

Lecture 11: Digital Logic

Announcements

- Midterm Grades Released
 - If you have any questions, email me or any of the TAs
 - Overall Class Average: ~81 %
- Project 3 released
 - Due August 11th 11:55pm, 2 Weeks
 - Due day before Final, so plan accordingly
 - No Extension
- Lectures until final:
 - Digital Logic (Chapter 4 in book)
- Rest of lecture time / recitation today:
 - Overview of Programming Assignment 3

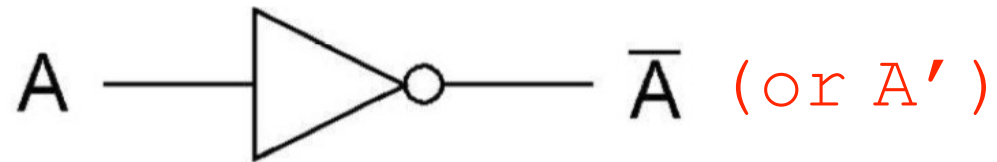
Logic Design

- How does your processor perform various operations?

Logic Gates

- Transition from representing information to implementing them
- Logic gates are simple digital circuits
 - Take one or more binary inputs
 - Produce a binary output
 - Truth table: relationship between the input and the output

Not Gate



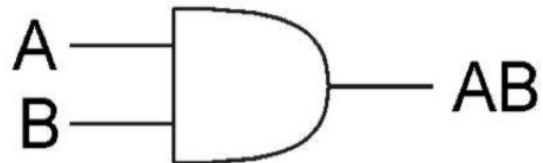
Truth table

In	Out
0	1
1	0

Simplest Gate

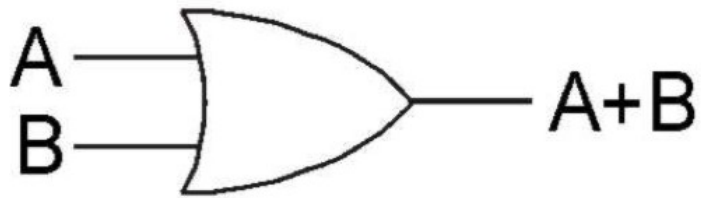
And Gate

A	B	C
0	0	0
0	1	0
1	0	0
1	1	1



AND

Or Gate



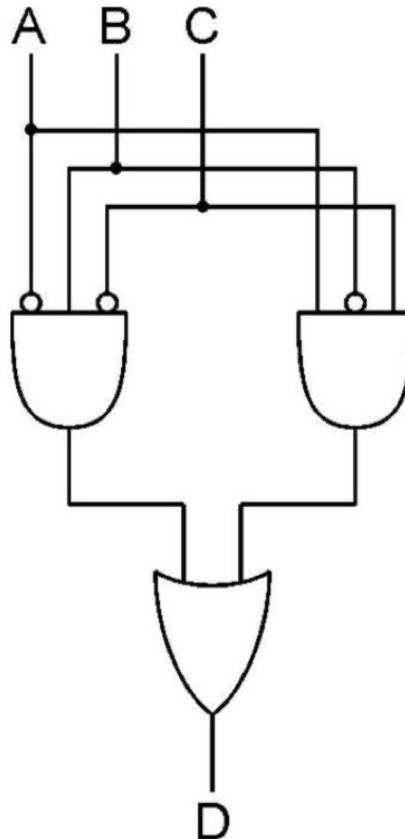
OR

A	B	C
0	0	0
0	1	1
1	0	1
1	1	1

Logical Completeness

- Can implement any truth table with **Not, Or, And** gates.

A	B	C	D
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

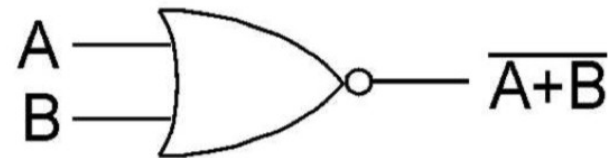


1. AND combinations that yield a "1" in the truth table.

2. OR the results of the AND gates.

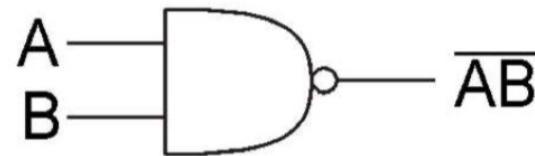
NAND and NOR gates

A	B	C
0	0	1
0	1	0
1	0	0
1	1	0



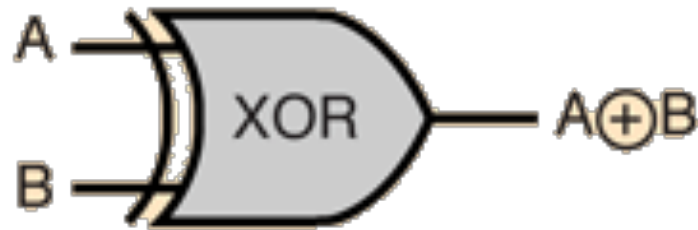
NOR

A	B	C
0	0	1
0	1	1
1	0	1
1	1	0



NAND

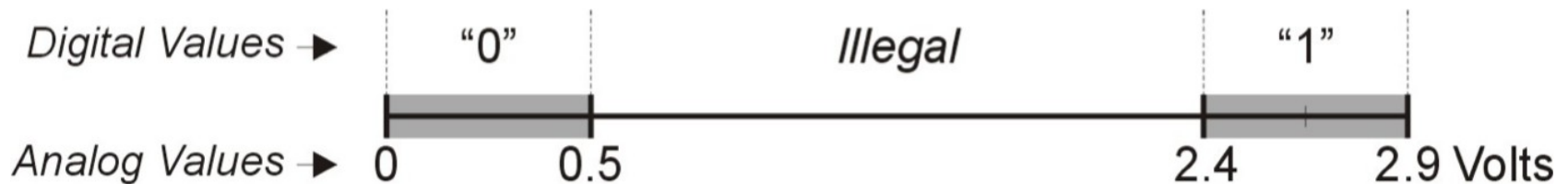
Xor Gate



A	B	Out
0	0	0
0	1	1
1	0	1
1	1	0

Beneath the Digital Abstraction

- Digital system uses discrete values
 - Represent it with continuous variables (voltage, etc)
 - Also must handle noise
- Transistors used to implement logical functions
- Voltage used to represent 0 or 1

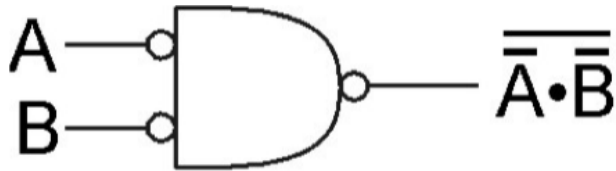


Transistor

- Microprocessors contain millions (billions) of transistors
 - Intel Pentium 4 (2000): 48 million
 - IBM/Apple PowerPC G5 (2003): 58 million
- A transistor acts as a switch
- Combined to implement logic functions (AND, OR, NOT..)
- Combined to build higher-level structures (Adder, Decoder..)
- Combined to build processor

DeMorgan's Law

- Converting AND to OR (and some NOT).



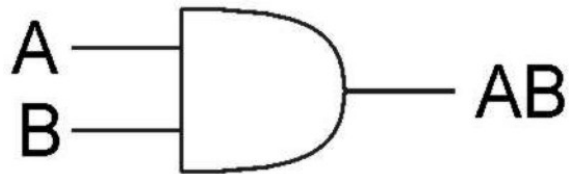
A	B	\overline{A}	\overline{B}	$\overline{A} \cdot \overline{B}$	$\overline{\overline{A} \cdot \overline{B}}$
0	0	1	1	1	0
0	1	1	0	0	1
1	0	0	1	0	1
1	1	0	0	0	1

- In general,
 - (1) $\overline{PQ} = \overline{P} + \overline{Q}$
 - (2) $\overline{\overline{P} + \overline{Q}} = PQ$

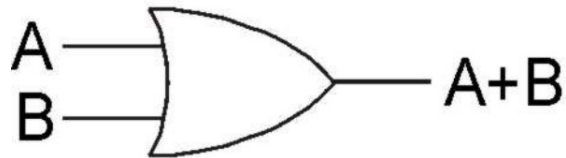
Recap

- 6 Widely used logic gates

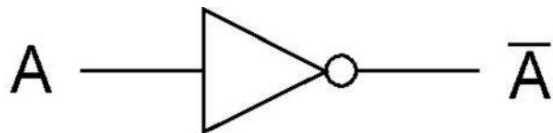
- And Gate



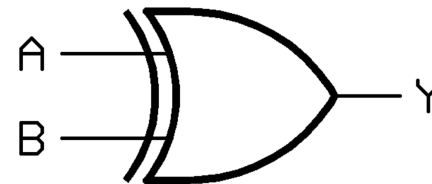
- Or Gate



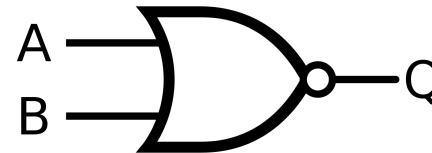
- Not Gate



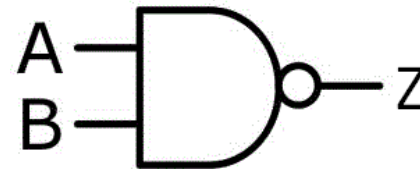
- Xor Gate



- Nor Gate



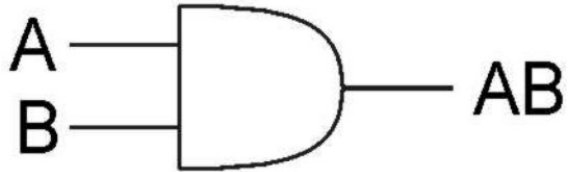
- Nand Gate



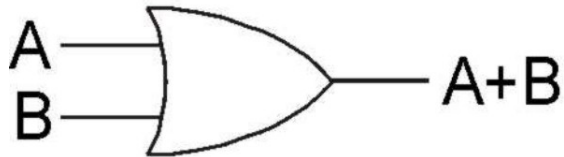
Transformation

- 6 Widely used logic gates

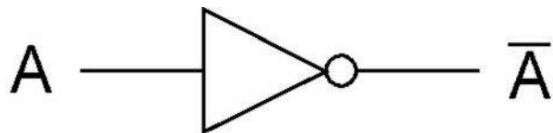
- And Gate



- Or Gate



- Not Gate



- Xor Gate

Can express any logic circuit with these as we have seen previously.

- Not Gate

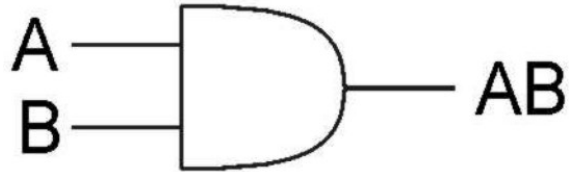


- Nand Gate

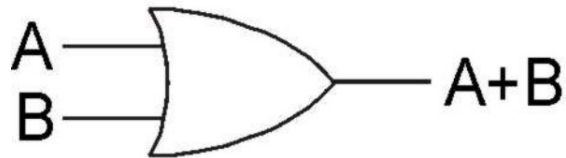


Transformation

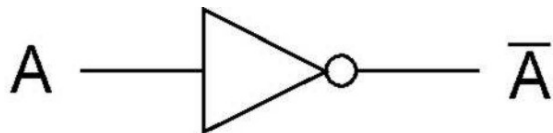
- 6 Widely used logic gates
- And Gate



- Or Gate



- Not Gate



- Xor Gate

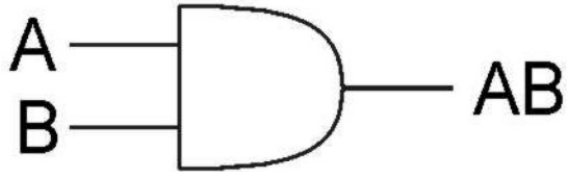
$$A\bar{B} + B\bar{A}$$

- Nor Gate

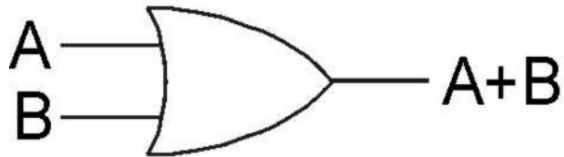
- Nand Gate

Transformation

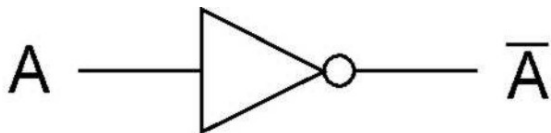
- 6 Widely used logic gates
- And Gate



- Or Gate



- Not Gate



- Xor Gate

$$A\bar{B} + B\bar{A}$$

- Nor Gate

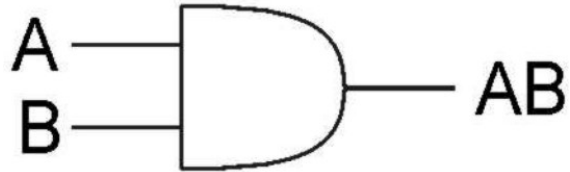
$$\overline{A+B}$$

- Nand Gate

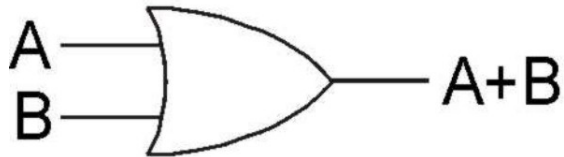
Transformation

- 6 Widely used logic gates

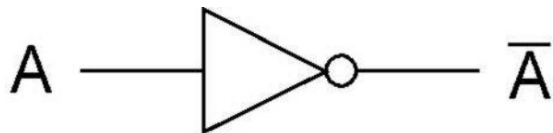
- And Gate



- Or Gate



- Not Gate



- Xor Gate

$$A\bar{B} + B\bar{A}$$

- Nor Gate

$$\overline{A+B}$$

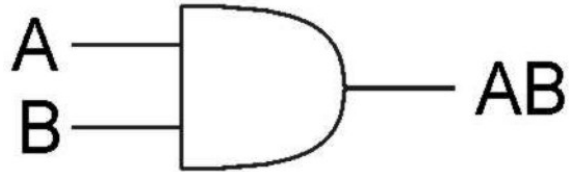
- Nand Gate

$$\overline{AB}$$

Transformation

- 6 Widely used logic gates

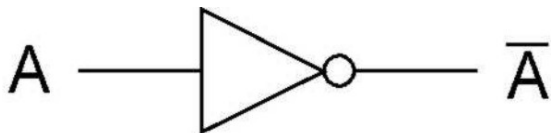
- And Gate



- Or Gate



- Not Gate



- Xor Gate

In fact...

You just need
these two

- NOT Gate

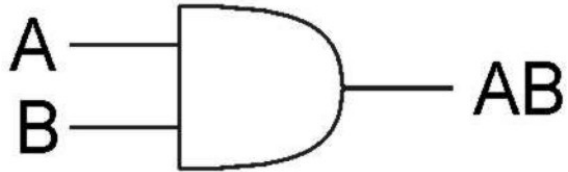
$$\overline{A + B}$$

- Nand Gate

$$\overline{AB}$$

Transformation

- 6 Widely used logic gates
- And Gate



- Or Gate

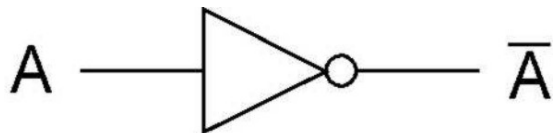
$\overline{\overline{A}B}$

- Xor Gate

- Nor Gate

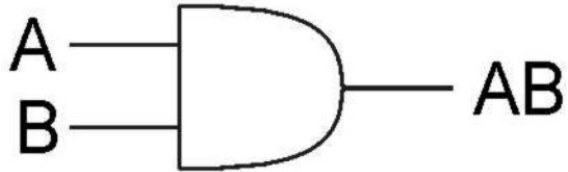
- Nand Gate

- Not Gate



Transformation

- 6 Widely used logic gates
- And Gate



- Or Gate

$\overline{\overline{A}B}$

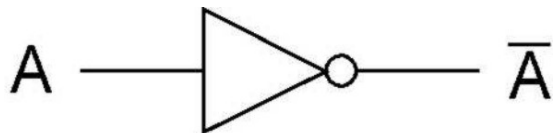
- Xor Gate

$\overline{\overline{A}B\overline{A}B}$

- Nor Gate

- Nand Gate

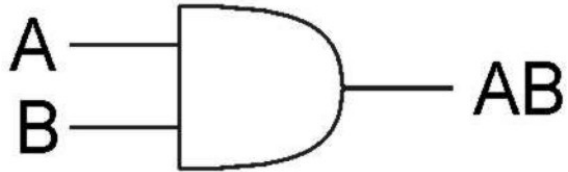
- Not Gate



Transformation

- 6 Widely used logic gates

- And Gate



- Or Gate

$$\overline{\overline{A} \overline{B}}$$

- Xor Gate

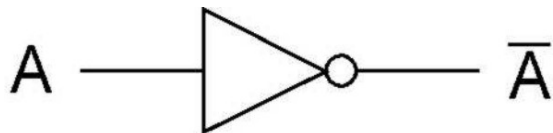
$$\overline{\overline{A} \overline{B} \overline{A} \overline{B}}$$

- Nor Gate

$$\overline{A} \overline{B}$$

- Nand Gate

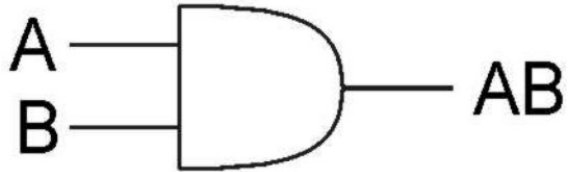
- Not Gate



Transformation

- 6 Widely used logic gates

- And Gate



- Or Gate

$$\overline{\overline{A} \overline{B}}$$

- Xor Gate

$$\overline{\overline{A} \overline{B} \overline{A} \overline{B}}$$

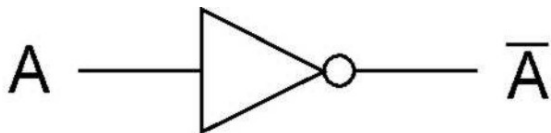
- Nor Gate

$$\overline{A} \overline{B}$$

- Nand Gate

$$\overline{A} \overline{B}$$

- Not Gate



Transformation

- 6 Widely used logic gates

- And Gate



- Or Gate



- Not Gate



- Xor Gate

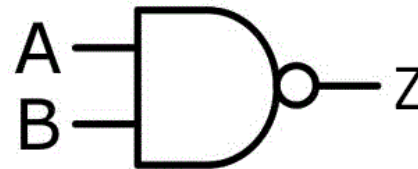
Side note:

Or You just need this Nand Gate to do anything

- NOT Gate



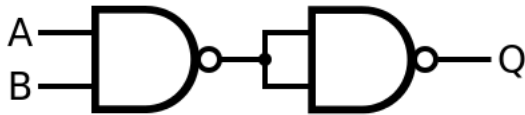
- Nand Gate



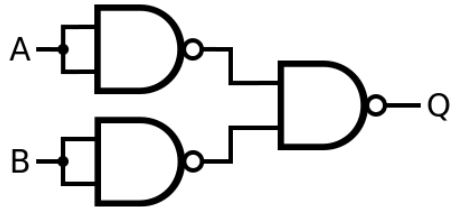
Transformation

- 6 Widely used logic gates

- And Gate



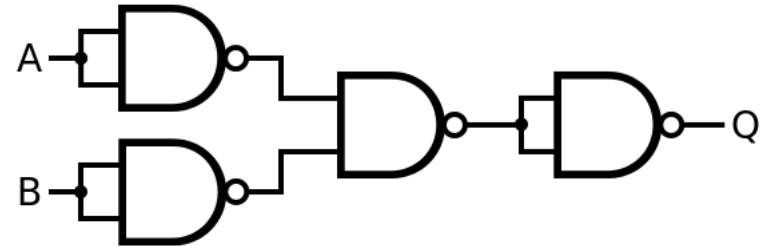
- Or Gate



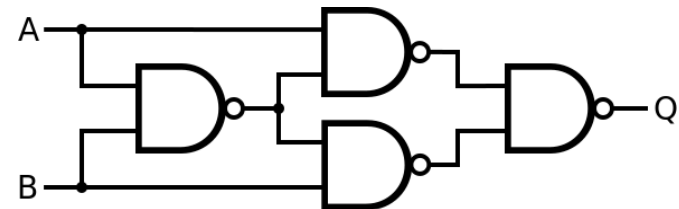
- Not Gate



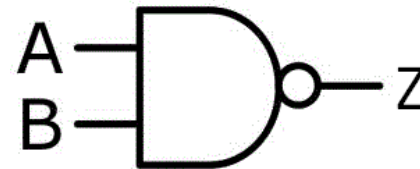
- Xor Gate



- Nor Gate



- Nand Gate



Transformation

- 6 Widely used logic gates

Side note:

Or You just need this
Nor Gate to do anything

- Or Gate



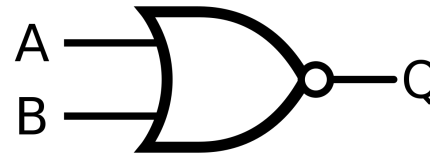
- Not Gate



- Xor Gate



- Nor Gate



- Nand Gate



Transformation

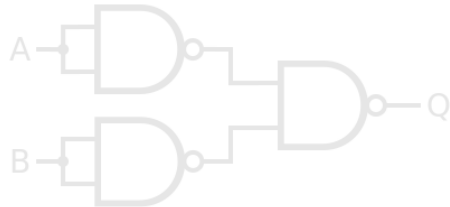
- 6 Widely used logic gates

- Xor Gate

- And Gate



- Or Gate

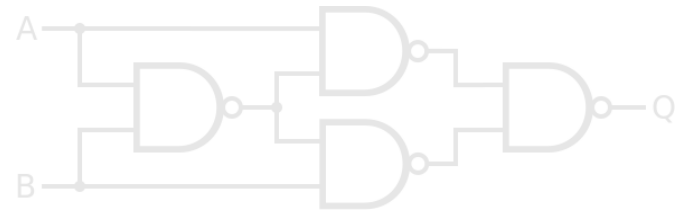


- Not Gate



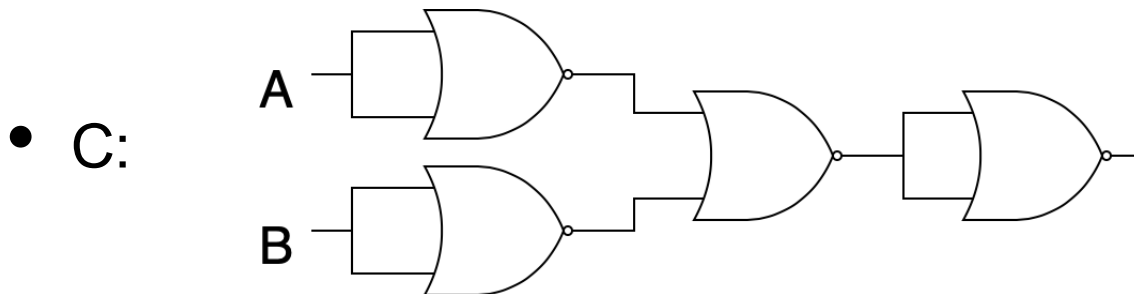
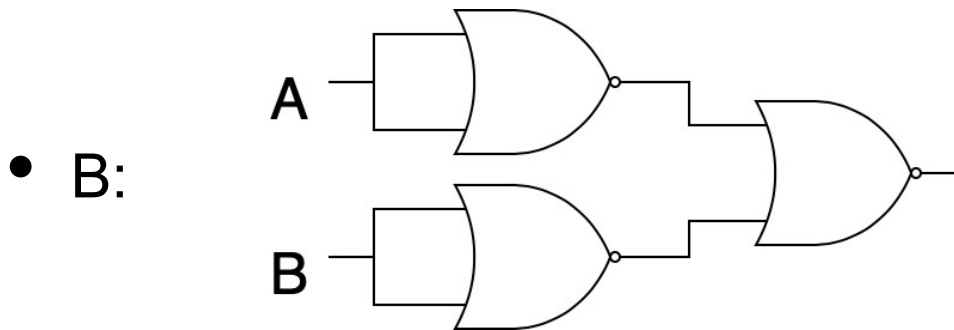
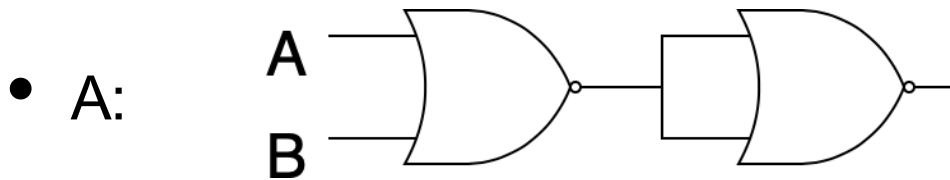
And many more: Imagine all the fun questions you'll be seeing in the final exam

- Nand Gate



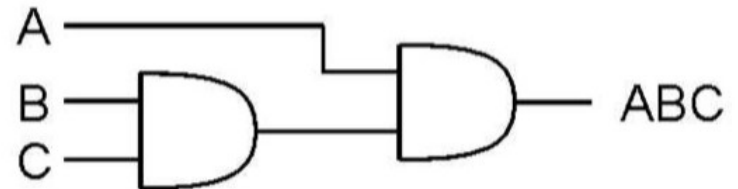
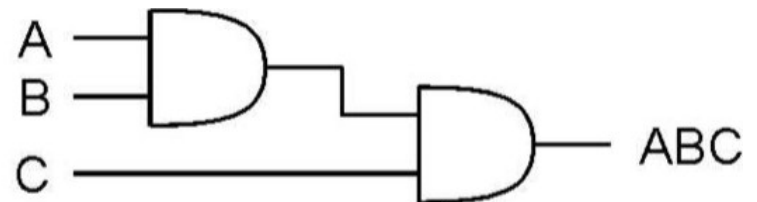
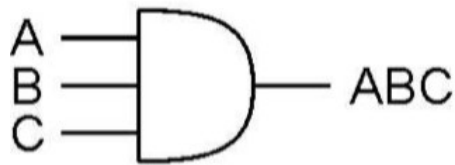
Question

- Which of these circuits using NOR gates is equivalent to an AND gate?



More than 2 Inputs?

- AND/OR can take any number of inputs:
 - AND = 1 if all inputs are 1
 - OR = 1 if any input is 1.
 - Similar for NAND/NOR, etc.



So ... Circuit Design?

- You have a circuit you want to build
- Derive a truth table for this circuit
- Derive Boolean expression for the truth table
- Then build the circuit based on the boolean expression
- The tricky part is how do you optimize this circuit?

Truth Table to Boolean Expression

sensor inputs

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

- Given a circuit, isolate the rows in which the output of the circuit is 1

Truth Table to Boolean Expression

sensor inputs

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

sensor inputs

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$\bar{A}BC = 1$

$A\bar{B}C = 1$

$AB\bar{C} = 1$

$ABC = 1$

- A product term that contains exactly one instance of every variable is called a **minterm**

Minterm

A	B	C	minterm
0	0	0	m0 $\bar{A}\bar{B}\bar{C}$
0	0	1	m1 $\bar{A}\bar{B}C$
0	1	0	m2 $\bar{A}B\bar{C}$
0	1	1	m3 $\bar{A}BC$
1	0	0	m4 $A\bar{B}\bar{C}$
1	0	1	m5 $A\bar{B}C$
1	1	0	m6 $AB\bar{C}$
1	1	1	m7 ABC

- A product term in which all variables appear once.
- Each minterm evaluates to 1 in exactly one case. All other case, it evaluates to 0.

Truth Table to Boolean Expression

sensor inputs

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

sensor inputs

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$\bar{A}BC = 1$

$A\bar{B}C = 1$

$AB\bar{C} = 1$

$ABC = 1$

- Given expressions for each row, build a larger Boolean expression. This is called a **sum-of-products (SOP)** form.

$$Output = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$

Truth Table to Boolean Expression

sensor inputs

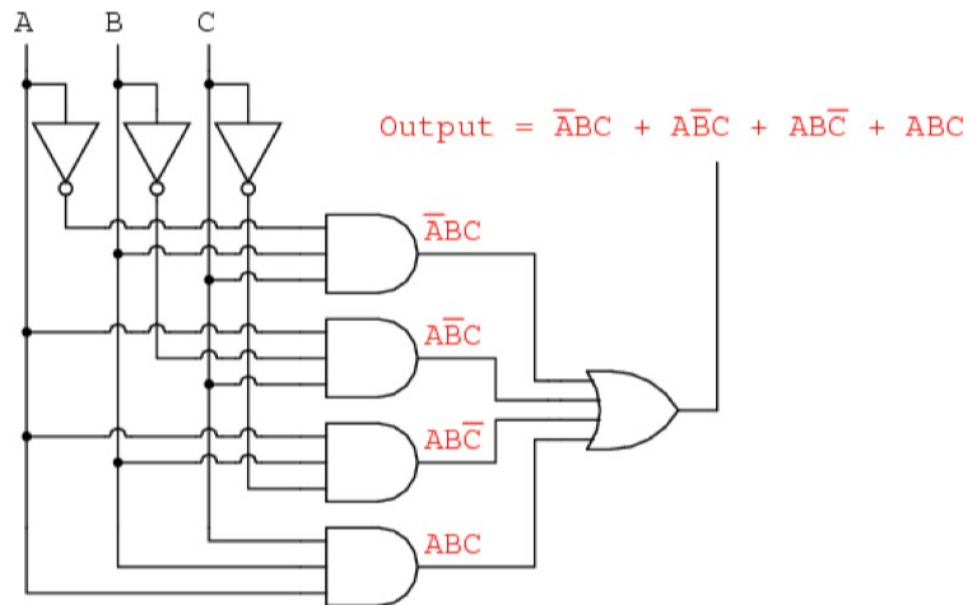
A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$\overline{A}BC = 1$$

$$A\overline{B}C = 1$$

$$AB\overline{C} = 1$$

$$ABC = 1$$



- Finally build the circuit
- However, SoP forms are usually not minimal. We must optimize.

Canonical Forms

- There are two canonical forms:
 - Sum of Products (SOP)

$$F = \overline{Y}Z + X\overline{Y}Z + XYZ$$

- Product of Sums (POS)

$$F = (Y + Z)(\overline{X} + Y + Z)(\overline{X} + \overline{Y} + Z)$$

Converting PoS to SoP

- Multiply through:

$$F = (Y + Z)(X + Y + Z)(X + Y + Z)$$

Converting PoS to SoP

- Multiply through:

$$F = (Y + Z)(X + Y + Z)(X + Y + Z)$$

$$F = (XY + Y + YZ + XZ + YZ + ZZ)(X + Y + Z)$$

Converting PoS to SoP

- Multiply through:

$$F = (Y + Z)(X + Y + Z)(X + Y + Z)$$

$$F = (XY + Y + YZ + XZ + YZ + ZZ)(X + Y + Z)$$

$$F = (Y + XZ + YZ)(X + Y + Z)$$

Converting PoS to SoP

- Multiply through:

$$F = (Y + Z)(\bar{X} + Y + Z)(\bar{X} + \bar{Y} + Z)$$

$$F = (\bar{X}Y + Y + YZ + \bar{X}Z + YZ + ZZ)(\bar{X} + \bar{Y} + Z)$$

$$F = (Y + \bar{X}Z + YZ)(\bar{X} + \bar{Y} + Z)$$

$$F = \bar{X}Y + Y\bar{Y} + YZ + \bar{X}Z + \bar{X}YZ + \bar{X}Z + \bar{X}YZ + Y\bar{Y}Z + YZ$$

Converting PoS to SoP

- Multiply through:

$$F = (Y + Z)(X + Y + Z)(X + Y + Z)$$

$$F = (XY + Y + YZ + XZ + YZ + ZZ)(X + Y + Z)$$

$$F = (Y + XZ + YZ)(X + Y + Z)$$

$$F = \cancel{XY} + \cancel{YY} + YZ + \cancel{XZ} + \cancel{XYZ} + \cancel{XZ} + \cancel{XYZ} + \cancel{YYZ} + YZ$$

$$F = \cancel{XY} + YZ + \cancel{XZ}$$

Converting SoP to PoS

- Complement, multiply through, then complement

$$F = \overline{Y}Z + X\overline{Y}Z + XYZ$$

Converting SoP to PoS

- Complement, multiply through, then complement

$$F = \overline{Y}Z + X\overline{Y}Z + XY\overline{Z}$$

$$\overline{F} = (Y + Z)(\overline{X} + Y + \overline{Z})(\overline{X} + \overline{Y} + Z)$$

Converting SoP to PoS

- Complement, multiply through, then complement

$$F = \overline{Y}Z + X\overline{Y}Z + XY\overline{Z}$$

$$\overline{F} = (Y + Z)(\overline{X} + Y + \overline{Z})(\overline{X} + \overline{Y} + Z)$$

$$\overline{F} = \overline{X}Y + YZ + \overline{X}Z$$

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$$F = (X + \overline{Y})(\overline{Y} + \overline{Z})(X + \overline{Z})$$

Algebraic Optimization

- Simply use the rules of Boolean logic

$$\overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$$

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$$\overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$$

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$$BC + A\overline{B}C + AB\overline{C}$$

Algebraic Optimization

- Simply use the rules of Boolean logic

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$$BC + A\overline{B}C + AB\overline{C}$$

$$B(C + A\overline{C}) + A\overline{B}C$$

Algebraic Optimization

- Simply use the rules of Boolean logic

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Algebraic Optimization

- Simply use the rules of Boolean logic

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$$B(C + A\overline{C}) + A\overline{B}C$$

$$B(C + A) + A\overline{B}C$$

$$AB + BC + A\overline{B}C$$

Algebraic Optimization

- Simply use the rules of Boolean logic

$$AB + BC + A\overline{B}C$$

Algebraic Optimization

- Simply use the rules of Boolean logic

$$AB + BC + A\bar{B}C$$

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Algebraic Optimization

- Simply use the rules of Boolean logic

$$AB + BC + A\bar{B}C$$

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$$BC + A(B + C)$$

Algebraic Optimization

- Simply use the rules of Boolean logic

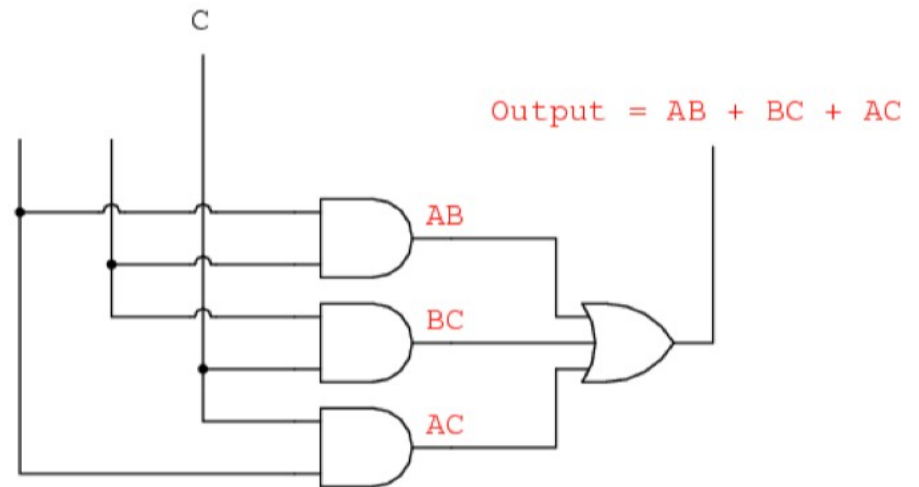
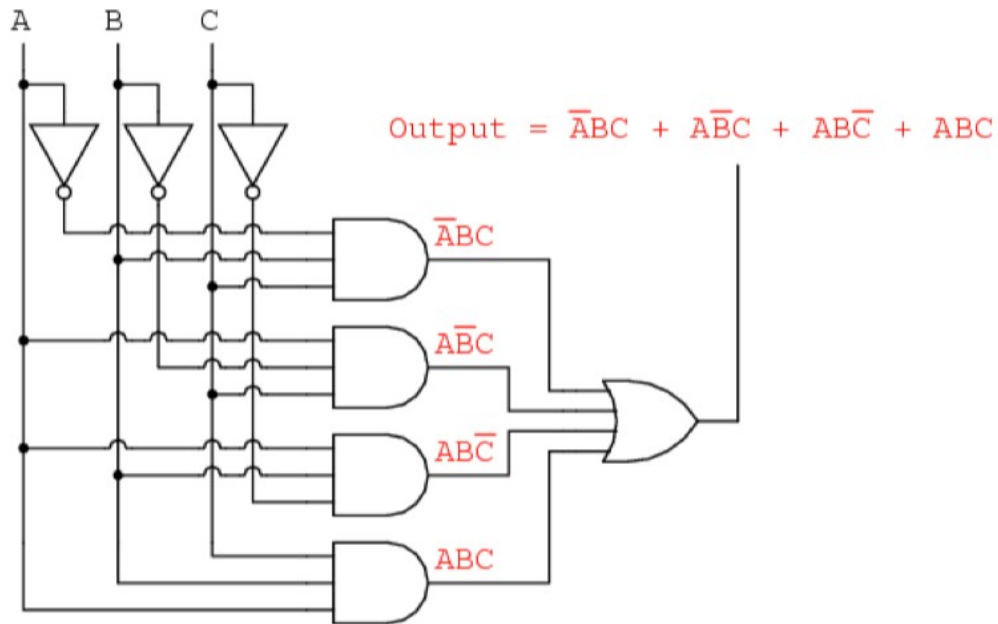
$$AB + BC + A\bar{B}C$$

$$BC + A(B + \bar{B}C)$$

$$BC + A(B + C)$$

$$AB + BC + AC$$

Algebraic Optimization



Question

- Simplify the following expression:

$$\overline{A}BC + \overline{A}BC + ABC + ABC$$

- A: A
- B: B
- C: C
- D: AC + BC
- E: A + C

Question

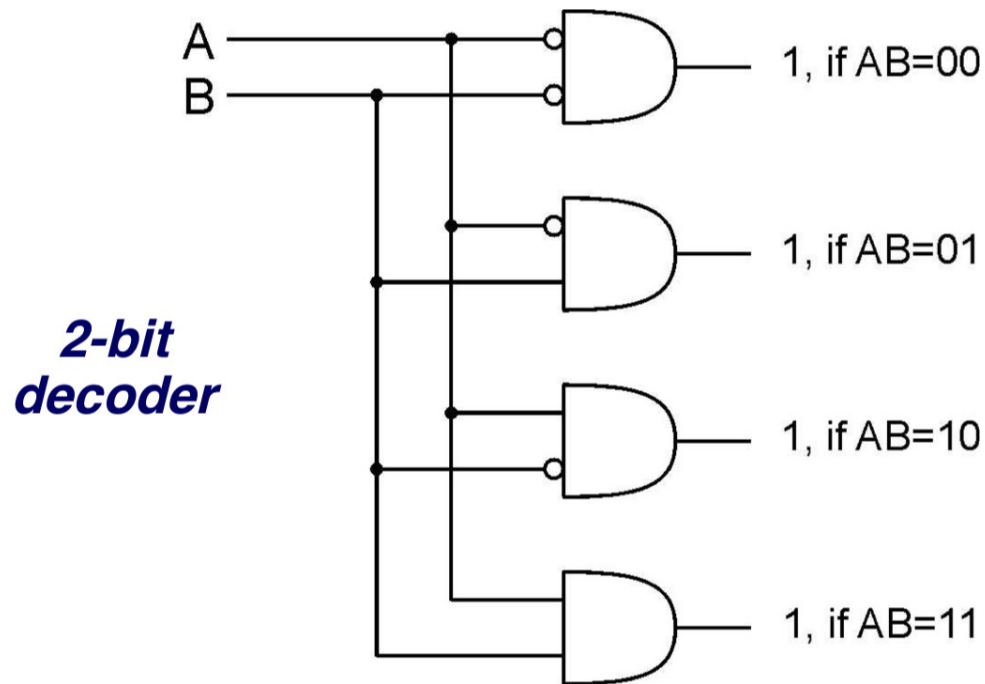
- Simplify the following expression:

$$\overline{A}BC + \overline{A}BC + ABC + ABC$$

- A: A
- B: B
- C: C
- D: AC + BC
- E: A + C

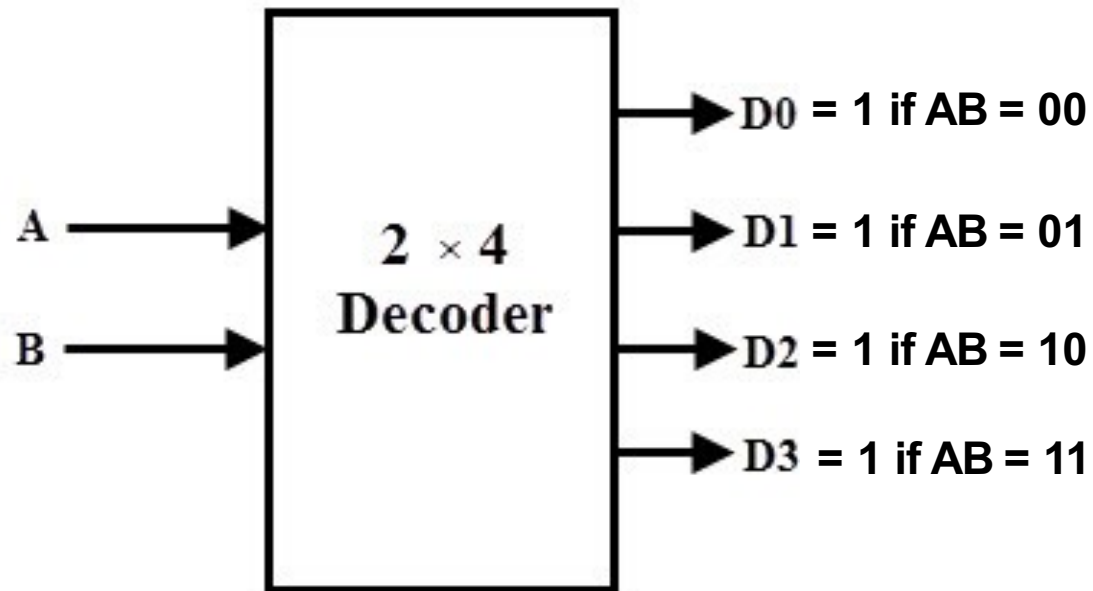
Decoder

- n inputs, 2^n outputs
- Exactly one output is 1 for a single possible input pattern



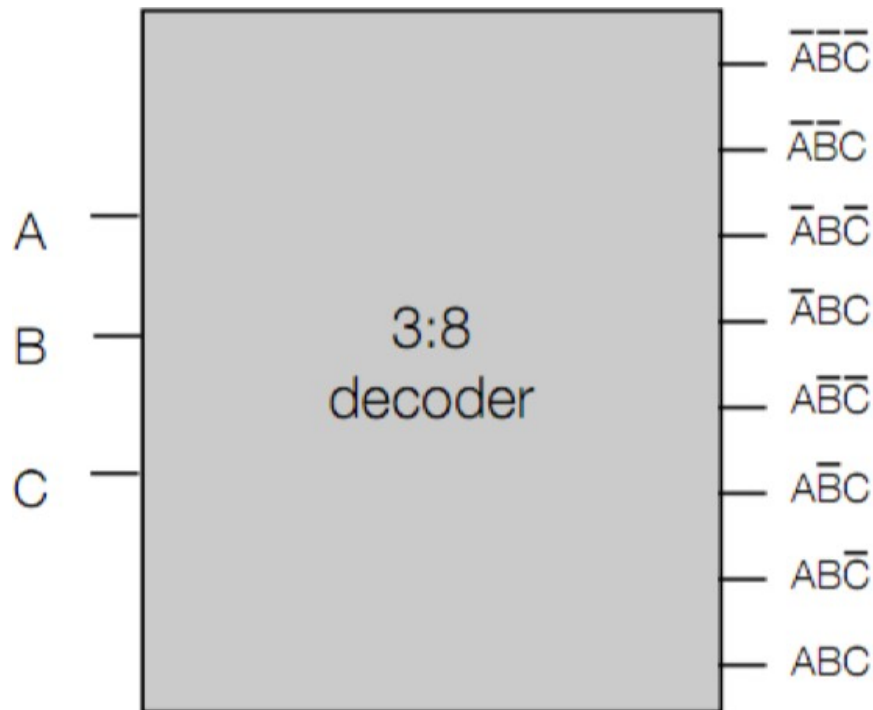
Decoder circuit

- n inputs, 2^n outputs
- Exactly one output is 1 for a single possible input pattern



Decoder Circuit

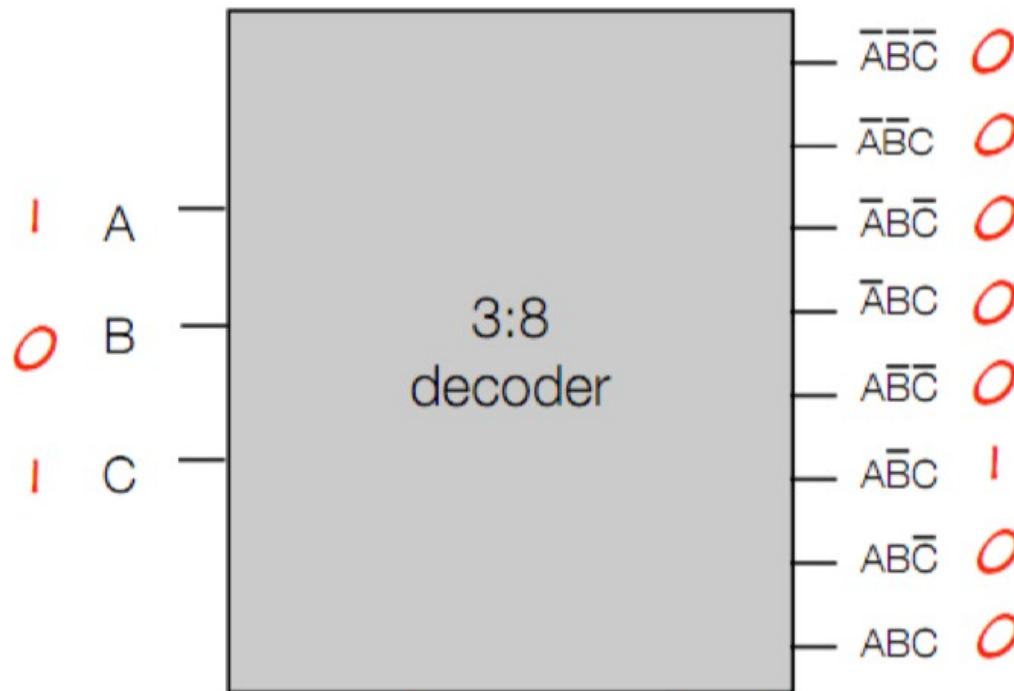
- Converts n -bit input to 2^n bit output



*"Standard" Decoder: i^{th} output = 1, all others = 0,
where i is the binary representation of the input (ABC)*

Decoder Circuit

- Converts n -bit input to 2^n bit output



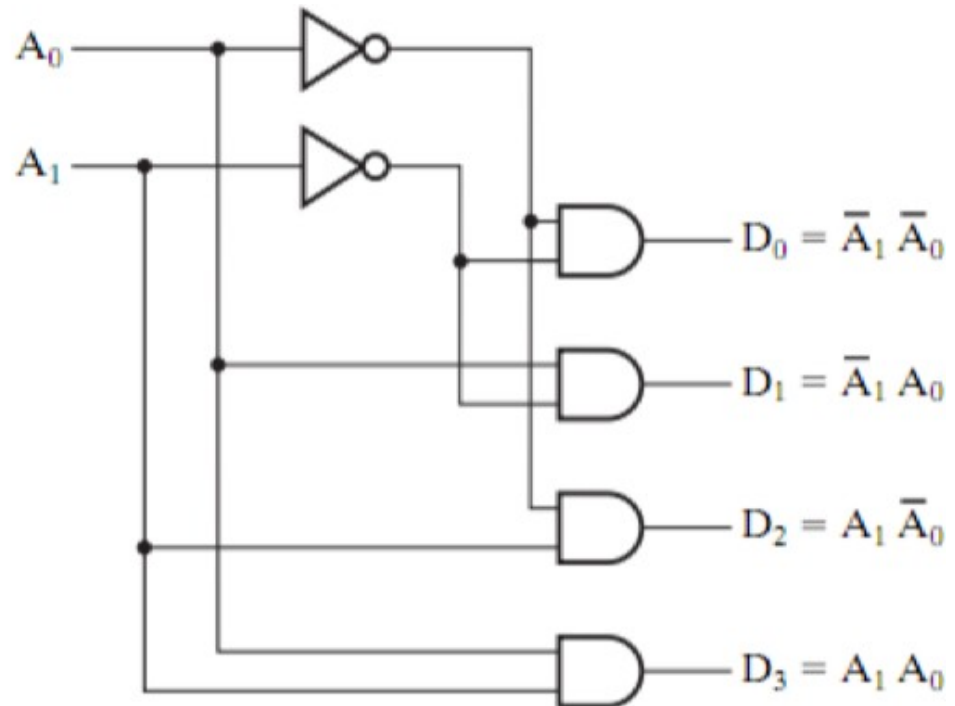
e.g., $ABC = 101$ ($i=5$)

"Standard" Decoder: i^{th} output = 1, all others = 0,
where i is the binary representation of the input (ABC)

Internal of 2:4 Decoder

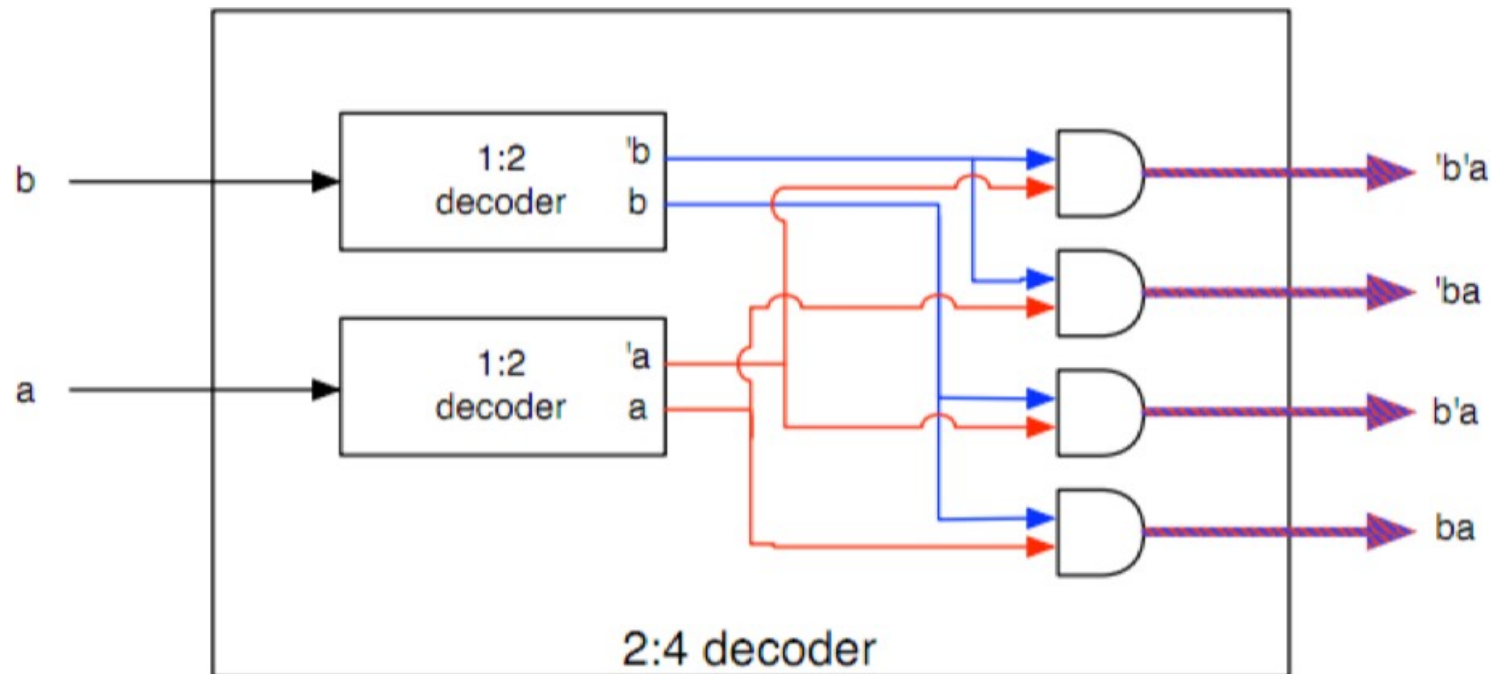
A_1	A_0	D_0	D_1	D_2	D_3
0	0	1	0	0	0
0	1	0	1	0	0
1	0	0	0	1	0
1	1	0	0	0	1

(a)

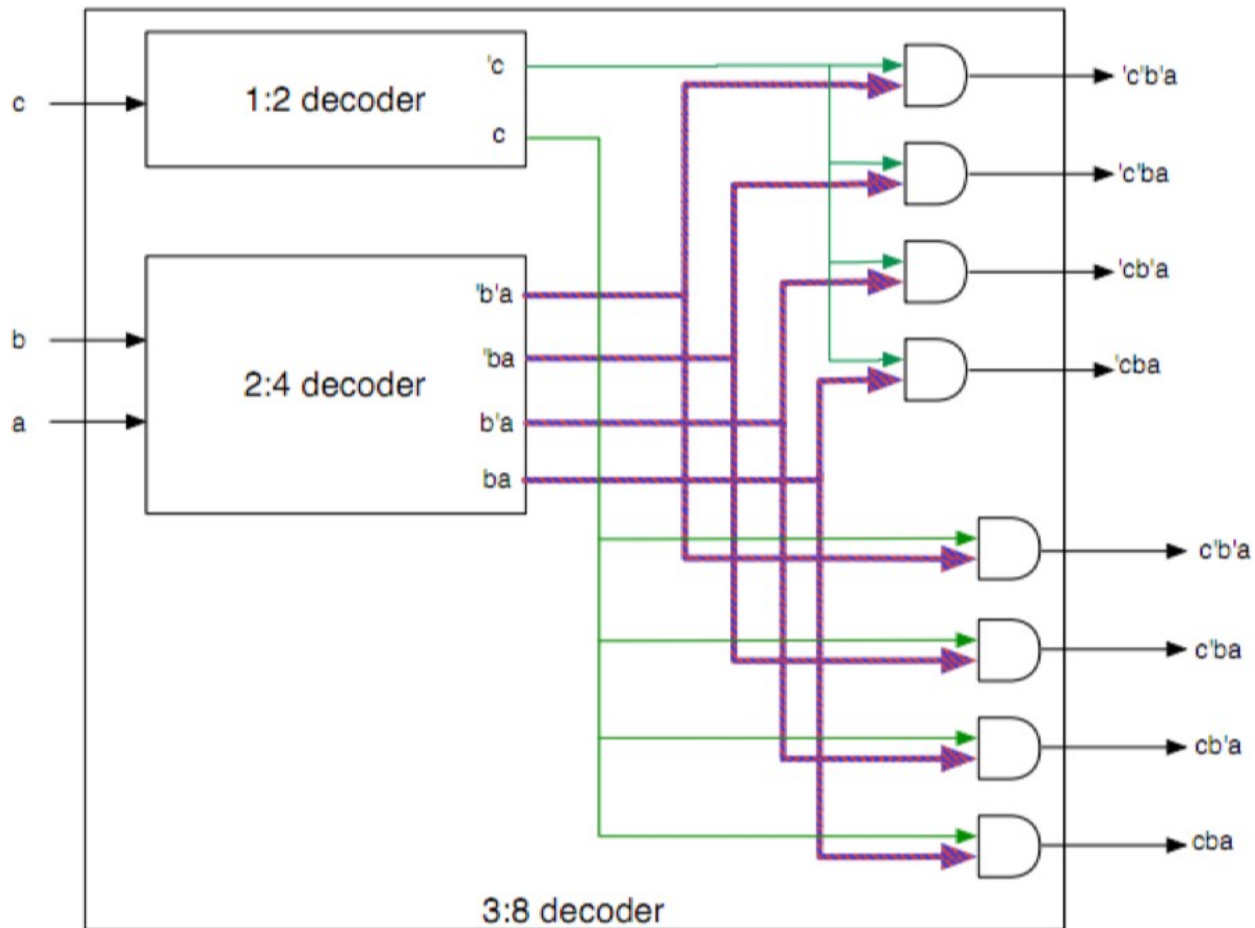


(b)

2:4 Decoder from 1:2 Decoders



3:8 Decoder from Smaller Decoders



Encoder

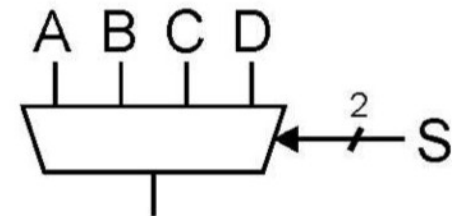
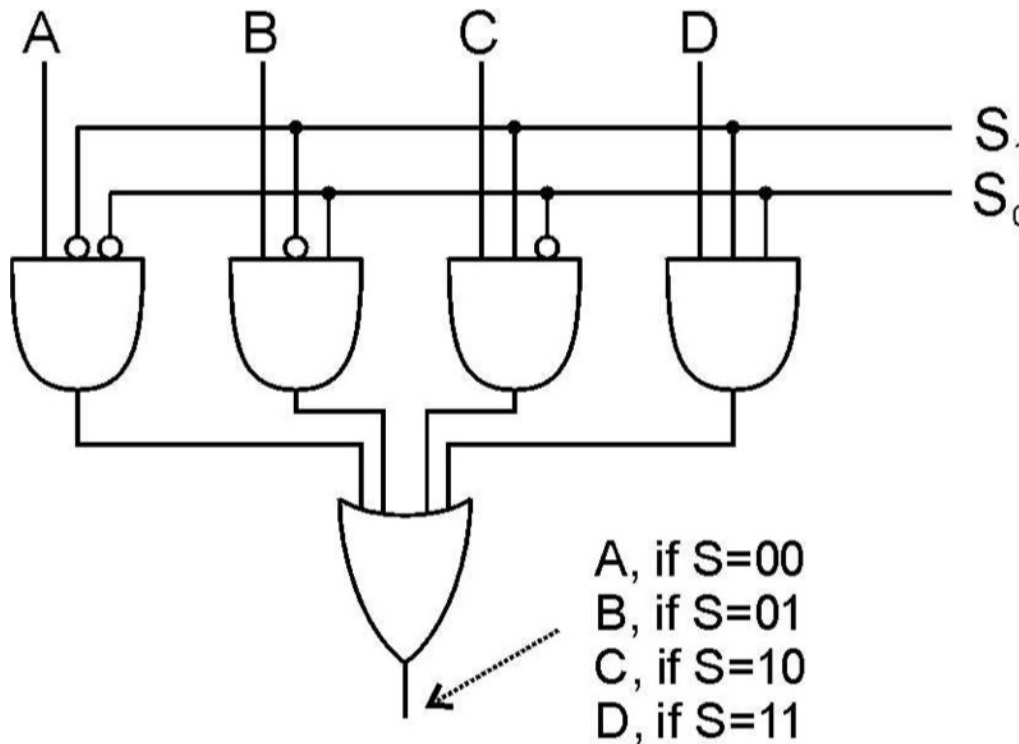
- Inverse of decoder:

TABLE 3-7
Truth Table for Octal-to-Binary Encoder

[illegible]

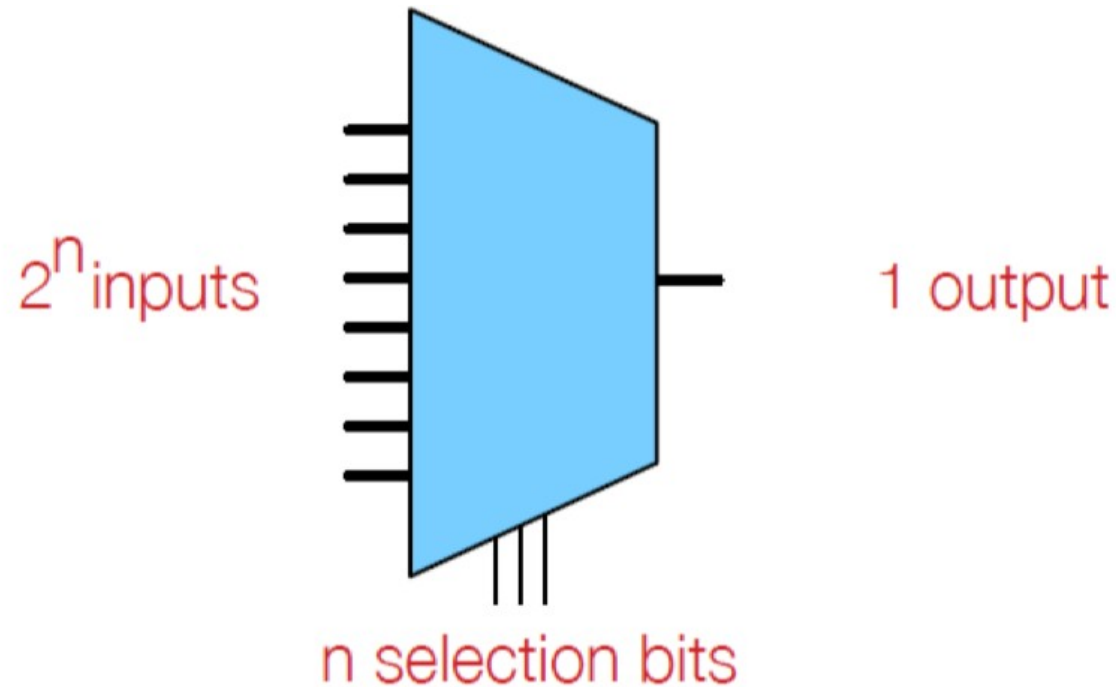
Multiplexer (MUX)

- 2^n inputs, n -bit selector, one output
- Output equals one of the inputs, depending on the selector



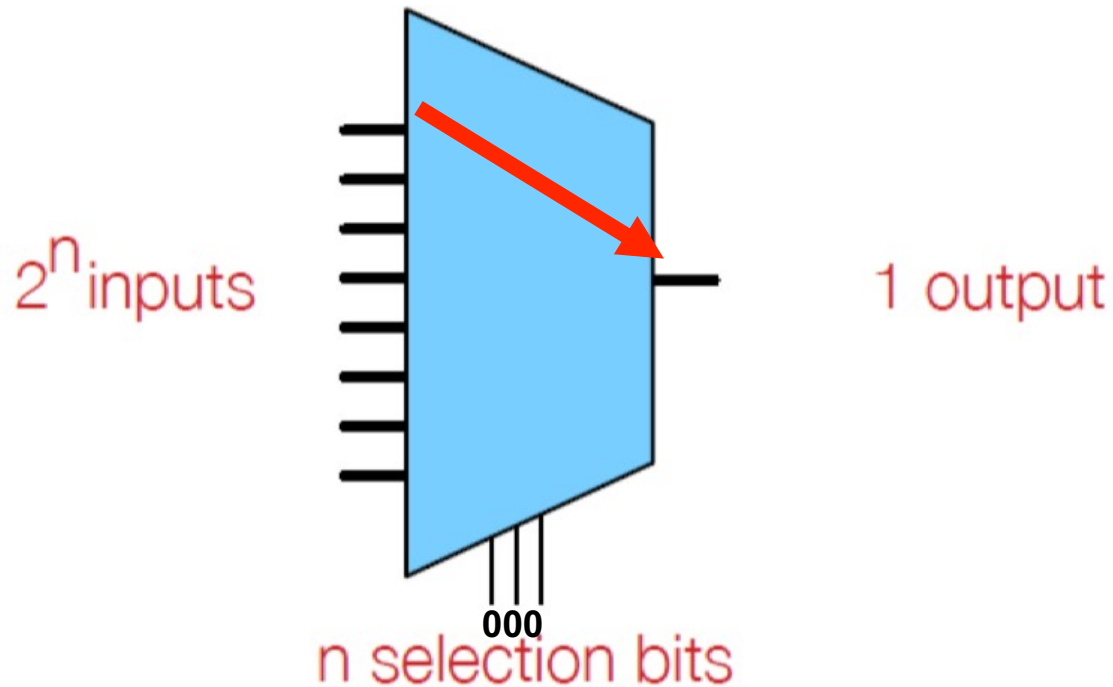
4-to-1 MUX

Multiplexer (MUX) Circuit



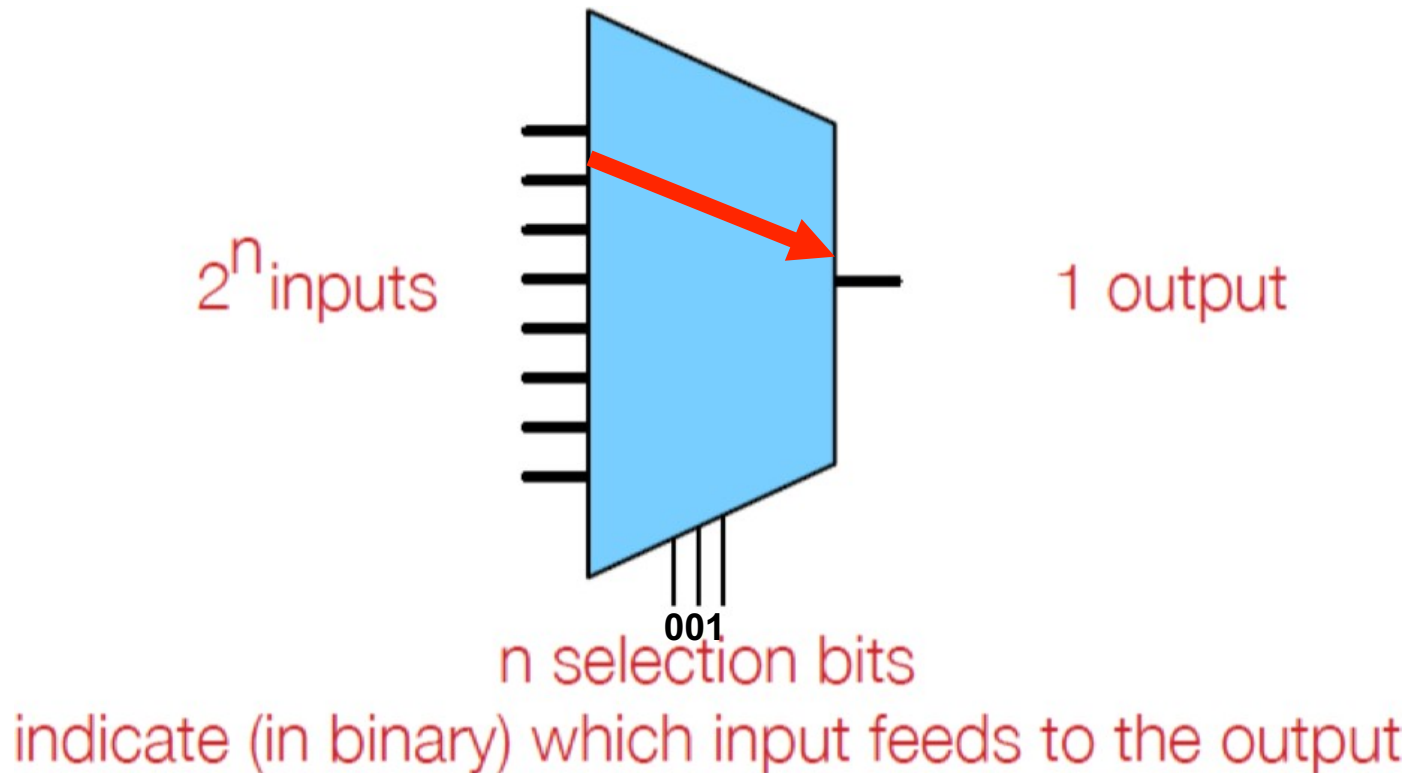
indicate (in binary) which input feeds to the output

Multiplexer (MUX) Circuit

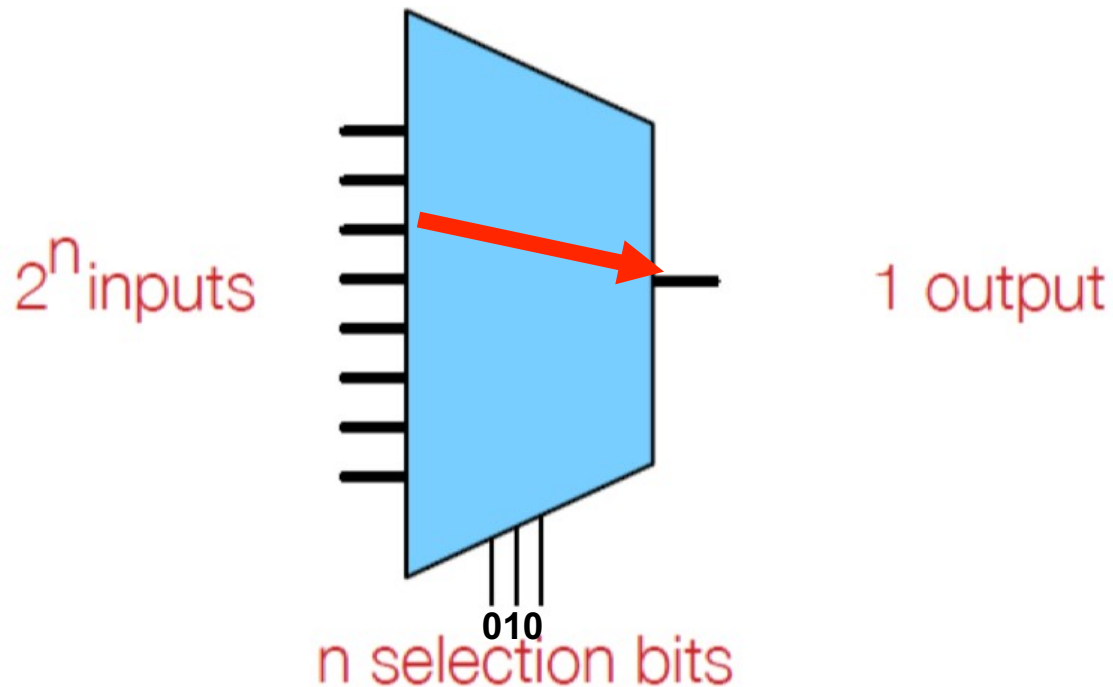


indicate (in binary) which input feeds to the output

Multiplexer (MUX) Circuit



Multiplexer (MUX) Circuit



indicate (in binary) which input feeds to the output

Functions with Decoders and Multiplexers

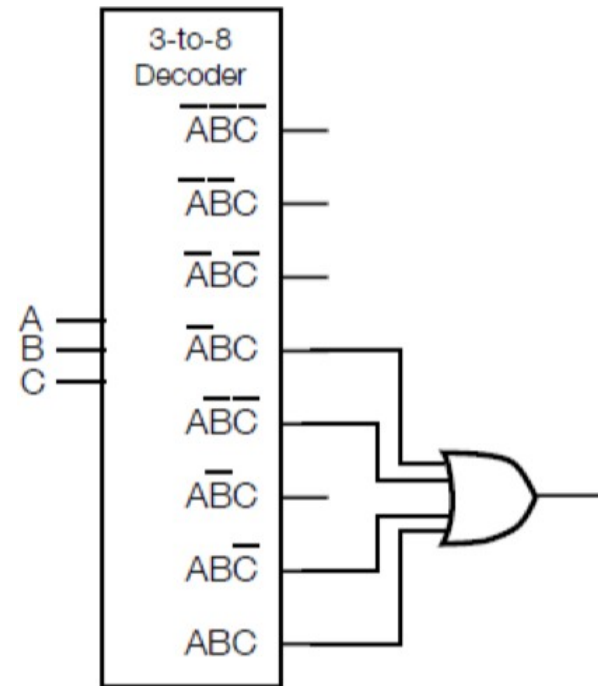
- e.g., $F = A\bar{C} + BC$

A	B	C	minterm	F
0	0	0	$\bar{A}\bar{B}\bar{C}$	0
0	0	1	$\bar{A}\bar{B}C$	0
0	1	0	$\bar{A}B\bar{C}$	0
0	1	1	$\bar{A}BC$	1
1	0	0	$A\bar{B}\bar{C}$	1
1	0	1	$A\bar{B}C$	0
1	1	0	$AB\bar{C}$	1
1	1	1	ABC	1

Functions with Decoders and Multiplexers

- e.g., $F = A\bar{C} + BC$

A	B	C	minterm	F
0	0	0	$\bar{A}\bar{B}\bar{C}$	0
0	0	1	$\bar{A}\bar{B}C$	0
0	1	0	$\bar{A}B\bar{C}$	0
0	1	1	$\bar{A}BC$	1
1	0	0	$A\bar{B}\bar{C}$	1
1	0	1	$A\bar{B}C$	0
1	1	0	$AB\bar{C}$	1
1	1	1	ABC	1

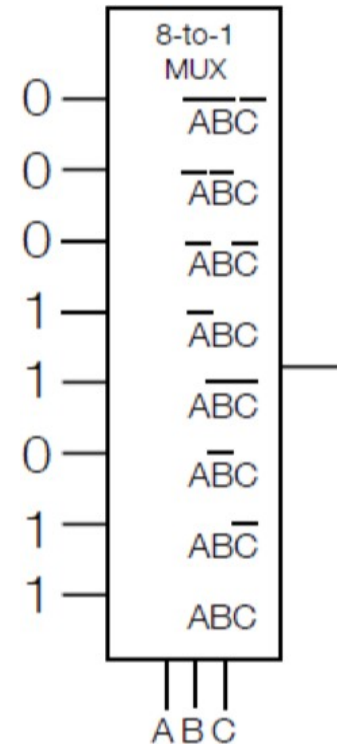


- OR minterms for which F should evaluate to 1

Functions with Decoders and Multiplexers

- e.g., $F = A\bar{C} + BC$

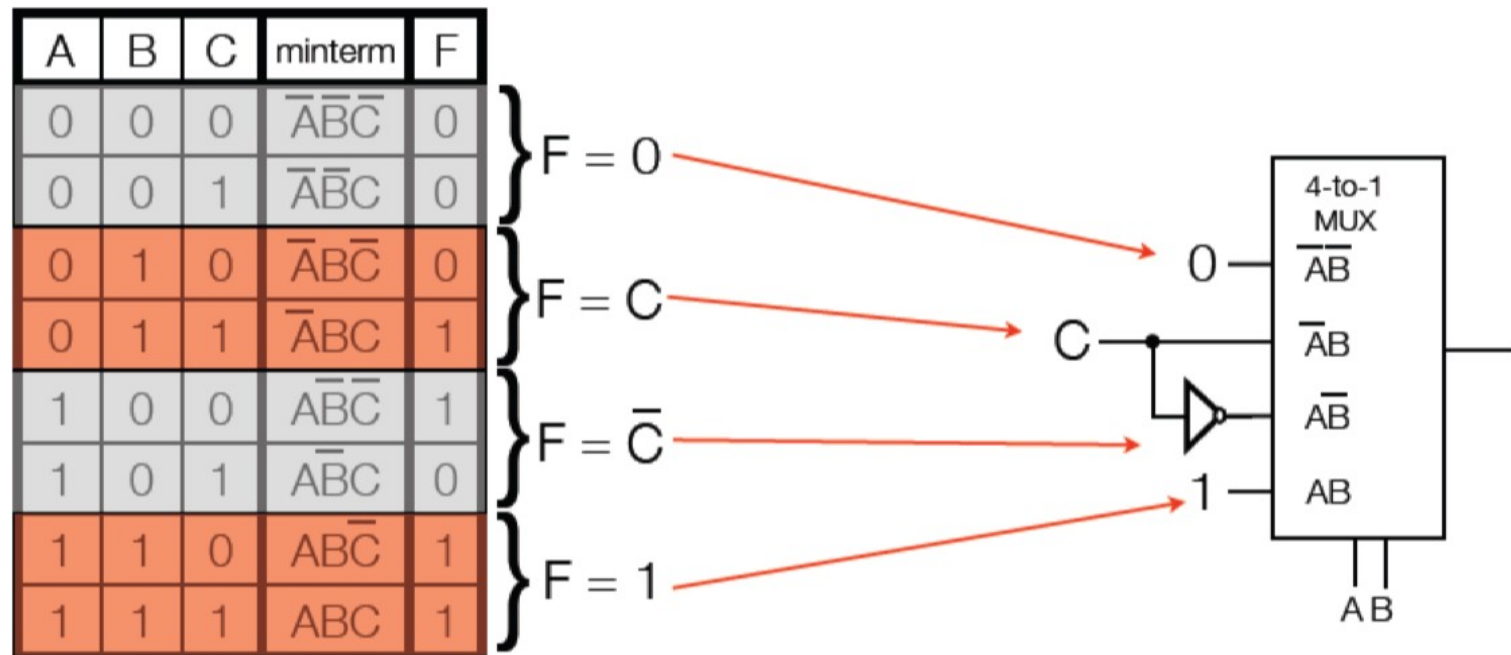
A	B	C	minterm	F
0	0	0	$\bar{A}\bar{B}\bar{C}$	0
0	0	1	$\bar{A}\bar{B}C$	0
0	1	0	$\bar{A}B\bar{C}$	0
0	1	1	$\bar{A}BC$	1
1	0	0	$A\bar{B}\bar{C}$	1
1	0	1	$A\bar{B}C$	0
1	1	0	$AB\bar{C}$	1
1	1	1	ABC	1



- Feed the value of F for each minterm in the input

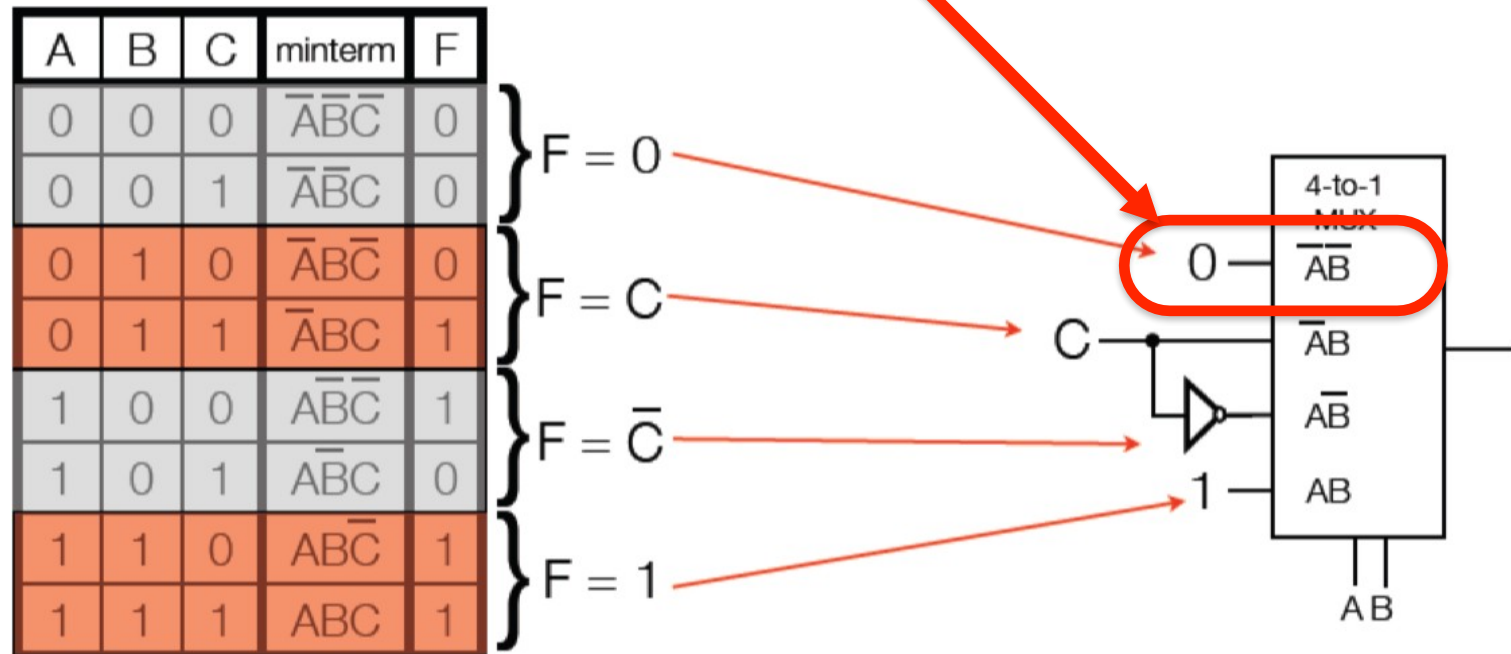
Using Smaller mux:

- We can use 4-to-1 mux with a trick:
- Every two rows have same A and B value. The output F depends on the value C.



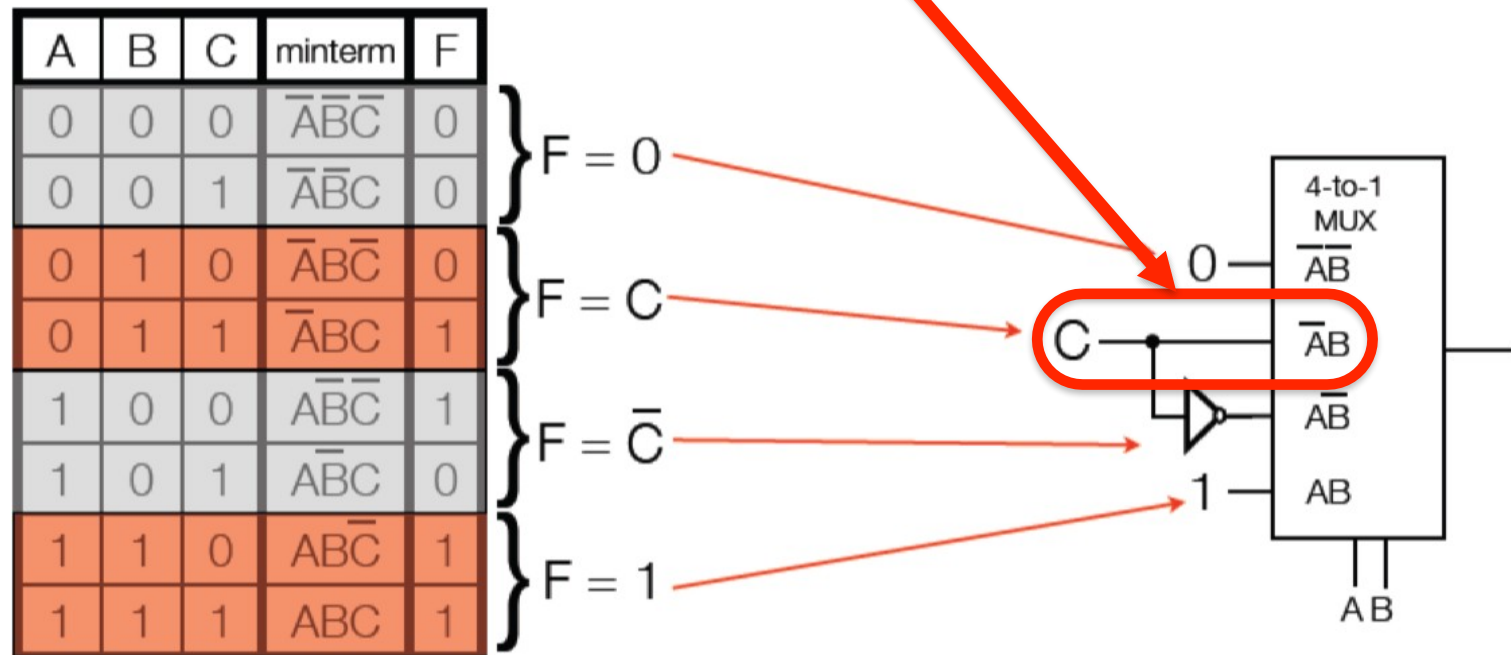
Using Smaller mux:

- We can use 4-to-1 mux with a trick:
- Every two rows have same A and B value. The output F depends on the value of C. If $\overline{A}B = 00$, then $F = 0$



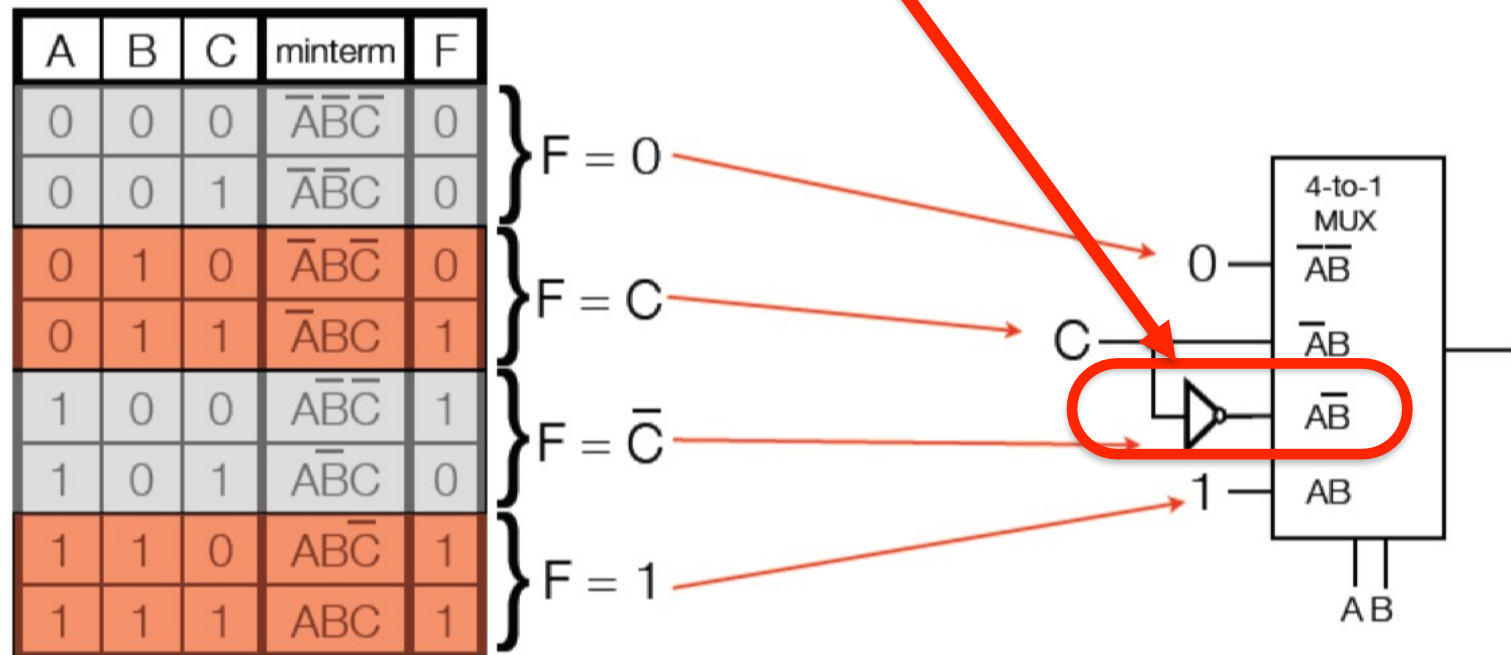
Using Smaller mux:

- We can use 4-to-1 mux with a trick:
- Every two rows have same A and B value. The output F depends on the value of C. If $AB = 01$, then $F = C$



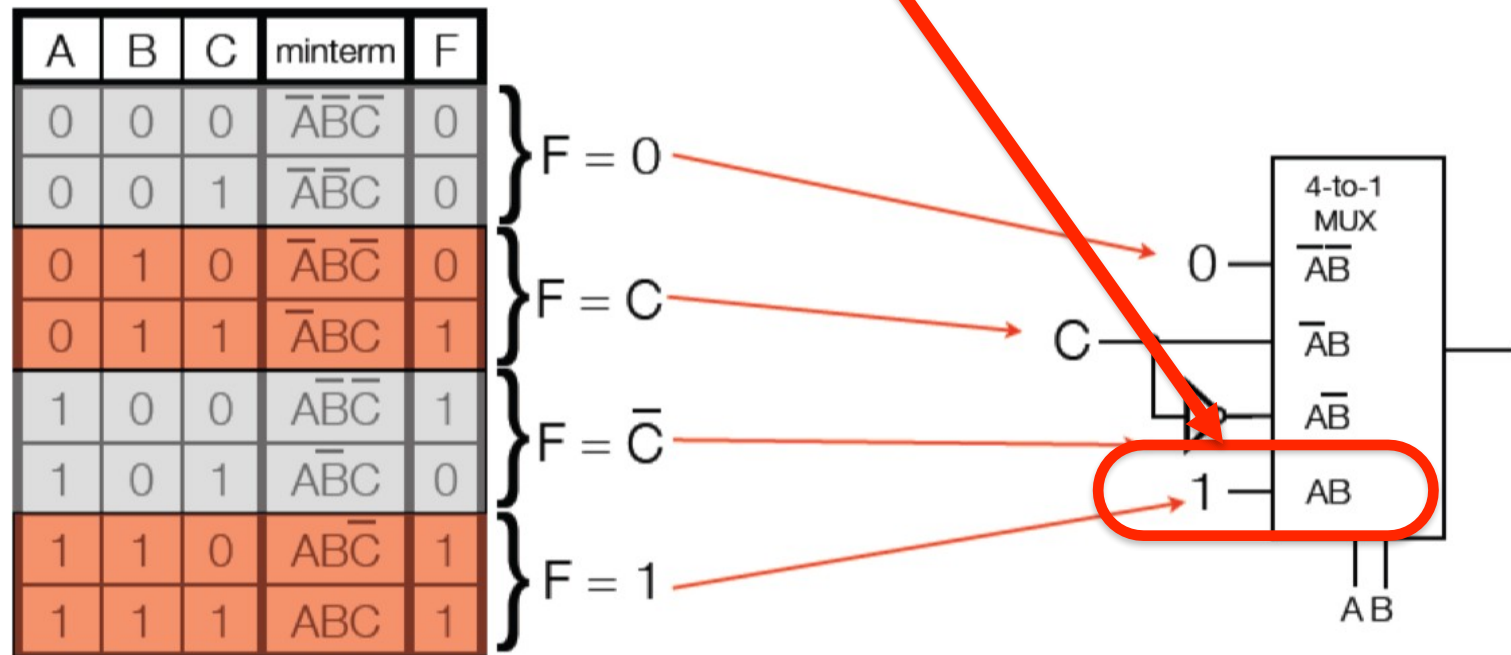
Using Smaller mux:

- We can use 4-to-1 mux with a trick:
- Every two rows have same A and B value. The output F depends on the value of C. If $AB = 10$, then $F = \bar{C}$



Using Smaller mux:

- We can use 4-to-1 mux with a trick:
- Every two rows have same A and B value. The output F depends on the value of C. If AB = 11, then F = 1



Another Example

$$F = \bar{A}C + \bar{B}\bar{C} + A\bar{C}$$

