I) Mutual exclusion - continuation
(Token ring)

Lemma 4 (Convergence) ME algorithm from any configuration reaches some legal configuration $C \in L$ in $O(n^3)$ steps where n - size of the ring.

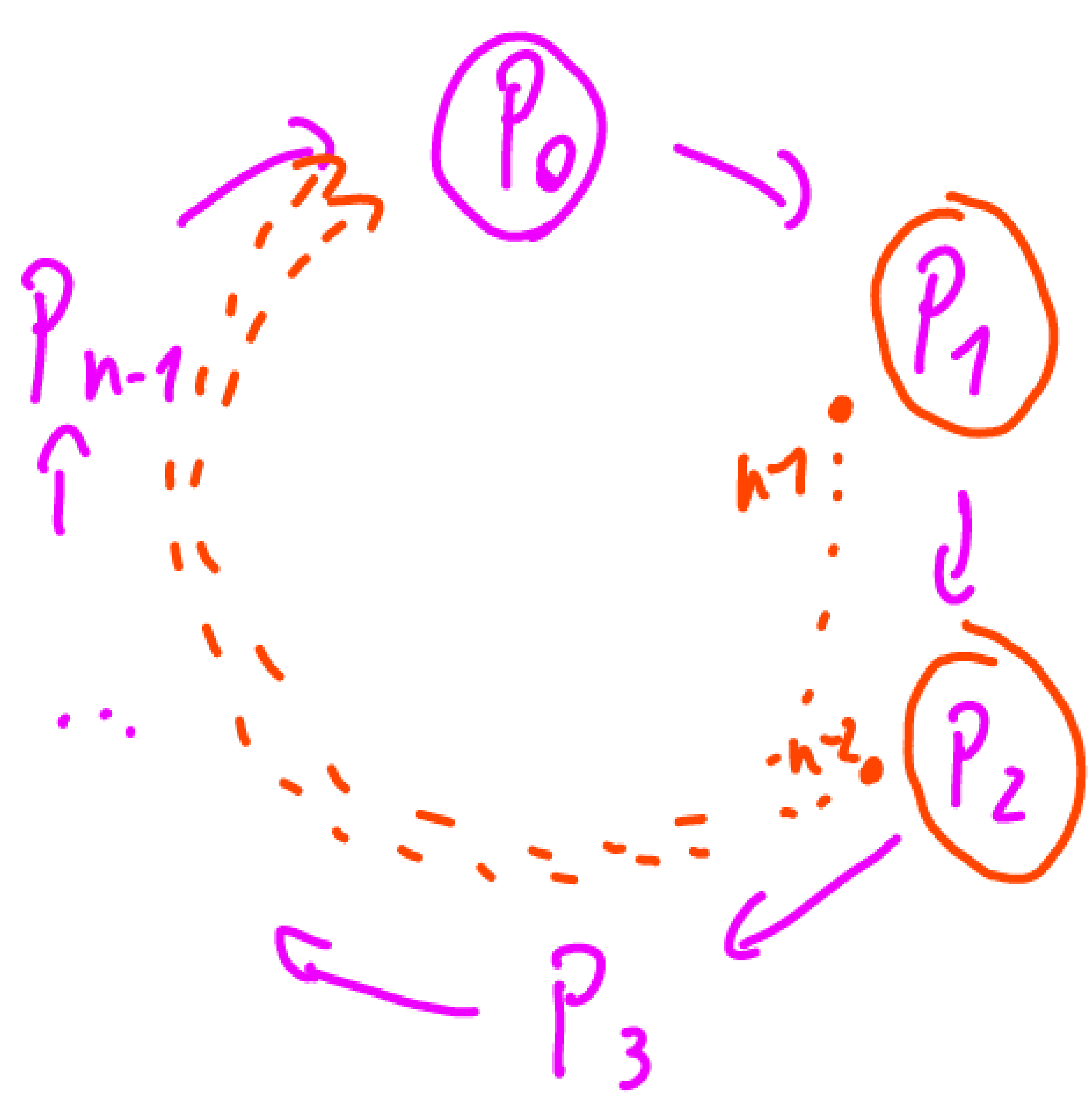
Pf Let us consider any initial configuration

a) At most $\frac{1}{2}n(n-1)$ steps can be done without step of P_0.

a.1) Let $F(t) = \sum_{i \in S(t)} (n-i)$, where

[we take all tokens we have and calculate their distance to P_0.]

$S(t) = \{i : i \geq 1, P_i \text{ in CS in step } t \text{ from initial configuration}\}$



a.2) $F(t)$ is a potential function $\quad 0 \leq F(t) \leq \sum_{1 \leq i \leq n-1} (n-i) = \frac{1}{2}n(n-1)$

$F(t) \in \mathbb{N}$

and $F(t)$ decreases with any step of any process, except P_0.

6) A legal configuration $r_0 = r_1 = \ldots = r_{n-1}$ is reached in $O(n^3)$ steps (it is not the only legal c)

6.1) For any configuration $c = (r_0, r_1, \ldots r_{n-1})$ there is at least one value $x \in \{0, 1, \ldots n\}$ which is <u>not</u> present in c.

6.2) r_0 takes value x after no more than n steps of P_0. Then x is unique value in the ring (other registers just copy values)

6.3) Since $r_0 = x$ is unique in the ring. P_0 will enter CS only when $r_0 = r_1 = \ldots = r_{n-1} = x$. This is legal configuration.

6.4) Thus we can compute an upper bound for total number of steps as:

$$n + n \frac{1}{2} n(n-1) + \frac{1}{2} n(n-1) = O(n^3)$$

steps of P_0 until $r_0 = x$ (from 6.2)

steps of $P_{i \neq 0}$ until $r_0 = x$ (from a and 6.2)

steps of $P_{i \neq 0}$ until next move of P_0 after $r_0 = x$ (from 6.3 and a)

Maximal matching (self-stabilizing)

Def: (Maximal matching)

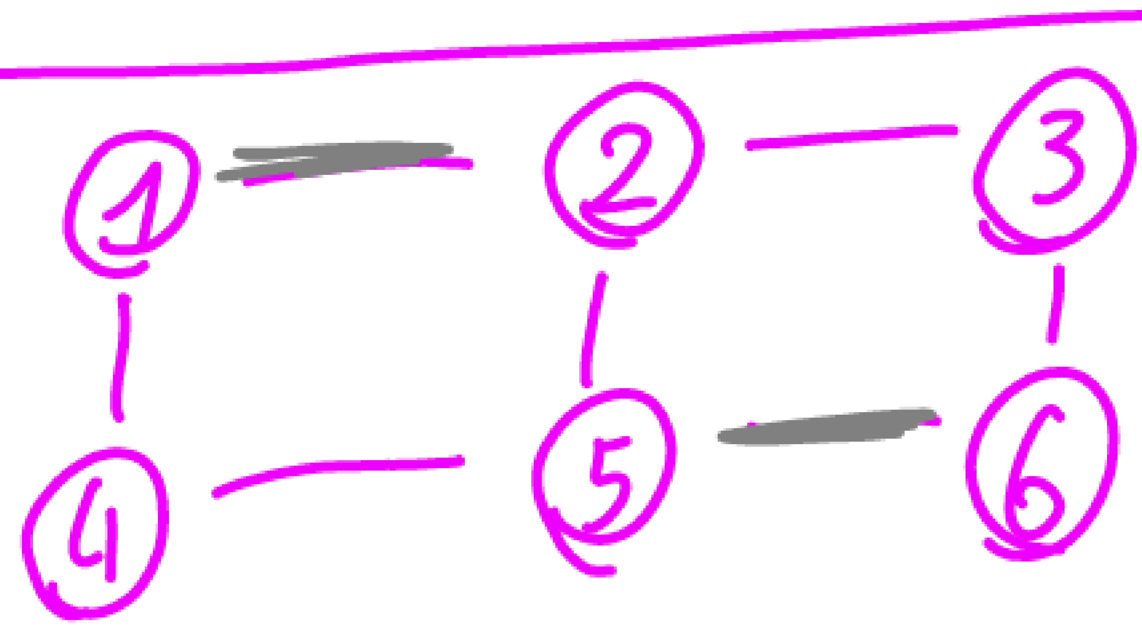
Let $G=(V,E)$ be undirected graph, i.e. $E \subseteq \{\{v_i,v_j\}: v_i,v_j \in V, i\neq j\}$

Maximal matching is a subset $M \subseteq E$ such that

a) each vertex $v \in V$ can be in at most one pair $\{v,u\} \in M$. (matching)

b) M cannot be extended by adding new edges (maximal)

Example



$M = \{\{1,2\},\{5,6\}\}$

watch out!
Maximal not necesary is maximum matching.

Maximum $= \{\{1,4\},\{2,5\},\{3,6\}\}$

Our requirements:

1) distributed algorithm – a node can communicate only with its neighbours to achive a global property

2) self-stabilizing algorithm – e.g. can adopt to chainges in topology.

The matching algorithm

1) each machine p has only one variable - pointer $pref_p \in N(p) \cup NULL$

where

$\cdot N(p)$ - set of neighbours of p in graph G.

2) possible states of p:

a) $married(p) \equiv pref_p = q \in N(p) \wedge pref_q = p \in N(q)$

b) $single(p) \equiv pref_p = NULL \wedge (\forall q \in N(p))(married(q))$

c) $free(p) \equiv pref_p = NULL \wedge (\exists q \in N(p))(\neg married(q))$

d) $wait(p) \equiv pref_p = q \in N(p) \wedge pref_q = NULL$

e) $chain(p) \equiv pref_p = q \in N(p) \wedge pref_q = r \in N(q) \wedge r \neq p$

$p \to q \to r$

Algorithm for machine p

do forever:

- if $pref_p = NULL \wedge (\exists q \in N(p))(pref_q = p)$: (*)
  - $pref_p = q$ // accept proposal
- if $pref_p = NULL \wedge (\forall q \in N(p))(pref_q \neq p) \wedge (\exists q \in N(p))(pref_q = NULL)$ (**)
  - $pref_p = q$ // propose
- if $pref_p = q \wedge pref_q \neq p \wedge pref_q \neq NULL$: (***)
  - $pref_p = NULL$ // unchain

in order to provide atomic operations in practice requires composing this algorithm on top of Token-Ring algorithm.

Specification S (Set of legal configuration): $(\forall p)(\text{married}(p) \vee \text{single}(p))$

Lemma 1 If the condition of specification S holds then set $M = \{\{p, \text{pref}_p\}, \text{pref}_p \neq NULL\}$ is maximal matching.

p-f (a) it is matching); indirect $(S \to M \Leftrightarrow \neg M \to \neg S)$:

$\{p,q\}, \{p,r\} \in M$, $q \neq r \to \neg S \equiv (\exists p)(\text{free}(p) \vee \text{wait}(p) \vee \text{chain}(p))$

b) is maximal; indirect proof:

assume that $M' = M \cup \{p,q\}$, $\{p,q\} \notin M$

$\Downarrow$

$\neg S$ for M

Lemma 2 (Corectness)

Specification S holds $\Longleftrightarrow$ a configuration is terminal.

p-f '$\rightarrow$' if S holds no action of the algorithm is possible

(*) possible if $(\exists p)(\text{wait}(p))$

(**) -11- $(\exists p)(\text{free}(p))$

(***) -11- $(\exists p)(\text{chain}(p))$

p-f '$\Leftarrow$' indirect proof: $\neg S \to$ configuration not terminal $\equiv$ an action of alg. is possible

- $(\exists p)(free(p) \vee wait(p) \vee chain(p))$
- $chain(p) \to$ (***) is possible
- $wait(p)$ for q $\to$ (*) is possible for q
- $free(p) \to (\exists q \in N(p))(\neg married(q) \wedge \neg single(q))$

p is free   p is not matched

a) $wait(q) \to$ (*) possible for neighbour of q
b) $chain(q) \to$ (***) – " – q
c) $free(q) \to$ (**) – " – q and p

there is always a move, so configuration is not terminal □.