(13) • continues and independent ran. var $X_1, X_2, \dots X_n$ all having same dist

→ PDF – $f(x)$

→ CDF – $F(x)$

We have to show that k-th order statistic $X_{k:n}$ has a distribution described by denisty function

$$f_k(x) = \frac{F(x)^{k-1} \cdot [1-F(x)]^{n-k} f(x)}{B(k, n-k+1)}$$

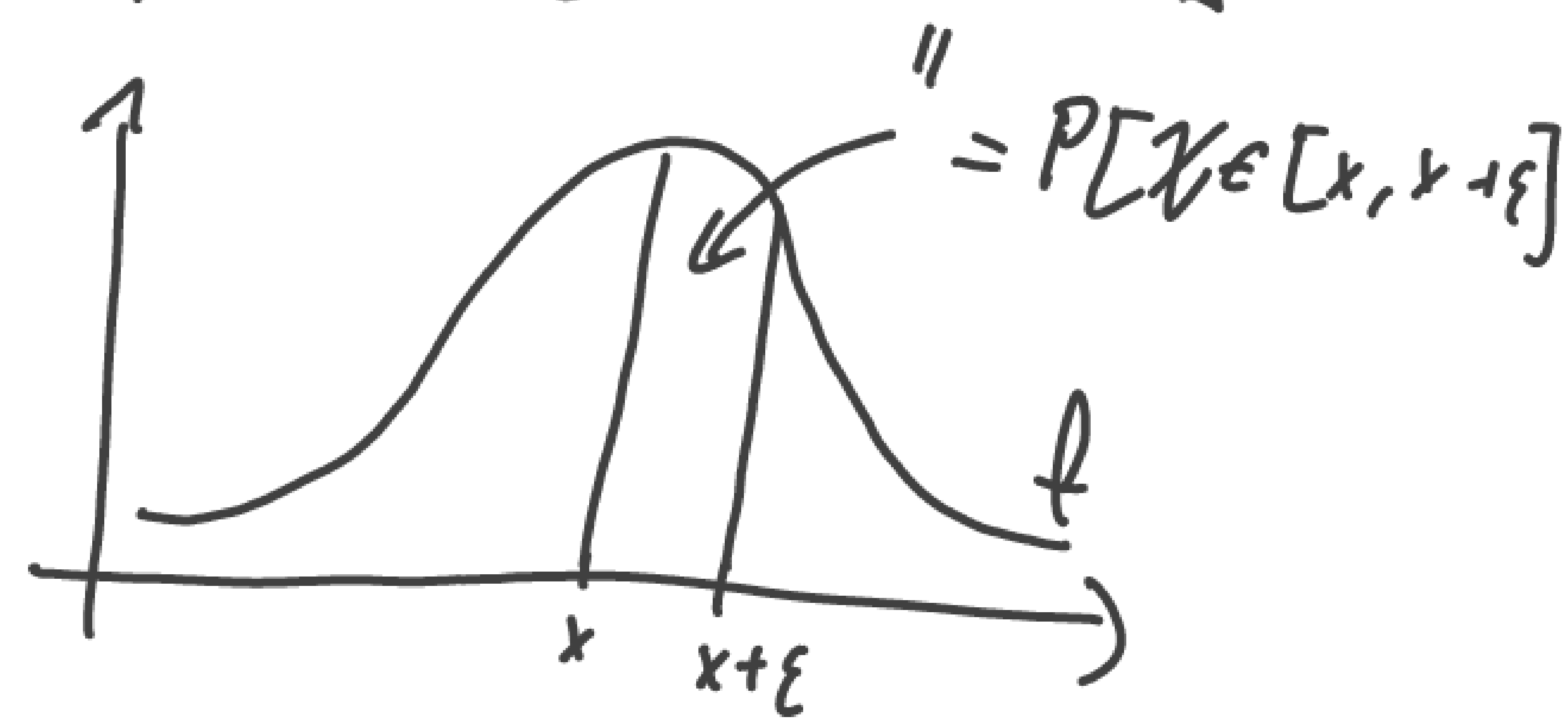
where $B(r,s)$ is beta function.

---

for continues rand. var we denote $f(x)\varepsilon \approx P[x \le X \le x+\varepsilon]$

$$\lim_{\varepsilon \to 0} f(x)\varepsilon = \lim_{\varepsilon \to 0} P[X \in [x, x+\varepsilon]]$$

$$\boxed{f(x) = \lim_{\varepsilon \to 0} \frac{P[X \in [x, x+\varepsilon]]}{\varepsilon}}$$



then for example

$P[X_{n:n} \in [x, x+\varepsilon]] = P[\text{one of the } X_i \in [x, x+\varepsilon] \wedge X_{\neq i} < x]$

$f_{k:n}(x) = P[X_{k:n} \in [x, x+\varepsilon]] =$

$= P[\text{one of the } X\text{'s} \in [x, x+\varepsilon] \wedge k-1 \text{ of others} < x \wedge n-k \text{ of other} > x]$

$= \sum_{i=1}^{n} P[X_i \in [x, x+\varepsilon] \wedge k-1 \text{ of others} < x \; n-k \text{ of others} > x]$

$\overset{\text{independent}}{=} n P[X_i \in [x, x+\varepsilon] \cdot P(\; - \; \text{''} \; - \;)$

$= n P[X_i \in [x, x+\varepsilon] \cdot \left(\binom{n-1}{k-1} P[X < x]^{k-1} \cdot P[X > x]^{n-k} \right)$

$= n f(x) \binom{n-1}{k-1} F(x)^{k-1} (1 - F(x))^{n-k}$

recall:

$$B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\,\Gamma(s)}{\Gamma(r+s)}$$

- $$B(k, n-k+1) = \frac{\Gamma(k)\,\Gamma(n-k+1)}{\Gamma(k+n-k+1)} = \frac{(k-1)!\,(n-k)!}{n!} = \frac{(k-1)!\,(n-k)!}{n\,(n-1)!}$$

$$= \frac{1}{n\binom{n-1}{k-1}}$$

$$f_k(x) = n\,f(x)\binom{n-1}{k-1}F(x)^{k-1}\,(1-F(x))^{n-k} = \frac{f(x)\,F(x)^{k-1}\,(1-F(x))^{n-k}}{B(k,n-k+1)}$$

(14) • $U_1, U_2, \ldots U_n$ - independent rand. variables

• $U_i \sim \text{Uniform}(0,1)$

show that $U_{k:n} \sim \text{Beta}(k, n-k+1)$ and $E[U_{k:n}] = \frac{k}{n+1}$

---

a) PDF of Beta distribution $f_{Beta}(\alpha, \beta) = \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$

and $B(\alpha,\beta) = \dfrac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$

We know from the previous task that density of $U_{k:n}$ is given by $f_{k:n}(x) = n f_X(x) \binom{n-1}{k-1} F_X(x)^{k-1} (1 - F_X(x))^{n-k}$

when we note that

$$f_{uni}(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases} \overset{a=0}{\underset{b=1}{=}} \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$$

$$F_{uni}(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a,b) \\ 1 & x \geq b \end{cases} \overset{a=0}{\underset{b=1}{=}} \begin{cases} 0 & x < 0 \\ x & x \in [0,1) \\ 1 & x \geq 1 \end{cases}$$

$\alpha = k$

$\beta = n-k+1$

$$f_k(x) = n \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k} =$$

$$= n \frac{(n-1)!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} =$$

$$= \left(\frac{(k-1)!(n-k)!}{n!} \right)^{-1} x^{k-1} (1-x)^{n-k} =$$

$$= B(k, n-k+1)^{-1} x^{k-1} (1-x)^{n-k} = \frac{x^{k-1}(1-x)^{n-k}}{B(k, n-k+1)}$$

b)

$$E[U_{k:n}] = \frac{k}{n+1}$$

$$E[U_{k:n}] = \int_0^1 x f_n(x)\,dx = \int_0^1 \frac{x^k (1-x)^{n-k}}{B(k, n-k+1)}\,dx =$$

$$= \frac{1}{B(k,n-k+1)} \cdot \int_0^1 x^k (1-x)^{n-k}\,dx = \frac{1}{B(k,n-k+1)}\, B(k+1, n-k+1) =$$

$$= \frac{1}{\frac{(k-1)!\,(n-k)!}{n!}} \cdot \frac{k!\,(n-k)!}{(n+1)!} = \frac{n!}{(k-1)!\,\cancel{(n-k)!}} \cdot \frac{k!\,\cancel{(n-k)!}}{(n+1)!} =$$

$$= \frac{k}{n+1}$$

(15) $U_{k:n}$ - k-th order statistic for n independent uniformly distributed rand var $\sim U(0,1)$.

a) $\hat{n}_k = \frac{k-1}{U_{k:n}}$ for $k \geq 2$ we have $E(\hat{n}_k) = n$

b) - " - for $k \geq 3$ we have $Var(\hat{n}_k) = \frac{n(n-k+1)}{k-2}$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \quad \text{for } X \sim g(x)$$

---

a)
$$E[\hat{n}_k] = E\left[\frac{k-1}{U_{k:n}}\right] = \int_0^1 \frac{k-1}{x} f_k(x)\, dx = (k-1)\int_0^1 \frac{1}{x} f_k(x)\, dx =$$

$$= (k-1)\int_0^1 \frac{1}{x} \cdot \frac{x^{k-1}(1-x)^{n-k}}{B(k,n-k+1)} dx = \frac{k-1}{B(k,n-k+1)} \int_0^1 x^{k-2}(1-x)^{n-k} dx =$$

$$= \frac{k-1}{B(k,n-k+1)} \cdot B(k-1,n-k+1) =$$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{(\alpha-1)!\,(\beta-1)!}{(\alpha+\beta-1)!}$$

$$= \frac{(k-1)\,n!}{(k-1)!\,(n-k)!} \cdot \frac{(k-2)!\,(n-k)!}{(n-1)!} = \frac{n!}{(n-1)!} = n \quad \blacksquare$$

---

b) $Var[\hat{n}_k] = E[\hat{n}_k^2] - E[\hat{n}_k]^2$ for $k \geq 3$

$$E[\hat{n}_k^2] = E\left[\left(\frac{k-1}{U_{k:n}}\right)^2\right] = \int_0^1 \frac{(k-1)^2}{x^2} f_k(x)\, dx = (k-1)^2 \int_0^1 \frac{1}{x^2} \frac{x^{k-1}(1-x)^{n-k}}{B(k,n-k+1)} =$$

$$= \frac{(k-1)^2}{B(k,n-k+1)} \cdot \underbrace{\int_0^1 x^{k-3}(1-x)^{n-k}}_{B(k-2,\,n-k+1)} = \frac{(k-1)^2\, n!}{(k-1)!\,(n-k)!} \cdot \frac{(k-3)!\,(n-k)!}{(n-2)!} =$$

$$\frac{(k-1)^2 n!}{(k-1)!(n-k)!} \cdot \frac{(k-3)!\,(n-k)!}{(n-2)!} = \frac{(k-1)\,n(n-1)(k-3)!}{(k-2)!} =$$

$$= \frac{(k-1)n(n-1)}{k-2} = E[\hat{n}_k^2]$$

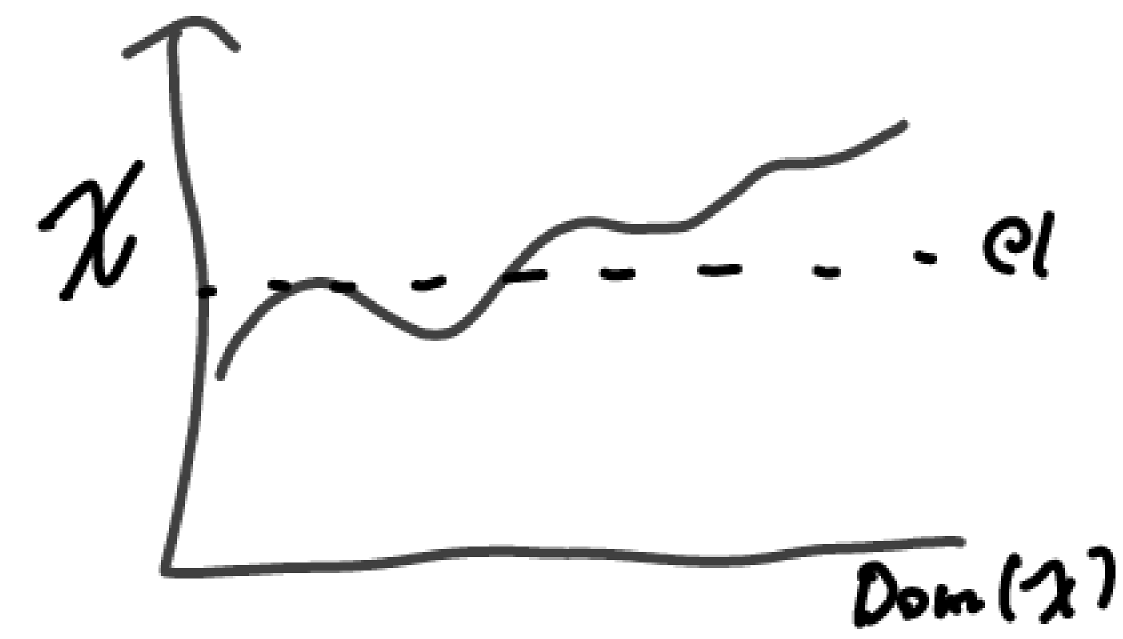
$$Var[\hat{n}_k] = E[\hat{n}_k^2] - E[\hat{n}_k]^2 = \frac{(k-1)n(n-1)}{k-2} - n^2 =$$

$$= \frac{n(nk-k-n+1) - n^2(k-2)}{k-2} = \frac{\underline{n^2k} - nk - \underline{n^2} + n - \underline{n^2k} + \underline{2n^2}}{k-2}$$

$$= \frac{n^2+n-nk}{k-2} = \frac{n(n-k+1)}{k-2}\,.$$

(16) $X \geq 0$, $a > 0$ then

$$P(X > a) \leq \frac{E[X]}{a}$$



Proof 1:

$$E[X] = \int_0^\infty x \cdot f_X(x)\, dx \geq \int_0^\infty a\, f_X(x)\, dx = a \int_0^\infty f_X(x)\, dx \geq$$

$$\geq a \int_a^\infty f_X(x)\, dx = a \cdot P(X > a)$$

$$E[X] \geq a \cdot P(X > a) \qquad // \text{ since } a > 0$$

$$\frac{E[X]}{a} \geq P(X > a)$$

Proof 2:

$$\mathbb{1}_a = \begin{cases} 0, & \text{if } x < a \\ a, & \text{if } x \geq a \end{cases}$$

$\mathbb{1}_a$ is 0-a rand var

then $E[X] \geq E[\mathbb{1}_a] = a \Pr[\mathbb{1}_a = a] + 0 \cdot \Pr[\mathbb{1}_a = 0] =$
$a \Pr[X \geq a]$

Markov inequality

"if $X > 0$ and $E[X]$ is small, then X is unlikely to be very large."

(17) Chebyshev inequality

"If the variance is small, then X is unlikely to be too far from the mean."

$$P[|X - E[X]| < a] \geq 1 - \frac{Var(X)}{a^2}$$

Proof

$$P[|X - E[X]| < a] = 1 - P[|X - E[X]| \geq a] =$$

$$= 1 - P[(X - E[X])^2 \geq a^2] \leq 1 - \frac{E[(X - E[X])^2]}{a^2} = 1 - \frac{Var(X)}{a^2}$$

positive rand var

$$P[Y \geq c] \leq \frac{E[Y]}{c}$$

(18) moment generating function $M_X(t) = E[e^{tX}]$

then n-th moment is achieved with $M_X^{(n)}(0) = E[X^n e^{0X}] = E[X^n]$

Chernoff inequality for sum of Bernoulli trials

- Let $X_1, X_2 \dots X_n$ be independent Bernoulli trials, such that

$$X_i = \begin{cases} 1 & \text{with } p_i \\ 0 & \text{w.t } 1-p_i \end{cases}$$

- Let $X = \sum_{i=1}^{n} X_i$
- Let $\mu = E[X]$

we want to prove that

a) $P[X \geq (1+\delta)\mu) \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$, for $\delta > 0$

b) $P[X \leq (1-\delta)\mu) \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}$, for $0 < \delta \leq 1$

Proof

(1) we note that $\mu = E[X] = E[\sum_{i=1}^{n} X_i] \underset{\substack{\text{linearity} \\ \text{of expec.}}}{=} \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p_i$

(2) by $P[X \geq (1+\delta)\mu]$ we mean $P[X \geq \delta\mu + \mu]$ thus the probability that X deviates from its μ by $\delta\mu$ or more

(3) theorem 1: if $X \wedge Y$ are independent, then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

proof:

$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX+tY}] = E[e^{tX} \cdot e^{tY}] = \leftarrow$ indepency

$= E[e^{tX}]E[e^{tY}] = M_X(t) \cdot M_Y(t)$

④ developing Chernoff bound
we start from computing moment generating function of X_i.

$$M_{X_i}(t) = E[e^{tX_i}] = p_i e^{t \cdot 1} + (1-p_i) e^{t \cdot 0} =$$

$$E[g(x)] = \int_x g(x) f(x) dx, \quad x \sim f(x)$$

$$= p_i e^t + (1-p_i) = 1 + p_i(e^t - 1) \leq e^{p_i(e^t-1)}$$

$\forall x \quad 1+y \leq e^y$ ⑥ task

⑤ then MGF for X

$$M_X(t) = M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t) \leq \prod_{i=1}^n e^{p_i(e^t-1)} =$$

$$= e^{\sum_{i=1}^n p_i(e^t-1)} = e^{(e^t-1)\sum_{i=1}^n p_i} = e^{(e^t-1)\mu}$$

⑥ let's go!

$$\Pr[X \geq (1+\delta)\mu] = \Pr[e^{tX} \geq e^{t(1+\delta)\mu}] \leq$$

Markov's ineq.

$$\leq \frac{E[e^{tX}]}{e^{t(1+\delta)\mu}} = \frac{M_X(t)}{e^{t(1+\delta)\mu}} \leq \frac{e^{(e^t-1)\mu}}{e^{t(1+\delta)\mu}}$$

⑦ to get a) we need to put $t = \ln(1+\delta)$, note, $t > 0 \wedge \delta > 0$.

we end up in $\Pr[X \geq (1+\delta)\mu] \leq \frac{e^{(e^t-1)\mu}}{e^{t(1+\delta)\mu}} \overset{t=\ln(1+\delta)}{=} \frac{e^{(e^{\ln(1+\delta)}-1)\mu}}{e^{\ln(1+\delta)(1+\delta)\mu}} =$

$$= \left(\frac{e^{1+\delta-1}}{(1+\delta)^{(1+\delta)}}\right)^\mu = \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu$$

$\triangle$

8 now exactly the same but different for b)

$$Pr[X \leq (1-\delta)\mu] = Pr[e^{tX} \leq e^{t(1-\delta)\mu}]$$

$$\leq \frac{E[e^{tX}]}{e^{t(1-\delta)\mu}} \leq \frac{e^{(e^t-1)\mu}}{e^{t(1-\delta)\mu}} \overset{\uparrow}{=} \frac{e^{(1-\delta-1)\mu}}{(1-\delta)e^{(1-\delta)\mu}} =$$

$t = \ln(1-\delta) < 0$ for $0 < \delta < 1$

$$= \left(\frac{e^{-\delta}}{(1-\delta)e^{(1-\delta)}}\right)^{\mu}$$

$\triangleright$

Notes for labs task 7

Chebyshev's

$$Pr[\,|X - E[X]| < \delta\,] > 1 - \frac{Var[X]}{\delta^2}$$

$X = \frac{\hat{n}}{n}$

$E[X] = 1$

$$Pr[\,|\frac{\hat{n}}{n} - 1| < \delta\,] > 1 - \frac{Var[\frac{\hat{n}}{n}]}{\delta^2}$$

$$// \frac{\hat{n}}{n} - 1 < \delta \wedge \frac{\hat{n}}{n} - 1 > -\delta$$

$$Pr[\,1-\delta < \frac{\hat{n}}{n} < 1+\delta\,] > 1 - \frac{Var[\frac{\hat{n}}{n}]}{\delta^2}$$

so we want $\alpha = \frac{Var[\frac{\hat{n}}{n}]}{\delta^2} \rightarrow \underline{\underline{\delta = \sqrt{\frac{\frac{1}{k}}{\alpha}}}}$

Chernoff's

$$Pr[\frac{1}{1+\varepsilon_1} \cdot \frac{k-1}{k} < \frac{\hat{n}_k}{n} < \frac{1}{1-\varepsilon_2} \cdot \frac{k-1}{k}\,] > 1 - f_k(\varepsilon_2) - f_k(\varepsilon_1)$$

- $f_k(x) = e^{xk}(1-x)^k$
- $\varepsilon_1 = \frac{\delta}{1-\delta}$
- $\varepsilon_2 = \frac{\delta}{1+\delta}$

$$\rightarrow \frac{1}{1+\varepsilon_1} \rightarrow \frac{1}{1+\frac{\delta}{1-\delta}} = \frac{1}{\frac{1-\delta+\delta}{1-\delta}} = \frac{1}{\frac{1}{1-\delta}}$$

$$= 1-\delta$$

$$\rightarrow \frac{1}{1-\varepsilon_2} \rightarrow \frac{1}{1-\frac{\delta}{1+\delta}} = \frac{1}{\frac{1+\delta-\delta}{1+\delta}} = 1+\delta$$

$$Pr[\,1-\delta < \frac{\hat{n}_k}{n} < 1+\delta\,] > 1 - f_k\left(\frac{\delta}{1+\delta}\right) - f_k\left(\frac{\delta}{1-\delta}\right) =$$

$$1 - f_k\left(\frac{\delta}{1+\delta}\right) - f_k\left(\frac{-\delta}{1-\delta}\right) =$$

$$e^{\frac{\delta}{1+\delta}k}\left(1-\frac{\delta}{1+\delta}\right)^k - e^{\frac{-\delta}{1-\delta}k}\left(1-\frac{-\delta}{1-\delta}\right)^k = \alpha$$

$$e^{\frac{\delta}{1+\delta}\cdot k}\left(\frac{1}{1+\delta}\right)^{k} - e^{\frac{-\delta}{1-\delta}k}\Big($$

$$\frac{1-\delta-\delta}{1-\delta} = \frac{1-2\delta}{1-\delta}$$

(19) $\Pr[|X-\mu| \geq \delta\mu] \leq 2e^{-\frac{\mu\delta^2}{3}}, \quad \delta \in (0,1)$

4.2 $\quad \Pr[X \geq (1+\delta)\mu] \leq e^{-\frac{\mu\delta^2}{3}}$

4.4 $\quad \Pr[X \leq (1-\delta)\mu] \leq e^{-\frac{\mu\delta^2}{2}}$

$$\Pr[|X-\mu| \geq \delta\mu] = \Pr[X \geq (1+\delta)\mu \lor X \leq (1-\delta)\mu] =$$

$$\left[\begin{array}{c} X-\mu \geq \delta\mu \lor X-\mu \leq -\delta\mu \\ X \geq \delta\mu+\mu \lor X \leq -\delta\mu+\mu \\ X \geq (1+\delta)\mu \lor X \leq (1-\delta)\mu \end{array}\right]$$

$$\Pr[X \geq (1+\delta)\mu] - \Pr[X \leq (1-\delta)\mu] \leq$$

$$e^{-\frac{\mu\delta^2}{3}} - e^{-\frac{\mu\delta^2}{2}} \leq e^{-\frac{\mu\delta^2}{3}} + e^{-\frac{\mu\delta^2}{3}} = 2e^{-\frac{\mu\delta^2}{3}}$$

4.4 $\quad \frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}} \leq e^{-\frac{\delta^2}{2}}$

$f(\delta) = -\delta - (1-\delta)\ln(1-\delta) + \frac{\delta^2}{2} \leq 0$

$f'(\delta) = \ln(1-\delta) + \delta$

$f''(\delta) = -\frac{1}{1-\delta} + 1$

$f'' < 0$ in $(0,1)$ and $f'(0) = 0$, we have

$f'(\delta) \leq 0$ in $[0,1)$, $f(\delta)$ nonincreasing

$f(0) = 0$, $f(\delta) \leq 0$ for $0 < \delta < 1$

20 S_n - number of heads in n-trials

- $E[S_n] = \frac{n}{2}$, $Var(X) = E[X^2] - E[X]^2$ - but 0-1 rand var $X = X^2$

a) Chebyshev's inequality $Pr[|X - E[X]| \geq a] \leq \frac{Var(X)}{a^2}$

$$\boxed{Pr\left[\left|S_n - \frac{n}{2}\right| \geq \frac{n}{4}\right] \leq \frac{4}{n}}$$

$S_n = \sum_{i=1}^{n} S_i$

- $E[S_i] = \frac{1}{2}$
- $E[S_n] = \frac{n}{2}$
- $Var[S_i] = E[X_i^2] - E[X_i]^2 = E[X_i] - E[X_i]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$
- $Var[S_n] = Var\left[\sum_{i=1}^{n} S_i\right] = \sum_{i=1}^{n} Var[S_i] = \frac{n}{4}$

$$Pr\left[\left|S_n - \frac{n}{2}\right| \geq \frac{n}{4}\right] = Pr\left[|S_n - E[S_n]| \geq \frac{n}{4}\right] \leq \frac{Var[S_n]}{\left(\frac{n}{4}\right)^2} =$$

$$= \frac{\frac{n}{4}}{\left(\frac{n}{4}\right)^2} = \frac{4}{n} \quad \square$$

6) Chernoff's inequality $P[|X-\mu| \geq \delta\mu] \leq 2e^{-\frac{\mu\delta^2}{3}}$

$$P_n\left[|S_n - \tfrac{n}{2}| \geq \tfrac{n}{4}\right] = P_n\left[|S_n - E[S_n]| \geq \delta E[S_n]\right] \leq$$

$$2e^{-\frac{E[S_n]\delta^2}{3}} \overset{\substack{\delta=\frac{1}{2} \\ E[S_n]=\frac{n}{2}}}{=} 2e^{-\frac{\frac{n}{2}\cdot\frac{1}{4}}{3}} = 2e^{-\frac{n}{24}}$$

(21) ...

if random() <= 2^{-c} $\quad \frac{1}{2}^{c}$

...

$$x_{10} = \sum_{i=1}^{n} 2^{-b_i}$$

$c = 1$	$\frac{1}{2}$	$0??...??$
$c = 2$	$\frac{1}{4}$	$00?...??$
$c = 3$	$\frac{1}{8}$	$000?...??$
$c = 4$	$\frac{1}{16}$	$0000?...??$

$$C_n = \begin{cases} 0 & \text{- heads} \\ 1 & \text{- tails} \end{cases}$$

$$[0,1) = [00...000_2, 11...111_2)$$

$$E[g(X)] = \int g(x) \cdot f(x)\, dx \quad X \sim f(x)$$

$E[2^{c_n}] = n+2$ and $Var[2^{c_n}] = \frac{n(n+1)}{2}$

a) C_n - number of heads in n trials $+1$

- $E[2^{C_n}] = n+2$

- $P(Now) = P(Now \cap Inc) + P(Now \cap Not\,Inc)$

- $P(A|B) = \frac{P(A \cap B)}{P(B)}$

① $2^{C_0} = 2$

② $E[2^{C_{n+1}}] = \sum_k 2^k \Pr[C_{n+1} = k] =$

$= \sum_k 2^k \Big(\underbrace{\Pr[C_{n+1} = k \mid C_n = k] \Pr[C_n = k]}_{\text{we don't increment}} + \underbrace{\Pr[C_{n+1} = k \mid C_n = k-1] \Pr[C_n = k-1]}_{\text{we increment}}$

$= \sum_k 2^k \Pr[C_{n+1} = k \mid C_n = k] \Pr[C_n = k] + \sum_k 2^k \Pr[C_{n+1} = k \mid C_n = k-1] \Pr[C_n = k-1]$

$= \sum_k 2^k \left(1 - \left(\frac{1}{2}\right)^k\right) \Pr[C_n = k] + \sum_k 2^k \left(\frac{1}{2}\right)^{k-1} \Pr[C_n = k-1] =$

$= \sum_k (2^k - 1) \Pr[C_n = k] + \sum_k 2 \cdot \Pr[C_n = k-1] =$

$= \sum_k 2^k \Pr[C_n = k] - \sum_k \Pr[C_n = k] + 2 \sum_k \Pr[C_n = k-1] =$

$= E[2^{C_n}] - 1 + 2 \cdot 1 = E[2^{C_n}] + 1$

so $E[2^{C_{n+1}}] = E[2^{C_n}] + 1$ but from assumption

$= n+2+1 = n+3$ $\square$

6) $Var[2^{C_n}] = E[(2^{C_n})^2] - E[2^{C_n}]^2$

we prove $E[4^{C_n}] = \frac{3n^2+9n+8}{2}$

$4^{C_0} = 4$

we start from computing $E[(2^{C_n})^2] =$

$E[(2^{C_{n+1}})^2] = E[4^{C_{n+1}}] = \sum_k 4^k Pr[C_{n+1}=k] =$

$= \sum_k 4^k Pr[C_{n+1}=k \mid C_n=k] \cdot Pr[C_n=k] + \sum_k 4^k Pr[C_{n+1}=k \mid C_n=k-1] Pr[C_n=k-1] =$

$= \sum_k 4^k \left(1-\left(\frac{1}{2}\right)^k\right) \cdot Pr[C_n=k] + \sum_k 4^k \left(\frac{1}{2}\right)^{k-1} \cdot Pr[C_n=k-1] =$

$= \sum_k 4^k \cdot Pr[C_n=k] + \sum_k 2^k Pr[C_n=k] + \sum_k 4 \cdot 2^{k-1} \cdot Pr[C_n=k-1] =$

$= E[4^{C_n}] - E[2^{C_n}] + 4E[2^{C_n}] = E[4^{C_n}] + 3(n+2)$

$\frac{n(n+1)}{2} = x - (n+2)^2$

$\frac{n^2+n}{2} = x - n^2 - 4n - 4$

$n^2+n = 2x - 2n^2 - 8n - 8$

$3n^2 + 9n + 8 = 2x$

$x = \frac{3n^2+9n+8}{2}$

$E[4^{C_{n+1}}] = \frac{3(n+1)^2 + 9(n+1) + 8}{2}$

$= \frac{3(n^2+2n+1) + 9n + 9 + 8}{2} =$

$= \frac{3n^2 + 6n + 3 + 9n + 17}{2}$

$= \frac{3n^2 + 15n + 20}{2}$

$E[4^{C_{n+1}}] = E[4^{C_n}] + 3(n+2) = \frac{3n^2+9n+8}{2} + 3(n+2) =$

$= \frac{3n^2 + 9n + 8 + 6n + 12}{2} = \frac{3n^2 + 15n + 20}{2}$ □

$\hat{R}_n = 2^{C_n} - 2$

- $E[\hat{R}_n] = E[2^{C_n} - 2] = E[2^{C_n}] - E[2] = n + 2 - 2 = n$
- $Var[\hat{R}_n] = E[\hat{R}_n^2] - E[\hat{R}_n]^2 =$

$$E[(2^{C_n} - 2)^2] = E[4^{C_n} - 4 \cdot 2^{C_n} + 4] =$$

$$= E[4^{C_n}] - 4E[2^{C_n}] + E[4] =$$

$$= \frac{3n^2 + 9n + 8}{2} - 4 \cdot (n+2) + 4 = \frac{3n^2 + 9n + 8 - 8n - 16 + 8}{2}$$

$$= \frac{3n^2 - n}{2}$$

$$= \frac{3n^2 - n}{2} - (n+2)^2 = \frac{3n^2 - n}{2} - n^2 - 4n - 4 =$$

$$= \frac{3n^2 - n - 2n^2 - 8n - 8}{2} = \frac{n^2 - 9n - 8}{2}$$