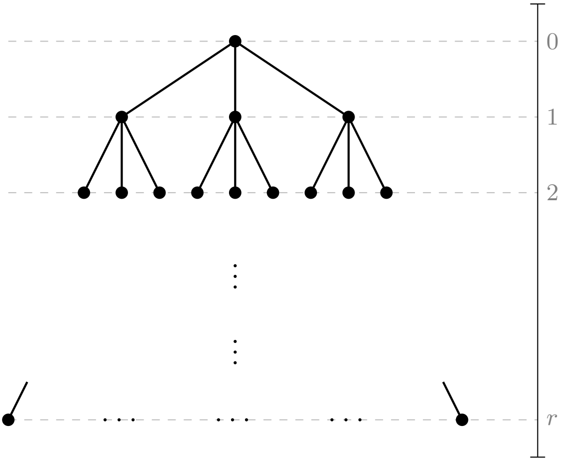
Time-evolution of a local operator Q is defined as $\Gamma^t_\Lambda(Q) := e^{iH_\Lambda t}\, Q\, e^{-iH_\Lambda t}$

The Gibbs state of a Hamiltonian H_Λ is defined as $\rho_\Lambda := e^{-\beta H_\Lambda} / \mathrm{Tr}\left[e^{-\beta H_\Lambda}\right]$

The correlation between two observables A and B is defined as $\mathrm{Cor}_\rho(A, B) := \mathrm{Tr}[\rho\, AB] - \mathrm{Tr}[\rho\, A]\, \mathrm{Tr}[\rho\, B]$

Δ-regular Tree Graph

$T_{\Delta,r} = (\mathcal{V}, \mathcal{E})$ is a Δ-regular tree graph of depth r.



Important concepts

(a) Decay of correlations (**DoC**): a property of a state that deals with how the correlation between two observables decays as the distance between them increases.

$$\mathrm{Cor}_\rho(A, B) \leq \|A\|\|B\|e^{-d/\xi}, \quad \text{exponential DoC.} \tag{1}$$

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(b) Local indistuinguishability (**LI**): a property of an interaction that says some local observable cannot distinguish between the full and truncated Gibbs state.

$$\mathrm{Tr}[\rho_\Lambda\, A] \approx \mathrm{Tr}[\rho_{\Lambda_l}\, A], \quad \Lambda_l \subset \mathcal{V}(G). \tag{2}$$

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(c) Local perturbations perturb locally (**LPPL**): a property of a Hamiltonian that says the effect a local perturbation has on the system is primarily local.

$$\mathrm{Tr}\left[\rho^{[H+V]} \, A\right] \approx \mathrm{Tr}\left[\rho^{[H]} \, A\right], \quad V \text{ local perturbation}. \tag{3}$$

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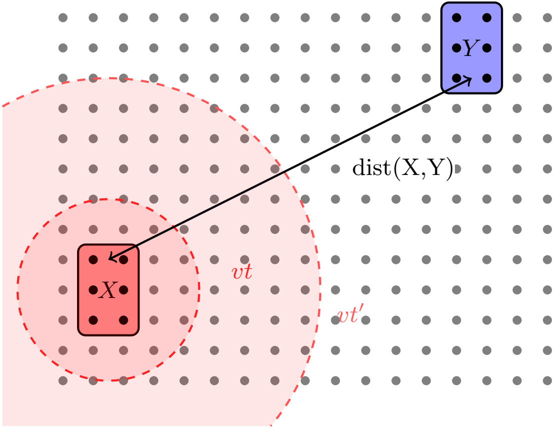
(d) Quantum Belief Propagation (**QBP**) equations: evolution-type equations for the unnormalised Gibbs state of $H(s) = H + sV$.

$$e^{-\beta H(s)} = \eta(s) \, e^{-\beta H} \, \eta^\dagger(s), \quad \eta(s) \text{ not unitary.} \tag{4}$$

(e) Lieb-Robinson bound (**LRB**): a property of a spin system on some graph.

$$\left\| \left[\Gamma^t_\Lambda(V), Q \right] \right\| \le \|V\| \, \|Q\| \, e^{\mu(vt-d)}. \tag{1}$$



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Let's use expansionals!

$$\eta(s) = \mathrm{E{\scriptsize XP}}_L \left(\int_0^s ; \mathrm{d}\lambda \ \Theta_{\beta,H,V}(\lambda) \right),$$
$$\Theta_{\beta,H,V}(\lambda) = -\frac{\beta}{2} \underbrace{\int_{-\infty}^{\infty} \mathrm{d}t \ f_\beta(t) \, \Gamma^t_{H(s)}(V)}_{\Phi_\beta^{H(s)}(V)}$$

LRB meets QBP

The existence of a Lieb-Robinson bound for $H(s)$ allows us to approximate the operators Φ and η. Moreover,

$$\left\|\eta_\Lambda(s) - \eta_{B_l(X)}(s)\right\| \leq \frac{\beta}{2}\, e^{\frac{\beta}{2}\|V\|}\, \|V\|\, g_{\mathrm{QBP}}\big(B_l(X), t\big).$$

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Assume the system has a DoC of the form $\mathrm{Cor}(V, Q) \leq \|V\|\|Q\| g_{\mathrm{DoC}}\big(d\big)$, then we can show LPPL

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$$\left|\mathrm{Tr}[\rho_\Lambda\, Q] - \mathrm{Tr}\left[\rho_{\Lambda\setminus X}\, Q\right]\right| \leq \|Q\|\|V\| g_{\mathrm{LI}}\big(\mathrm{dist}(X, Y)\big). \tag{3}$$

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With LI we can prove conditions needed for $g_{\mathrm{DoC}}\big(\,\cdot\,\big)$.

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Use LRB to truncate $\Gamma^t_H(V)$ to a local region.

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The cost of calculating $\mathrm{Tr}_{B_l(X)}[\rho^{(B_l(X))}\, \Gamma^t_{B_l(X)}(V)]$ is $2^{O(\mathrm{Vol}(l))}$.

Applying it to the tree

$T_{\Delta,r} = (\mathcal{V}, \mathcal{E})$. QBP: $e^{-\beta(H+V)} = \eta(1)\, e^{-\beta H}\, \eta(1)^\dagger$. Define $H_j = \sum_{i=0}^{j-1} \Phi_i$ (spiral)

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$$\rho_{j+1} := \frac{e^{-\beta(H_j+\Phi_j)}}{Z_j}, \quad Z_j = \mathrm{Tr}\left[e^{-\beta(H_j+\Phi_j)}\right] \implies Z_n = \prod_{j=1}^{n} \mathrm{Tr}\left[\rho_j\, \eta_j\, \eta_j^\dagger\right].$$

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Find approximations to ρ_j and $\mathcal{E}_j$ based on balls of radius a and l.
(Dropping j subscript)

$$\left|\mathrm{Tr}[\rho\, \mathcal{E}] - \mathrm{Tr}\left[\widetilde{\rho}_{B_a(X)}\, \widetilde{\mathcal{E}}_{B_l(X)}\right]\right| \leq \beta \int_0^1 \mathrm{d}\lambda\, \mathrm{Cor}_{\rho(\lambda)}(H_\partial, \widetilde{\mathcal{E}}_{B_l(X)}) + \left\|\mathcal{E} - \widetilde{\mathcal{E}}_{B_l(X)}\right\|. \quad (5)$$

Conclusion

- We have shown how the concepts of QBP, LRB, DoC, LPPL, and LI are linked.
- Importantly, we show the QBP can be well-approximated on a tree.
- The cost of calculating a trace marginal is exponential in the volume of the region.
- Under an appropriate DoC for the tree, a value for l and a can be found such that the free energy can be efficiently approximated to within ϵ.