(29) $S_1, S_2, \dots S_n$ - ind. rand. var , $S_i \sim Exp(\lambda_i)$, $\lambda_i > 0$

previously:

$S_{min} = \min\{S_1, S_2, \dots S_n\} \sim Exp(\Lambda)$, $\Lambda = \lambda_1 + \lambda_2 + \dots \lambda_n$

$Pr[S_{min} = S_i] = \frac{\lambda_i}{\Lambda}$ we show it by:

$$Pr[S_{min} = S_i] = Pr[\min\{S_1, S_2, \dots S_n\} = S_i] =$$

$$= \int_0^\infty Pr[x = S_i] \cdot \prod_{\substack{j \neq i \\ j=1}}^{n} Pr[x < S_j] \, dx =$$

$$= \int_0^\infty \lambda_i e^{-\lambda_i x} \cdot \prod_{\substack{j \neq i \\ j=1}}^{n} \left(e^{-\lambda_j x}\right) dx =$$

$$= \lambda_i \int_0^\infty e^{-(\lambda_1 + \dots + \lambda_n)x} \, dx = \lambda_i \int_0^\infty e^{-\Lambda x} dx =$$

$$= \lambda_i \left| -\frac{e^{-\Lambda x}}{\Lambda} \right|_0^\infty = \lambda_i \left(0 - \left(-\frac{1}{\Lambda}\right)\right) = \frac{\lambda_i}{\Lambda}$$

(30) X - rand. var takes value from $[0, \infty)$

we have to show that

$$X \sim Exp(\lambda) \leftrightarrow \underbrace{(\forall_{t,s} \geq 0)(Pr[X > t+s \mid X > s] = Pr[X > t])}_{MLP}$$

'$\to$': $X \sim Exp(\lambda) \to MLP$

$$\forall_{t,s>0} \; Pr[X > t+s \mid X > s] = \frac{Pr[X > t+s \wedge X > s]}{Pr[X > s]} = \frac{Pr[X > t+s]}{Pr[X > s]} =$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = Pr[X > t] \quad \triangle$$

$$Pr[X > x] = 1 - Pr[X \leq x] = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x} \text{ for } x > 0$$

'$\leftarrow$' $MLP \to X \sim Exp(\lambda)$

we begin by showing $\circledast = MLP \to G(x) = G(1)^x$

① $(\forall_{t,s} \geq 0)\,(G(t+s) = G(t)G(s))$

$$G(t+s) = Pr[X > t+s] =$$

$$= \underbrace{Pr[X > t+s \mid X > s] \cdot Pr[X > s]}_{MLP} + \underbrace{Pr[X > t+s \mid X \leq s]}_{0} \cdot Pr[X \leq s] =$$

$$= Pr[X > t] \cdot Pr[X > s] = G(t) \cdot G(s)$$

② We show that we can use $G(\cdot)$ for $\forall x \in \mathbb{Q}$

$$\cdot\, m, n \in \mathbb{N}^+ : G\left(\frac{m}{n}\right)^n = G(\underbrace{\tfrac{1}{n} + \tfrac{1}{n} + \ldots + \tfrac{1}{n}}_{m})^n \overset{①}{=} \left(G\left(\tfrac{1}{n}\right)^m\right)^n =$$

$$= \left(G\left(\tfrac{1}{n}\right)^n\right)^m = G(1)^m \to$$

$$G\left(\frac{m}{n}\right) = G(1)^{\frac{m}{n}}$$

③ $\exists q_n, r_n \in \mathbb{Q}$ such that $q_n \leq x \leq r_n$ $\forall x \in \mathbb{R}$ and

$\lim\limits_{n\to\infty} q_n = x$ and $\lim\limits_{n\to\infty} r_n = x$

④ Since $G(x) = \Pr[X > x]$ is not-increasing we know that

$$q_n \leq x \leq r_n$$

$$G(q_n) \leq G(x) \leq G(r_n)$$

$$G(1)^{q_n} \leq G(x) \leq G(1)^{r_n}$$

$$n\to\infty \downarrow \quad G(1)^x \leq G(x) \leq G(1)^x$$

then $G(x) = G(1)^x$ $\forall x \in \mathbb{R}$

(31) We toss a coin until r-tails

tails - p
heads - $q = 1-p$

$X \sim$ Negative Binomial Distr(r, p)

X = number of heads before r tails.

PMF:

$$P_r[X=x] = \binom{x-1}{r-1}(1-p)^{x-r}p^r \quad \text{for } x = r, r+1, r+2, \dots$$

Expected value

$$M(t) = E(e^{tX}) = \frac{(pe^t)^r}{(1-(1-p)e^t)^r} \quad \text{for } (1-p)e^t < 1$$

$$M'(t) = \frac{d}{dx}\frac{p^r e^{tr}}{(1-(1-p)e^t)^r} = p^r \frac{d}{dt}\frac{\overbrace{e^{tr}}^{=f(t)}}{\underbrace{(1-(1-p)e^t)^r}_{=g(t)}} =$$

$$= p^r \frac{f'(t)g(t) - g'(t)f(t)}{g(t)^2} =$$

$$= p^r \frac{(e^{tr}\cdot(rt)' \cdot (1-(1-p)e^t)^r - r(1-(1-p)e^t)^{r-1} \cdot (1-(1-p)e^t)' \cdot e^{tr}}{(1-(1-p)e^t)^{2r}}$$

$$= p^r \frac{(e^{tr}\cdot r\cdot(1-(1-p)e^t)^r - r(1-(1-p)e^t)^{r-1} \cdot ((p-1)e^t + 0)e^{tr}}{(1-(1-p)e^t)^{2r}}$$

$$= p^r \frac{r(e^{tr}(1-(1-p)e^t)^r - r(1-(1-p)e^t)^{r-1} \cdot (p-1)e^t e^{tr}}{(1-(1-p)e^t)^{2r}}$$

$$M'(0) = p^r \frac{r e^0(1-(1-p)e^0)^r - r(1-(1-p)e^0)^{r-1} \cdot (p-1)e^0 e^0}{(1-(1-p)e^0)^{2r}}$$

$$E[X] = M'(0) = p^r \frac{e^0 r(1-(1-p)e^0)^r - r(1-(1-p)e^0)^{r-1}\cdot(p-1)e^0 e^0}{(1-(1-p)e^0)^{2r}} =$$

$$= p^r \frac{rp^r - rp^{r-1}\cdot(p-1)}{p^{2r}} = p^r \frac{rp^r - rp^r + rp^{r-1}}{p^{2r}}$$

$$= \frac{rp^{r-1}}{p^r} = \frac{r}{p}$$

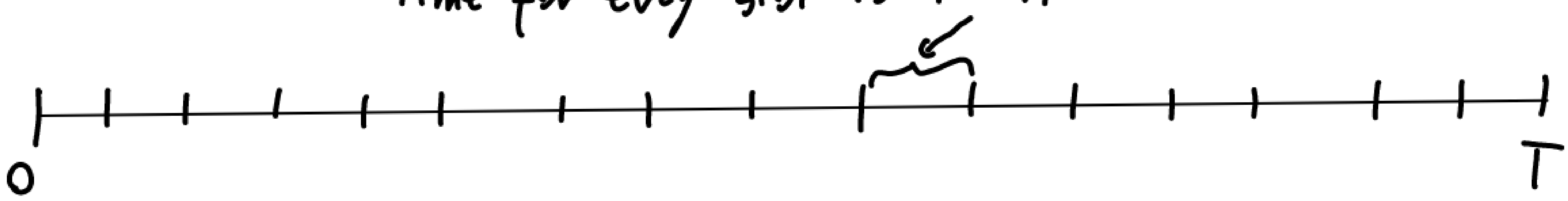
$$Var[X] = E[X^2] - E[X]^2 = M''(0) - [M'(0)]^2 = \ldots$$

$$= \ldots = \frac{r(1-p)}{p^2}$$

(32) discrete Poisson distribution

intuition having n-time slots for simple Bernoulli trial B_i with $Pr[B_i = 1] = \underline{\underline{\lambda \cdot t}}$

time for every slot is $t = \frac{T}{n}$



then $X_{T,n} = B_1 + B_2 + \ldots + B_n$, $X_{T,n} \sim Bin(n, \lambda t)$

but when we $\lim\limits_{n \to \infty} X_{T,n}$ we get $X \sim Poiss(\lambda T)$

because $\lim\limits_{n \to \infty} n \cdot \lambda t = \lim\limits_{n \to \infty} n \cdot \lambda \frac{T}{n} = \lambda T$

---

with parameter μ:

$$Pr[X = j] = \frac{e^{-\mu} \mu^j}{j!}$$

- using Taylor expansion $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$

  we can check that:

$$\sum_{j=0}^{\infty} Pr[X = j] = \sum_{j=0}^{\infty} \frac{e^{-\mu} \mu^j}{j!} = e^{-\mu} \sum_{j=0}^{\infty} \frac{\mu^j}{j!} = e^{-\mu} \cdot e^{\mu} = 1$$

---

what's the expected value?

$$E[X] = \sum_{j=0}^{\infty} j \cdot Pr[X = j] = \sum_{j=0}^{\infty} j \cdot \frac{e^{-\mu} \mu^j}{j!} = \sum_{j=1}^{\infty} \frac{e^{-\mu} \mu^j}{(j-1)!} =$$

$$= \mu \sum_{j=1}^{\infty} \frac{e^{-\mu} \mu^{j-1}}{(j-1)!} = \mu \sum_{j=0}^{\infty} \frac{e^{-\mu} \mu^j}{j!} = \mu \cdot 1 = \mu$$

Binomial Limit Theorem.

Let $X_n \sim Bin(n, p_n) \wedge \lim_{n\to\infty} n \cdot p_n = \mu > 0$ then $(*)$

$$(\forall j \in \mathbb{N})(\lim_{n\to\infty} Pr[X_n = j] = \frac{e^{-\mu}\mu^j}{j!}$$

<u>proof</u>

$$\lim_{n\to\infty} Pr[X_n = j] = \lim_{n\to\infty} \binom{n}{k} p_n^k (1-p_n)^{n-k} \overset{(*)}{=}$$

$$= \lim_{n\to\infty} \frac{n(n-1)\dots(n-k+1)}{k!} \left(\frac{\mu}{n}\right)^k \left(1-\frac{\mu}{n}\right)^{n-k} =$$

$$= \lim_{n\to\infty} \frac{n^k + O(n^{k-1})}{k!} \frac{\mu^k}{n^k} \left(1-\frac{\mu}{n}\right)^{n-k} =$$

$$= \lim_{n\to\infty} \frac{\mu^k}{k!}\left(1-\frac{\mu}{n}\right)^{n-k} = \lim_{n\to\infty} \frac{\mu^k}{k!} \underbrace{\left(1-\frac{\mu}{n}\right)^n}_{\xrightarrow{n\to\infty} e^{-\mu}} \underbrace{\left(1-\frac{\mu}{n}\right)^{-k}}_{\xrightarrow{n\to\infty} 1}$$

$$= e^{-\mu}\frac{\mu^k}{k!} \quad \square$$

(33) X - number of balls thrown untill all bins are not empty.

a) Using the approximation of the number of balls in a given urn by Poisson distribution, show that for large values of n, we have

$$Pr[X > n\ln(n) + cn] \approx 1 - e^{-e^{-c}}$$

b) determine the smallest value of c such that for large values of n, the value of X lies in $[n\ln(n) - cn, n\ln(n) + cn]$ in 99% of casses.

---

a) # balls in each bin is Poisson rand. var X_i, with $E[X_i] = \ln(n) + c$
so expected total # balls is $m = n(\ln(n) + c)$. Also

$Pr[j\text{-th bin empty}] = e^{-(\ln(n)+c)} = \frac{e^{-c}}{n}$

all bins are independent, $\mathcal{E}$ - event that all bins are not empty

$$\lim_{n\to\infty} Pr(\mathcal{E}) = \left(1 - \frac{e^{-c}}{n}\right)^n \underset{n\to\infty}{\approx} e^{-e^{-c}}$$

$$Pr[\mathcal{E}] = Pr[\mathcal{E} \mid |X - m| \le \sqrt{2m\ln m}] \cdot Pr[|X - m| \le \sqrt{2m\ln m}] +$$
$$+ Pr[\mathcal{E} \mid |X - m| > \sqrt{2m\ln m}] \cdot Pr[|X - m| > \sqrt{2m\ln m}]$$

1) $Pr[|X - m| > \sqrt{2m\ln m}]$ is $o(1)$ $\Leftrightarrow$ p-b that in Poisson case the number balls thrown deviates singnificointly from its mean m is $o(1)$.

2) $|Pr[\mathcal{E} \mid |X - m| \le \sqrt{2m\ln m}] - Pr[\mathcal{E} \mid X = m]| = o(1)$ $\Leftrightarrow$
the difference between our experiment coming up with exactly m or almost m balls is ass. negligible differen in the p-b $\mathcal{E}$.

using 1), 2) we get

$$\Pr(\mathcal{E}) = \Pr[\mathcal{E} \mid |X - m| \le \sqrt{2m\ln m}) \cdot (1 - o(1)) + o(1) =$$
$$= \Pr[\mathcal{E} \mid X = m)(1 - o(1)) + o(1), \text{ hence}$$

$$\lim_{n\to\infty} \Pr[\mathcal{E}] = \lim_{n\to\infty} \Pr[\mathcal{E} \mid X = m]$$

so conditioning $X = m$ is like throwing m balls to n bins randomly

---

1) using Chernoff bound $\Pr(X \ge x) \le e^{x - m - x\ln(\frac{x}{m})}$, we put $x = m + \sqrt{2m\ln m}$, getting

$$\Pr[X > m + \sqrt{2m\ln m}] \le e^{\sqrt{2m\ln m} - (m + \sqrt{2m\ln m})\ln(1 + \sqrt{2m\ln m}/m)}$$
$$\le e^{-(\ln m + \sqrt{2m\ln(m)}(\ln m/m))} = o(1)$$

simillar argument for $x < m$ so $\Pr[|X - m| > \sqrt{2m\ln m}] = o(1)$

---

2) $\Pr[\mathcal{E} \mid X = k]$ is increasing in k.

$$\Pr(\mathcal{E} \mid X = m - \sqrt{2m\ln m}) \le \Pr[\mathcal{E} \mid |X - m| \le \sqrt{2m\ln m}]$$
$$\le \Pr[\mathcal{E} \mid X = m + \sqrt{2m\ln m}$$

hence we get bound for

$$\Pr[\mathcal{E} \mid |X - m| \le \sqrt{2m\ln m}] - \Pr[\mathcal{E} \mid X = m] \le$$
$$\le \Pr[\mathcal{E} \mid X = m + \sqrt{2m\ln m}] + \Pr[\mathcal{E} \mid X = m - \sqrt{2m\ln m}]$$
$$= o(1) \quad \square$$

6) determine the smallest value of c such that for large values of n, the value of X lies in $[n\ln(n) - cn,\ n\ln(n) + cn]$ in 99% of casses.

$$\mu = E[X] = n\ln(n) + \Theta(n)$$

using Chebyshev

$$\Pr[|X - \mu| \geq cn] \leq \frac{\pi^2}{6c^2}$$

$$\Pr[\mu - cn \geq X \leq \mu + cn] \leq 1 - \frac{\pi^2}{6c^2}$$

$$\Pr[n\ln(n) - cn \geq X \leq n\ln(n) + cn] \leq 1 - \frac{\pi^2}{6c^2}$$

$$1 - \frac{\pi^2}{6c^2} = \frac{99}{100}$$

$$\frac{1}{100} = \frac{\pi^2}{6c^2}$$

$$6c^2 = \pi^2 \cdot 100$$

$$c = \sqrt{\frac{\pi^2 \cdot 100}{6}}$$

$$\Pr[|X-\mu| \geq \delta\mu] \leq 2e^{-\frac{\delta^2\mu}{3}}$$

$$|X-\mu| < \delta\mu \quad \wedge \quad X-\mu > -\delta$$

$$\mu = n\ln(n)$$

$$-\delta\mu < X-\mu < \delta\mu$$

$$-\delta\mu+\mu < X < \delta\mu+\mu$$

$$\begin{cases} -\delta\mu+\mu = n\ln(n) - cn \\ -\delta\mu = -cn \\ \delta = \frac{cn}{n\ln(n)} = \frac{c}{\ln(n)} \end{cases}$$

$$\begin{cases} \delta\mu+\mu = n\ln(n) + cn \\ \delta\mu = cn \\ \delta = \frac{cn}{n\ln(n)} \end{cases}$$

$$1-2e^{\left(\frac{cn}{n\ln n}\right)^2 \cdot \frac{n\ln n}{3}} = 99\%$$