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A New Ant Colony Optimization Algorithm for the Lower Bound of Sum Coloring Problem

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Abstract We consider an undirected graph $G = (V, E)$, the minimum sum coloring problem (MSCP) asks to find a valid vertex coloring of G, using natural numbers (1, 2, ...), the aim is to minimize the total sum of colors. In this paper we are interested in the elaboration of an approximate solution for the minimum sum coloring problem (MSCP), more exactly we try to give a lower bound for MSCP by looking for a decomposition of the graph based on the metaheuristic of ant colony optimization (ACO). We test different instances to validate our approach.

Keywords Minimum sum coloring problem · Lower bound for MSCP · Ant colony optimization

1 Introduction

In this work we seek to improve the lower bound of the minimum sum coloring problem (MSCP) given in the literature. The sum coloring problem is derived from the graph coloring one, which is an NP-hard problem. This latter has various applications like the timetable problems [7], design and operation of flexible manufacturing systems [26], frequency assignment [15], register allocation [5], scheduling problem [29], etc. Thus, several heuristic and metaheuristic approaches have been developed to produce good coloring with a reasonable execution time. Lucet et al. [21] proposed an exact method based on a linear-decomposition. Bloechliger and Zufferey describe a reactive tabu implementation [2]. The most powerful heuristics are the hybrid algorithms combining local search with a population-based approach [13, 14].

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The minimum sum coloring problem consist in finding a graph coloring, by assigning integers at each vertex, so that the sum of colors is minimal. It was demonstrated by Kubicka and Schwenk [19] that MSCP is NP-complete. The minimum sum coloring problem can model a number of interesting problems including those from VLSI design, scheduling and resource allocation [1, 22]. The theoretical results existing in the literature for MSCP were demonstrated for particular graphs [18], but numerically, the only results are the following:

In 2007, Kokosinski and Kwarciany [17] gave the upper bounds by using a parallel genetic algorithm based on GPX and CEX crossover with a number of iterations between 5000 and 10000. In 2010, Moukrim et al. presented a lower bound for MSCP by studying several approaches based on the extraction of partial graphs [23], so they gave a coloring for the complementary graph using a greedy algorithm MRLF. Li et al. [20] demonstrated that the complexity of MRLF is $o(n^3)$, so based on this result they improved the upper bound of MSCP given in [17].

In 2011, Douiri and Elbernoussi presented upper bounds by combining a genetic algorithm with a local heuristic (DBG) [12].

The sections of this paper will be distributed as follows: In Section 2, we quote the definitions related to the graph coloring problem (GCP) and the minimum sum coloring problem (MSCP). In Section 3 we treat the lower bound of the sum coloring graph. Section 4 describes our resolution approach, the principle of ACO metaheuristic is presented, and the implementation of different steps of ACO method to resolve our problem were detailed. Section 5 reports some computational results.

2 Definitions

We consider an undirected graph $G = (V, E)$, where V is the set of vertices and E denotes the set of edges. The coloring of G, which assigns k colors to the vertices of the graph is called k-coloring. A k-coloring is an assignment $c : V \longrightarrow \{1, ..., k\}$ such as $c(x) \neq c(y), \forall (x, y) \in E$. The value $c(x)$ associated with vertex x is called color of x.

This application allows us to decompose the vertex set V into k classes, each one is associated with a color.

$$\text{Let } V = \{V_1, V_2, ..., V_k\}, \text{ with } card(V) = \sum\nolimits_{i\in(1,...,k)} card(V_i)$$

The smallest number of different used colors for a valid coloring is called the chromatic number, and denoted $\chi(G)$. It is well known that the k-coloring problem is NP-complete and the searching of $\chi(G)$ coloring is NP-hard [16], thus heuristic approaches are inevitable in practice.

MSCP consists in finding a valid coloring so that $\sum_{v\in V} c(v)$ is minimal, this sum is denoted $\sum(G) = min_c \sum_{v\in V} c(v)$ see Fig. 1. The smallest number of colors used to color G in the MSCP problem is called the strength of G and denoted $s(G)$. It is trivial that $s(G)$ is lower bounded by $\chi(G)$:

$$\chi(G) \leq s(G) \tag{1}$$

If we decompose the graph G into independent subsets $V_1, V_2, ..., V_k$, we obtain a valid k-coloring by assigning to each subset V_i the color i, $1 \leq i \leq k$ then $\sum(G) = \sum_{i\in(1,...,k)} i.card(V_i)$, where $card(V_1) \geq card(V_2) \geq .... \geq card(V_k)$.

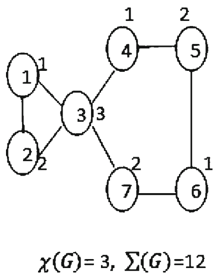

Fig. 1 A valid coloring of the vertices using three colors

For any graph $G = (V, E)$, we give below the lower bounds in the existing literature. Kokosinski and Kwarciany proposed two theoretical lower bounds in [17]:

$$card(V) + (s(G)(s(G) - 1))/2 \leq \sum(G) \tag{2}$$

$$card(V) + (\chi(G)(\chi(G) - 1))/2 \leq \sum(G) \tag{3}$$

To prove Eq. 2, it suffices to see that the coloring sum with exactly $s(G)$ colors is at least equal to $\sum_{i=1}^{s(G)} i + (n - s(G)) = \frac{s(G)(s(G)+1)}{2} = n + \frac{s(G)(s(G)-1)}{2}$, inequality (3) is deducted directly from Eqs. 1 and 2.

Thomassen et al. [27] proposed the following interesting tight bound on the chromatic sum of graph depending on the number of edges:

$$\lceil\sqrt{8.card(E)}\rceil \leq \sum(G). \tag{4}$$

3 Lower Bound of MSCP

Consider an undirected graph $G = (V, E)$. We determine a partial graph $G' = (V, E')$ $(E' \subseteq E)$ by decomposing G into disjoint cliques. As each pair of vertices in a clique is connected by an edge, so each vertex receives a different color, and therefore, for any clique of size w the clique coloring sum is equal to $w(w+1)/2$. Assume that G is partitioned into k cliques of size w_i with $card(V) = \sum_{i\in(1,\ldots,k)} card(w_i)$, we obtain a graph $G' = (V, E')$ which has a graph coloring sum equal to $\sum_{v\in V} c(v) = \sum_{i\in(1,\ldots,k)} w_i(w_i + 1)/2$. This value will be a lower bound of the coloring sum of G:

$$\sum\nolimits_{i\in(1,\ldots,k)} w_i(w_i + 1)/2 \leq \sum(G)$$

3.1 Remarks

1. The quality of the lower bound depends on the way to decompose the graph. There are several decompositions: tree decomposition and path decomposition studied by [3]. Our approach is based on clique decomposition. The graphs (A), (B) and (C) see Fig. 2 correspond to different decompositions of the graph in Fig. 1. For this example we notice that the coloring sum of the decomposition into cliques for the graph (A) is better than the others ($\sum(G') = 12$) as a lower bound.

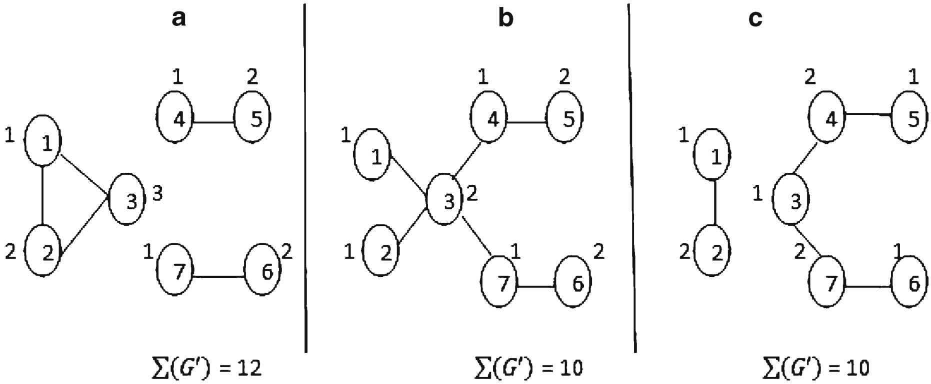

Fig. 2 Partial graphs of G

2. In Fig. 3 we decompose the graph G into cliques in two different ways, we get the following coloring sum $\sum(G'_1) = 16$ and $\sum(G'_2) = 13$. This leads to the observation that decomposition of graphs into larger cliques is correlated with bigger values of the corresponding graph coloring sums.

4 Idea of Resolution

The purpose is to decompose the graph in the form of cliques. For this reason, we build the complementary graph G^c of the original graph G. We seek then a valid coloring, using an algorithm, that will provide stable sets of G^c. This approach allows us to decompose G into cliques. Our goal is to maximize the bound $\sum w_k(w_k + 1)/2$, where w_k is the stable set size associated to the k color given by coloring G^c.

4.1 The G^c Coloring Using ACO

After the construction of the complementary graph G^c an ant colony optimization algorithm (ACO) is applied to color the graph by a coloring without conflicts.

ACO is a population-based metaheuristic introduced by Dorigo [8] in 1992 and developed after in [10, 11]. Ants are capable to resolve collectively complex problems. For that purpose, they communicate between them in a local and indirect way,

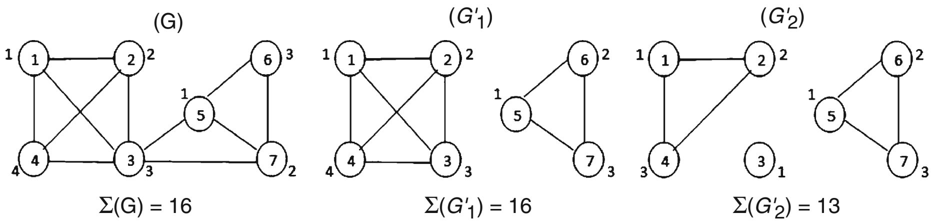

Fig. 3 A graph G and two decomposition into cliques of G

with a volatile hormone called pheromone. During their progression, ants deposit a pheromone trail. Then they choose their path randomly with a probability depending on the quantity of pheromone previously deposited. This mechanism, which allows ants to solve complex problems collectively, is originally based on algorithms of artificial ants. This metaheuristic has solved several NP-hard combinatorial optimization problems, such as, the quadratic assignment problem [9], the traveling salesman problem [4], network design [24]. Costa and Hertz [6] adapted the principles of Ant System to solve the graph coloring problem, Walkowiak presented the graph coloring problem which is equivalent to the coloring problem of a map using a minimum number of colors. He described the main characteristics of ant algorithms and gave two examples of ant algorithms applications to the graph coloring problem [28], Salari and Eshghi [25] proposed a modification of an ACO algorithm which is conforms to Max–Min Ant System structure for the graph coloring problem using a local search heuristic.

The behavior of artificial ants is inspired from real ants: they deposit pheromone on the components of the graph and they choose their paths in relation to pheromone trails previously deposited, these traces evaporate over time. This indirect communication provides an information about the quality of the followed paths and their attractive power of ants in future iterations.

At each solution construction step the ant has to decide to which neighboring vertex to move. This decision is made probabilistically based on the pheromone values and some heuristic information that especially helps finding a good solution in the beginning of the algorithm when all pheromone values are equal. Each ant constructs its solution. It deposits an adequate amount of pheromone onto traversed edges. Additionally a certain amount of pheromone evaporates after each iteration. The evaporation can be adjusted with a parameter, called evaporation rate ρ. Our ACO algorithm is applied to find a conflict-free coloring for G^c.

The individual $p =$ (c(1),c(2), ..., c(N)) corresponds to an assignment of k colors to all vertices of the graph G^c. The size of every individual is equal to the number of G vertices. The first step consists in initializing the tracks of pheromone, thereafter, in each cycle, every ant constructs a solution (coloring) and the pheromone trails are updated. The algorithm stops when the maximum number of cycles is reached. Each time, an i-th vertex is chosen, and a set of candidate colors noted C is built by using a constructive algorithm, the i-th vertex receives the color l from the set C with probability:

$$p(i \leftarrow l) = \frac{\tau_i(l)^{\alpha} . \eta_i^{\beta}}{\sum\limits_{l \in C} \tau_i(l)^{\alpha}} \tag{5}$$

- α and β are two parameters.
- $\tau_i(l)$ is the pheromone amount deposed on color l.

$$\eta_i = \frac{1}{nclr_i} \tag{6}$$

- $nclr_i$ is the current number of colors used before coloring the i-th vertex.

The construction of the candidate set of valid colors for the i-th vertex, is based on the connections of i-th vertex and the current number of used colors. In the Fig. 4a the vertex {5} is related to the vertices {2, 3}. Therefore the candidate set of colors

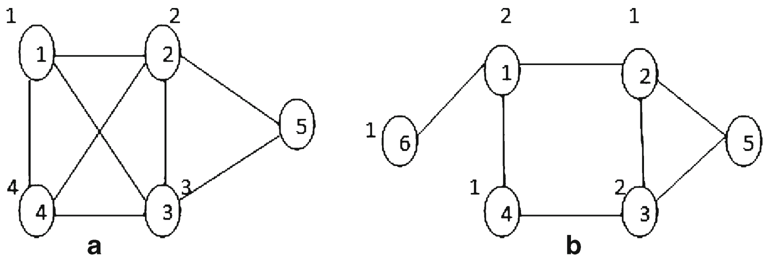

Fig. 4 Construction of the set C

is $C = \{1, 4\}$, the vertex $\{5\}$ receives a color of C with a probability depending on τ and η. For the Fig. 4b the vertex $\{5\}$ is related to the vertices $\{2, 3\}$. The vertex $\{5\}$ can take neither color 1 nor color 2 i.e. $C = \emptyset$, so the vertex $\{5\}$ receives a new color noted $(nclr + 1)$.

4.2 Update

The management of the updates is made in two stages, a local update and a global update.

Once the ant k has generated a solution (coloring), the amounts of pheromone deposited by each ant on each color is updated following this rule:

$$\tau_i(l) \leftarrow (1 - \rho_{\mathrm{loc}}).\tau_i(l) + \Delta\tau_i(l) \tag{7}$$

and

$$\Delta\tau_i(l) \leftarrow \Delta\tau_i(l) + \frac{L_{\mathrm{loc}}}{nclr_k} \tag{8}$$

where

- $\Delta\tau_i(l)$ is the pheromone addition by the ant k on the color l assigned to the i-th vertex.
- L_{loc} is a constant.
- ρ_{loc} is an evaporation factor.
- $nclr_k$ is the number of colors used by the ant k.

At each iteration T, and after that each ant has built its solution, a global update of pheromone quantities is made for all colors intervening in the best coloring.

This global update use a new factor ρ_{gl} as follows:

$$\tau_i(l) \leftarrow (1 - \rho_{\mathrm{gl}}).\tau_i(l) + \Delta\tau_i(l) \tag{9}$$

and

$$\Delta\tau_i(l) \leftarrow \Delta\tau_i(l) + \frac{L_{\mathrm{gl}}}{nclr_{\mathrm{best}}} \tag{10}$$

- $\Delta\tau_i(l)$ is the pheromone addition on each colors of the best coloring at iteration T.

- L_{gl} is a constant.
- $nclr_{\text{best}}$ is the smallest number of colors found by all ants at each iteration T.

The global update is done in order to favor the colors assigned to clique's vertices of maximum size. In fact we try to found a solution given by ants with fewer colors ($nclr_{\text{best}}$). So for the complementary graph G^c we have fewer independent sets, and therefore a smaller number of cliques of G with larger sizes, see remark in Section 3.1.

At each iteration, we seek to found the ant that gives a minimum colors number ($nclr_{\text{best}}$). Thereafter we increase the pheromone quantity inversely proportional to the number ($nclr_{\text{best}}$) on this solution's colors for each vertex see Eqs. 9 and 10. Consequently we favor more these colors for a future choice.

Algorithm 1

Require: Graph $G = (V, E)$, α, β, ρ_{loc}, ρ_{gl}, L_{loc}, L_{gl}, nbAnts, $T_{\max}$
Ensure: A coloring G^c
Begin
Give the complementary graph G^c
Initialize pheromone trails
$T = 0$
while $T \leq T_{\max}$ **do**
  for $k = 1$ to $nb\ Ants$ **do**
    $c(1) \leftarrow 1$
    $nclr = 1$ the current number of colors used
    for $i = 2$ to $card(V)$ **do**
      Apply the constructive algorithm, and give the set of candidate colors C
      if $C \neq \emptyset$ **then**
        Choose a color in C for the i-th vertex with the probability:
        $p(i \leftarrow l) = \frac{\tau_i(l)^\alpha . \eta_i^\beta}{\sum_{l \in C} \tau_i(l)^\alpha}$
      else
        $c(i) \leftarrow nclr + 1$
      end if
    end for
    The local update of pheromone quantities of every ant:
    $\Delta\tau_i(l) \leftarrow \Delta\tau_i(l) + \frac{L_{\text{loc}}}{nclr_k}$
    $\tau_i(l) \leftarrow (1 - \rho_{\text{loc}}).\tau_i(l) + \Delta\tau_i(l)$
  end for
  The global update of pheromone quantities after each iteration:
  $\Delta\tau_i(l) \leftarrow \Delta\tau_i(l) + \frac{L_{\text{gl}}}{nclr_{\text{best}}}$
  $\tau_i(l) \leftarrow (1 - \rho_{\text{gl}}).\tau_i(l) + \Delta\tau_i(l)$
  $T \leftarrow T + 1$
end while
End

Having colored the graph $G^c = (V, E')$ by the ACO algorithm (Algorithm 1) each independent set of G^c corresponds to a clique of the graph G. We were able to decompose the graph $G = (V, E)$ in the form of cliques. We can calculate the value $\sum_{i \in (1,\ldots,k)} w_i(w_i + 1)/2 \leq \sum(G)$, where w_i is the size of the clique i.

Table 1 Computational results for $\sum(G)$

Graph	n	Card(E)	$\chi(G)$	$w_{\max}$	UB	UB_{kok}	LB_{th}	LB_{Mkr}	LB_{Ant}	No. of successes	Time (s)
Anna	138	493	11	11	279	281	193	**272**	**272**	2/20	47.61
David	87	406	11	11	241	243	142	**234**	**234**	4/5	35.43
Huck	74	301	11	11	243	243	129	**243**	**243**	4/5	31.07
Jean	80	254	10	10	217	218	125	**216**	**216**	3/20	32.94
Queen5.5	25	160	5	5	75	75	36	**75**	**75**	5/5	6.19
Queen6.6	36	290	7	6	138	138	57	**126**	**126**	5/5	9.39
Queen7.7	49	476	7	7	196	196	70	**196**	**196**	5/5	13.37
Queen8.8	64	728	9	8	302	302	100	**288**	**288**	5/5	19.54
Miles250	128	387	8	8	343	347	156	**316**	**316**	4/5	44.25
Miles500	128	1170	20	20	755	762	318	**677**	**677**	2/20	53.72
Games120	120	638	9	9	446	460	156	**442**	**442**	4/5	48.32
Myciel3	11	20	4	2	21	21	**17**	16	16	5/5	2.93
Myciel4	23	71	5	2	45	45	33	**34**	**34**	5/5	5.26
Myciel5	47	236	6	2	93	93	62	**70**	**70**	5/5	8.79
Myciel6	95	755	7	2	189	189	116	**142**	**142**	5/5	27.92
Myciel7	191	2360	8	2	381	382	219	**286**	**286**	5/5	119.76
Fpsol2.i.1	496	11654	65	43	3405	*	2576	*	**2590**	7/10	476.18
Inithx.i.1	864	18707	54	32	3679	*	2295	*	**2801**	4/10	711.85
Mug88-1	88	146	4	3	190	*	94	*	**163**	10/10	37.46
Mug88-25	88	146	4	3	187	*	94	*	**162**	10/10	46.10
Mug100-1	100	166	4	3	211	*	106	*	**187**	8/10	55.14
Mug100-25	100	166	4	3	214	*	106	*	**185**	5/10	59.52
2-Inser 3	37	72	4	2	62	*	43	*	**55**	10/10	11.78
3-Inser 3	56	110	4	2	92	*	62	*	**84**	10/10	19.07
Zeroin.i.2	211	3541	30	30	1013	*	646	*	**1003**	4/10	155.98
Zeroin.i.3	206	3540	30	30	1007	*	641	*	**997**	6/10	149.24

Bold entries signifies best results

5 Results

From different instances imported from the DIMACS library (http://mat.gsia.cmu.edu/COLOR04/), we give experimental results for the lower bound of MSCP, to compare with the theoretical results given by the relations (3) and (4), and also by Moukrim et al. [23]. Our algorithm for the graph coloring sum has been implemented on a PC windows 7, AMD Athlon(tm) X2 dual-core QL-65 (2cpus) 2.1GHz with 4GB RAM. The numerical results obtained by our algorithm are shown in Table 1. For each instance, we indicate the number of vertices n, the number of edges card(E), the chromatic number $\chi(G)$, the size $w_{\max}$ of the maximal clique found after a decomposition into cliques, the theoretical lower bounds LB_{th}, as well as the upper bound given by a parallel genetic algorithm UB_{kok} [17], UB the best upper bound in [12], LB_{Mkr} the lower bounds given in [23], and LB_{Ant} the lower bounds found by our algorithm. We also indicate the number of successful tries, the symbol '*' means that the information is not available. For each graph, we set the number of ants to 30. The parameters α and β were tested by several values between 1 and 10, and we finally chose the best experimental parameters $\alpha = 8$ and $\beta = 4$. The same way was used for determining the best values for the evaporation rate ρ and for parameters L_{loc}, L_{gl}. In Figs. 5 and 6 we show on an example (myciel7) the approaches taken to fix parameters $\rho_{\text{loc}} = 0.75$ and $\rho_{\text{gl}} = 0.2$, the same tests were conducted for several graphs in Table 1 and the same values were achieved. For L_{loc} and L_{gl}, they were

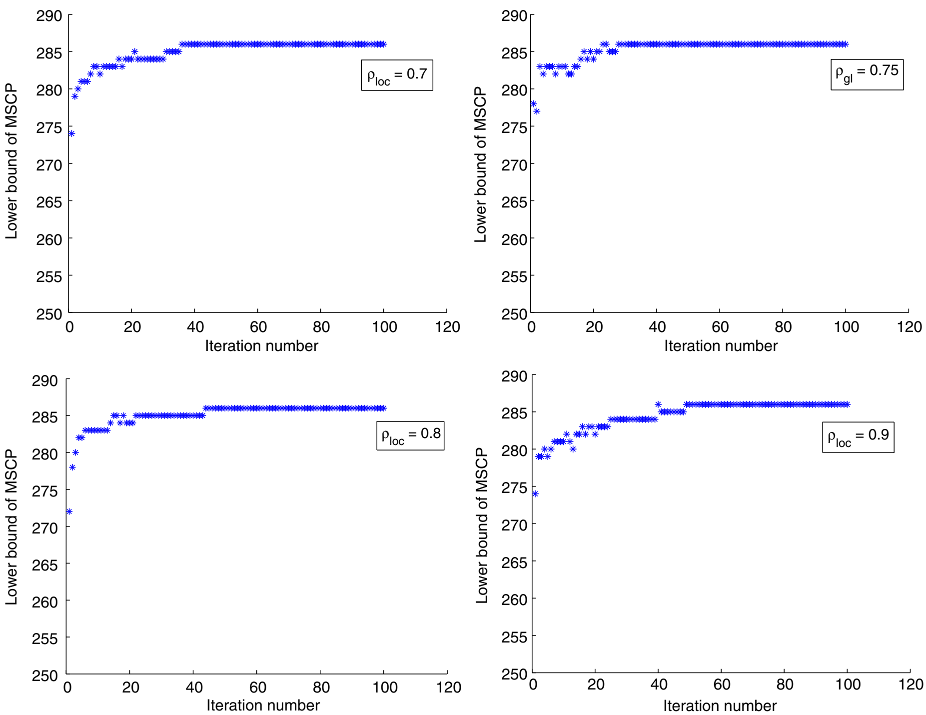

Fig. 5 Comparison of convergence solutions based on different values of ρ_{loc} for myciel7 by setting $\rho_{\text{gl}} = 0.2$

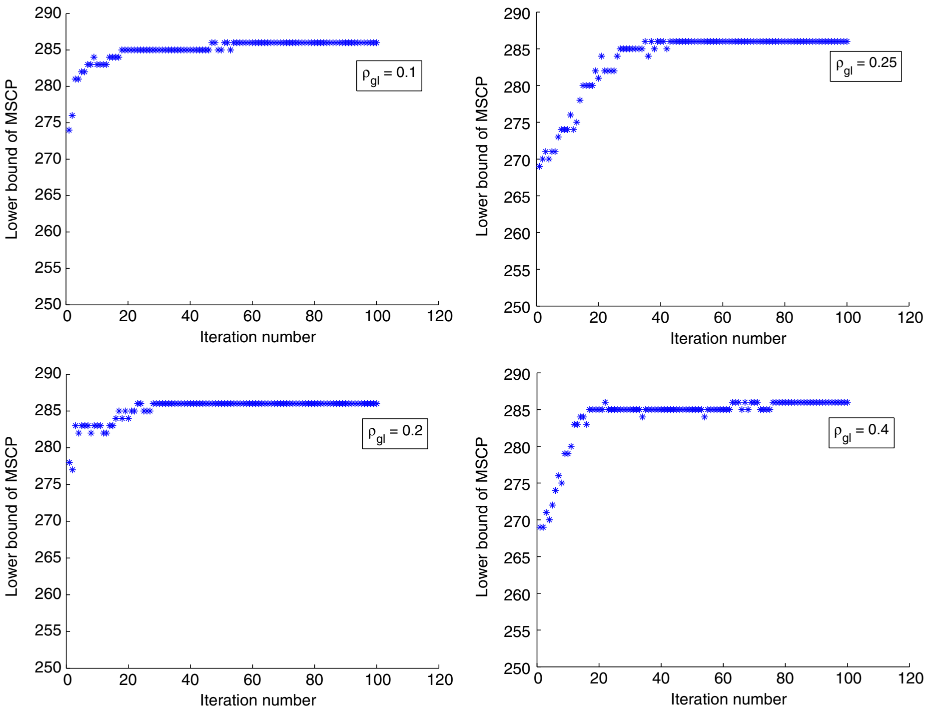

Fig. 6 Comparison of convergence solutions based on different values of ρ_{gl} for myciel7 by setting $\rho_{loc} = 0.75$

tested by different values between 10 and 20, and for each instance the best value was taken. The maximum iterations number T_{max} is taken between 40 and 100.

Most of these instances have been easily resolved, the lower bound found for myciel3 instance are identical to that in [23] and do not improve the theoretical bound.

In fact, we use a decomposition of myciel3 graph in the form of cliques. The decomposition of 11 vertices of this graph that gives the best lower bound is (2, 2, 2, 2, 2, 1), five cliques with size equal to 2 and one with size 1. To obtain the lower theoretical bound 17, it is necessary to have a decomposition under the shape (3, 2, 2, 2, 1, 1) which is impossible, because the myciel3 graph contains no clique with size 3.

6 Conclusion

In this paper we proposed a contribution concerning the study of the lower bound of the minimum sum coloring problem (MSCP). For this, we based ourselves on the coloring of the complementary graph. We apply an ant colony algorithm (ACO) adapted to this problem, to obtain a cliques decomposition of the original graph. This clique decomposition provides better lower bound for MSCP than other graph decompositions. Our approach has improved theoretical bounds given in column

(LB_{th}) except in the case myciel3. For the first 16 instances the lower bounds in [23] were obtained, these results are listed in columns LB_{Mkr} and LB_{Ant}. To show the effectiveness of our algorithm we tested 10 other instances untreated in the literature for the case of lower bound, but studied in [12] for upper bound (see column UB). For these instances we have improved all theoretical bounds.

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