Lecture 2

Question: How many rounds do we need to elect a leader with probability $\geq 1-\frac{1}{f}$ (for some fixed f e.g. $f=10^6$)?

Lemma 5

Let $f>1$ and let $S_1, \dots, S_r$ be independent copies of the same experiment such that $\forall i \in \{1,\dots,r\}$ $\Pr[S_i] = p \geq \lambda$.
Then $r \geq \frac{-\ln f}{\ln(1-\lambda)} \Rightarrow \Pr[S_1 \cup S_2 \cup \dots \cup S_r] \geq 1-\frac{1}{f}$

Proof:

① $\Pr[S_1 \cup S_2 \cup \dots \cup S_r] \geq 1-\frac{1}{f} \Leftrightarrow 1-(1-p)^r \geq 1-\frac{1}{f} \Leftrightarrow$

$\Leftrightarrow (1-p)^r \leq \frac{1}{f} \Leftrightarrow r \cdot \ln(1-p) \leq -\ln f \Leftrightarrow r \geq \frac{-\ln f}{\ln(1-p)}$

② $r \geq \frac{-\ln f}{\ln(1-\lambda)} \overset{③}{\Rightarrow} r \geq \frac{-\ln f}{\ln(1-p)} \overset{①}{\Rightarrow} \Pr[S_1 \cup \dots \cup S_r] \geq 1-\frac{1}{f}$

③ $p \geq \lambda \rightarrow 1-p \leq 1-\lambda \rightarrow \ln(1-p) \leq \ln(1-\lambda) \rightarrow \frac{1}{\ln(1-p)} \geq$

$\geq \frac{1}{\ln(1-\lambda)} \rightarrow \frac{-\ln f}{\ln(1-p)} \leq \frac{-\ln f}{\ln(1-\lambda)} \leq r$ $\square$

for example

- $f = 10^2 \rightarrow r = \left\lceil \frac{-\ln f}{\ln(1-\lambda)} \right\rceil = 6$ with p_b $\geq 1-\frac{1}{10^2}$
- $f = 10^6 \rightarrow r = 16$

Conclusion

$\#\text{slots} \leq \left\lceil \frac{-\ln f}{\ln(1-\lambda)} \right\rceil$ with p_b $\geq 1-\frac{1}{f}$

Lower bound

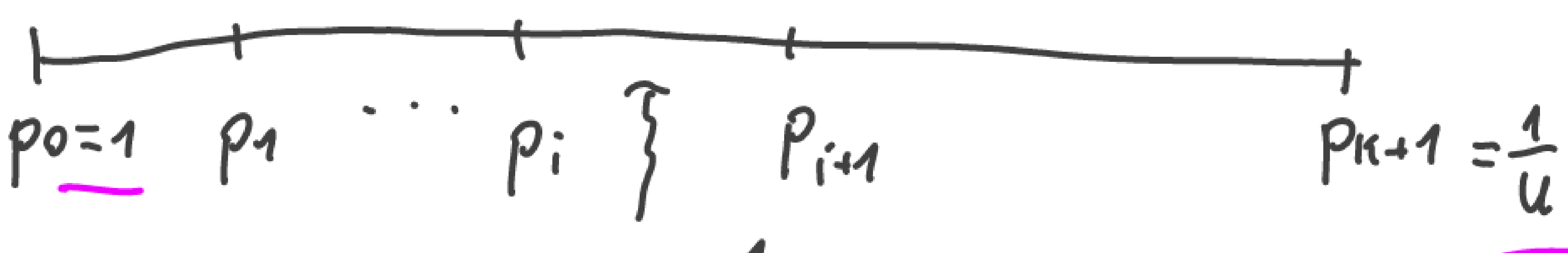
Question: What is the shortest vector $\bar{p}_k = (p_1, \dots, p_k)$ such that we can elect a leader with p-b $\geq 1 - \frac{1}{f}$?

Remarks:

1. Without loss of generality we can assume that
$$p_1 \geq p_2 \geq p_3 \geq \dots \geq p_k$$
2. In an optimal vector we have $\forall i \in \{1, 2, \dots k\}$ $p_i \geq \frac{1}{k}$
3. Additionaly $p_0 = 1$, $p_{k+1} = \frac{1}{u}$

Lemma 6 (With a hole inside ☺)

There is $i \in \{0, 1, \dots k\}$ such that $\frac{p_i}{p_{i+1}} \geq u^{\frac{1}{k+1}}$



$$n^* = \frac{1}{\sqrt{p_i \cdot p_{i+1}}}$$

$$\frac{1}{n^*} = \sqrt{p_i \cdot p_{i+1}} \quad \text{// geometric mean}$$

Proof:

Let us implicitly assume that $\forall i \in \{0, \dots k\}$ $\frac{p_i}{p_{i+1}} \leq u^{\frac{1}{k+1}}$

then we have

$$\frac{p_0}{\cancel{p_1}} \cdot \frac{\cancel{p_1}}{\cancel{p_2}} \cdots \frac{\cancel{p_k}}{p_{k+1}} = \frac{p_0}{p_{k+1}} = \frac{1}{\frac{1}{u}} = u$$

contradiction $\square$

Lemma 7

Let $S_{\bar{p}_k, n}$ denote an event that in one of k slots the leader was elected. Then for any vector $\bar{p}_k$ there is $n^* \in \{2, \dots, u\}$ such that $P_n[S_{\bar{p}_k, n^*}] \leq 1 - \left(1 - \frac{3e}{u^{\frac{1}{2(k+1)}}}\right)$

Proof:

- for any $\bar{p}_k$ there is i s.t. $\frac{p_i}{p_{i+1}} \geq \frac{1}{u^{\frac{1}{k+1}}}$ (Lemma 6)
- we choose $n^* = \left\lceil \frac{1}{\sqrt{p_i \cdot p_{i+1}}} \right\rceil$
- we know that $P_n[S_{\bar{p}_k, n^*}] = 1 - \prod_{i=1}^{k} \left(1 - n^* \cdot p_i (1 - p_i)^{n^* - 1}\right)$
- to show that $\forall i \in \{1, \dots k\}$ $n^* p_i (1 - p_i)^{n^* - 1} \leq \frac{3e}{u^{\frac{1}{2(k+1)}}}$

Theorem

If $k \leq \frac{\ln u}{2 \ln(3e)} - 1$, $f > 1$ and $\forall n \in \{2, \dots u\}$ $P_n[S_{\bar{p}_k, n}] \geq 1 - \frac{1}{f}$

then $k \geq \frac{\frac{1}{2} \ln u}{\ln \ln u + \ln\left(\frac{3e}{2} \cdot \frac{f}{f-1}\right)} \in O\left(\frac{\ln u}{\ln \ln u}\right)$