① $\mathcal{X} \sim Geo(p)$

$P(\mathcal{X}=k) = p \cdot (1-p)^{k-1}$; • $\mathcal{X}$ is r.v. representing the number of trials until the first success occurs.
• p is the probability of success.

$$E[\mathcal{X}] = \sum_{k=1}^{\infty}\left(k \cdot P(\mathcal{X}=k)\right) = \sum_{k=1}^{\infty} k \cdot p \cdot (1-p)^{k-1} =$$

$$= p \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = p\left(1 + 2(1-p)^1 + 3(1-p)^2 + \dots\right)$$

$$= p\left(\sum_{k=1}^{\infty}(1-p)^{k-1} + \sum_{k=2}^{\infty}(1-p)^{k-1} + \sum_{k=3}^{\infty}(1-p)^{k-1} + \dots\right) =$$

$p \in [0,1]$ thus $1-p \in [0,1]$, so we can use fact that

$$\sum_{i=1}^{\infty} r^{i-1} = \frac{1}{1-i}$$

$$= p\left(\frac{1}{1-(1-p)} + \frac{1-p}{1-(1-p)} + \frac{(1-p)^2}{1-(1-p)} + \dots\right) =$$

$$= 1 + (1-p) + (1-p)^2 + \dots$$

$$= \sum_{k=1}^{\infty}(1-p)^{k-1} = \frac{1}{1-(1-p)} = \frac{1}{p}$$

② $X \sim Geo(p)$

$$Var(X) = E[X^2] - E[X]^2 = E[X(X-1)] + E[X] - E[X]^2 =$$
$$= E[X(X-1)] + \frac{1}{p} - \frac{1}{p^2} = \odot$$

$$E[X^2] = E[X^2 - X + X] =$$
$$= E[X(X-1)] + E[X]$$

$$E[X(X-1)] = \sum_{k=1}^{\infty} k(k-1)\, p(1-p)^{k-1} =$$
$$= p\sum_{k=1}^{\infty} k(k-1)(1-p)^{k-1} \overset{q=1-p}{=} p\sum_{k=1}^{\infty} k(k-1)\, q^{k-1} = p\frac{d}{dq}\left(\sum_{k=1}^{\infty}(k-1)q^{k}\right) =$$
$$= p\frac{d}{dq}\left(\sum_{k=2}^{\infty}(k-1)q^{k}\right) = p\frac{d}{dq}\left(q^2\sum_{k=2}^{\infty}(k-1)q^{k-2}\right) = p\frac{d}{dq}\left(q^2\frac{d}{dq}\left(\sum_{k=2}^{\infty}q^{k-1}\right)\right)$$
$$= p\frac{d}{dq}\left(q^2\frac{d}{dq}\left(\sum_{k=1}^{\infty}q^{k}\right)\right) = p\frac{d}{dq}\left(q^2\frac{d}{dq}\left(\sum_{k=0}^{\infty}q^{k} - 1\right)\right) =$$ $q \in [0,1]$
$$= p\frac{d}{dq}\left(q^2\frac{d}{dq}\left(\frac{1}{1-q} - 1\right)\right) = p\frac{d}{dq}\left(\frac{q^2}{(1-q)^2}\right) = p\left(\frac{-2q}{(q-1)^3}\right) =$$
$$= p\left(\frac{-2(1-p)}{(1-p-1)^3}\right) = p\left(\frac{-2(1-p)}{-p^3}\right) = \frac{2(1-p)}{p^2}$$

$$\odot = \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2(1-p) + p - 1}{p^2} = \frac{2 - 2p + p - 1}{p^2} =$$
$$= \frac{1-p}{p^2}$$

③ $p \in [0,1]$, $n \geq k \geq 1$, $n, k \in \mathbb{N}$ argmax(·)?

a) $f(p) = np(1-p)^{n-1} =$

- $\alpha(p) = np$
- $\beta(p) = (1-p)^{n-1} = \gamma(\delta(p))$

$\gamma(x) = x^{n-1}$

$\delta(x) = 1-x$

$f'(p) = (\alpha(p) \cdot \beta(p))' =$

$= \alpha'(p)\beta(p) + \alpha(p)\beta'(p) =$

$\beta'(p) = (\gamma(\delta(p)))' = \gamma'(\delta(p)) \cdot \delta'(p) =$

$= (n-1)(1-p)^{n-2} \cdot (-1) = \underline{(1-n)(1-p)^{n-2}}$

$n \cdot (1-p)^{n-1} + np \cdot (1-n)(1-p)^{n-2} =$

$= n(1-p)(1-p)^{n-2} + np(1-n)(1-p)^{n-2} =$

$= (1-p)^{n-2}(n-np) + (np-n^2p)(1-p)^{n-2} =$

$= (1-p)^{n-2}(n-np+np-n^2p) = (1-p)^{n-2}(n-n^2p)$

$f'(p) = 0$

$(1-p)^{n-2}(n-n^2p) = 0$

$(1-p)^{n-2} = 0 \quad \vee \quad n-n^2p = 0$

$\qquad\qquad\qquad\qquad\quad n(1-np) = 0$

$p = 1$

$n = 0 \qquad np = 1$

$\qquad\qquad \to p = \frac{1}{n}$

We should also calculate $f''(p)$ to check if it's min or max but we know how this function looks.

6) $f(n) = np(1-p)^{n-1}$

- $\alpha(n) = np$
- $\beta(n) = (1-p)^{n-1}$

$$f'(n) = \alpha'(n)\beta(n) + \alpha(n)\beta'(n) =$$

$$= p\cdot(1-p)^{n-1} + np\cdot\ln(1-p)(1-p)^{n-1} =$$

$$= p(1-p)^{n-1}(1+n\ln(1-p))$$

$$f'(n) \overset{?}{=} 0$$

$$p(1-p)^{n-1}(1+n\cdot\ln(1-p)) = 0$$

$$p=0 \lor p=1 \quad \lor \quad 1+n\cdot\ln(1-p) = 0$$

$$\Leftrightarrow n = -\frac{1}{\ln(1-p)}$$

c) $f(k) = \binom{n}{k} p^k (1-p)^{n-k}$

...

④ $(1+x)^r \geq 1 + rx$ for $x \geq -1,\ r \geq 1$

see next task ⑤...

$\nabla =$ $(1+x)^{\frac{p}{q}} \leq 1 + \frac{p}{q}x$ since $\frac{p}{q} < 1 \wedge x \geq -1$

$(1+x) \leq \left(1+\frac{p}{q}x\right)^{\frac{q}{p}}$ let $y = \frac{p}{q}x$, note that since $x \geq -1$

$y \geq -\frac{p}{q} \geq -1$

$\left(1+\frac{q}{p}y\right) \leq (1+y)^{\frac{q}{p}}$

let $\frac{q}{p} = s$ (s is arbitrary rational number ≥ 1)

(y — " " — ≥ -1)

$(1+y)^s \geq 1 + sy$ $\square$

⑤ $(1+x)^r \le 1+rx$, $x \ge -1$, $r \in (0,1)$ fun proof

Let $r = \frac{p}{q}$, $p,q \in \mathbb{N}^+$ $p<q$

Let's consider such series:

$$\underbrace{1, 1, \ldots, 1, \underbrace{(1+x), (1+x), \ldots (1+x)}_{p}}_{q} =$$

GeometricMean( ) $= \sqrt[q]{1 \cdot 1 \cdot \ldots \cdot 1 \cdot (1+x) \cdot \ldots (1+x)} = (1+x)^{\frac{p}{q}}$

ArithmeticMean( ) $= \dfrac{1+1+\ldots+1+(1+x)\ldots(1+x)}{q} = 1+\frac{p}{q}x$

then using well known fact that for any n non-negative numbers $(x \ge -1)$
GM ≤ AM we conclude that

$$(1+x)^{\frac{p}{q}} \le 1+\frac{p}{q}x \quad \text{so afte substituting } r=\frac{p}{q}$$

$$(1+x)^r \le 1+rx \quad \square$$

⑥ $1+x \leq e^{x}$, $x \in \mathbb{R}$

Let's define $f(x) = e^{x} - 1 - x$, we will prove that this function is always non negative.

$f'(x) = e^{x} - 1$, $f'(0) = 1 - 1 = 0$

$f''(x) = e^{x}$, since 2nd derivative is >0 for all x, f' is strictly increasing.

$f'(x) \geq 0$ for $x \geq 0$ and $f'(x) \leq 0$ for $x \leq 0$, thus

$f(x)$ has a minimum in $x = 0$, namely $f(0) = 0$.

So $f(x) \geq 0$, what means that $e^{x} - 1 - x \geq 0 \Leftrightarrow 1 + x \leq e^{x}$ ▫

⑦ $\frac{x}{e^x} < \frac{1.5}{x^2}$, $x > 0$

x^2 is always > 0

$\frac{x^3}{e^x} < \frac{3}{2}$ we will prove that, by showing that $f(x) = \frac{x^3}{e^x}$

1) has global max value $\frac{27}{e^3}$, at $x = 3$

2) $\lim_{x \to \infty} f(x) = 0$

1) $f'(x) = -e^{-x}(x-3)x^2$, $-e^{-x}(x-3)x^2 = 0$

$x = 0$ $\vee$ $x = 3$

$x \notin D$

$f''(x) = e^{-x}x(x^2-6x+6)$, $f''(3) = e^{-3} \cdot 3(3^2 - 6 \cdot 3 + 6) < 0$, thus

f has a local maximum in $x = 3$.

$f(3) = \frac{27}{e^3} \approx 1.344 < 1.5$

2) $\lim_{x \to \infty} \frac{x^3}{e^x} = \lim_{x \to \infty} \frac{27}{e^x} = 0$

using L'Hospital's rule three times

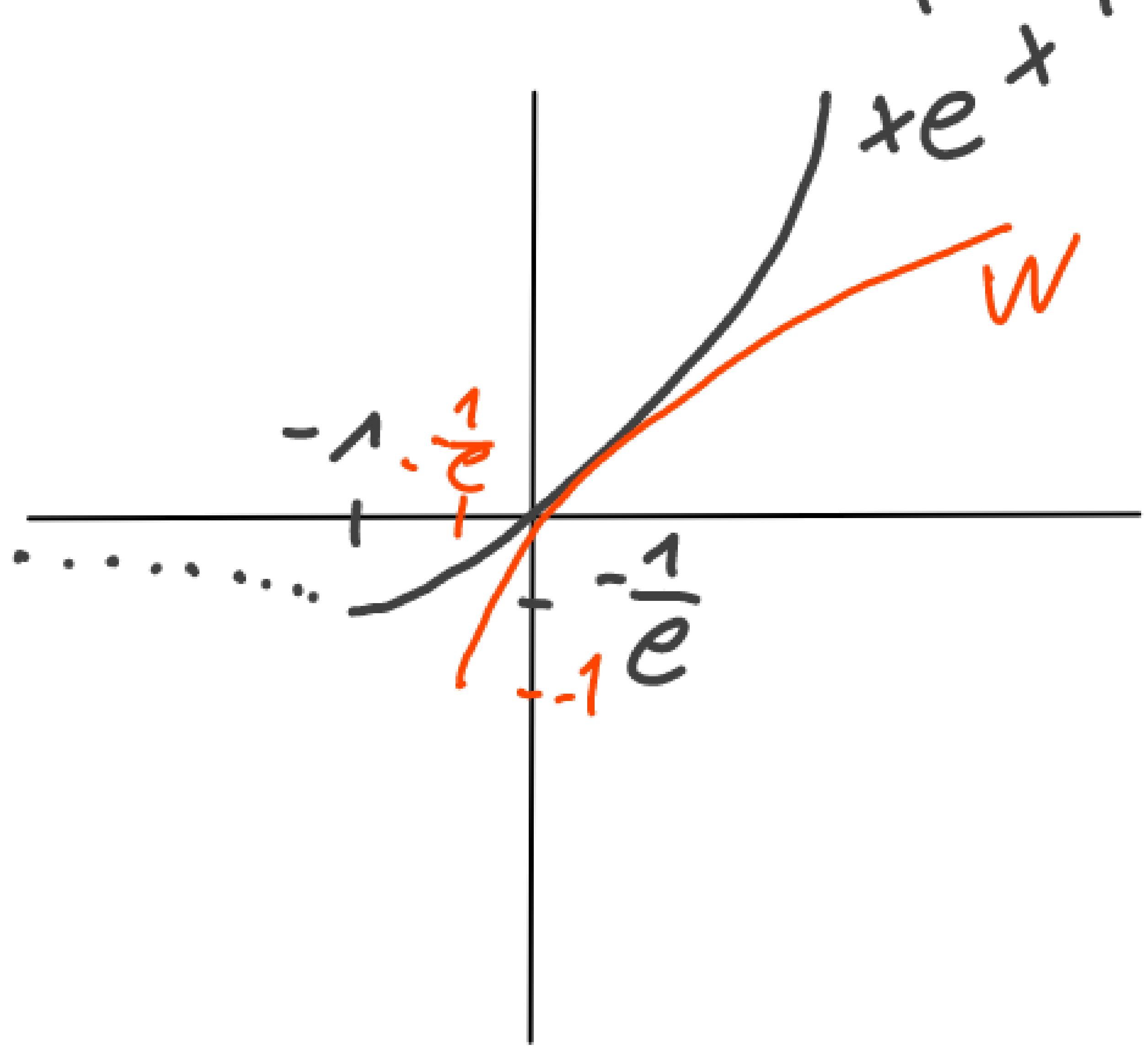
8 $f_i(n) = n\frac{1}{2^i}\left(1-\frac{1}{2^i}\right)^{n-1}$

a) $f_i(2^{i-1}) = 2^{i-1}\frac{1}{2^i}\left(1-\frac{1}{2^i}\right)^{2^{i-1}-1} = \frac{1}{2}\left(1-\frac{1}{2^i}\right)^{2^{i-1}-1}$

...

⑨ Lambert W function

$W(x)$ is the inverse of $f(x) = xe^x$. $\quad x \in \mathbb{R}$



$D_W: [-\frac{1}{e}, +\infty)$

$Im_W: [-1, +\infty)$

note that $f(x)$ on $(-\infty, -1)$ is decreasing thus whe defining it's inverse we cut the domain$_f$.

with such definition of W, we have:

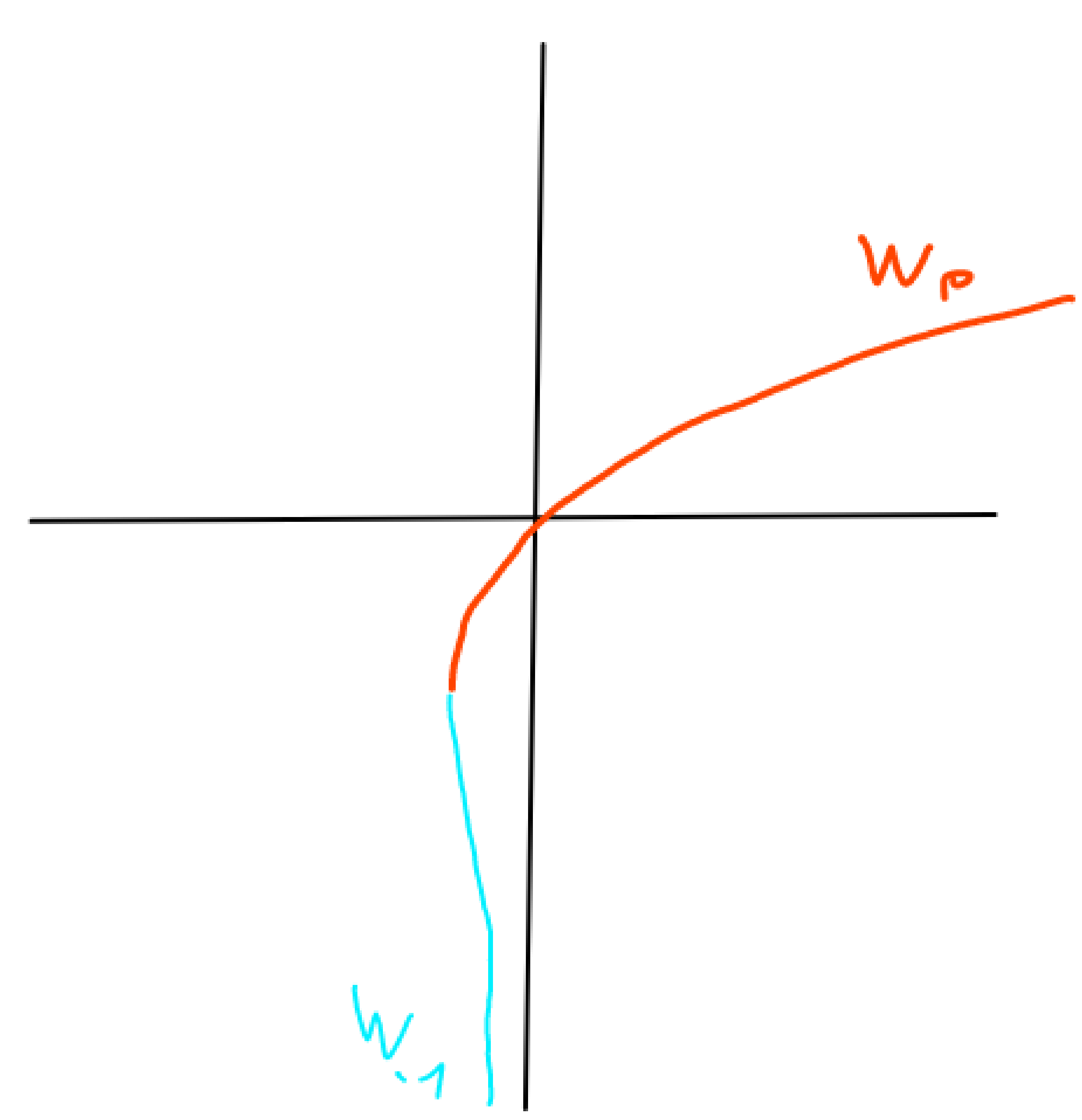
- $W(xe^x) = x \quad , x \geq -1$
- $W(x)e^{W(x)} = x \quad , x \geq -\frac{1}{e}$

But Lambert W function may be also considered with complex numbers, Then for each $k \in \mathbb{Z}$ there is one branch $W_k: \mathbb{C} \to \mathbb{C}$, these functions have following property:

for any $w, z \in \mathbb{C}$:

$$we^w = z \iff w = W_k(z) \text{ for some } k.$$

for real numbers branch W_{-1}, W_0 are enough.



⑩ $3^x = x^3$ using Lambert W function

a) we want to transform above equation to form $ze^z = w$.

$$3^x = x^3 \quad / \ln$$

$$x \cdot \ln(3) = 3 \cdot \ln(x)$$

$$\frac{\ln(3)}{3} = \frac{\ln(x)}{x}$$

$$\ln(x) \cdot x^{-1} = \frac{\ln(3)}{3}$$

$$\ln(x) \cdot e^{-\ln(x)} = \frac{\ln(3)}{3}$$

$$-\ln(x) \cdot e^{-\ln(x)} = \frac{\ln(3)}{3} \quad / W_k(\cdot)$$

$$W_k\left(-\ln(x) \cdot e^{-\ln(x)}\right) = W_k\left(\frac{\ln(3)}{3}\right)$$

$$-\ln(x) = W_k\left(\frac{\ln(3)}{3}\right)$$

$$x = e^{-W_k\left(\frac{\ln(3)}{3}\right)}$$

for two real branches

- $k = -1$: $x = 3$
- $k = 0$: $x \approx 2.478\ldots$

b) this function is called Product Log. It sort of makes sense considering that the inverse of e^x is natural log, then the inverse of $x \cdot e^x$ is product log.

11. $\ln(x) - \ln(\ln(x)) < W_0(x) \le \ln(x) - \frac{1}{2}\ln(\ln(x))$, $x \ge e$

so, we have $\ln x - 1\ln\ln x < W_0(x) \le \ln x - \frac{1}{2}\ln\ln x$

let's follow the article:

1. let's define $f(x) = \ln x - \frac{1}{p}\ln\ln x - W_0(x)$, for $p \in (0,2]$
   (substraction of $W_0(x)$ from left and right side)

2. for $x \ge e$, we have

$$\frac{d}{dx} f(x) = \frac{p\ln x - 1 - W_0(x)}{px(1+W_0(x))\ln x}$$

3. for $p=2$:

$$\frac{d}{dx} f_{p=2}(x) = \frac{(\ln(x) - W_0(x)) + (\ln x - 1)}{2x(1+W_0(x))\ln x}$$

4. $\frac{d}{dx} f_{p=2}(x) > 0$ for $x > e$ $\rightarrow$ $f_{p=2}(x) > f_{p=2}(e) = 0$, in the case of $x = e$ $f(x) = 0$ giving equality, what proves right hand side.

5. for $p \in (0,2)$:

$$\frac{d}{dx} f(e) = \frac{p-2}{2ep}$$ what is < 0, thus $-\frac{1}{2}$ is the best coefficient.

6. left side; for $x \ge e$: $\cdot \ln(W_0(x)) = \ln x - W_0(x)$
   $\cdot \ln(W_0(x)) \le \ln\ln(x)$ } because $W_0(x) < \ln x$

then $\ln x - W_0(x) \le \ln\ln(x) \Leftrightarrow \ln x - 1\ln\ln x - W_0(x) \le 0$

7. -1 comes from the fact that $\lim_{x \to \infty}(-(\ln x - 1\ln\ln x - W_0(x))) = 0$

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I hope you won't make it here.