Distributed Algorithms 2022/2023
(practical exercise)

Leader election

1 — Find the expected value for the random variable $X \sim Geo(p)$.

2 — Find the variance of the random variable $X \sim Geo(p)$.

3 — Let $p \in [0,1]$ and $n \geq k \geq 1$, $n, k \in \mathbb{N}$. For what value of the argument a function f takes the maximum value?

a. $f(p) = np(1-p)^{n-1}$,
b. $f(n) = np(1-p)^{n-1}$,
c. $f(k) = \binom{n}{k} p^k (1-p)^{n-k}$.

4 — Prove that $(1+x)^r \geq 1 + rx$ for $x \geq -1, r \geq 1$.

5 — Prove that $(1+x)^r \leq 1 + rx$ for $x \geq -1, r \in (0,1)$.

6 — Prove that $1 + x \leq e^x$ for $x \in \mathbb{R}$.

7 — Prove that $\frac{x}{e^x} < \frac{1.5}{x^2}$ for $x > 0$.

8 — Let $f_i(n) = n\frac{1}{2^i}\left(1 - \frac{1}{2^i}\right)^{n-1}$. Prove that functions $f_i(2^{i-1})$, $f_{i+1}(2^{i-1})$ are decreasing and function $f_{i-1}(2^i)$ is increasing for $i \geq 2$.

9 — Present the definition of the Lambert W functions family and draw its real branches.

10 — Use Lambert W function to analytically determine real solutions to the equation

$$3^x = x^3 \, .$$

What is Lambert's W function called in Mathematica?

11 — Have a look at this paper and show that for $x \geq e$

$$\ln x - \ln\ln x < W_0(x) \leq \ln x - \frac{1}{2}\ln\ln x \, .$$

12 — Completion of the lecture proof. Check that if $K \geq 1, f > 1, u \geq 2$ and

$$3e(K+1)u^{\frac{-1}{2(K+1)}} \geq 1 - \frac{1}{f} \qquad \text{then} \qquad K \geq \frac{\ln u}{2W_0(\frac{3e}{2}\frac{f}{f-1}\ln u)} - 1 \, .$$

Good luck,
J.L.