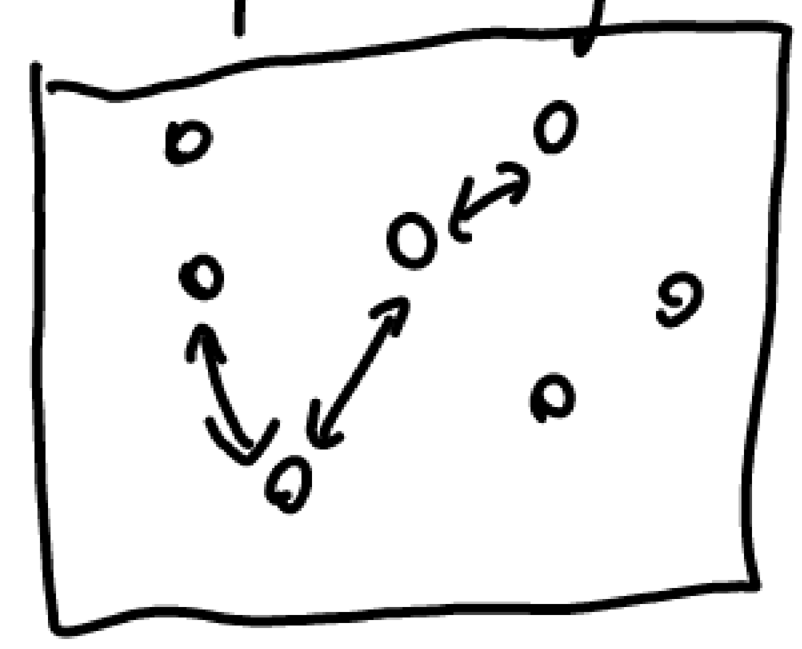
Lecture 1

01.03.2023

Leader election

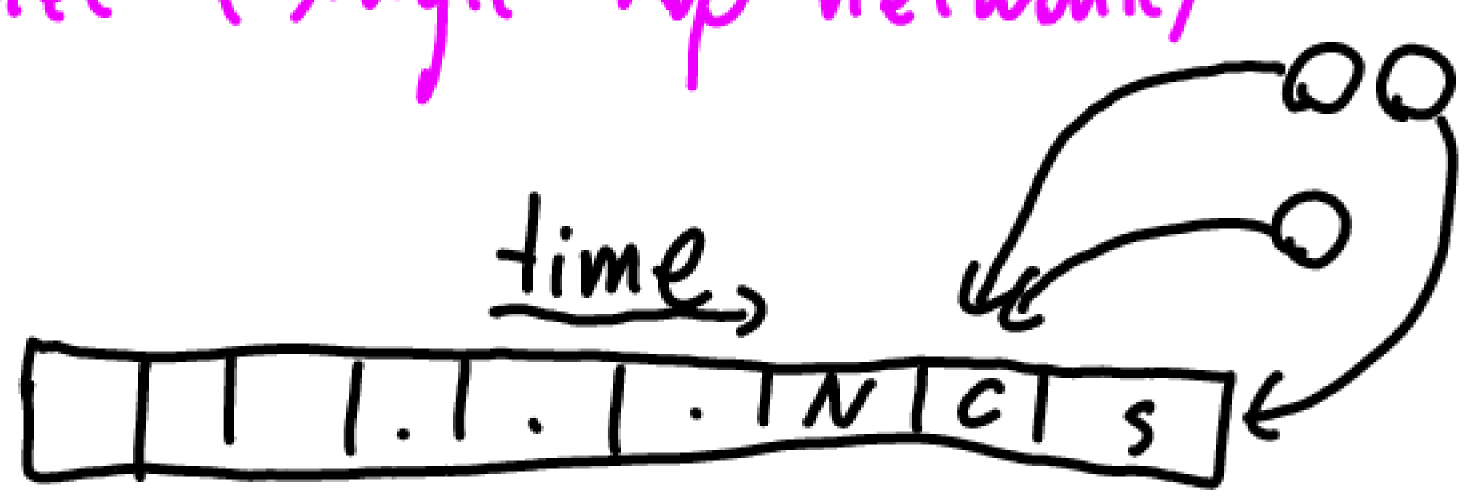
(I) Wireless Ad-Hoc networks (e.g. Internet of Things, sensor networks)



Beeping model

1. $n \geq 2$, n - can be unknown
2. All devices should be indistinguishable (full - symmetry)
3. Shared communication channel (single-hop network)
4. Time is devided into slots
5. Collision detection:

   slot $\in$ {NULL, SINGLE, COLLISON}



fixed slots (synchronization)

The problem: Select the leader that will have an exclusive access to the channel as quick as possible

General idea: Use randomness to break the symmetry

II Leader election algorithm

ELECTION($\vec{p}$):

- $i \leftarrow 0$
- slot $\leftarrow$ NULL
- while (slot $\neq$ SINGLE)
  - $i \leftarrow i+1$
  - each device beeps with prob p_i
  - update the value of slot ← device that sends in a SINGLE slot is a leader
- return i

$\vec{p} = (p_1, p_2, \dots)$

$p_i \in [0,1]$

Scenario 1 (we know n)

Lemma 1 If n is known then setting $p_i = \frac{1}{n}$ maximize prob. of leader election in slot i.

Proof: SCC_i - an event that success in slot i

$$Pr[SCC_i] = \binom{n}{1} p_i^1 (1-p_i)^{n-1} \equiv f(p_i), \quad p_i \in [0,1]$$

Remark For $p_i = \frac{1}{n}$ we have:

$$Pr[SCC_i] = n \cdot \frac{1}{n} \cdot \left(1 - \frac{1}{n}\right)^{n-1} > \left(1 - \frac{1}{n}\right)^n \xrightarrow{n \to \infty} \frac{1}{e}$$

Lemma 2 Let $\vec{p} = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots)$ and let L be a random variable denoting the number of slots until a leader is elected.

Then $E[L] < e$.

Proof: $L \sim Geo(p)$, $p = Pr[SCC_i] > \frac{1}{e}$

geometric dist

$$E[L] = \frac{1}{p} < e \quad \wedge \quad Var[L] = \frac{1-p}{p^2}$$

Lemma 3 Let S_t be an event that the leader is elected in t initial slots. Then we will show that $Pr[S_t] \geq 1 - e^{-\frac{t}{e}}$

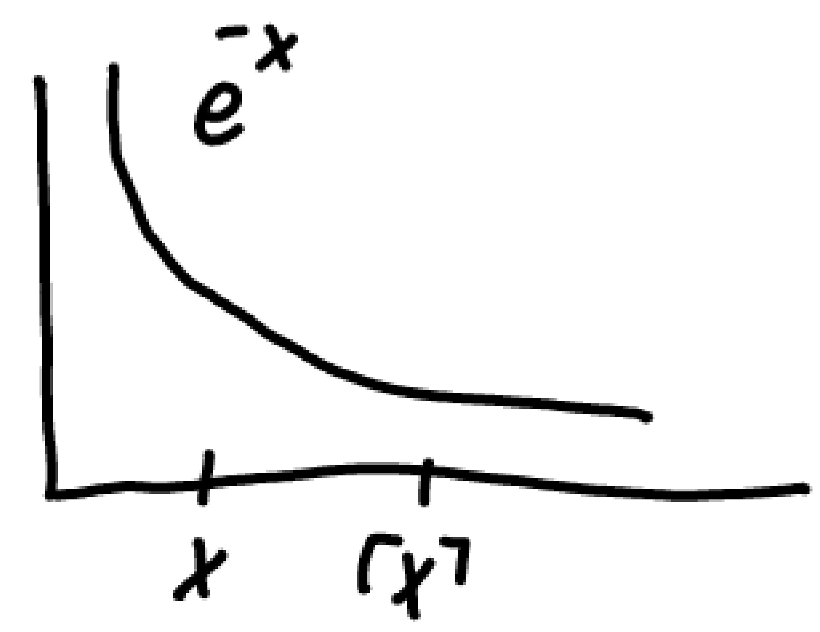
Proof $F_t \equiv \neg S_t$

$$Pr[F_t] = \underbrace{Pr[\neg SCC_i]^t}_{\text{not selecting in } t \text{ initial slots}} = (1-p)^t < \left(1-\frac{1}{e}\right)^t \leq e^{-\frac{t}{e}}$$

- $p > \frac{1}{e} \to (1-p) < \left(1-\frac{1}{e}\right)$
- $1+x \leq e^x$ for $x \in \mathbb{R}$

Remark

Let $t = \lceil e \cdot \ln n \rceil$, then $Pr[F_t] < e^{-\frac{\lceil e \cdot \ln n \rceil}{e}} \leq e^{-\ln n} = \frac{1}{n}$



$$\Downarrow$$

$$Pr[S_t] \geq 1 - \frac{1}{n}$$

For example $n = 1000 \to t = 19 \to Pr[S_t] \geq 99{,}9\%$

Scenario II (known upper bound $u \geq n \geq 2$) (if we don't know anything about n, we can use very large u)

Idea: We will show that $\vec{p}_u = (\underbrace{p_1, p_2, \dots p_m}_{\text{a round of length } m}, p_1, p_2, \dots p_m, p_1 \dots)$

$p_i = \frac{1}{2^i}$, $m = \lceil \log_2 u \rceil + 1$, so we have something like $\left(\frac{1}{2}, \frac{1}{4}, \dots \frac{1}{u}, \dots\right)$

we will show that this is almost optimal.

the number of slots to elect a leader with high probability

$O(\log u)$ ← our solution

gap

$\Omega\left(\frac{\log u}{\log \log u}\right)$ ← lower bound

Notation

- $P_i(n) = \binom{n}{1}\frac{1}{2^i}\left(1-\frac{1}{2^i}\right)^{n-1}$ ← prob. of success in slot i for n devices (we don't know $\underline{n}$)
- $S_{m,n}$ ← event that success in one round of length m for n devices
- $Pr[S_{m,n}] = 1 - \prod_{i=1}^{m}(1-P_i(n)) \geq 1 - \underbrace{(1-P_x(n))(1-P_y(n))(1-P_z(n))}_{\text{we chose 3 best slots for each } n}$

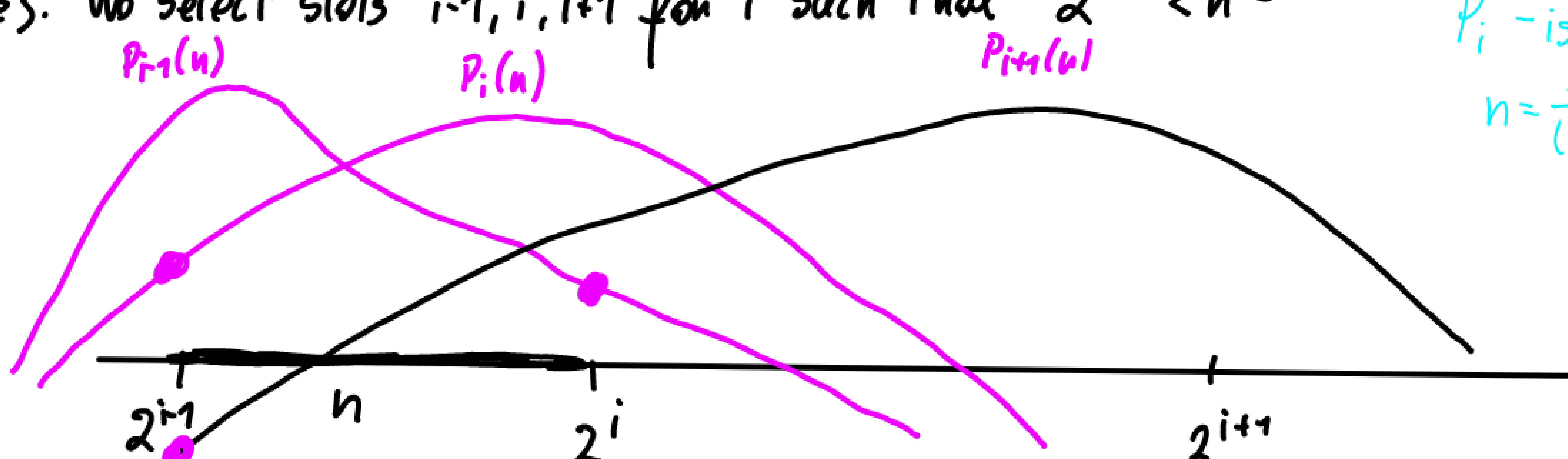
Lemma 4 For any $n \in \{2,3,\dots,u\}$ we have $Pr[S_{m,n}] \geq \underbrace{1-\left(1-\frac{1}{4}\right)\left(1-\frac{1}{2}e^{-\frac{1}{2}}\right)\left(1-\frac{1}{4}e^{-\frac{3}{4}}\right)}_{\lambda \approx 0.579}$

Proof

$n=2$: $Pr[S_{m,n}] \geq 1-(1-P_1(2))(1-P_2(2)) = 1-\frac{5}{16} \approx 0{,}68 > \lambda$ □

$n \geq 3$: We select slots $i-1, i, i+1$ for i such that $2^{i-1} < n \leq 2^i$

(One maximum)
P_i - is unimodule
$n = \frac{1}{\ln\left(1-\frac{1}{2^i}\right)} \approx 2^i$



we look for smallest values in this interval

$$\Pr[S_{m,n}] \geq 1-(1-P_{i-1}(n))(1-P_i(n))(1-P_{i+1}(n))$$

we pick minimal values

$$\geq 1-(1-P_{i-1}(2^i))(1-P_i(2^{i-1}))(1-P_{i+1}(2^{i-1}))$$

- $P_{i-1}(2^i)$ increasing for $i \geq 2$: so minimal value is for smallest i
$$P_{i-1}(2^i) \geq P_1(4) = \frac{1}{4}$$
- $P_i(2^{i-1})$ decreasing for $i \geq 2$: so minimal value is for $i \to \infty$
$$P_i(2^{i-1}) \geq \lim_{i\to\infty}(2^{i-1}) = \lim_{i\to\infty}\left(2^{i-1}\cdot\frac{1}{2^i}\left(1-\frac{1}{2^i}\right)^{2^{i-1}}\right) =$$
$$= \frac{1}{2}e^{-\frac{1}{2}}$$
- $P_{i+1}(2^{i-1})$ decreasing ...
$$P_{i+1}(2^{i-1}) \geq \frac{1}{4}e^{-\frac{1}{4}}$$

Conclusion

Let R denote the number of rounds untill a leader is elected.

Note $R \sim Geo(q)$, $q = \Pr[S_{m,n}] > \lambda \approx 0.579$

Thus we have that $E[R] = \frac{1}{q} \leq \frac{1}{\lambda} < 2$, $Var[R] = \ldots \Rightarrow O(\log n)$

average number of slots $\leq 2 \log_2 n$