

Numerical solution of PDEs - NSPDE

Advanced Numerical Analysis - ANA

LECTURE 7

Last Lecture:

- Weak formulations:
 - different boundary conditions
 - general elliptic differential operators
 - well-posedness by Lax-Milgram Lemma
- The method of Galerkin

Today's lecture:

- Cea's lemma
- The Finite Element Method
- Error analysis for two-points BVPs
- FEM in higher-dimensions
- Conditioning of the FEM system

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quarteroni

Computer practical

- FD in two dimensions

Method Galerkin analysis

Given • $(V, (\cdot, \cdot))$ Hilbert

• $\mathcal{B} : V \times V \rightarrow \mathbb{R}$

• $F \in V'$

{ bilinear
continuous
coercive }
(s)
(d)

Then, Lax-Milgram, there exists a unique

$u \in V : \quad \mathcal{A}(u, v) = F(v) \quad \forall v \in V \quad (\text{WP})$

Galerkin method :

- Consider sequence $\{V_n\}_n$ of subspaces of V
- Restrict (WP) in V_n :
find $u_n \in V_n : \quad \mathcal{A}(u_n, v_n) = F(v_n) \quad \forall v_n \in V_n$
(GM)

analysis :

by Lax-Milgram, $\exists! u_n \in V_n$ sol of (GM) and
a priori bound holds :

$$\|u_n\|_V \leq \frac{1}{\lambda} \|F\|_V, \quad (\text{STABILITY})$$

Also, we have Galerkin orthogonality :

$$\mathcal{A}(u - u_n, v) = 0 \quad \forall v \in V_n \quad (\text{FULL CONSISTENCY})$$

Convergence :

Lemma of Céa: Under assumptions of LM

$$\|u - u_n\|_V \leq \frac{\gamma_d}{d} \inf_{v_n \in V_n} \|u - v_n\|$$

(quasi-optimality
property)

Proof: apply Galerkin orthog with $v = v_n - u_n$

for any $v_n \in V_n$

$$= \mathcal{A}(u - u_n, v_n - u_n) = 0$$

$$\begin{aligned} &= \mathcal{A}(u - u_n, v_n - u) + \mathcal{A}(u - u_n, u - u_n) \\ &= \mathcal{A}(u - u_n, v_n - u) + \mathcal{A}(u - u_n, u - u_n) \end{aligned}$$

$$\Rightarrow \begin{aligned} &\geq \mathcal{A}(u - u_n, u - v_n) = \mathcal{A}(u - u_n, u - u_n) \geq \\ &\quad | \text{ continuity of } \mathcal{A} \quad \text{coercivity of } \mathcal{A} \\ &\leq \gamma \|u - u_n\| \|u - v_n\| \quad \& \|u - u_n\|^2 \leq \end{aligned}$$

$$\Rightarrow \|u - u_n\| \leq \frac{\gamma}{\lambda} \|u - v_n\| \quad \forall v_n \in V_n \quad \square$$

Convergence now follows, for instance, if

- Fix $\{\varphi_j\}_{j=1}^{\infty}$ complete orthonormal system for V

- Define $V_n = \text{span } \{\varphi_1, \dots, \varphi_n\}$

- Then, or $\lim_{n \rightarrow \infty} V_n$ is dense in V

We have $\forall u \in V$, $\forall \varepsilon > 0 \exists n:$

$$\exists w_n \in V_n : \|u - w_n\| \leq \varepsilon$$

This + (a) \Rightarrow convergence

or $\inf_{w_n \in V_n} \|u - w_n\| \rightarrow 0$

FEN, spectral, etc. are instances
of practical choices of subspaces!

Finite Element Method (FEM)

FEN for two-points BVP.

model problem: reaction-diffusion

$$\begin{cases} \Delta u = -(\alpha u')' + c u = f & \Omega = (0, 1) \\ u(0) = 0 = u(1) \end{cases}$$

$$\alpha(x) \geq \alpha_0 > 0, \quad c \geq 0, \quad f \in L^2(0, 1)$$

Weak-formulation: $u \in H_0^1(\Omega) := V$

$$A(u, v) := (\alpha u', v') + (c u, v) = (f, v) \quad \forall v \in V$$
$$= \int_0^1 \alpha u' v' \quad = F(v)$$

Since prob. is symmetric have more refine analysis (general)

Proposition (the Symmetric case):

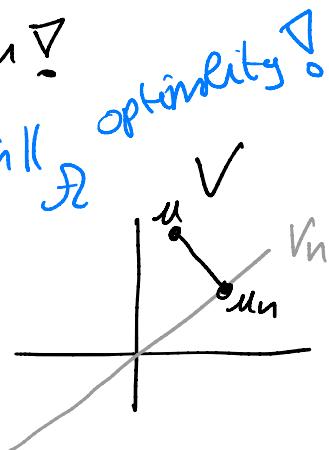
Order less. of (e), if \mathcal{R} is symmetric, then

$$\|u - u_n\|_{\mathcal{V}} \leq \sqrt{\delta_2} \min_{v_n \in V_h} \|u - v_n\|_{\mathcal{V}}$$

\nearrow better constant!

$$\|u - u_n\|_{\mathcal{R}} = \min_{v_n \in V_h} \|u - v_n\|_{\mathcal{R}}$$

where $\|w\|_{\mathcal{R}} = \mathcal{R}(w, w)^{1/2}$



Proof: exercise

ooo back to model problem and FEM

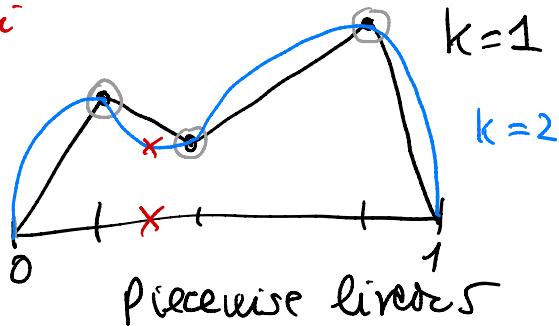
- partition $\Omega = (0, 1)$ by $\begin{cases} x_0 = 0 \\ x_i = x_{i-1} + h_i \\ i = 1, \dots, N \end{cases}$

and let $h = \max_i h_i$

Define the space of Finite Elements or

$$V_h = \{v \in C_0(\bar{\Omega}) : v|_{I_i} \in P^k(I_i), \forall i=1, \dots, n\}$$

$$I_i = (x_{i-1}, x_i)$$



FEM (k): Find $u_h \in V_h$: $\mathcal{A}(u_h, v_h) = F(v_h)$
for $v_h \in V_h$

Here $P^k(\omega)$, $\omega \in \mathbb{R}^d$

Space of polynomials of degree $\leq d$

algebraic formulation of FEM:

• fix a basis $\{\varphi_m\}_{m=1}^{\dim V_h}$

• write $u_h = \sum_m v_m \varphi_m$

• test with basis:

$$\sum_m U_m \left[\underbrace{(\alpha \varphi_m^1, \varphi_\ell^1)}_{S_{\ell m}} + \underbrace{(\gamma \varphi_m, \varphi_\ell)}_{M_{\ell m}} \right] = (f, \varphi_\ell) \quad \forall \ell = 1, \dots, \dim V_h$$

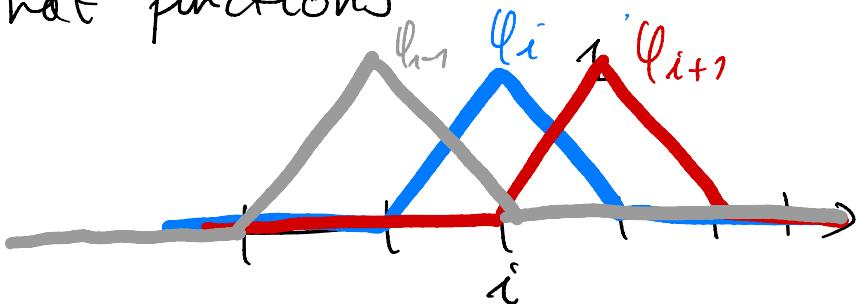
$$U = \{U_m\}_{m=1}^{\dim V_h}, F = \{F_\ell\}, F_\ell = (f, \varphi_\ell)$$

$$\Rightarrow \begin{cases} A U = F & \leftarrow \text{load vector} \\ A = S + M & \begin{matrix} \leftarrow \text{stiffness} \\ \leftarrow \text{mass} \end{matrix} \\ & \text{matrix} \end{cases}$$

Crucial issue: practical

choice of the FE basis ??

$k=1$: hat functions



- fix nodal values or degrees of freedom (DOF) and then the corresponding Lagrange basis $\{\phi_i\}$:

$$\phi_i(x_j) = \delta_{ij} \quad \text{the hat functions}$$

- note $\text{supp } \phi_i \cap \text{supp } \phi_j = \emptyset$

$$\text{if } |i-j| > 1$$

$\Rightarrow A$ is sparse !

Practical implementation

- record implementations of the basis functions
- loop over "elements" (that is, the intervals composing mesh) to evaluate local contributions to the integrals defining FEM

For instance

$$\begin{aligned} S_{lm} &= \int_0^1 \alpha(x) \varphi_m(x) \varphi_l(x) \\ &= \sum_i \int_{I_i} \alpha(x) \varphi_m(x) \varphi_l(x) \end{aligned}$$

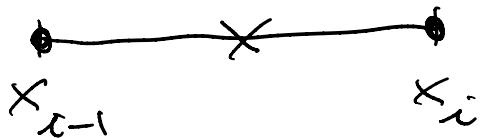
such such integral is in general evaluated by quadrature?

$$\rightarrow x \sum_i \sum_j w_j \alpha(x_j) \varphi_m(x_j) \varphi_l(x_j)$$

(w_j, x_j) a given quadrature formula

Second example: $k=2$

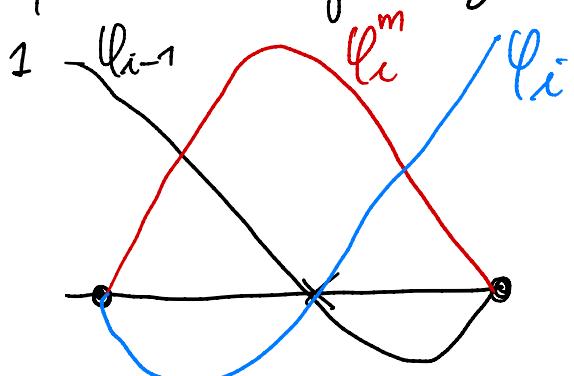
- on each interval, use $P_2(I_i)$



three DOF: end points x_{i-1}, x_i

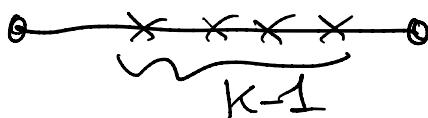
$$\text{mid } x_i^m := \frac{x_{i-1} + x_i}{2}$$

- define Lagrange basis



$$\dim V_h = V_h^2 = (H-1) + N$$

In general $V_h^k = (N-1) + N(k-1)$



- dim and stencil grows but higher order polynomials give higher order methods ?

A priori analysis of FE

Recall general result for Galerkin:

by Cea, we have

$$\|u - u_h\|_V \leq \sqrt{\delta_2} \inf_{v_h \in V_h} \|u - v_h\|_V$$

with $V = H_0^1(0, 1)$

$$\|v\|_V = \left(\int_0^1 (v')^2 dx \right)^{1/2}$$

H_0^1 -norm

$$\text{or} \quad = \left(\int_0^1 (v')^2 dx + \int_0^1 v^2 dx \right)$$

H^1 norm

and also

$$\|u - u_h\|_H = \min_{v_h \in V_h} \|u - v_h\|_H$$

$$\|w\|_H = \sqrt{(w, w)}^{1/2} = \left(\int_0^1 \alpha(w)^2 + c w^2 \right)^{1/2}$$

Issue: characterize $\inf_{v_h \in V_h} \|u - v_h\|_V$

$\forall u \in V$, in terms of h, k .

Clearly, $\inf_{v_h \in V_h} \|u - v_h\|_V \leq \|u - I_h u\|_V$

sore "special" element from V_h

$\forall u \in V$.

Note, can write

$$\|u - I_n u\|_V^2 = \sum_{i=1}^n \|u - I_n u\|_{I_i}^2$$

• recall, $\forall I_i \quad v_h|_{I_i} \in P_k(I_i)$

\Rightarrow local approx by polynomials
question

Def (FEM interpolant): Define on each I_i the FE interpolant

by setting $I_n u|_{I_i} \in P^k(I_i)$

such that $I_n u|_{I_i}(x_j) = u(x_j)$

$\forall x_j$ "node" within I_i ↴ endpt of h

Theorem (Q, LT) $\|v - I_n v\|_{L^2(I_i)} \leq C_I h_i^{k+1} |v|_{H^{k+1}(I_i)}$

local
poly
interp

$$\|v - I_n v\|_{L^2(I_i)} \leq C_I h_i^k |v|_{H^{k+1}(I_i)} \quad \forall v \in H^{k+1}(I_i)$$

\Rightarrow global bound follows:

$$\|v - I_h v'\|_{L^2(\Omega)}^2 = \sum_{i=1}^N \|v - I_h v'\|_{L^2(I_i)}^2$$

$$\leq C_I \sum_{i=1}^N h_i^{2k} |v|_{H^{k+1}(I_i)}^2$$

if $v \in H^{k+1}(I_i)$

also,

$$\leq C_I h^{2k} |v|_{H^{k+1}(\Omega)}^2$$

if $v \in H^{k+1}(\Omega)$

Theorem (convergence and rates) :

$$\|u - u_h'\|_{L^2(\Omega)} \leq C h^k |u|_{H^{k+1}(\Omega)}$$

$\exists C > 0$ indep. of h .

if $u \in H^{k+1}(\Omega)$.

$$(\text{obs}_h) \leq C \left(\left(\sum_i h_i^k |u|_{H^{k+1}(I_i)}^2 \right)^{1/2} \right)$$

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{k+1} |u|_{H^{k+1}(\Omega)}$$

$$(\text{obs}_h) \leq ch \left(\sum_i h_i^k |u|_{H^{k+1}(I_i)}^2 \right)^{1/2}$$

↑ global h !

If $u \in H^s(\Omega)$ $s \leq k+1$

then $\|u - u_h\|_{L^2(\Omega)} \leq ch^{s-1} |u|_{H^s(\Omega)}$

again, with C independent of h .
 (bounds explicit in k are also available)