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Offset-free MPC explained: novelties, subtleties, and applications

Gabriele Pannocchia *,** **Marco Gabiccini** * **Alessio Artoni** *

* *Department of Civil and Industrial Engineering, University of Pisa, Italy*
** *Corresponding author, e-mail: gabriele.pannocchia@unipi.it*

Abstract: This paper presents an updated and comprehensive description of offset-free MPC algorithms for nonlinear (and linear) discrete-time systems, with the intended objectives of clarifying the main concepts, showing new results, highlighting subtleties by means of challenging applications. First, the offset-free tracking problem for nonlinear systems is presented, putting a strong accent on the role of the disturbance model and observer, and then novel and stronger offset-free estimation results are presented. Next, recent advances in linear offset-free MPC are described, which show the equivalence of the velocity form algorithm (so far considered an alternative method) to a particular disturbance model and observer. Then, the concepts of offset-free estimation are exploited to design an offset-free economic MPC algorithm, which can asymptotically achieve the highest economic performance despite persistent model errors and disturbances. Extensive application results are presented to show the benefits of offset-free MPC algorithms over standard ones, and to clarify misconceptions and design errors that can prevent constraint satisfaction, closed-loop stability, and offset-free performance.

Keywords: Nonlinear discrete-time systems, linear discrete-time systems, model predictive control, state and disturbance estimation.

1. INTRODUCTION

The role of feedback in the design of effective control systems is central. In traditional PID control, for instance, the error between reference and actual *measured* output is the *sole* variable that determines the computed control action. In Model Predictive Control (MPC), an optimal control problem is solved at each decision time (mainly) because a new state *measurement* (or estimate) becomes available. Feedback is necessary to reduce the effect of disturbances and to cope with unavoidable modeling errors. Nonetheless, the way in which feedback is used to achieve offset-free tracking in the presence of persistent errors or disturbances appears to be often a question of personal preference among possible different methods.

In conventional linear (PID) control design, the offset-free goal is achieved by integrating the tracking error. In the linear quadratic regulation (LQR) framework, the use of *disturbance observers* was proposed and discussed in (Davison and Smith, 1971; Kwakernaak and Sivan, 1972). In the context of MPC, several methods have been proposed. In the first industrial implementations, such as Dynamic Matrix Control [see e.g. (Prett and Garcia, 1988, Sec. 5.2)], an output correction term defined as the difference between the measured and the predicted output was adopted. This method is probably the simplest disturbance model, usually referred to as *output disturbance model* (Rawlings, 2000), and it is known to be applicable only to open-loop stable systems. In the context of state-space linear systems, a wider range of approaches are known and applied. General formulations based on disturbance models and observers have been extensively discussed (Muske and Badgwell, 2002; Pannocchia and Rawlings, 2003; Maeder et al., 2009), although an important issue on which disturbance model is more effective was matter of investigation until Rajamani et al. (2009) presented an important result on the equivalence of different disturbance models. Alternative offset-free approaches are based on a *velocity form* linear model, in which the input (and usually also the state) are replaced by their variation between two time steps [see e.g. (Pannocchia and Rawlings, 2001; Wang, 2004; González et al., 2008; Betti et al., 2012, 2013) and references therein]. The concepts of linear offset-free MPC have been extended to nonlinear MPC in (Morari and Maeder, 2012).

The general goal of this paper is to shed some light on this aspect of MPC theory and design, which is often overlooked in academic papers but is fundamental for actual implementation. The rest of the paper is organized as follows. In Section 2, we present a comprehensive description of the available results on offset-free nonlinear MPC, and then we show new results on the asymptotic convergence of the estimator. In Section 3, the offset-free linear MPC design is discussed to show that the velocity-form model is a special case of disturbance model/observer, and not an alternative method as commonly believed. In Section 4, we extend the concepts of offset-free nonlinear MPC to design an economic MPC algorithm that is able to cope with persistent errors while still achieving the optimal ultimate economic performance. Three application examples are extensively presented in Section 5 to emphasize the results of the previous sections, and to highlight possible pitfalls of the offset design. Finally, we summarize the main achievements and sketch possible research directions in Section 6.

Notation. The fields of real (nonnegative real), integer (nonnegative integer), and complex numbers are denoted by $\mathbb{R}$ ($\mathbb{R}_{\geq 0}$), $\mathbb{I}$ ($\mathbb{I}_{\geq 0}$) and $\mathbb{C}$, respectively. The n-dimensional vector space with components in $\mathbb{R}$ is denoted by $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, the Euclidean norm is denoted by $|x|$. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all its eigenvalues lie strictly inside the unit circle. Given a sequence $\mathbf{x} = \{x(0)\ x(1)\ \ldots\}$ and $a, b \in \mathbb{I}_{\geq 0}$, $a < b$, we define: $\|\mathbf{x}\|_{a:b} = \max_{k\in\mathbb{I}, a\leq k\leq b} |x(k)|$. A time-invariant

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discrete-time system is written as $x(k+1) = f(x(k),u(k))$ or simply $x^+ = f(x,u)$ when it is not necessary to specify the current time k. A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to the class $\mathscr{K}$ if it is continuous, zero at zero, and strictly increasing. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to the class $\mathscr{KL}$ if for each $k \geq 0$ the function $\beta(\cdot,k) \in \mathscr{K}$, and for each $s \in \mathbb{R}_{\geq 0}$ there holds $\lim_{k\to\infty} \beta(s,k) = 0$.

2. OFFSET-FREE NONLINEAR MPC: DEFINITIONS, KNOWN AND NOVEL RESULTS

2.1 Plant, nominal model and constraints

In the paper we are concerned with the control of time-invariant dynamical systems in the form:

$$\begin{aligned} x_p^+ &= f_p(x_p,u,w_p) \\ y &= h_p(x_p,v_p) \end{aligned} \tag{1}$$

in which $x_p, x_p^+ \in \mathbb{R}^n$ denote the current and successor *plant* states, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output, $w_p \in \mathbb{R}^{n_w}$ and $v_p \in \mathbb{R}^{n_v}$ denote plant state and output disturbances. The plant output is measured at each time $k \in \mathbb{I}$.

Functions $f_p : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ and $h_p : \mathbb{R}^n \times \mathbb{R}^{n_v} \to \mathbb{R}^p$ are not known precisely but are assumed to be continuous. However, in order to design a Model Predictive Control algorithm for (1), a time-invariant dynamical system *model* is known:

$$\begin{aligned} x^+ &= f(x,u) \\ y &= h(x) \end{aligned} \tag{2}$$

in which $x, x^+ \in \mathbb{R}^n$ denote the current and successor *model* states. The functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be continuous.

Let $w \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$ be defined as:

$$w := f_p(x_p,u,w_p) - f(x,u), \quad v := h_p(x_p,v_p) - h(x) \tag{3}$$

The next assumption is considered throughout the paper.

Assumption 1. (General). Disturbances are bounded in compact sets: $w \in \mathbb{W}$, $v \in \mathbb{V}$.

Input and output are required to satisfy the following constraints at all times:

$$g_u(u) \leq 0, \qquad g_y(y) \leq 0 \tag{4}$$

in which $g_u : \mathbb{R}^m \to \mathbb{R}^{q_u}$ and $g_y : \mathbb{R}^p \to \mathbb{R}^{q_y}$ are convex functions defining the following compact convex sets:

$$\mathbb{U} := \{u \in \mathbb{R}^m \mid g_u(u) \leq 0\}, \quad \mathbb{Y} := \{y \in \mathbb{R}^p \mid g_y(y) \leq 0\} \tag{5}$$

Moreover, induced by the model (2) and the output constraint set, the following state constraint set is defined:

$$\mathbb{X} := \{x \in \mathbb{R}^n \mid g_y(h(x)) \leq 0\} \tag{6}$$

Remark 2. If the state is measurable, it follows that $h_p(x_p,v_p) = x_p$ and $v_p = 0$ at all times. Clearly, in such cases, the model output function is chosen as $h(x) = x$ and $\mathbb{Y}$ also represents the constraint set for the state.

Given a sequence of inputs $\mathbf{u} = \{u(0),u(1),\ldots\}$ and an initial state z, we define $\mathbf{y}_{z,\mathbf{u}} = \{y(0),y(1),\ldots\}$ as the corresponding sequence of outputs generated by system (2). We also denote by $x(k;z,\mathbf{u})$ the solution at time k of system (2). Next, we recall the following definitions (Rawlings and Mayne, 2009, Sec.4.2).

Definition 3. (Observability). System (2) is observable if there exists a finite $N_o \in \mathbb{I}$ and $\gamma(\cdot) \in \mathscr{K}$ such that for any two initial states z_1 and z_2 and control sequence $\mathbf{u}$, and all $k \geq N_o$

$$|z_1 - z_2| \leq \gamma(\|\mathbf{y}_{z_1,\mathbf{u}} - \mathbf{y}_{z_2,\mathbf{u}}\|_{0:k}) \tag{7}$$

Definition 4. (Asymptotic stability of the estimate). Let $\hat{x}(k)$ be an estimate of $x(k;x_0,\mathbf{u})$ obtained for system (2), given a sequence of output measurements $\mathbf{y}_{x_0,\mathbf{u}} = \{y(0),\ldots,y(k)\}$ and a (prior) estimate of the initial state $\bar{x}_0$. The estimate is asymptotically stable if there exists $\beta \in \mathscr{KL}$ such that for all initial state x_0 and prior estimate $\bar{x}_0$ and $k \in \mathbb{I}_{\geq 0}$ there holds:

$$|x(k;x_0,\mathbf{u}) - \hat{x}(k)| \leq \beta(\|x_0 - \bar{x}_0\|,k) \tag{8}$$

2.2 Offset-free tracking problem definition

The controlled output $y_c \in \mathbb{R}^{p_c}$ is defined as function of the measured output:

$$y_c = r(y) \tag{9}$$

and $\bar{y}_c \in \mathbb{Y}$ denotes its desired setpoint. The offset-free problem is to design an output feedback MPC law $u = \kappa(y)$ such that:

G1: Input and output constraints are satisfied at all times.
G2: The closed-loop system reaches an equilibrium.
G3: The following condition holds true:

$$\lim_{k\to\infty} y_c(k) = \bar{y}_c \tag{10}$$

For these goals (in particular G2 and G3) to be attainable the following additional assumption is necessary.

Assumption 5. Disturbances are asymptotically constant, i.e. there exist $\bar{w} \in \mathbb{W}$ and $\bar{v} \in \mathbb{V}$ such that:

$$\lim_{k\to\infty} w(k) = \bar{w}, \qquad \lim_{k\to\infty} v(k) = \bar{v} \tag{11}$$

2.3 State augmentation and estimation

Offset-free MPC algorithms are generally based on an *augmented model* (Muske and Badgwell, 2002; Pannocchia and Rawlings, 2003; Maeder et al., 2009; Maeder and Morari, 2010). The general augmented model can be written as:

$$\begin{aligned} x^+ &= F(x,d,u) \\ d^+ &= d \\ y &= H(x,d) \end{aligned} \tag{12}$$

in which $d \in \mathbb{R}^{n_d}$ is the so-called *disturbance state* or simply disturbance. Functions $F : \mathbb{R}^n \times \mathbb{R}^{n_d} \times \mathbb{R}^m :\to \mathbb{R}^n$ and $H : \mathbb{R}^n \times \mathbb{R}^{n_d} :\to \mathbb{R}^p$ are assumed to be continuous, and *consistent* with the nominal model, i.e. for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ there holds:

$$F(x,0,u) = f(x,u), \qquad H(x,0) = h(x) \tag{13}$$

Remark 6. The disturbance d follows an integral dynamics.

We make the following assumption on the augmented system.

Assumption 7. The augmented model (12) is observable.

Remark 8. The augmented model (12) is observable only if the nominal model (2) is observable. Conversely, given an observable model (2), it is in general possible to define an augmented model (12), with $n_d \leq p$ disturbance states, that is observable.

At each time k, given the measurement of the output $y(k)$, an observer for (12) is used to estimate the augmented state $(x(k),d(k))$. For simplicity of exposition, we focus on a 'steady-state Kalman filter like' estimator, in which only the current measurement of $y(k)$ is used to update the prediction of $(x(k),d(k))$ made at the previous decision time. Alternative methods based on Moving Horizon Estimation (MHE) use instead a sequence of previous output measurements, and can be applied in a similar manner.

Without introducing the double-index notation, we define $\hat{x}(k)$ and $\hat{d}(k)$ as the *filtered* estimate of $x(k)$ and $d(k)$ obtained using the output measurement at time k. We instead use the symbols $\hat{x}^*(k)$, $\hat{d}^*(k)$ and $\hat{y}^*(k)$ to denote the *predicted* estimate of $x(k)$, $d(k)$ and $y(k)$, respectively, obtained at time $k-1$ using the augmented model (12), i.e.:

$$\begin{aligned} \hat{x}^*(k) &= F(\hat{x}(k-1), \hat{d}(k-1), u(k-1)) \\ \hat{d}^*(k) &= \hat{d}(k-1) \\ \hat{y}^*(k) &= H(\hat{x}^*(k), \hat{d}^*(k)) \end{aligned} \tag{14}$$

Having defined the output prediction error as:

$$e(k) = y(k) - \hat{y}^*(k) \tag{15}$$

the filtering relations can be written as follows:

$$\begin{aligned} \hat{x}(k) &= \hat{x}^*(k) + \kappa_x(e(k)) \\ \hat{d}(k) &= \hat{d}^*(k) + \kappa_d(e(k)) \end{aligned} \tag{16}$$

We make the following assumption on the observer.

Assumption 9. (Nominal observer). The functions $\kappa_x : \mathbb{R}^p \to \mathbb{R}^n$ and $\kappa_d : \mathbb{R}^p \to \mathbb{R}^{n_d}$ are continuous and satisfy:

$$\kappa_x(0) = 0, \qquad \kappa_d(e) = 0 \Leftrightarrow e = 0 \tag{17}$$

Moreover, relations (14)–(16) form an asymptotically stable observer for the augmented system (12).

Remark 10. From condition (17) in Assumption 9, it follows in general that $n_d \geq p$. This observation and Remark 8 lead to the choice $n_d = p$.

The issues associated with the use of $n_d < p$ are highlighted later in Section 5.

2.4 Target calculation, optimal control problem and receding horizon implementation

Given the current estimate of the augmented state $(\hat{x}(k), \hat{d}(k))$, an offset-free MPC algorithm needs to compute the equilibrium target that ensures exact tracking of the controlled variable. Depending on input/output dimensions and on the system dynamics, such a target may not be unique. Hence, in the general case we solve the following target problem denoted by $\mathbb{P}_s(\bar{y}_c, \hat{d}(k))$:

$$\min_{x,u,y} \ell_s(y,u) \tag{18a}$$

subject to:

$$x = F(x, \hat{d}(k), u) \tag{18b}$$

$$y = H(x, \hat{d}(k)) \tag{18c}$$

$$r(y) = \bar{y}_c \tag{18d}$$

$$y \in \mathbb{Y}, \qquad u \in \mathbb{U} \tag{18e}$$

in which $\ell_s : \mathbb{R}^p \times \mathbb{R}^m :\to \mathbb{R}$ is the *steady-state cost function*. We assume that (18) is feasible and we denote its (unique) solution as $(x_s(k), u_s(k), y_s(k))$.

Remark 11. Problem $\mathbb{P}_s(\bar{y}_c, \hat{d}(k))$ is parametric in the controlled variable setpoint and in the current disturbance estimate. It needs therefore to be solved at each decision time, and not only when the setpoint changes.

Let $\mathbf{x} = \{x_0\ x_1\ \cdots\ x_N\}$ and $\mathbf{u} = \{u_0\ u_1\ \cdots\ u_{N-1}\}$ be, respectively, a state sequence and an input sequence. For $i = 0, \ldots, N-1$, let $y_i = H(x_i, \hat{d}(k))$ be the model output corresponding to a state x_i and disturbance estimate $\hat{d}(k)$. Furthermore, let $\tilde{x}_i := x_i - x_s(k)$ for $i = 0, \ldots, N$, and $\tilde{u}_i := u_i - u_s(k)$ for $i = 0, \ldots, N-1$. The following finite-horizon optimal control problem, denoted by $\mathbb{P}(\hat{x}(k), \hat{d}(k), x_s(k), u_s(k), y_s(k))$, is posed:

$$\min_{\mathbf{x},\mathbf{u}} \sum_{i=0}^{N-1} \ell(\tilde{x}_i, \tilde{u}_i) + V_f(\tilde{x}_N) \tag{19a}$$

subject to:

$$x_0 = \hat{x}(k) \tag{19b}$$

$$x_{i+1} = F(x_i, \hat{d}(k), u_i) \tag{19c}$$

$$H(x_i, \hat{d}(k)) \in \mathbb{Y}, \qquad u_i \in \mathbb{U} \tag{19d}$$

$$\tilde{x}_N \in \mathbb{X}_f \tag{19e}$$

in which $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is a strictly positive definite convex function. $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\mathbb{X}_f$ are, respectively, a terminal cost function and a terminal set, designed according to the usual stabilizing conditions (Rawlings and Mayne, 2009, Sec. 2.5).

Assuming that problem (19) is feasible, its solution is denoted by $(\mathbf{x}^0(k), \mathbf{u}^0(k))$ and the associated receding horizon implementation is given by:

$$u(k) = u_0^0(k) \tag{20}$$

Remark 12. In problem $\mathbb{P}(\cdot)$, the disturbance estimate $\hat{d}(k)$ is a fixed parameter. Consequently, $\mathbb{P}(\cdot)$ can be rewritten as a regulation problem in the deviation variables $\{\tilde{x}_i, \tilde{u}_i\}$.

Remark 13. Following the ideas of (Limon et al., 2008), the target problem $\mathbb{P}_s(\cdot)$ and the dynamic optimization problem $\mathbb{P}(\cdot)$ could be merged together in a single optimization problem (Betti et al., 2013).

2.5 Asymptotic stability and offset-free tracking

In order to establish offset-free tracking, it is necessary that the closed-loop system reaches an asymptotically stable equilibrium. Establishing conditions under which this occurs is very hard because of the combined presence of state estimator, target calculation and dynamic optimization. Indeed, almost all available methods for offset-free MPC assume that an asymptotically stable equilibrium has been reached, and then show that offset-free control is attained at such an equilibrium. One noticeable exception is the robust offset-free linear MPC algorithm proposed in (Betti et al., 2013), which is designed under the assumption that the full state is measurable at each time. Such an algorithm uses the so-called velocity form model (Pannocchia and Rawlings, 2001), which is reviewed later in Section 3 and for which equivalence to a particular disturbance model and observer has been recently established in (Pannocchia, 2015).

We next present the main offset-free result and a streamlined proof. Similar results and arguments have been shown in (Muske and Badgwell, 2002; Pannocchia and Rawlings, 2003; Maeder et al., 2009; Morari and Maeder, 2012).

Theorem 14. Assume that the target calculation problem (18) and the regulation problem (19) are feasible at all times, and that the closed-loop system reaches an equilibrium with input u_∞ and output y_∞. It follows that:

$$r(y_\infty) = \bar{y}_c \tag{21}$$

Proof. If the closed loop reaches a steady state, stability of the observer implies that the augmented state estimate also reaches a steady state. Let $(\hat{x}_\infty, \hat{d}_\infty)$ be such a steady state, and let $\hat{y}^*_\infty := H(\hat{x}_\infty, \hat{d}_\infty)$. From (14) and (16), we can write

$\hat{d}^*_\infty = \hat{d}_\infty = \hat{d}^*_\infty + \kappa_d(y_\infty - \hat{y}^*_\infty)$, which implies $\kappa_d(y_\infty - \hat{y}^*_\infty) = 0$. From (17), it then follows $y_\infty = \hat{y}^*_\infty = \hat{y}_\infty$.

We now turn our attention to the target calculation problem $\mathbb{P}_s(\bar{y}_c, \hat{d}_\infty)$, and denote its solution as $(x_{s,\infty}, u_{s,\infty}, y_{s,\infty})$. At steady state, the first element of the optimal control sequence is given by u_∞, and the corresponding first element of the optimal state sequence is $\hat{x}_\infty$. Closed-loop stability of the equilibrium and positive definiteness of the cost function $\ell(\cdot)$ imply that $(\hat{x}_\infty, u_\infty) = (x_{s,\infty}, u_{s,\infty})$, which, coupled with (18c)–(18d) and the previous relations, imply that $r(y_\infty) = r(\hat{y}_\infty) = r(y_{s,\infty}) = \bar{y}_c$. □

We next present a novel result on the asymptotic stability and offset-free estimation under state feedback.

Theorem 15. Assume that the state is measurable (i.e. $y = x$), and that the augmented model and observer functions are:

$$F(x,u,d) := f(x,u) + B_d d, \quad H(x,d) := x, \qquad \kappa_x(e) = e, \quad \kappa_d(e) = K_d e \tag{22}$$

with $B_d \in \mathbb{R}^{n\times n}$ invertible, and $B_d K_d = I$.

(1) The observer is nominally asymptotically stable.
(2) For any w respecting Assumption 5, there holds:

$$\lim_{k\to\infty} x(k) - \hat{x}^*(k) = 0, \qquad \lim_{k\to\infty} \hat{d}^*(k) = B_d^{-1}\bar{w} \tag{23}$$

Proof. (Observability and stability of the observer) To prove this result, we consider that the "actual" augmented state evolves as:

$$\begin{aligned} x(k+1) &= f(x(k),u(k)) + B_d d(k) \\ d(k+1) &= d(k) \end{aligned} \tag{24}$$

Recall that in this case the prediction error is $e(k) = x(k) - \hat{x}^*(k)$. Since $\kappa_x(e) = e$, it follows that $\hat{x}(k) = x(k)$ for all $k \in \mathbb{I}$. Therefore, the "predicted" augmented state is given by:

$$\begin{aligned} \hat{x}^*(k+1) &= f(x(k),u(k)) + B_d \hat{d}(k) \\ \hat{d}^*(k+1) &= \hat{d}(k) \end{aligned} \tag{25}$$

whereas the filtered disturbance estimate, $\hat{d}(k)$, is given by:

$$\hat{d}(k) = \hat{d}^*(k) + K_d e(k) = \hat{d}^*(k) + K_d(x(k) - \hat{x}^*(k)) \tag{26}$$

By combining (24), (25) and (26), and defining $e_d(k) := d(k) - \hat{d}^*(k)$, we obtain:

$$\begin{bmatrix} e^+ \\ e_d^+ \end{bmatrix} = \begin{bmatrix} -B_d K_d & B_d \\ -K_d & I \end{bmatrix} \begin{bmatrix} e \\ e_d \end{bmatrix} = \begin{bmatrix} -I & B_d \\ -K_d & I \end{bmatrix} \begin{bmatrix} e \\ e_d \end{bmatrix} = \mathscr{M}_1 \begin{bmatrix} e \\ e_d \end{bmatrix} \tag{27}$$

It is easy to see that the matrix $\mathscr{M}_1$ in (27) is nilpotent of degree 2, i.e. $\mathscr{M}_1^2 = 0$, which proves the nominal asymptotic stability of the observer of the augmented system.

(Asymptotic zero estimation offset) We study the estimation error with respect to the process state, which evolves as:

$$x(k+1) = f(x(k),u(k)) + w(k) \tag{28}$$

We again have that $\hat{x}(k) = x(k)$ for all $k \in \mathbb{I}$. Combining (28), (25), (26) and $B_d K_d = I$, we obtain:

$$\begin{bmatrix} e(k+1) \\ \hat{d}^*(k+1) \end{bmatrix} = \begin{bmatrix} -I & -B_d \\ K_d & I \end{bmatrix} \begin{bmatrix} e(k) \\ \hat{d}^*(k) \end{bmatrix} + \begin{bmatrix} w(k) \\ 0 \end{bmatrix} = \mathscr{M}_2 \begin{bmatrix} e(k) \\ \hat{d}^*(k) \end{bmatrix} + \begin{bmatrix} w(k) \\ 0 \end{bmatrix} \tag{29}$$

The matrix $\mathscr{M}_2$ is also nilpotent of degree 2, so that $\begin{bmatrix} e(k) \\ \hat{d}^*(k) \end{bmatrix} \to \begin{bmatrix} e_\infty \\ \hat{d}^*_\infty \end{bmatrix}$ as $k \to \infty$. Since K_d is invertible, the limit values are:

$$e_\infty = 0, \qquad \hat{d}^*_\infty = B_d^{-1}\bar{w} \tag{30}$$

which completes the proof. □

3. OFFSET-FREE LINEAR MPC: DISTURBANCE MODELS, WHAT ELSE?

In this section we restrict our attention to the case in which the nominal model (2) is linear and the controlled variable is a linear combination of the output, i.e.

$$f(x,u) := Ax + Bu, \qquad h(x) := Cx, \qquad r(y) := Dy \tag{31}$$

It is assumed that the following condition holds true:

$$\text{rank} \begin{bmatrix} A-I & B \\ DC & 0 \end{bmatrix} = n + p_c \tag{32}$$

which ensures that problem (18) can admit a solution.

We present the general disturbance model and the velocity form model. Then, we show that the latter method is equivalent to the use of a particular case of disturbance model/observer, covering and strengthening the results recently reported in (Pannocchia, 2015), which also showed that the method proposed in (Tatjewski, 2014) is a particular case of disturbance model/observer.

3.1 Disturbance model and observer

The augmented model functions in (12) read as follows:

$$F(x,u,d) := Ax + Bu + B_d d, \qquad H(x,d) := Cx + C_d d \tag{33}$$

and the observer functions are linear, in the form:

$$\kappa_x(e) := K_x e, \qquad \kappa_d(e) := K_d e, \tag{34}$$

in which $K_x \in \mathbb{R}^{n\times p}$, $K_d \in \mathbb{R}^{n_d \times p}$ and $\text{rank}(K_d) = n_d$.

Compactly, we define the augmented state $\xi := \begin{bmatrix} x \\ d \end{bmatrix}$, the augmented model matrices:

$$A_a := \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix}, \quad B_a := \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C_a := [C \ \ C_d] \tag{35}$$

and write the augmented model dynamics as:

$$\begin{aligned} \xi^+ &= A_a \xi + B_a u \\ y &= C_a \xi \end{aligned} \tag{36}$$

Similarly, let $K_a := \begin{bmatrix} K_x \\ K_d \end{bmatrix}$ be the augmented observer gain matrix, and let the filtered estimate of the augmented state be denoted as $\hat{\xi} := \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}$. For the linear augmented model (36), the general observer relations (14)–(16) can be written as:

$$\hat{\xi}(k) = A_a \hat{\xi}(k-1) + B_a u(k-1) + K_a e(k) \tag{37}$$

Remark 16. Asymptotic stability of the augmented observer requires that the characteristic matrix $(A_a - K_a C_a A_a)$ is Hurwitz.

The next proposition summarizes the main results regarding the design of offset-free linear MPC algorithms based on a disturbance model (Pannocchia and Rawlings, 2003).

Proposition 17. Consider the augmented matrices given in (35). The following results hold true:

(1) The pair (C_a, A_a) is detectable (observable) if and only if (C, A) is detectable (observable) and

$$\text{rank} \begin{bmatrix} A-I & B_d \\ C & C_d \end{bmatrix} = n + n_d \tag{38}$$

(2) There exist matrices (B_d, C_d) such that condition (38) holds if and only if $n_d \le p$.

(3) If $n_d = p$, it follows that any asymptotically stable observer gain matrix is such that K_d is invertible and the offset-free Assumption 9 holds true.

3.2 Velocity form

Another known method to design a linear offset-free MPC algorithm is to use the so-called velocity form model (Pannocchia and Rawlings, 2001; Wang, 2004; González et al., 2008; Betti et al., 2012, 2013), which is obtained from the nominal model (31) by defining the state increment $\delta x(k) := x(k) - x(k-1)$, the input increment as $\delta u(k) := u(k) - u(k-1)$, and rewriting the nominal model as follows:

$$\begin{aligned} \delta x^+ &= A\delta x + B\delta u \\ y^+ &= y + C\delta x^+ = y + CA\delta x + CB\delta u \end{aligned} \tag{39}$$

If we define the augmented state of the velocity form model as $\xi_\delta := \begin{bmatrix} \delta x \\ y \end{bmatrix}$ and the associated matrices:

$$A_\delta := \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix}, \quad B_\delta := \begin{bmatrix} B \\ CB \end{bmatrix}, \quad C_\delta := [0 \ I] \tag{40}$$

the velocity form model can be written as:

$$\begin{aligned} \xi_\delta^+ &= A_\delta \xi_\delta + B_\delta \delta u \\ y &= C_\delta \xi_\delta \end{aligned} \tag{41}$$

Remark 18. The second block component (y) of the velocity form state is measurable, but the first block component (δx) may not be measurable. So, in general, an observer is also needed for this model form to estimate δx.

Let $K_\delta := \begin{bmatrix} K_{\delta x} \\ K_y \end{bmatrix}$ be the associated observer gain matrix for the velocity form model. The general observer relations (14)–(16) for the velocity form model can be written compactly as:

$$\hat{\xi}_\delta(k) = A_\delta \hat{\xi}_\delta(k-1) + B_\delta \delta u(k-1) + K_\delta e(k) \tag{42}$$

We remark that K_δ should be chosen such that $(A_\delta - K_\delta C_\delta A_\delta)$ is Hurwitz.

The next proposition (easily proved) summarizes the main properties of the velocity form model.

Proposition 19. Consider the velocity form model matrices in (40). The following results hold true:

(1) The pair (C_δ, A_δ) is detectable if and only if (C,A) is detectable.
(2) The pair (A_δ, B_δ) is stabilizable if and only if (A,B) is stabilizable and condition (32) holds true.

Since y is measured, it is customary to use a deadbeat observer for this second component of the state ξ_δ, i.e. $K_y = I$, so that from (42) it follows $\hat{y}(k) = y(k)$. In this case, the only matrix to choose is $K_{\delta x}$, and the following result holds true. The next result (easily proved) explains the conditions that $K_{\delta x}$ should satisfy to ensure stability of the observer.

Proposition 20. Let $K_y = I$. The velocity form observer characteristic matrix $(A_\delta - K_\delta C_\delta A_\delta)$ is Hurwitz if and only if $K_{\delta x}$ is chosen such that $(A - K_{\delta x} CA)$ is Hurwitz.

Remark 21. We notice by inspection that the observer characteristic matrix $(A_\delta - K_\delta C_\delta A_\delta)$ has p eigenvalues at the origin.

3.3 Velocity form equivalent disturbance model

We now show that the velocity form model, with deadbeat observer gain $K_y = I$, is equivalent to a particular disturbance model/observer. This new result, first shown in (Pannocchia, 2015), is here reported and proved in a slightly different form.

Theorem 22. Consider the velocity form model (41) and observer (42), with a stable output deadbeat observer gain $K_\delta = \begin{bmatrix} K_{\delta x} \\ I \end{bmatrix}$. This is equivalent to using the following disturbance model and observer gains:

$$B_d = K_{\delta x}, \quad C_d = I - CK_{\delta x}, \quad K_x = K_{\delta x}, \quad K_d = I \tag{43}$$

Proof. By expanding and rearranging the various terms in (42), and recalling (40), we obtain:

$$\begin{aligned} \delta\hat{x}(k) &= (A - K_{\delta x} CA)\delta\hat{x}(k-1) + (B - K_{\delta x} CB)\delta u(k-1) \\ &\quad + K_{\delta x}(y(k) - \hat{y}(k-1)) \\ \hat{y}(k) &= y(k) \end{aligned}$$

which can also be rewritten as:

$$\begin{aligned} \delta\hat{x}(k) &= (A - K_{\delta x} CA)\delta\hat{x}(k-1) + \\ &\quad (B - K_{\delta x} CB)\delta u(k-1) + K_{\delta x}(y(k) - y(k-1)) \end{aligned} \tag{44}$$

Now consider the evolution of the augmented system with matrices given in (43), which can be written:

$$\begin{aligned} \hat{x}(k) &= A\hat{x}(k-1) + Bu(k-1) \\ &\quad + K_{\delta x}(y(k) - C(A\hat{x}(k-1) + Bu(k-1))) \end{aligned} \tag{45}$$

By rewriting (45) at time $k-1$ and taking the difference of both sides, we obtain:

$$\begin{aligned} \delta\hat{x}(k) &= A\delta\hat{x}(k-1) + B\delta u(k-1) + K_{\delta x}(y(k) - y(k-1)) \\ &\quad - K_{\delta x} C(A\delta\hat{x}(k-1) + B\delta u(k-1))) \end{aligned} \tag{46}$$

which can be rewritten as

$$\begin{aligned} \delta\hat{x}(k) &= (A - K_{\delta x} CA)\delta\hat{x}(k-1) + \\ &\quad (B - K_{\delta x} CB)\delta u(k-1) + K_{\delta x}(y(k) - y(k-1)) \end{aligned} \tag{47}$$

The proof is completed by comparing (47) with (44). □

3.4 Equivalence in the state feedback case

If the state is measurable, the previous relations between the velocity form and disturbance model methods become even stronger. Since the state is measurable, it may be desirable/possible to use a deadbeat observer for the state, i.e. such that $\hat{x}(k) = x(k)$.

When the state is measurable, the augmented matrices of the disturbance model method read as follows:

$$A_a = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix}, \quad B_a = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C_a = [I \ C_d] \tag{48}$$

with $B_d \in \mathbb{R}^{n \times n}$, $C_d \in \mathbb{R}^{n \times n}$ (i.e. $n_d = n$), and K_a is chosen such that $(A_a - K_a C_a A_a)$ is Hurwitz. The general detectability condition (38) is consequently replaced by:

$$\text{rank} \begin{bmatrix} A - I & B_d \\ I & C_d \end{bmatrix} = 2n \tag{49}$$

A state/disturbance deadbeat observer is then designed by satisfying the following restrictions:

$$\text{rank}(B_d) = n, \quad C_d = 0, \quad K_x = I, \quad K_d = B_d^{-1} \tag{50}$$

Remark 23. Using (50), it is straightforward to see that $A_a - K_a C_a A_a = \begin{bmatrix} 0 & 0 \\ -B_d^{-1} A & 0 \end{bmatrix}$, i.e. the observer characteristic matrix has all its eigenvalues at zero.

In the velocity form model, the augmented matrices are still defined in (40), and K_δ should be chosen such that $(A_\delta - K_\delta C_\delta A_\delta)$ is Hurwitz. In the state feedback case, the state/output deadbeat observer is obtained by choosing $K_{\delta x} = I$ and $K_y = I$. With such a choice, we see $A_\delta - K_\delta C_\delta A_\delta = \left[\begin{smallmatrix} 0 & -I \\ 0 & 0 \end{smallmatrix}\right]$, i.e. the observer characteristic matrix has all its eigenvalues at zero.

We now present the equivalence results.

Theorem 24. Assume the state is measurable, and consider the velocity form model (41) and observer (42), with a state/output deadbeat observer gain $K_\delta = \begin{bmatrix} I \\ I \end{bmatrix}$. This model is a particular case of disturbance model with:

$$B_d = I, \quad C_d = 0, \quad K_x = I, \quad K_d = I \tag{51}$$

Proof. The result is a direct consequence of the equivalence relations (43), with $K_{\delta x} = I$ and $C = I$. □

4. OFFSET-FREE ECONOMIC MPC

We discuss in this section how the fundamentals of offset-free design can be embedded into an economic MPC framework, in order to ensure that the best economic performance can be achieved despite the presence of persistent discrepancies between the actual process and the MPC model.

4.1 A brief overview of economic MPC

To set up a framework that is consistent with the previous results on output feedback offset-free MPC, we consider an economic stage cost function defined in terms of measurable quantities, namely input and output. We denote such a cost function as $\ell_e(y,u)$, with $\ell_e : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}$. At each decision time, let $\hat{x}(k)$ be the estimate of the current state. Let $\mathbf{x} = \{x_0 \ x_1 \ \cdots \ x_N\}$ and $\mathbf{u} = \{u_0 \ u_1 \ \cdots \ u_{N-1}\}$ be, respectively, a state sequence and an input sequence. Then, the economic MPC solves the following optimization problem, denoted by $\mathbb{P}_e(\hat{x}(k), x_s)$:

$$\min_{\mathbf{x},\mathbf{u}} \sum_{i=0}^{N-1} \ell_e(h(x_i), u_i) \tag{52a}$$

subject to:

$$x_0 = \hat{x}(k) \tag{52b}$$
$$x_{i+1} = f(x_i, u_i) \tag{52c}$$
$$g_u(u_i) \le 0, \qquad g_y(h(x_i)) \le 0 \tag{52d}$$
$$x_N = x_s \tag{52e}$$

in which x_s represents the feasible equilibrium state such that $\ell_e(\cdot)$ is minimized. As usual, only the first input of the optimal solution to (52) is sent to the plant, i.e. $u(k) = u_0^0(k)$, and at the next decision time a new problem $\mathbb{P}_e(\cdot)$ is solved for the new current state estimate.

4.2 An offset-free economic MPC algorithm

If the actual plant equals model (2) and the state is measured, under mild conditions, it can be shown that the closed-loop system state converges asymptotically to the equilibrium x_s (Diehl et al., 2011; Rawlings et al., 2012). On the other hand, if the actual plant (1) differs from the nominal model (2), besides the issues of robust stability and constraint satisfaction, the closed-loop system may converge to an equilibrium that is not optimal, i.e. in which $\ell_e(\cdot)$ is not minimized. For instance, under Assumption 5, the best plant equilibrium is:

$$\min_{\bar{x}_p, \bar{u}, \bar{y}} \ell_e(\bar{y}, \bar{u}) \tag{53a}$$

subject to:

$$\bar{x}_p = f(\bar{x}_p, \bar{u}) + \bar{w} \tag{53b}$$
$$\bar{y} = h(\bar{x}_p) + \bar{v} \tag{53c}$$
$$\bar{u} \in \mathbb{U}, \qquad \bar{y} \in \mathbb{Y} \tag{53d}$$

Clearly, it would be desirable to achieve this same equilibrium input/output even if the actual plant (1) is not known precisely. To this aim, we augment the nominal system (2) and define the augmented model (12), whose current state $(\hat{x}, \hat{d})$ is estimated from the output measurement by the observer (14)–(16).

Then, the (current) best equilibrium steady state is given by:

$$(x_s(k), u_s(k), y_s(k)) := \arg\min_{x,u,y} \ell_e(y,u) \tag{54a}$$

subject to:

$$x = F(x, \hat{d}(k), u), \qquad y = H(x, \hat{d}(k)) \tag{54b}$$
$$g_u(u) \le 0, \qquad g_y(y) \le 0 \tag{54c}$$

Remark 25. Since $\hat{d}(k)$ is not necessarily constant, the equilibrium problem (54) needs to be solved at each decision time.

Consequently, we modify problem (52) into an offset-free economic MPC problem:

$$\min_{\mathbf{x},\mathbf{u}} \sum_{i=0}^{N-1} \ell_e(H(x_i, \hat{d}(k)), u_i) \tag{55a}$$

subject to:

$$x_0 = \hat{x}(k) \tag{55b}$$
$$x_{i+1} = F(x_i, \hat{d}(k), u_i) \tag{55c}$$
$$g_u(u_i) \le 0, \qquad g_y(H(x_i, \hat{d}(k))) \le 0 \tag{55d}$$
$$x_N = x_s(k) \tag{55e}$$

Finally, the first input of the optimal solution to (55) is sent to the plant, and at the next decision time the whole procedure (augmented state estimation, equilibrium calculation, economic MPC problem) is solved again.

We now present the main result on offset-free economic MPC, which is stated and proved assuming state feedback.

Theorem 26. Under the assumptions of Theorem 15, assume that equilibrium problem (54) and the economic MPC problem (55) are feasible at all times, and that the closed-loop system reaches an equilibrium with input and output (u_∞, y_∞). The achieved ultimate cost $\ell(y_\infty, u_\infty)$ is the same as that in (53).

Proof. If the closed-loop reaches an equilibrium with input and output (u_∞, y_∞), stability of the observer implies that the augmented state estimate also reaches a steady state. Let $(\hat{x}_\infty, \hat{d}_\infty)$ be the augmented state estimate at the reached equilibrium, and let $(x_{s,\infty}, u_{s,\infty}, y_{s,\infty})$ be the solution to (54) given the equilibrium disturbance estimate $\hat{d}_\infty$. It follows that the applied input u_∞, which is the first element of the optimal control sequence solution to (55), is equal to $u_{s,\infty}$.

Under the assumptions of Theorem 15, we observe that $\bar{w} = \lim_{k\to\infty} w(k) = \lim_{k\to\infty} B_d \hat{d}(k) = B_d \hat{d}_\infty$, and therefore: $f(x,u) + \bar{w} = F(x, \hat{d}_\infty, u)$ (and obviously $h(x) = x = H(x, \hat{d}_\infty)$). Hence, problem (54) and problem (53) are identical, and this completes the proof. □

5. ILLUSTRATIVE EXAMPLES

We present several numerical examples to highlight the concepts and results discussed in this paper. Simulations are performed using the Python interface to the open-source CasADi framework (Andersson, 2013), and the arising NLPs/QPs are solved with IPOPT (Wächter and Biegler, 2006).

5.1 Offset-free nonlinear MPC of a cart-pole system in the presence of obstacles

Problem description and control task. The first system we consider is a mechanical one and its schematic is depicted in Fig. 1. It consists of one prismatic (translational) joint and one revolute joint. This system is usually referred to as cart-pole. The configuration of the system is $q = [q_1\ q_2]^T$, where $q_1 > 0$ is the horizontal displacement of the cart and q_2 is the rotation angle of the pole around the hinge O, taken with respect to the vertical direction. The state of the system is therefore $x = [x_1\ x_2\ x_3\ x_4]^T = [q_1\ q_2\ \dot{q}_1\ \dot{q}_2]^T$. The center of mass G of the pole is at a distance r from O. The cart has mass M, and the pole has mass m and moment of inertia J_G with respect to the center of mass G. The gravitational acceleration $g = 9.81$ m/s^2 acts in the $-y$ direction of a world frame. A horizontal force $u > 0$ acts in the $+x$ direction of a world frame on the cart and represents the only control input applied to the system. The goal of the MPC regulator is to apply a proper force u to move the system from the initial state $x_0 = [0\,0\,0\,0]^T$ to the final state $x_N = [q_1^f\ 0\,0\,0]^T$, where q_1^f represents a position *beyond* the two obstacles jutting downward from the ceiling (Fig. 1). The novelty with respect to the classical cart-pole control problem is represented by the introduction of such obstacles. The requirement that the controller has to steer the pole to its final state while *always* maintaining a minimum clearance from the obstacles represents an additional, non trivial, difficulty, as the joint angle q_2 must belong to $[-\pi/2, \pi/2]$.

The Lagrangian dynamics of the system can be written in the standard form as:

$$B(q)\ddot{q} + n(q,\dot{q}) = Pu \tag{56}$$

where $B(q) \in \mathbb{R}^{2\times 2}$ is the symmetric, positive definite *inertia matrix*, $n(q,\dot{q}) \in \mathbb{R}^2$ is the vector collecting *Coriolis*, *centrifugal* and *gravitational* terms, $P \in \mathbb{R}^2$ is the actuation vector, and $u \in \mathbb{R}$ is the horizontal force applied to the cart.

For the cart-pole system, the explicit expressions of the above quantities are:

$$B(q) = \begin{bmatrix} M_t & -mlc_2 \\ -mlc_2 & J_O \end{bmatrix}, \quad n(q,\dot{q}) = \begin{bmatrix} ml\dot{q}_2^2 s_2 \\ -mgls_2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{57}$$

where $c_2 = \cos q_2 = \cos x_2$ and $s_2 = \sin q_2 = \sin x_2$, and the positions $M_t = M + m$, $J_O = J_G + ml^2$ were made. It is worth noting that the above equations are equivalent to: (i) the translational dynamic equilibrium of the whole system along $+x$, and (ii) the rotational dynamic equilibrium around hinge O.

The explicit nonlinear equations in state space form appear as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ -a(x_2)mls_2\left(J_O x_4^2 - mglc_2\right) \\ -a(x_2)mls_2\left(mlx_4^2 c_2 - M_t g\right) \end{bmatrix} + a(x_2) \begin{bmatrix} 0 \\ 0 \\ J_O \\ mlc_2 \end{bmatrix} u \tag{58}$$

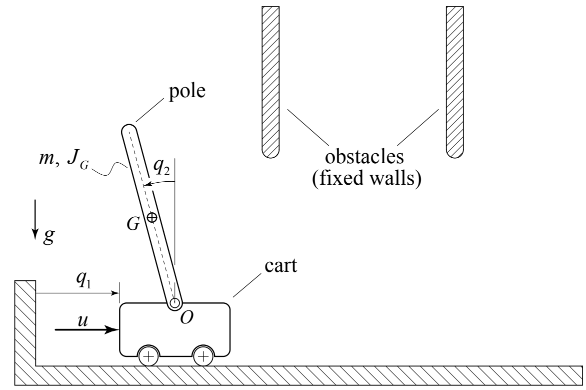

Fig. 1. Schematic of the cart-pole system with obstacles.

where we defined $a(x_2) = [J_O M_t - (ml\cos x_2)^2]^{-1}$.

Results and discussion. We first consider the case in which the full state of the system is measurable, i.e. $h(x) = x$ in (2), and we compare two MPCs:

- NMPC0 uses no disturbance model: $F(x,d,u) = f(x,u)$ and $H(x,d) = h(x) = x$, with a linear deadbeat observer: $\kappa_x(e) = e$.
- NMPC1 uses a state disturbance model: $F(x,d,u) = f(x,u) + d$ and $H(x,d) = h(x) = x$, with a deadbeat augmented linear observer: $\kappa_x(e) = e$ and $\kappa_d(e) = e$.

For both controllers we use a discretization time $T_s = 0.12$ s, an MPC horizon with $N = 25$, and a quadratic cost function. The optimal control problem is solved in both state and control sequences, according to a direct multiple shooting scheme in which integration of the model is performed using a fixed step $(T_s/10)$ Runge-Kutta algorithm. Control force is constrained in the range $[-100,\ 100]$ N, and during the simulation an external unmeasured disturbance force is superimposed to test the disturbance rejection capabilities of the controllers.

Closed-loop simulation results are reported in Fig. 2: the top panel shows the horizontal displacement and rotation angle, along with the system inputs (control force and unmeasured disturbance), vs. time. The bottom panel reports the prediction errors on these two states. An accompanying video showing the cart-pole system behavior is available in (Pannocchia et al., 2015). From these results we see that, due to persistent prediction errors, NMPC0 is not able to stabilize the cart-pole system in the final position. On the other hand, NMPC1 succeeds in this task thanks to the quick suppression of the prediction error.

We next consider the more difficult case in which only two states, cart displacement and pole rotation angle, are measured.

- NMPC2 uses a linear (input) disturbance model with one disturbance state: $F(x,d,u) = f(x,u) + B_d d$ and $H(x,d) = h(x) = x$, in which B_d is chosen as the Jacobian matrix of $f(x,u)$ with respect to the input u at the final equilibrium. A linear observer, $\kappa_x(e) = K_x e$ and $\kappa_d(e) = K_d e$, is designed as a steady-state Kalman filter for the linearized augmented model at the final equilibrium.
- NMPC3 uses a linear (input/output) disturbance model with two disturbance states: $F(x,d,u) = f(x,u) + B_d d$ and $H(x,d) = h(x) + C_d d = x + C_d d$, in which the first column of B_d is chosen as the Jacobian matrix of $f(x,u)$ with respect to the input u at the final equilibrium, whereas the second column of B_d and C_d are chosen so that

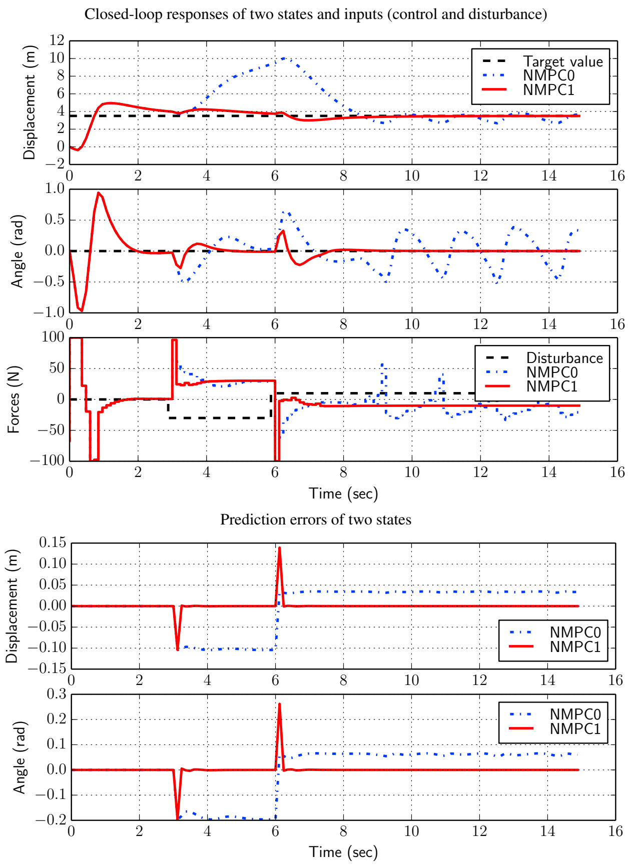

Fig. 2. State feedback control of the cart-pole: comparison of standard vs. offset-free NMPC.

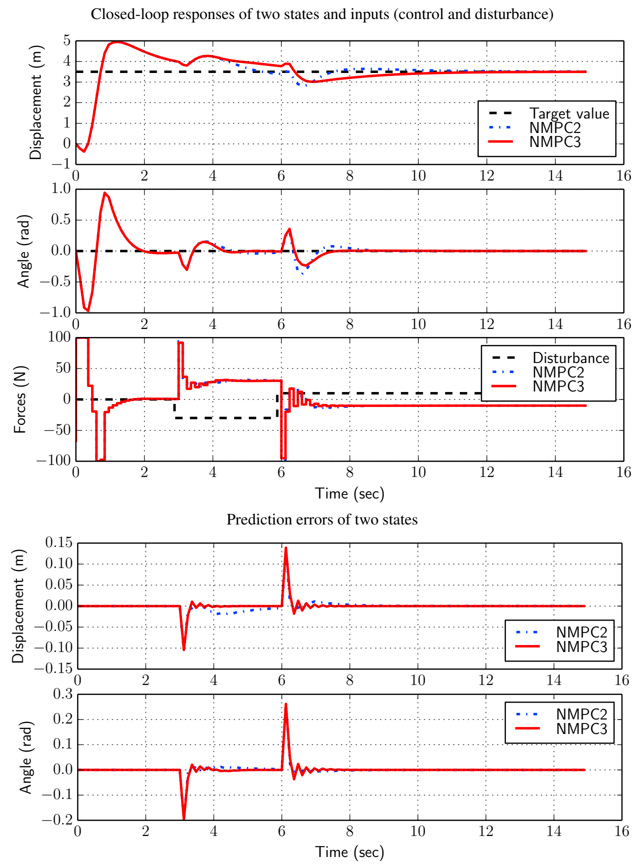

Fig. 3. Output feedback control of the cart-pole: comparison of two offset-free NMPCs.

the augmented system is observable. A linear observer, $\kappa_x(e) = K_x e$ and $\kappa_d(e) = K_d e$, is designed as a steady-state Kalman filter for the linearized augmented model at the final equilibrium.

Results are reported in Fig. 3. We notice that both MPCs succeed in the task with rather similar performances. It should be noticed that NMPC2 is using only one disturbance state even if two measurements are available, and hence Assumption 9 does not hold for NMPC2. Nonetheless, it is able to suppress offset because the actual disturbance enters the system with the same dynamics of the input. At any steady state this effect is correctly modeled by the chosen linear (input) disturbance model, but different disturbances may not be rejected by NMPC2. On the other hand, NMPC3 uses two disturbances, and therefore it is capable of ensuring offset-free reconstruction by zeroing the prediction error for any asymptotically constant disturbance. It should be mentioned that the design of an observable augmented system is nontrivial for this example because of the presence of two integrators in the discretized model dynamics (at any equilibrium). For instance, a linear output disturbance model with $B_d = 0$ and $C_d = I$ is not expected to be observable for this case.

5.2 Offset-free linear MPC of a nonisothermal CSTR

Process description and control objectives. The second example is a nonisothermal continuous stirred tank reaction (CSTR) in which an irreversible, exothermic reaction A→B occurs in the liquid phase. The process is described by the following set of ODEs, in which the state $x = [c\ T\ h]^T$ (reactant concentration, reactor temperature, liquid level) and $u = [T_c\ F]^T$ (coolant temperature and outlet volumetric flow rate), and the model parameters are taken from (Rawlings and Mayne, 2009, p. 52).

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix} \frac{F_0 c_0 - u_2 x_1}{\pi r^2 x_3} - k_0 \exp(-\frac{E}{R x_2}) x_1 \\ \frac{F_0 (T_0 - x_2)}{\pi r^2) x_3} - \frac{\Delta H}{\rho C_p} k_0 \exp(-\frac{E}{R x_2}) x_1 + \frac{2 U_0}{r \rho C_p}(u_1 - x_2) \\ \frac{F_0 - u_2}{\pi r^2} \end{bmatrix} \tag{59}$$

The input is constrained $u \in [295, 305]\,\mathrm{K} \times [0, 0.25]\,\mathrm{m}^3/\mathrm{min}$, and the control task is to regulate the outlet concentration at $\bar{c} = 0.500\,\mathrm{kmol/m}^3$ and the liquid level at $\bar{h} = 0.659\,\mathrm{m}$. This operation is particularly difficult because: (i) this is an open-loop unstable equilibrium, (ii) we use a *linear model* $x^+ = f(x, u)$ in all MPCs, while the controlled process (59) is highly nonlinear because of the exothermic reaction, (iii) from time 5 min to 6 min the inlet flow rate increases linearly acting as unmeasured disturbance. All MPCs use a horizon of $N = 50$, a discretization time of 0.2 min, and a quadratic cost function.

Results and discussion. We consider the state feedback case, i.e. $y = h(x) = x$, and define the controlled variable as $y_c = r(y) = [x_1\ x_3]^T$. We compare two linear MPCs:

- LMPC0 uses a (partial) state disturbance model: $F(x, d, u) = f(x, u) + \begin{bmatrix} d_1 \\ 0 \\ d_2 \end{bmatrix}$, with deadbeat augmented state observer $K_x = I$, $K_d = \left[\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}\right]$.
- LMPC1 uses a (full) state disturbance model: $F(x, d, u) = f(x, u) + d$ with deadbeat augmented state observer $K_x = I$, $K_d = I$.

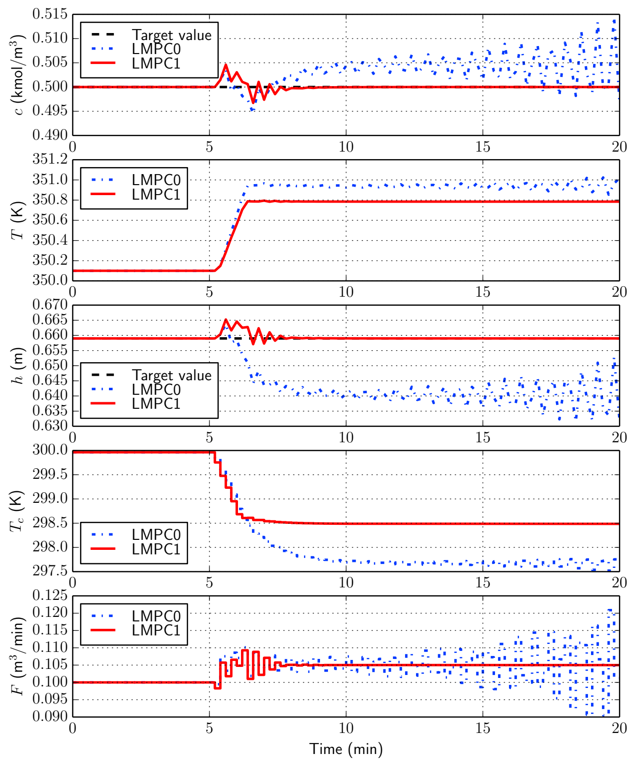

Fig. 4. State feedback control of the CSTR: comparison of states and inputs using linear MPCs.

We remark that both augmented models are observable and the associated deadbeat observers are nominally stable. However, we notice that in LMPC0, disturbances are added only to the two controlled states (x_1 and x_3), but not to the other measured state (x_2), whereas in LMPC1 a disturbance is added to each state. Hence Assumption 9 does not hold for LMPC0.

Closed-loop results (states and inputs) are shown in Fig. 4. From these results, we see that LMPC0 is unable to achieve offset-free control in the two controlled variables when the inlet flow rate disturbance occurs, and indeed the closed-loop system becomes unstable due to the inherent mismatch between the nominal linear model and the nonlinear controlled process (59). Depending on the observer and controller tuning, we could obtain a stable closed-loop behavior with LMPC0, but still there would be offset because of the use of two disturbances in the presence of three measurements. It is also possible to find an augmented observer gain (K_x, K_d) in a way that offset-free control is achieved, but this observer gain would depend on the MPC cost function parameters (Pannocchia and Rawlings, 2003; Maeder et al., 2009). Such a practice is not recommended, in general, because it requires simultaneous retuning of the observer whenever the controller cost function parameters (state and input penalties) change. On the other hand, LMPC1 is perfectly able to achieve offset-free control given that it uses the *correct* number of disturbances in its augmented model. Finally, given the equivalence results discussed in Sections 3.3 and 3.4, an MPC based on the velocity form model would behave identically to LMPC1 because it would (inherently) use three disturbances.

5.3 Offset-free economic MPC of a nonisothermal CSTR

The last example that we present concerns the same process of Section 5.2, but in this case the nonlinear nominal model

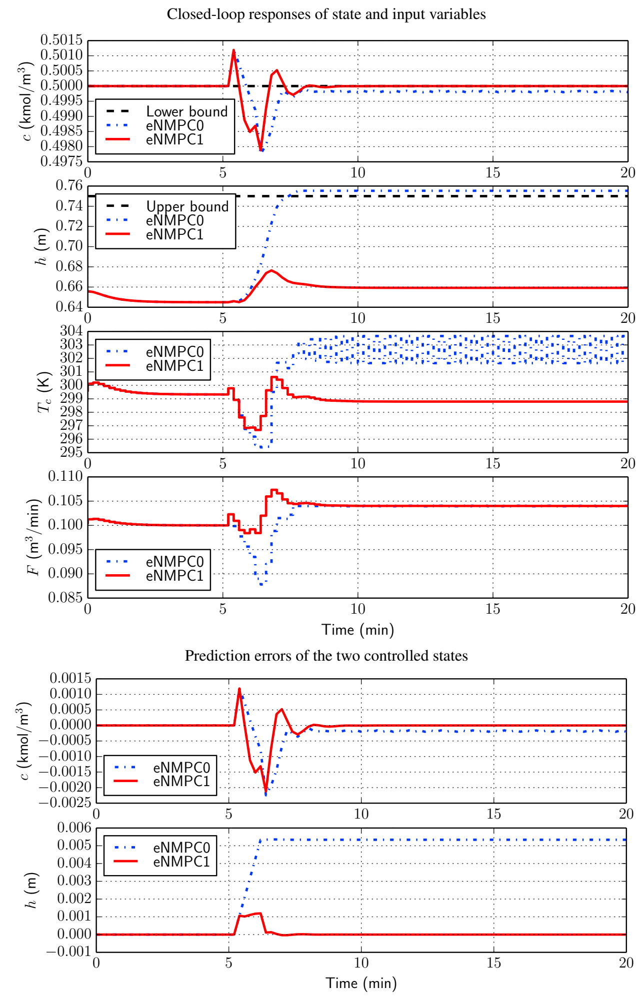

Fig. 5. Output feedback control of the CSTR: comparison of standard vs. offset-free economic NMPCs.

(59) is used in the design of economic nonlinear MPCs, as described in Section 4. We assume to measure only x_1 and x_3, i.e. $y = [x_1 \;\; x_3]^T$, and the economic cost function is $\ell_e(y,u) = y_1$, so that the reactant concentration has to be minimized, while the following output constraints are imposed:

$$0.5 \leq y_1 \leq 1.0, \quad 0.5 \leq y_2 := x_3 \leq 0.75 \qquad (60)$$

As in the previous section, from time 5 min to 6 min the inlet flow rate increases linearly, acting as an unmeasured disturbance. All MPCs use a horizon of $N = 50$ with a discretization time of 0.2 min.

Two output feedback economic NMPCs are compared:

- eNMPC0 uses no disturbance model: $F(x,d,u) = f(x,u)$ and $H(x,d) = h(x) = x$ and a linear state observer: $\kappa_x(e) = K_x e$, in which K_x is chosen as a steady-state Kalman filter gain at the nominal equilibrium.
- eNMPC1 uses a linear (input) disturbance model: $F(x,d,u) = f(x,u) + B_d d$ and $H(x,d) = h(x) = x$, in which B_d is chosen as the Jacobian matrix of $f(x,u)$ with respect to the input u at the nominal equilibrium. A linear observer, $\kappa_x(e) = K_x e$ and $\kappa_d(e) = K_d e$, is designed as a steady-state Kalman filter for the linearized augmented model at the nominal equilibrium.

Closed-loop results are reported in Fig. 5: the top panel shows controlled and inputs variables, whereas the bottom panel shows the prediction error on the two controlled variables. From these results we observe that eNMPC0 makes the closed-loop system unstable due to the fact that output constraints are permanently violated. In fact, both output prediction errors do not vanish at steady state, whereas in eNMPC1 both output prediction errors are quickly eliminated, and consequently the actual output constraints are satisfied. From this example, we can appreciate how offset-free prediction of the outputs is necessary to make sure that an economic controller does not destabilize the closed-loop system in the attempt to recover from a current output that does not satisfy the constraints.

6. CONCLUSIONS

This paper has described the latest advances in the design of offset-free MPC algorithms. We presented a self-contained summary of the available results for nonlinear MPC, based on the use of disturbance models and observers, and we extended the existing asymptotic convergence results. Then we focused on linear MPC and showed that a commonly known method based on the velocity form model is indeed a particular case of disturbance model, and not an alternative route to offset-free tracking. We also extended the concept of offset-free estimation to the design of economic MPC for systems with persistent errors/disturbances. Challenging examples of nonlinear processes (controlled by nonlinear MPC, linear MPC, and economic MPC) have been included to highlight the significance of the presented results, and also to emphasize specific subtleties related to the number of used disturbances and to the process dynamics, which may result in an incorrect design.

A final note is reserved to future directions in this research area. On the one hand, it may be useful to explore the issue of disturbance modeling (i.e. where to put the disturbances) in nonlinear system, coupled with nonlinear observers (MHE, most notably). On the other, robust stability questions for offset-free (linear, nonlinear, economic) MPC are still wide open.

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