① $X \sim Geo(p)$

$P(X=k) = p \cdot (1-p)^{k-1}$; • X is r.v. representing the number of trials until the first success occurs.
• p is the probability of success.

$$E[X] = \sum_{k=1}^{\infty}(k \cdot P(X=k)) = \sum_{k=1}^{\infty} k \cdot p \cdot (1-p)^{k-1} =$$

$$= p \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = p(1 + 2(1-p)^1 + 3(1-p)^2 + \dots)$$

$$= p\left(\sum_{k=1}^{\infty}(1-p)^{k-1} + \sum_{k=2}^{\infty}(1-p)^{k-1} + \sum_{k=3}^{\infty}(1-p)^{k-1} + \dots\right) =$$

$p \in [0,1]$ thus $1-p \in [0,1]$, so we can use fact that

$$\sum_{i=1}^{\infty} r^{i-1} = \frac{1}{1-i}$$

$$= p\left(\frac{1}{1-(1-p)} + \frac{1-p}{1-(1-p)} + \frac{(1-p)^2}{1-(1-p)} + \dots\right) =$$

$$= 1 + (1-p) + (1-p)^2 + \dots$$

$$= \sum_{k=1}^{\infty}(1-p)^{k-1} = \frac{1}{1-(1-p)} = \frac{1}{p}$$

② $X \sim Geo(p)$

$$Var(X) = E[X^2] - E[X]^2 = E[X(X-1)] + E[X] - E[X]^2 =$$

$$= E[X(X-1)] + \frac{1}{p} - \frac{1}{p^2} = \odot$$

$$E[X^2] = E[X^2 - X + X] =$$
$$= E[X(X-1)] + E[X]$$

$$E[X(X-1)] = \sum_{k=1}^{\infty} k(k-1)\, p(1-p)^{k-1} =$$

$$= p\sum_{k=1}^{\infty} k(k-1)(1-p)^{k-1} \overset{q=1-p}{=} p\sum_{k=1}^{\infty} k(k-1)\, q^{k-1} = p\frac{d}{dq}\left(\sum_{k=1}^{\infty}(k-1)\, q^{k}\right) =$$

$$= p\frac{d}{dq}\left(\sum_{k=2}^{\infty}(k-1)\, q^{k}\right) = p\frac{d}{dq}\left(q^2\sum_{k=2}^{\infty}(k-1)\, q^{k-2}\right) = p\frac{d}{dq}\left(q^2\frac{d}{dq}\left(\sum_{k=2}^{\infty} q^{k-1}\right)\right)$$

$$= p\frac{d}{dq}\left(q^2\frac{d}{dq}\left(\sum_{k=1}^{\infty} q^{k}\right)\right) = p\frac{d}{dq}\left(q^2\frac{d}{dq}\left(\sum_{k=0}^{\infty} q^{k} - 1\right)\right) = \quad q \in [0,1]$$

$$= p\frac{d}{dq}\left(q^2\frac{d}{dq}\left(\frac{1}{1-q} - 1\right)\right) = p\frac{d}{dq}\left(\frac{q^2}{(1-q)^2}\right) = p\left(\frac{-2q}{(q-1)^3}\right) =$$

$$= p\left(\frac{-2(1-p)}{(1-p-1)^3}\right) = p\left(\frac{-2(1-p)}{-p^3}\right) = \frac{2(1-p)}{p^2}$$

$$\odot = \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2(1-p)+p-1}{p^2} = \frac{2-2p+p-1}{p^2} =$$

$$= \frac{1-p}{p^2}$$

③ $p \in [0,1]$, $n \geq k \geq 1$, $n,k \in \mathbb{N}$ argmax$(\cdot)$?

a) $f(p) = np(1-p)^{n-1} =$

- $\alpha(p) = np$
- $\beta(p) = (1-p)^{n-1} = \gamma(\delta(p))$

$\gamma(x) = x^{n-1}$

$\delta(x) = 1-x$

$f'(p) = (\alpha(p) \cdot \beta(p))' =$

$= \alpha'(p)\beta(p) + \alpha(p)\beta'(p) =$

$\beta'(p) = (\gamma(\delta(p)))' = \gamma'(\delta(p)) \cdot \delta'(p) =$

$= (n-1)(1-p)^{n-2} \cdot (-1) = \underline{(1-n)(1-p)^{n-2}}$

$n \cdot (1-p)^{n-1} + np \cdot (1-n)(1-p)^{n-2} =$

$= n(1-p)(1-p)^{n-2} + np(1-n)(1-p)^{n-2} =$

$= (1-p)^{n-2}(n-np) + (np-n^2p)(1-p)^{n-2} =$

$= (1-p)^{n-2}(n-np+np-n^2p) = (1-p)^{n-2}(n-n^2p)$

$f'(p) = 0$

$(1-p)^{n-2}(n-n^2p) = 0$

$(1-p)^{n-2} = 0 \quad \vee \quad n-n^2p = 0$

$\downarrow \qquad\qquad n(1-np) = 0$

$p = 1$

$n = 0 \qquad np = 1$

$\to p = \frac{1}{n}$

We should also calculate $f''(p)$ to check if it's min or max but we know how this function looks.

6) $f(n) = np(1-p)^{n-1}$

- $\alpha(n) = np$
- $\beta(n) = (1-p)^{n-1}$

$$f'(n) = \alpha'(n)\beta(n) + \alpha(n)\beta'(n) =$$

$$= p\cdot(1-p)^{n-1} + np\cdot\ln(1-p)(1-p)^{n-1} =$$

$$= p(1-p)^{n-1}(1+n\ln(1-p))$$

$$f'(n) \overset{?}{=} 0$$

$$p(1-p)^{n-1}(1+n\cdot\ln(1-p)) = 0$$

$$p=0 \lor p=1 \quad \lor \quad 1+n\cdot\ln(1-p) = 0$$

$$\Downarrow$$

$$n = -\frac{1}{\ln(1-p)}$$

c) $f(k) = \binom{n}{k} p^k (1-p)^{n-k}$

...

④ $(1+x)^r \geq 1+rx \quad$ for $x \geq -1,\ r \geq 1$

see next task ⑤...

$\triangledown = (1+x)^{\frac{p}{q}} \leq 1+\frac{p}{q}x \qquad$ since $\frac{p}{q} < 1 \wedge x \geq -1$

$(1+x) \leq \left(1+\frac{p}{q}x\right)^{\frac{q}{p}} \qquad$ let $y = \frac{p}{q}x$, note that since $x \geq -1$

$$y \geq -\frac{p}{q} \geq -1$$

$\left(1+\frac{q}{p}y\right) \leq (1+y)^{\frac{q}{p}}$

let $\frac{q}{p} = s$ (s is arbitrary rational number ≥ 1)

(y — 〃 〃 — ≥ -1)

$(1+y)^s \geq 1+sy$ □

⑤ $(1+x)^r \leq 1+rx \quad , \; x \geq -1 \; , \; r \in (0,1)$ fun proof

Let $r = \frac{p}{q}$, $p, q \in \mathbb{N}^+$ $p < q$

Let's consider such series:

$$\underbrace{1, 1, \ldots, 1, \underbrace{(1+x), (1+x), \ldots (1+x)}_{p}}_{q} =$$ 

$$\text{GeometricMean}(\ldots) = \sqrt[q]{1 \cdot 1 \cdot \ldots \cdot 1 \cdot (1+x) \cdot \ldots (1+x)} = (1+x)^{\frac{p}{q}}$$

$$\text{ArithmeticMean}(\ldots) = \frac{1+1+\ldots+1+(1+x)\ldots(1+x)}{q} = 1 + \frac{p}{q}x$$

then using well known fact that for any n non-negative numbers $(x \geq -1)$
GM ≤ AM we conclude that

$$(1+x)^{\frac{p}{q}} \leq 1 + \frac{p}{q}x$$ so afte substituting $r = \frac{p}{q}$

$$(1+x)^r \leq 1 + rx$$

⑥ $1+x \leq e^{x}$, $x \in \mathbb{R}$

Let's define $f(x) = e^{x} - 1 - x$, we will prove that this function is always non negative.

$f'(x) = e^{x} - 1$, $f'(0) = 1 - 1 = 0$

$f''(x) = e^{x}$, since 2nd derivative is >0 for all x, f' is strictly increasing.

$f'(x) \geq 0$ for $x \geq 0$ and $f'(x) \leq 0$ for $x \leq 0$, thus

$f(x)$ has a minimum in $x = 0$, namely $f(0) = 0$.

So $f(x) \geq 0$, what means that $e^{x} - 1 - x \geq 0 \Leftrightarrow 1 + x \leq e^{x}$ □

⑦ $\frac{x}{e^x} < \frac{1.5}{x^2}$, $x > 0$

x^2 is always > 0

$\frac{x^3}{e^x} < \frac{3}{2}$ we will prove that, by showing that $f(x) = \frac{x^3}{e^x}$

1) has global max value $\frac{27}{e^3}$, at $x = 3$

2) $\lim_{x \to \infty} f(x) = 0$

1) $f'(x) = -e^{-x}(x-3)x^2$, $-e^{-x}(x-3)x^2 = 0$

$x = 0 \quad \vee \quad x = 3$

$x \notin D$

$f''(x) = e^{-x}x(x^2-6x+6)$, $f''(3) = e^{-3} \cdot 3(3^2-6 \cdot 3+6) < 0$, thus

f has a local maximum in $x = 3$.

$f(3) = \frac{27}{e^3} \approx 1.344 < 1.5$

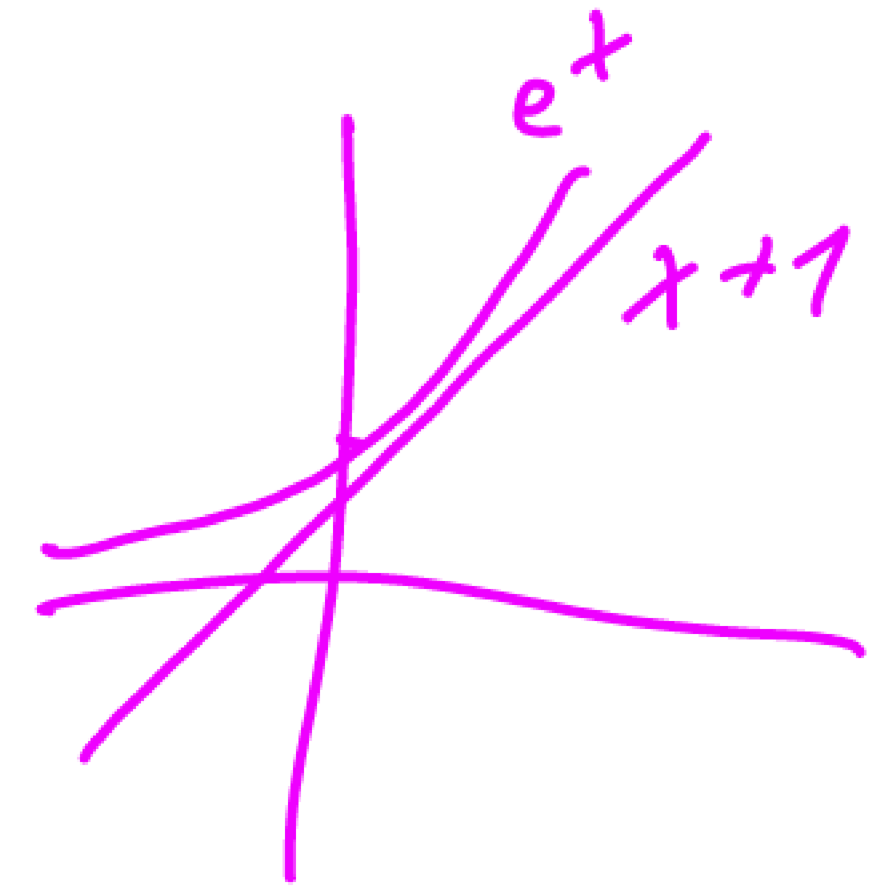
2) $\lim_{x \to \infty} \frac{x^3}{e^x} = \lim_{x \to \infty} \frac{27}{e^x} = 0$

using L'Hospital's rule three times

⑧ $f_i(n) = n\frac{1}{2^i}\left(1-\frac{1}{2^i}\right)^{n-1}$

$i \geq 2$

$e^x \geq x+1$



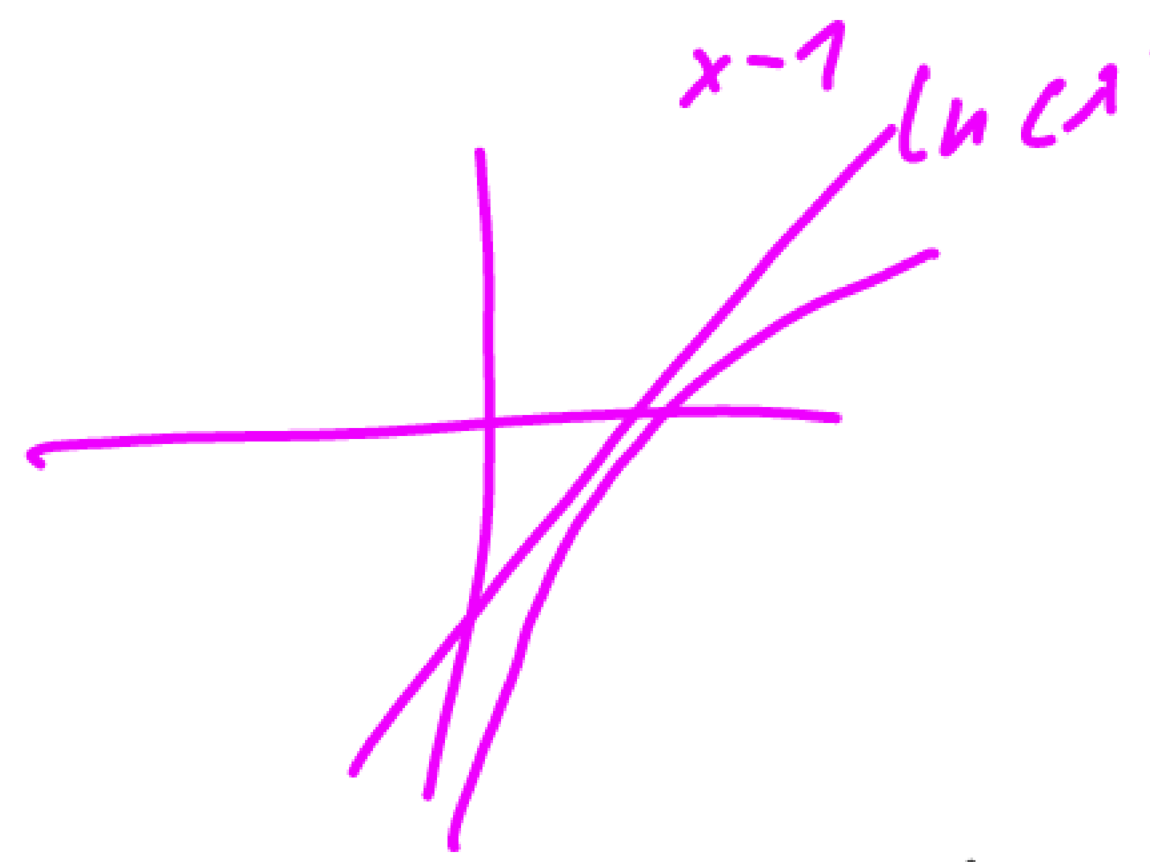
a) $f_i(2^{i-1}) = \frac{1}{2} g_2(2^{i-1})$

b) $f_{i+1}(2^{i-1}) \downarrow = \frac{1}{4} g_4(2^{i-1})$

c) $f_{i-1}(2^i) \nearrow = 2\, g_{\frac{1}{2}}(2^i)$

$g_\alpha(x) = \left(1-\frac{1}{\alpha x}\right)^{x-1}$

$\frac{d}{dx} g_\alpha(x) = \frac{d}{dx}\left(1-\frac{1}{\alpha x}\right)^{x-1} =$



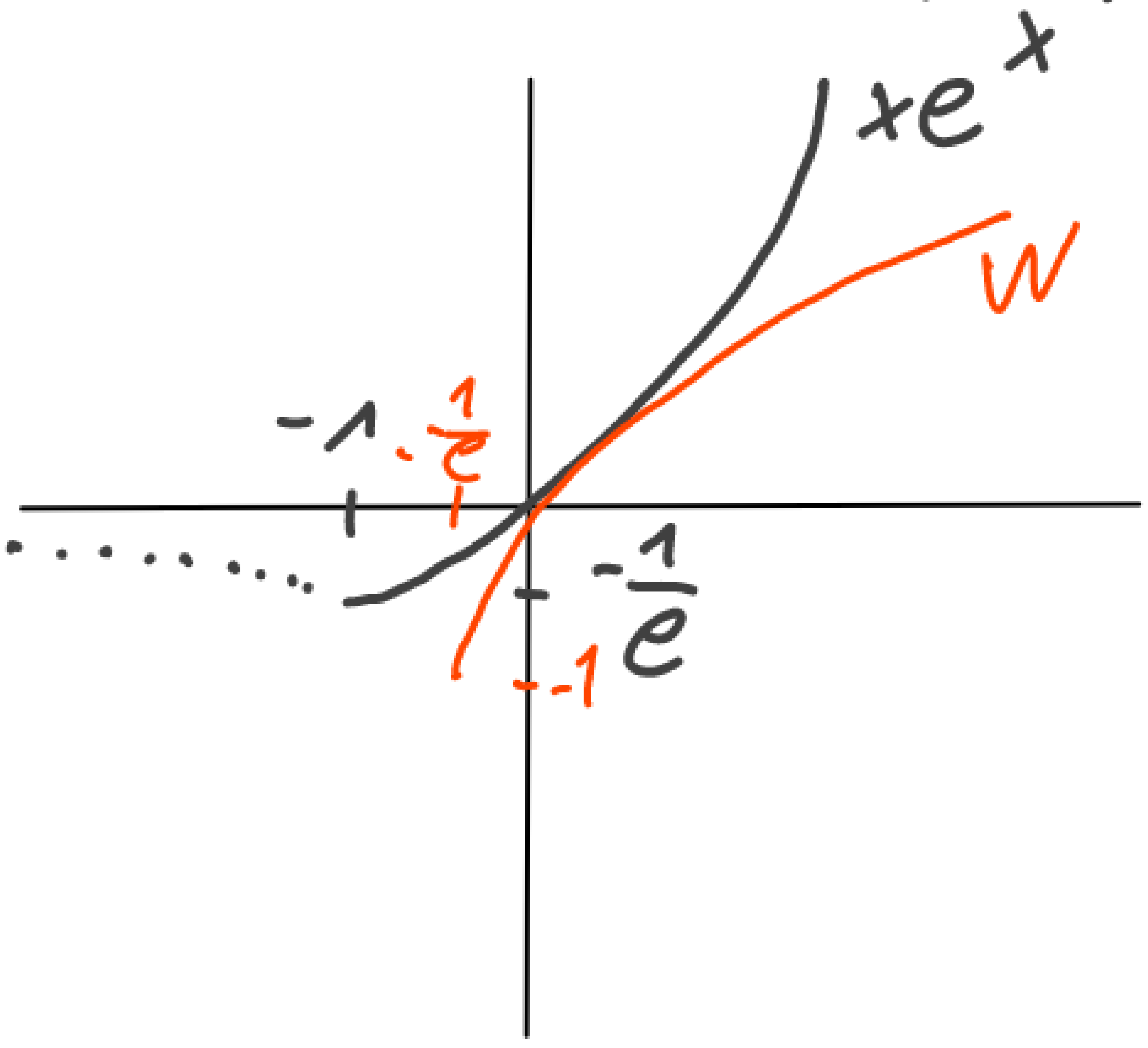
$\frac{\partial}{\partial x}\left(f(x)^{g(x)}\right) = \frac{\partial}{\partial x} e^{g(x)\ln f(x)}$

$= f(x)^{g(x)} \cdot \frac{d}{dx}\left(g(x)\ln(f(x))\right)$

$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$

⑨ Lambert W function

$W(x)$ is the inverse of $f(x) = xe^x$. $\quad x \in \mathbb{R}$



$D_W: [-\frac{1}{e}, +\infty)$

$Im_W: [-1, +\infty)$

note that $f(x)$ on $(-\infty, -1)$ is decreasing thus whe defining it's inverse we cut the domain$_f$.

with such definition of W, we have:

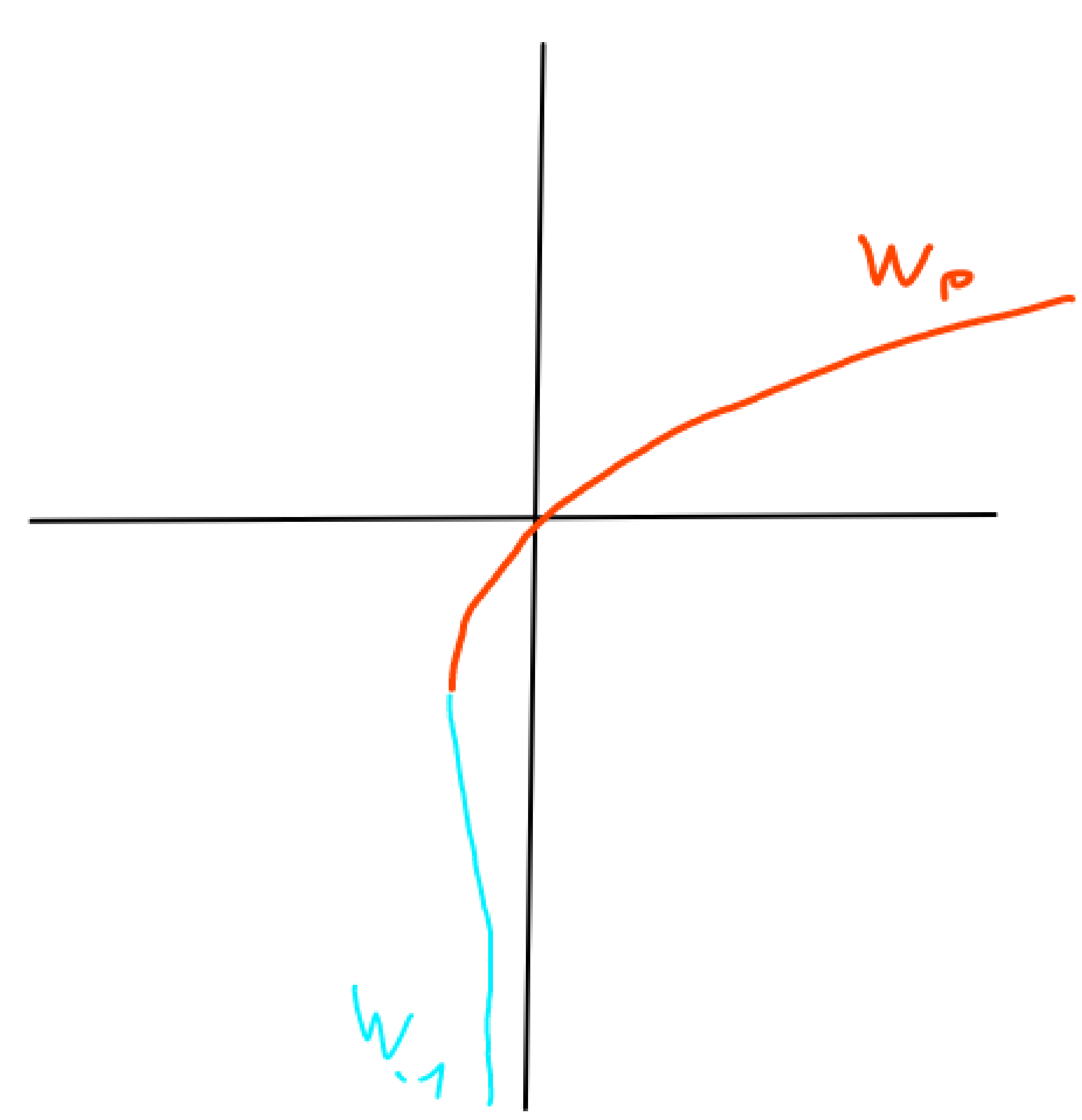
- $W(xe^x) = x \quad , x \geq -1$
- $W(x)e^{W(x)} = x \quad , x \geq -\frac{1}{e}$

But Lambert W function may be also considered with complex numbers, Then for each $k \in \mathbb{Z}$ there is one branch $W_k : \mathbb{C} \to \mathbb{C}$, these functions have following property:

for any $w, z \in \mathbb{C}$:

$$we^w = z \iff w = W_k(z) \text{ for some } k.$$

for real numbers branch W_{-1}, W_0 are enough.



(10) $3^x = x^3$ using Lambert W function

a) we want to transform above equation to form $ze^z = w$.

$$3^x = x^3 \quad / \ln$$

$$x \cdot \ln(3) = 3 \cdot \ln(x)$$

$$\frac{\ln(3)}{3} = \frac{\ln(x)}{x}$$

$$\ln(x) \cdot x^{-1} = \frac{\ln(3)}{3}$$

$$\ln(x) \cdot e^{-\ln(x)} = \frac{\ln(3)}{3}$$

$$-\ln(x) \cdot e^{-\ln(x)} = -\frac{\ln(3)}{3} \quad / W_k(\cdot)$$

$$W_k\left(-\ln(x) \cdot e^{-\ln(x)}\right) = W_k\left(-\frac{\ln(3)}{3}\right)$$

$$-\ln(x) = W_k\left(-\frac{\ln(3)}{3}\right)$$

$$x = e^{-W_k\left(-\frac{\ln(3)}{3}\right)}$$

for two real branches

- $k = -1$: $x = 3$
- $k = 0$: $x \approx 2.478\ldots$

b) this function is called Product Log. It sort of makes sense considering that the inverse of e^x is natural log, then the inverse of $x \cdot e^x$ is product log.

11 $\ln(x) - \ln(\ln(x)) < W_0(x) \le \ln(x) - \frac{1}{2}\ln(\ln(x))$, $x \ge e$

so, we have $\ln x - 1 \ln\ln x < W_0(x) \le \ln x - \frac{1}{2}\ln\ln x$

let's follow the article:

1. let's define $f(x) = \ln x - \frac{1}{p}\ln\ln x - W_0(x)$, for $p \in (0,2]$
(substraction of $W_0(x)$ from left and right side)

2. for $x \ge e$, we have

$$\frac{d}{dx} f(x) = \frac{p\ln x - 1 - W_0(x)}{p x (1 + W_0(x)) \ln x}$$

3. for $p=2$:

$$\frac{d}{dx} f_{p=2}(x) = \frac{(\ln(x) - W_0(x)) + (\ln x - 1)}{2x(1+W_0(x))\ln x}$$

4. $\frac{d}{dx} f_{p=2}(x) > 0$ for $x > e$ $\rightarrow$ $f_{p=2}(x) > f_{p=2}(e) = 0$, in the case of $x = e$ $f(x) = 0$ giving equality, what proves right hand side.

5. for $p \in (0,2)$:

$$\frac{d}{dx} f(e) = \frac{p-2}{2ep}$$ what is < 0, thus $-\frac{1}{2}$ is the best coefficient.

6. left side; for $x \ge e$: $\ln(W_0(x)) = \ln x - W_0(x)$
$\ln(W_0(x)) \le \ln\ln(x)$ } because $W_0(x) < \ln x$

then $\ln x - W_0(x) \le \ln\ln(x) \Leftrightarrow \ln x - 1\ln\ln x - W_0(x) \le 0$

7. -1 comes from the fact that $\lim_{x\to\infty}(-(\ln x - 1\ln\ln x - W_0(x))) = 0$

Proof: $W_0'(x) = \dfrac{W_0(x)}{x(1+W_0(x))}$

$$W_0(x)e^{W_0(x)} = x \quad // \frac{d}{dx}$$

$$W_0' e^{W_0} + W_0 \cdot W_0' e^{W_0} = 1$$

$$W_0' \cdot e^{W_0} \left(1 + W_0\right) = 1 \qquad e^{-W_0(x)} = \frac{W_0(x)}{x}$$

$$W_0' = \frac{e^{-W_0}}{1+W_0} = \frac{W_0}{x(1+W_0)}$$

(12) $k \geq 1,\ f > 1,\ u \geq 2,\ 3e(k+1)u^{\frac{-1}{2(k+1)}} \geq 1 - \frac{1}{f}$

then

$$3e(k+1)u^{\frac{-1}{2(k+1)}} \geq 1 - \frac{1}{f}$$

$$3e(k+1) \geq \frac{f-1}{f} u^{\frac{1}{2(k+1)}} \quad // f > 1$$

$$3e\frac{f}{f-1} \geq \frac{1}{k+1} u^{\frac{1}{2(k+1)}}$$

$$\frac{3e}{2}\frac{f}{f-1} \geq \frac{1}{2(k+1)} u^{\frac{1}{2(k+1)}}$$

$$\frac{3e}{2}\frac{f}{f-1}\ln u \geq \frac{\ln u}{2(k+1)} e^{\frac{\ln u}{2(k+1)}} \qquad / W_0$$

$$2W_0\left(\frac{3e}{2}\frac{f}{f-1}\ln u\right) \geq \frac{\ln u}{(k+1)}$$

$$\frac{2W_0\left(\frac{3e}{2}\frac{f}{f-1}\ln u\right)}{\ln u} \geq \frac{1}{k+1}$$

$$k \leq \frac{\ln u}{2W_0\left(\frac{3e}{2}\frac{f}{f-1}\ln u\right)} - 1$$