Two views on the flow conservation law

1 PROOF

Let $G = (V, E)$ ne a flow network, and f a flow.
e is the source, s is the sink.
For any node $u \notin \{e, s\}$, the first form of the flow conservation law writes :

$$\sum_{v \in V} f(u, v) = 0 \tag{1}$$

We then separate the sum in three parts :

$$\sum_{v \in V} f(u, v) = \sum_{v \in V, f(u,v) > 0} f(u, v) + \sum_{v \in V, f(u,v) < 0} f(u, v) + \sum_{v \in V, f(u,v) = 0} f(u, v) \tag{2}$$

But we have that :

$$\sum_{v \in V, f(u,v) = 0} f(u, v) = 0 \tag{3}$$

Hence,

$$\sum_{v \in V} f(u, v) = \sum_{v \in V, f(u,v) > 0} f(u, v) + \sum_{v \in V, f(u,v) < 0} f(u, v) \tag{4}$$

But using 1 :

$$\sum_{v \in V, f(u,v) > 0} f(u, v) + \sum_{v \in V, f(u,v) < 0} f(u, v) = 0 \tag{5}$$

Or :

$$\sum_{v \in V, f(u,v) > 0} f(u, v) = - \sum_{v \in V, f(u,v) < 0} f(u, v) \tag{6}$$

But with the flow **antisymmetry**, since $f(u, v) = -f(v, u)$:

$$- \sum_{v \in V, f(u,v) < 0} f(u, v) = \sum_{v \in V, f(u,v) < 0} f(v, u) \tag{7}$$

What's more, $f(u, v) < 0$ if and only if $f(v, u) > 0$.
Hence :

$$\sum_{v \in V, f(u,v) < 0} f(v, u) = \sum_{v \in V, f(v,u) > 0} f(v, u) \tag{8}$$

Finally :

$$\boxed{\sum_{f(u,v) > 0} f(u, v) = \sum_{f(v,u) > 0} f(v, u)} \tag{9}$$