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# SPC: SUBSPACE PREDICTIVE CONTROL

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Abstract: Subspace identification has proven to be an excellent system identification method under peculiar industrial situations. Model predictive control on the other hand also turned out to be a very competitive method, especially in chemical industry. In practise, the identification and the control of a system are almost always considered as two separate problems. In the present paper some remarkable analogies between subspace identification and model predictive control are uncovered. Both methods can be combined in a very elegant way to form a numerically robust and easily implementable control/identification algorithm. This is the reason why we refer to it as subspace predictive control. The main result is that the system identification step and the controller design are done simultaneously. Starting from input and output measurements of the unknown system, only a QR-decomposition followed by a SV-decomposition are required to find the controller parameters. *Copyright © 1999 IFAC*

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## 1. INTRODUCTION

In the late eighties and early nineties a new identification method, called subspace system identification, was developed. Since then, many paper have been published in this area. Among others we cite (Moonen *et al.*, 1989), (Verhaegen and Dewilde, 1992), (Viberg *et al.*, 1997), (De Moor *et al.*, 1998) and the book (Van Overschee and De Moor, 1996). Subspace identification proved to

be a very efficient system identification method, especially under peculiar industrial conditions such as high order systems with multiple inputs and outputs. Many papers appeared where subspace identification was successfully applied (see e.g. (Favoreel *et al.*, 1998) for different examples). The main advantages of subspace identification over more classical methods, such as prediction error methods, are that they are non-iterative (thus no convergence problems) and numerically robust (since only based on basic linear algebra techniques). Moreover, MIMO systems can be treated as easily as SISO systems.

Model predictive control (MPC) on the other hand is a general name for a whole class of model-

based control methods that appeared in the late seventies and early eighties (Richalet *et al.*, 1978), (Cutler and Ramaker, 1979), (Garcia *et al.*, 1989). Since MPC has its origins in industry it is no use to say that the number of successful applications, especially in chemical industry, is uncountable.

In this paper we propose a new method, based on results from the field of subspace system identification on the one hand and model predictive control on the other hand, to combine the identification and the predictive control design of linear systems in one single operation. In the literature, the identification and the control of a system is hardly ever considered as one big problem. The classical way of controlling a system is indeed that of making a model of the system after which this model can be used to design a controller. The main result of the present paper is that the system identification step and the calculation of the controller parameters are replaced by a QR-decomposition followed by a singular value decomposition (SVD) of a matrix that is constructed from input and output measurements of the unknown system only.

We will consider linear time-invariant systems of the form:

$$x_{k+1} = Ax_k + Bu_k + Ke_k, \qquad (1)$$
$$y_k = Cx_k + Du_k + e_k. \qquad (2)$$

where the inputs $u_k \in I\!\!R^m$, the outputs $y_k \in I\!\!R^l$ and the states $x_k \in I\!\!R^n$. The noise sequence $e_k$ is supposed to be zero-mean Gaussian white with variance $\mathbf{E}[e_p e_q^T] = S\delta_{pq}$.

The outline of the paper is as follows. In Section 2 we recall some basic results of subspace system identification. The main ideas are explained and the importance of the QR-decomposition and SVD are illustrated. Section 3 contains the main results of the paper. It describes how subspace identification and model predictive control can be combined in a very elegant way.

## 2. BRIEF OVERVIEW OF LINEAR SUBSPACE SYSTEM IDENTIFICATION

The problem treated in linear subspace identification can be described as follows:

> Given measurements of the inputs $u_k$ and the outputs $y_k$ of an unknown linear time-invariant system (1)-(2), *find* an estimate of the system matrices $A, B, C, D, K$ and $S$.

In the following sections we recall the basic results of subspace identification.

### 2.1 *Matrix input-output equations*

The starting point of subspace identification is the following set of matrix input-output equations:

$$Y_f = \Gamma_N X_f + H_N U_f + H_N^s E_f, \qquad (3)$$
$$Y_p = \Gamma_M X_p + H_M U_p + H_M^s E_p \qquad (4)$$

where the different terms will be explained below. If the available measurements of the inputs and the outputs are $u_k, y_k$ for $k \in \{1, \ldots, j+M+N\}$, then we can construct the following data block Hankel matrices:

$$U_p = \begin{pmatrix} u_1 & u_2 & \ldots & u_j \\ u_2 & u_3 & \ldots & u_{j+1} \\ \ldots & \ldots & \ldots & \ldots \\ u_M & u_{M+1} & \ldots & u_{j+M-1} \end{pmatrix}, \qquad (5)$$

$$U_f = \begin{pmatrix} u_{M-1} & u_{M+2} & \ldots & u_{M+j} \\ u_{M+2} & u_{M+3} & \ldots & u_{M+j+1} \\ \ldots & \ldots & \ldots & \ldots \\ u_{M+N} & u_{M+N+1} & \ldots & u_{M+N+j} \end{pmatrix} \qquad (6)$$

where the indices $p$ and $f$ stand for *past* and *future*. In a similar way we can also define the block Hankel matrices $Y_p, Y_f$ and $E_p, E_f$ containing the outputs $y_k$ and the measurement noise $e_k$ respectively. The number of columns in these data matrices $j$ should be much larger (typically 100 times) than the number of block rows $N$ and $M$. We will also use the following short-hand notation:

$$W_p = \begin{pmatrix} Y_p \\ U_p \end{pmatrix}.$$

The past and future state sequences are defined as:

$$X_p = \begin{pmatrix} x_1 & x_2 & \ldots & x_j \end{pmatrix},$$
$$X_f = \begin{pmatrix} x_{M+1} & x_{M+2} & \ldots & x_{M+j} \end{pmatrix}.$$

Further we also have the following system related matrices:

$$\Gamma_q = \begin{pmatrix} C \\ CA \\ \ldots \\ CA^{q-1} \end{pmatrix},$$

$$H_q = \begin{pmatrix} D & 0 & \ldots & 0 \\ CB & D & \ldots & 0 \\ \ldots & \ldots & \ldots \ldots \\ CA^{q-2}B & CA^{q-3}B & \ldots & D \end{pmatrix},$$

$$H_q^s = \begin{pmatrix} I_l & 0 & \ldots & 0 \\ CK & I_l & \ldots & 0 \\ \ldots & \ldots & \ldots \ldots \\ CA^{q-2}K & CA^{q-3}K & \ldots & I_l \end{pmatrix},$$

where $q \in \mathbb{N}_0$, $\Gamma_q$ is the extended observability matrix, $H_q$ and $H_q^s$ the block Toeplitz matrices

containing the impulse response of the system to the deterministic input $u_k$ and the stochastic input $e_k$ respectively.

A very commonly used tool in subspace identification is the *orthogonal projection* of the row space of matrices. The projection of the row space of $\mathcal{A} \in {I\!\!R}^{p \times j}$ into the row space of $\mathcal{B} \in {I\!\!R}^{q \times j}$ is denoted by $\mathcal{A}/\mathcal{B}$ and defined as:

$$\mathcal{A}/\mathcal{B} = \mathcal{A}\mathcal{B}^{\dagger}\mathcal{B}.$$

### 2.2 Basics of subspace identification

Although there exist several subspace identification methods in the literature (Van Overschee and De Moor, 1996), (Verhaegen and Dewilde, 1992), (Viberg *et al.*, 1997) they all have two main steps in common.

**Step 1:** The first step of the subspace identification problem can be interpreted as follows: *given* the past inputs and outputs $W_p$ and the future inputs $U_f$, *find* a prediction of the future outputs $Y_f$. Inspired by (3)-(4) we use a linear predictor of the form:

$$\widehat{Y}_f = L_w W_p + L_u U_f. \tag{7}$$

The least squares prediction $\widehat{Y}_f$ of $Y_f$ can be found as the solution of the following least squares problem:

$$\min_{L_w, L_u} ||Y_f - \left( L_w \ L_u \right) \begin{pmatrix} W_p \\ U_f \end{pmatrix}||_F^2.$$

Usually, the problem of finding $\widehat{Y}_f$ is formulated in terms of the orthogonal projection of the row space of $Y_f$ into the row space spanned by $W_p$ and $U_f$:

$$\widehat{Y}_f = Y_f / \begin{pmatrix} W_p \\ U_f \end{pmatrix} = Y_f \begin{pmatrix} W_p \\ U_f \end{pmatrix}^{\dagger} \begin{pmatrix} W_p \\ U_f \end{pmatrix}. \tag{8}$$

The implementation of this projection can be done in a numerically very efficient way (Verhaegen and Dewilde, 1992) by making the following QR-decomposition (which is a standard Matlab command):

$$\begin{pmatrix} W_p \\ U_f \\ Y_f \end{pmatrix} = \begin{pmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{pmatrix}. \tag{9}$$

By posing:

$$L = \left( R_{31} \ R_{32} \right) \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}^{\dagger}$$

it is easy to show that (8) can be written as:

$$\widehat{Y}_f = L \begin{pmatrix} W_p \\ U_f \end{pmatrix}.$$

We then know that:

$$L_w = L(:, 1 : N(m + l)),$$
$$L_u = L(:, N(m + l) + 1 : end).$$

These two matrices will play a very important role in the next section where the link between subspace identification and predictive control is made.

**Step 2:** The second step then consists in calculating the SVD of $L_w$. If the number of columns in the data block Hankel matrices $Y_f, U_f, W_p$ is infinite ($j = \infty$) this is a rank deficient matrix of order $n$. Due to the noise however, $L_w$ will not be rank-deficient in practise. To get rid of a part of the noise, a singular value decomposition is used to approximate $L_w$ by a rank deficient matrix:

$$L_w = \left( U_1 \ U_2 \right) \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix}$$
$$\approx U_1 S_1 V_1^T \tag{10}$$

where the rank is determined by inspecting the number of dominant singular values $S_1$. This is an approximation of the order $n$ of the system. Important is that, under the assumption that the number of columns in the data block Hankel matrices $Y_f, U_f, W_p$ is infinite ($j = \infty$) there exists a direct link between $L_w$ and the observability matrix $\Gamma_N$ and the state sequence $X_f$:

$$\Gamma_N = U_1 S_1^{1/2},$$
$$\widehat{X}_f = S_1^{1/2} V_1^T W_p.$$

In (Van Overschee and De Moor, 1996) it is proven that $\widehat{X}_f$ is a Kalman filter estimate of the state sequence $X_f$.

The main point where the different subspace identification algorithms diverge is the way the state space matrices $A, B, C, D$ and $K, S$ are calculated from $\Gamma_N$ and/or $\widehat{X}_f$. One can distinguish three classes of algorithms: those that use the observability matrix $\Gamma_N$, those that use the estimate $\widehat{X}_f$ of the state sequence $X_f$ and finally those that use both.

## 3. SUBSPACE IDENTIFICATION BASED IMPLEMENTATION OF MPC

### 3.1 Main results

Let us start this section by stating a general predictive control problem.

**Predictive Control Problem**

**Given** *a future reference output $r_k$ for $k \in \{1, \dots, N\}$ and measurements of the inputs $u_k$ and the outputs $y_k$ of the unknown system (1)-(2),* **find** *the input sequence $u_f = (u_1, \dots u_N)$ such that the following quadratic cost function $J$ is minimized:*

$$J = \sum_{k=1}^{N} (\widehat{y}_k - r_k)^T Q_k (\widehat{y}_k - r_k) + u_k^T R_k u_k \quad (11)$$

*where $\widehat{y}_k$ is the k-step-ahead predicted output, $u_k$ the future input and $Q_k \in \mathbb{R}^{l \times l}$ and $R_k \in \mathbb{R}^{m \times m}$ user-defined weighting matrices.*

The first term in this equation penalizes the fact that the predicted outputs $\widehat{y}_k$ are different from the reference outputs $r_k$ while the second term keeps the energy in the inputs $u_k$ at a reasonable level. Only the first calculated input $u_1$ is implemented. At every time-step the above problem is thus solved with the newly obtained measurement of the output. In the literature this is commonly called the receding horizon principle.

We call the *forward horizon* the number of time steps $N$ over which the system output $\widehat{y}_k$ is predicted and the control input $u_k$ is calculated. The *backward horizon* is the number of past input and output data points $M$ that are used to predict the future outputs $\widehat{y}_k$. We assume that the last measured output and implemented input are $y_0$ and $u_0$ respectively. The forward horizon therefore starts at time step 1 and ends at time step $N$ (see Figure 1).
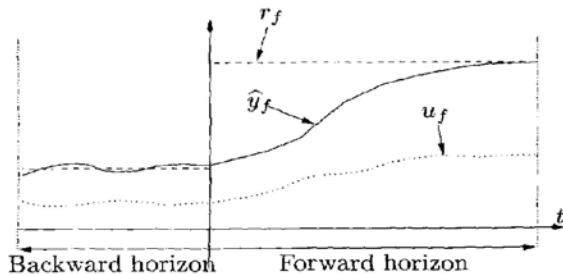


Figure 1. **Principle of MPC. The aim is to find the future control sequence $u_f$ that minimizes the performance criterion $J$. The prediction of the forward horizon is done on the basis of the knowledge of the backward horizon.**

Defining the future input sequence $u_f$, the corresponding estimated future output sequence $\widehat{y}_f$ and the future reference output trajectory $r_f$ as:

$$u_f = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}, \quad \widehat{y}_f = \begin{pmatrix} \widehat{y}_1 \\ \widehat{y}_2 \\ \vdots \\ \widehat{y}_N \end{pmatrix}, \quad r_f = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix},$$

the control criterion (11) can also be written as:

$$J = (\widehat{y}_f - r_f)^T Q (\widehat{y}_f - r_f) + u_f^T R u_f$$

where the matrices $Q \in \mathbb{R}^{Nl \times Nl}$ and $R \in \mathbb{R}^{Nm \times Nm}$ are defined as:

$$Q = \begin{pmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_N \end{pmatrix}, R = \begin{pmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_N \end{pmatrix}.$$

From (7) we then know that the output sequence $y_f$ can be expressed as:

$$\widehat{y}_f = L_w w_p + L_u u_f$$

where

$$w_p = \begin{pmatrix} y_p \\ u_p \end{pmatrix}$$

with $u_p \in \mathbb{R}^{Nm}$ and $y_p \in \mathbb{R}^{Nl}$ the $M$ last known values of the inputs and the outputs:

$$y_p = \begin{pmatrix} y_{-M+1} \\ \vdots \\ y_{-1} \\ y_0 \end{pmatrix}, u_p = \begin{pmatrix} u_{-M+1} \\ \vdots \\ u_{-1} \\ u_0 \end{pmatrix}.$$

The performance criterion $J$ can now be written as:

$$J = (L_w w_p + L_u u_f - r_f)^T Q (L_w w_p + L_u u_f - r_f) + u_f^T R u_f.$$

Putting the trace of the derivative of $J$ with respect to the input sequence $u_f$ to zero:

$$\text{tr}\, \frac{\delta J}{\delta u_f} = 0$$

the SPC control law becomes:

$$u_f = (R + L_u^T Q L_u)^{-1} L_u^T Q (r_f - L_w w_p). \quad (12)$$

Only the first element $u_1$ will be implemented:

$$u_1 = -L_w^c w_p + L_r^c r_f$$

where $L_u^c$ and $L_r^c$ are defined as:

$$L_r^c = \left\{ (R + L_u^T Q L_u)^{-1} \right\} (1:m,:) L_u^T Q,$$
$$L_w^c = L_r^c L_w.$$

**Remarks:**

- At first it might seem not so obvious to fit the SPC control law (12) into the general form of a state feedback controller:

$$u_1 = -K_c \widehat{x}_1$$

where $K_c$ is the feedback gain and $\widehat{x}_1$ the estimated state of the system. To see the link we have to go back to subspace system identification theory. In Section 2 we have seen that the term $L_w w_p$ has the nice property that it can be written as the product between the observability matrix $\Gamma_N$ and a specific Kalman filter estimate $\widehat{x}_1$ of the system:

$$L_w w_p = \Gamma_N \widehat{x}_1$$

where $\Gamma_N = U_1 S_1^{1/2}$ and $\widehat{x}_1 = S_1^{1/2} V_1^T w_p$. See also (Van Overschee and De Moor, 1996) or (De Moor et al., 1998) for more details. Therefore we have that the feedback gain and the Kalman filter state estimate of the SPC-controller are:

$$K_c = L_r^c U_1 S_1^{1/2},$$
$$\widehat{x}_1 = S_1^{1/2} V_1^T w_p.$$

- The classical way of implementing an MPC-controller consists in mainly three steps. In the first step one performs a system identification which allows for the calculation of the system matrices $A, B, C, D, K$ and $S$. These are then used to design a state observer. In the last step the state feedback gain $K_c$ is calculated. In the SPC-approach presented here, these three different steps are replaced by a QR-decomposition (9) and a singular value decomposition (10) of a matrix only containing input and output data. Moreover, the system matrices $A, B, C, D, K$ and $S$ do not have to be calculated explicitly. The only parameters that are of importance are $L_u$ and $L_w$. It is also worth noting that different numerically robust techniques exist for an efficient and fast calculation of QR and SV-decompositions. This makes them particularly attractive for on-line use.

### 3.2 Algorithm

The final subspace predictive control algorithm can now be implemented as in Figure 2.

### 4. CONCLUSIONS

In this paper we have shown that subspace identification and model predictive control can be

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**Subspace Predictive Control algorithm**

(1) Construct the data block Hankel matrices $Y_f, U_f$ and $W_p$ from the available input and output measurements.

(2) Solve the following least squares problem (preferably with a QR-decomposition) for the unknown parameters $L_w$ and $L_u$:

$$\min_{L_w, L_u} ||Y_f - \begin{pmatrix} L_w & L_u \end{pmatrix} \begin{pmatrix} W_p \\ U_f \end{pmatrix} ||_F^2.$$

(3) Approximate $L_w$ by a rank-$n$ matrix by taking the singular value decomposition i.e.:

$$L_w = U_1 S_1 V_1^T.$$

An estimate of the system order $n$ can be found by inspecting the number of dominant singular values. This also gives an idea of the length of the backward and forward horizons $M$ and $N$.

(4) Construct the controller inputs:

$$w_p = ( y_{-M+1}^T \cdots y_{-1}^T \ y_0^T |$$
$$u_{-M+1}^T \cdots u_{-1}^T \ u_0^T )^T .$$

(5) Implement the first input $u_1$ of the SPC-control sequence $u_f$:

$$u_f = (R + L_u^T Q L_u)^{-1} L_u^T Q (L_w w_p - r_f).$$

(6) To implement the following control step repeat from step 4 with the newly measured output $y_1$.

Figure 2. **Algorithm for the implementation of an SPC-controller. The first three steps only have to be calculated once. They mainly consist of a QR-decomposition and a singular value decomposition. The remaining steps should be applied at every control step.**

combined in a very elegant way in order to obtain a numerically robust, fast and easy implementable control/identification algorithm. Starting from input-output data of the system, a QR-decomposition followed by a SV-decomposition are sufficient to obtain the system parameters and the model predictive controller parameters.

*fication of nonlinear systems*, IWT-Eureka SINOP-SYS: *Model-based structural monitoring using in-operation system identification*), FWO-Vlaanderen: *Analysis and design of matrix algorithms for adaptive signal processing, system identification and control, based on concepts from continuous time system theory and differential geometry, Numerical algorithms for subspace system identification: Extension towards specific applications*, FWO-Onderzoeksgemeenschappen: *Identification and Control of Complex Systems, Advanced Numerical Methods for Mathematical Modeling*); **Belgian Federal Government**( Interuniversity Attraction Pole IUAP IV/02: *Modeling, Identification, Simulation and Control of Complex Systems*, Interuniversity Attraction Pole IUAP IV/24: IMechS: Intelligent Mechatronic Systems); **European Commission:** (Human Capital and Mobility: *SIMONET: System Identification and Modeling Network, SCIENCE-ERNSI: European Research Network for System Identification.*)

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