General Concepts

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1 Fanelli

1.1 Seasonality modelling in commodity and energy markets

Motivation and scope. Many commodities (especially energy) exhibit recurring patterns in *levels* and/or in *volatility*. Typical examples are winter–winter or summer–summer demand cycles, and, for electricity, strong intraweek patterns (weekdays vs. weekends) that show up as an approximately 7-day periodicity in autocorrelation diagnostics.

1.1.1 Seasonality in spot price models

Deterministic seasonal component as a Fourier series. A standard way to represent a smooth periodic seasonal component is a finite Fourier expansion (harmonic regression). Let t denote time measured in years (or in fractions of a trading year), and let $Y(t)$ be the deterministic seasonal component at time t. A widely used parametrisation is

$$Y(t) \;=\; \alpha_0 + \sum_{i=1}^{N} \Big[\alpha_i \cos(2\pi i t) + \beta_i \sin(2\pi i t)\Big], \tag{2.12}$$

where $i = 1$ corresponds to annual periodicity, $i = 2$ to semi-annual, $i = 3$ to quarterly, etc.

Estimation by OLS. The coefficients can be obtained by ordinary least squares by treating the trigonometric terms as non-stochastic regressors:

$$Y(t) \;=\; \alpha_0 + \sum_{i=1}^{N} \Big[\alpha_i \cos(2\pi i t) + \beta_i \sin(2\pi i t)\Big] + \varepsilon(t), \qquad \varepsilon(t) \sim \mathcal{N}(0, \sigma^2), \tag{2.13}$$

where σ^2 is estimated from regression residuals.

Practical time scaling. When working with daily data, it is common to convert the observation index into a fraction of a commercial year (e.g. 252 trading days), so that t increases by $1/252$ per trading day and the trigonometric terms represent annual harmonics.

Additive decomposition with deterministic seasonality. In several spot modelling approaches (in particular for electricity), one separates a deterministic seasonal function from the stochastic component by writing

$$s(t) \;=\; f(t) + x(t), \tag{1}$$

where $f(t)$ captures seasonality at different frequencies (daily/weekly/monthly/yearly) and $x(t)$ captures the unpredictable dynamics. This decomposition is often embedded in richer dynamics such as regime-switching models.

1.1.2 Seasonality in forward/futures valuation via spot decomposition

Deterministic seasonal component inside a spot-factor model. A common modelling template for storable or partially storable commodities is to decompose the (log-)spot into components and allow for an explicit deterministic seasonal term:

$$S(t) \;=\; s(t) + X(t) + L(t), \tag{2}$$

where $X(t)$ and $L(t)$ represent short- and long-term components, and $s(t)$ is a deterministic seasonal component. Forward/futures prices are then obtained by applying the chosen valuation relationship (e.g. cost-of-carry / factor-model valuation).

1.2 Seasonality modelling in the forward curve

1.2.1 Seasonality through a seasonal cost-of-carry relationship

Non-seasonal level proxy and monthly seasonal premia. One approach assumes liquid maturities up to one year (or an integer number of years) and separates (i) a non-seasonal market level from (ii) seasonal premia attached to calendar months. Consider the geometric average of observed forward prices at time t:

$$F(t) \;=\; \left(\prod_{T=1}^{N} F(t,T)\right)^{1/N}, \qquad N = 12k,\; k = 1, 2, \ldots, \tag{3}$$

which serves as a non-seasonal proxy for the overall futures level. The seasonal cost-of-carry representation is

$$F(t,T) \;=\; F(t)\exp\Big(\big[s(T) - \gamma(t)\big](T-t)\Big), \tag{4}$$

where $s(T) = s(M)$ with $M \in \{1, \ldots, 12\}$ is a deterministic monthly seasonal premium (high-demand / low-supply periods), and $\gamma(t, T-t)$ is a stochastic premium (stochastic cost-of-carry) indexed by time-to-maturity.

1.2.2 Seasonality through volatility structures (HJM-style view)

Normalising by spot volatility and PCA. A second route is to attribute seasonality to the *volatility structure* of forwards. One can represent the forward dynamics (under a pricing measure) as

$$\frac{dF(t,T)}{F(t,T)} \;=\; \sigma_S(t) \sum_{i=1}^{n} \sigma_i(t,T)\, d\widetilde{W}_i(t), \tag{5}$$

where $\sigma_S(t)$ denotes spot volatility and $\sigma_i(t,T)$ are maturity-dependent volatility loadings. Operationally, an estimation workflow is:

- estimate $\sigma_S(t)$ (e.g. rolling-window spot volatility);
- normalise forward returns by dividing by $\sigma_S(t)$;
- apply PCA to the normalised series to extract common factors.

Monthly seasonality via dummy variables in the forward volatility. A direct and flexible way to encode monthly seasonality is to let the instantaneous forward volatility depend on the calendar month through dummy variables. For instance, in a setting where the instantaneous forward volatility is denoted $\sigma_P(t,u)$ (electricity) or $\sigma_F(t,u)$ (gas), one can set

$$\sigma_P(t,u) \;=\; \sum_{i=1}^{12} \Big(\sigma_2^{(i)} + \sigma_1^{(i)} e^{-\alpha_i(u-t)}\Big) D_i, \tag{5.12}$$

where D_i is the month-i dummy, $\sigma_1^{(i)}$ is a short-term volatility level, $\sigma_2^{(i)}$ is a long-term volatility level, and α_i controls the maturity decay, hence shaping a month-dependent Samuelson-type effect.

From instantaneous volatility to delivery-period volatility. When the traded contract delivers over $[\tau_1, \tau_2]$, a natural link is to average the instantaneous volatility over the delivery window, e.g.

$$\Sigma_P(t,\tau_1,\tau_2) \;=\; \frac{1}{\tau_2-\tau_1} \int_{\tau_1}^{\tau_2} \sigma_P(t,u)\, du, \tag{5.10}$$

and similarly for gas.

Monthly seasonality for gas forwards (analogous form). For gas, an analogous monthly-dummy specification is used:

$$\sigma_F(t,u) \;=\; \sum_{i=1}^{12} \Big(\beta_2^{(i)} + \beta_1^{(i)} e^{-\delta_i(u-t)}\Big) D_i, \tag{5.16}$$

with $(\beta_1^{(i)}, \beta_2^{(i)}, \delta_i)$ month-specific parameters.

1.2.3 Path-dependent volatility with an explicit seasonal term

Seasonal + path-dependent decomposition. A further specification for capturing seasonality in futures rates is to define the forward volatility as

$$\sigma(t,T) \ := \ S(t,T) + X(t,T), \tag{3.9}$$

where $S(t,T)$ is a seasonal term and $X(t,T) = [f(t)]^u$ (with $u \in \mathbb{R}$) is a path-dependent term.

Monthly seasonal term as a dummy-variable exponential decay. A concrete parametric form for the seasonal term is

$$S(t,T) \ = \ \sum_{i=1}^{12} \Big(\sigma_2^{(i)} + \sigma_1^{(i)} e^{-\lambda_i (T-t)}\Big) D_i, \tag{6}$$

where D_i selects the calendar month, and $(\sigma_1^{(i)}, \sigma_2^{(i)}, \lambda_i)$ shape the month-specific volatility level and its decay with maturity, hence encoding monthly seasonality directly in the term structure of volatility.

Alternative smooth seasonal functional forms. Instead of dummy variables, one can encode seasonality with a smooth trigonometric form, e.g.

$$S(t) \ = \ s_1 + s_2 \cos\Big(s_3 + \frac{2\pi t}{12}\Big), \tag{3.10}$$

with constants (s_1, s_2, s_3) estimated from data and t measured in months (or rescaled accordingly).

1.3 Seasonality handling in empirical workflows and applications

1.3.1 Detecting and motivating weekly seasonality in electricity

Empirical diagnostics for electricity spot data often reveal strong dependence at lag 7 (weekly periodicity), consistent with systematic weekday/weekend differences in demand and price formation. This motivates explicitly accounting for weekly seasonality (in levels and/or volatilities) before fitting residual dynamics (diffusion + jumps, regime switching, etc.).

1.3.2 De-trending and de-seasonalising forward price time series

Mean level function with annual cosine term. A practical de-seasonalisation step for (log) forward prices is to fit a low-dimensional mean function combining a trend and a yearly seasonal component. For a forward (or swap/forward proxy) log-price series $\log X(t, \tau_1, \tau_2)$ one can use

$$\log X(t, \tau_1, \tau_2) \ = \ a_0 + a_1 t + a_2 \cos(2\pi t), \tag{5.17}$$

assuming (for daily data) 252 trading days per year, so that the cosine term captures annual variation. Parameters (a_0, a_1, a_2) are estimated by least squares.

Removing seasonality prior to modelling residual dynamics. After estimating the mean function, the de-seasonalised (and detrended) series is obtained by subtracting the fitted values from the observed log-prices. The resulting residuals/returns can then be analysed for jumps and other stylised facts, motivating jump-diffusion components and recursive jump-filtering procedures on the de-seasonalised returns.

1.4 Commodity swaps

Role and economic interpretation. Commodity swaps are mid- to long-term OTC instruments used to *lock in a fixed commodity price* over a predefined horizon. They are typically *financially settled* and expose the parties to *counterparty credit risk* unless collateralisation/netting is explicitly incorporated.

1.4.1 Plain vanilla swap: structure and cashflows

Fixing dates and legs. Consider fixing (and payment) dates $T_1 < \cdots < T_n$. At each T_i:

- the *fixed leg* pays a contractual fixed price K (times the contracted volume);
- the *floating leg* pays the realised commodity price $S(T_i)$ (often an index fixing).

By convention, the *swap buyer* pays fixed and receives floating; the *swap seller* does the opposite.

Per-period settlement rule. Let Q_i be the contracted volume for period i (monthly/quarterly/annual are common). The net settlement at T_i is proportional to

$$\text{Difference}_i \;=\; Q_i\big(K - S(T_i)\big),$$

so that:

- if $K > S(T_i)$ the buyer pays the difference;
- if $K < S(T_i)$ the seller pays the difference.

Replication viewpoint and mark-to-market effects. At inception, a plain vanilla swap is typically struck so that its value is zero. Economically, it can be interpreted as (i) a strip of forward contracts on the commodity plus (ii) an implicit borrowing/lending position at the *implied forward rate*. Consequently, the swap market value changes over time as forward prices and interest rates move; moreover, once the first payment has occurred, the remaining position generally acquires nonzero value even if the curve and rates do not move, due to the cashflow already exchanged.

1.4.2 Fair fixed price (swap rate) and valuation

Discounted value in terms of forward prices. Fix a valuation time $t \leq T_1$ and assume a constant risk-free rate r (continuous compounding). Using forward prices $F(t, T_i)$ for delivery at T_i, the discounted value of the floating-minus-fixed stream can be written as

$$X(t) \;=\; \sum_{i=1}^{n} e^{-r(T_i - t)} \big(F(t, T_i) - K \big). \tag{1.2}$$

The *fair swap price* (swap rate) is obtained by imposing $X(t) = 0$ and solving for K, yielding

$$K \;=\; \frac{\sum_{i=1}^{n} e^{-r(T_i - t)} F(t, T_i)}{\sum_{i=1}^{n} e^{-r(T_i - t)}}. \tag{7}$$

Hence, K is a discount-factor-weighted average of the relevant forward prices.

Continuous-delivery analogue. If delivery (and settlement) occurs continuously over a period $[T_1, T_2]$, the same logic gives

$$X(t) = \int_{T_1}^{T_2} e^{-r(T-t)} \big(F(t, T) - K \big) \, dT, \qquad K = \frac{\int_{T_1}^{T_2} e^{-r(T-t)} F(t, T) \, dT}{\int_{T_1}^{T_2} e^{-r(T-t)} \, dT}, \qquad t \leq T_1.$$

1.4.3 Swaps in electricity: delivery-period contracts

From instantaneous forwards to traded swaps/futures. In electricity markets, contracts typically settle over a *delivery period* $[\tau_1, \tau_2]$ rather than at a single maturity. Let $F(t, u)$ denote the instantaneous forward price for delivery at time u. The price of an electricity swap/future delivering continuously over $[\tau_1, \tau_2]$ can be represented as an integral over the forward curve:

$$F_s(t, \tau_1, \tau_2) \;=\; \int_{\tau_1}^{\tau_2} \widehat{w}(u, \tau_1, \tau_2) \, F(t, u) \, du, \tag{3.8}$$

where $\widehat{w}(\cdot, \tau_1, \tau_2)$ is a weight function linking delivery-period prices to pointwise forwards.

Discounting-based weights. A convenient choice is obtained by taking $w(u) = e^{-ru}$ and normalising over the delivery window:

$$\widehat{w}(u, \tau_1, \tau_2) \;=\; \frac{w(u)}{\int_{\tau_1}^{\tau_2} w(v) \, dv} \;=\; \frac{re^{-ru}}{e^{-r\tau_1} - e^{-r\tau_2}}.$$

This produces a properly normalised weighting of the instantaneous forward curve consistent with discounting.

1.5 Forward and futures contracts in commodity and energy markets

1.5.1 Forward contracts: definition, payoff, and mark-to-market value

Contract definition (OTC forward). A *forward contract* is a bilateral OTC agreement to buy or sell a given quantity of a commodity at a fixed future delivery date T for a predetermined contract price K. Standard features are: (i) privately negotiated terms, (ii) typically no intermediate cash flows before delivery, (iii) physical (or financial) settlement at T, and (iv) presence of *counterparty credit risk*.

Payoff at delivery. For a long forward (buyer of the contract), the payoff at time T is

$$\Pi_T^{\mathrm{long}} \;=\; S(T) - K,$$

while for a short forward it is $K - S(T)$, where $S(T)$ denotes the spot price at delivery.

Arbitrage-free value before maturity. Fix $t \leq T$. Consider a forward entered at time 0 with delivery price K. Its *fair value* at time t (mark-to-market) is obtained by the standard offsetting strategy: at time t enter an opposite forward maturing at T at the current forward price $F(t,T)$, so that physical deliveries cancel at T. No-arbitrage then implies

$$X(t) \;=\; e^{-r(T-t)}\big(F(t,T) - K\big), \tag{8}$$

where r is the (effective) risk-free interest rate over $[t,T]$.

Delivery over multiple dates or a delivery period. In commodities (and especially in energy) a contract may entail delivery over multiple dates $T_1 < \cdots < T_n$ (or continuously over a period). Then the contract can be decomposed into a strip of single-date forwards. With a fixed price K, the present value at time t is

$$X(t) \;=\; \sum_{i=1}^{n} e^{-r(T_i-t)}\big(F(t,T_i) - K\big). \tag{1.1}$$

The *fair fixed price* (i.e. the delivery price that makes the contract worth zero at t) is the discount-factor-weighted average of the relevant forward prices:

$$K \;=\; \frac{\sum_{i=1}^{n} e^{-r(T_i-t)} F(t,T_i)}{\sum_{i=1}^{n} e^{-r(T_i-t)}}. \tag{9}$$

For continuous delivery over $[T_1, T_2]$ the sums become integrals and one obtains the analogous ratio of discounted integrals.

1.5.2 Futures contracts: exchange trading and the futures–forward distinction

Definition (exchange-traded futures). A *futures contract* is a standardised forward traded on an exchange. In addition to the buyer and seller, there is a *clearing house* acting as central counterparty.

Margins and daily settlement (mark-to-market). Futures require an initial margin and are *marked to market daily*: each trading day a settlement price is set and gains/losses are credited/debited via a variation margin mechanism. This greatly reduces credit risk because losses are realised and margined continuously rather than accumulating until delivery.

Main differences between forwards and futures. Key contrasts are:

- **Trading venue and standardisation:** futures are exchange-traded and standardised; forwards are OTC and customised.
- **Credit risk:** forwards entail counterparty risk; futures are (to a first approximation) credit-risk free due to clearing and margining.
- **Cash-flow timing:** forwards settle at delivery; futures settle incrementally through daily mark-to-market.
- **Forward vs. futures price:** equality holds only under restrictive conditions; in particular, forward and futures prices coincide when the interest rate is constant (and effectively independent of the underlying).

1.5.3 Forward and futures valuation under a risk-neutral measure

Risk-neutral expectation representation. Commodity pricing models are often calibrated to panels of futures prices. For derivative pricing, the risk-neutral (or risk-adjusted) distribution of future spot prices is central. Under the risk-neutral measure Q, the time-t forward/futures price for maturity T satisfies

$$F(t,T) \;=\; \mathbb{E}_t^Q\big[S(T)\big]. \tag{3.1}$$

This identity is a modelling cornerstone: specifying a tractable spot dynamics under Q yields $F(t,T)$ by conditional expectation; conversely, one may model $F(t,T)$ directly.

1.5.4 Spot–forward relationship for storable commodities (cost-of-carry)

Storage, financing, and no-arbitrage bounds. For *storable* commodities, the link between spot and forward prices is governed by the economics of buying, storing, and financing the physical commodity. A commodity is stored only if the present value of selling at a future time is at least as large as selling today.

With spot $S(t)$ and forward $F(t,T)$, no-arbitrage implies that the forward price must compensate the holder for:

- the **financing cost** (interest rate),
- the **storage cost** (physical carry),
- and is reduced by the **convenience yield** (benefit of holding inventory).

Cost-of-carry formula with storage cost and convenience yield. Let $r(t)$ be the (continuously compounded) risk-free rate, $m(t)$ the storage cost rate, and $c(t)$ the convenience yield rate. The cost-of-carry relationship is

$$F(t,T) \;=\; S(t)\exp\Big(\big[r(t)+m(t)-c(t)\big](T-t)\Big). \tag{10}$$

Defining $\gamma(t) = c(t) - m(t)$ (net convenience yield), this can be written as

$$F(t,T) \;=\; S(t)\exp\big((r(t)-\gamma(t))(T-t)\big).$$

Storage costs act analogously to a negative dividend: they push the forward curve upward relative to the interest-rate growth of the spot.

1.5.5 Energy-specific features: electricity and the breakdown of spot–forward replication

Electricity as a flow commodity and delivery-period settlement. Electricity is a *flow* commodity with very limited storability. Market contracts are written on delivery over a *period* (e.g. week, month, quarter, year) rather than at a single maturity. During the delivery period the contract is settled against the realised system (day-ahead) spot prices, so electricity "forwards" are economically close to *swaps* exchanging a floating spot price over a period against a fixed price.

Consequences of non-storability. Non-storability implies that the classical storable-commodity spot–forward replication argument breaks down. The market is incomplete, and spot-forward hedging is not straightforward. Additionally, the presence of delivery periods restricts the class of spot models that allow analytical pricing of traded contracts.

Direct modelling of the forward/futures curve (HJM-style approach). Because of these constraints, one often models forward/futures prices *directly* in a Heath–Jarrow–Morton style. In its simplest risk-neutral form, forward curve dynamics may be specified as

$$\frac{dF(t,T)}{F(t,T)} \;=\; \sigma(t,T)\, d\widetilde{W}(t), \tag{11}$$

with a maturity-dependent volatility function $\sigma(t,T)$.

Link between instantaneous forwards and traded delivery-period contracts. Let $F(t, u)$ denote the *instantaneous* forward price for delivery at time u. The price of an electricity swap/future delivering continuously over $[\tau_1, \tau_2]$ can be expressed as an integral over the forward curve:

$$F_s(t, \tau_1, \tau_2) \;=\; \int_{\tau_1}^{\tau_2} \widehat{w}(u, \tau_1, \tau_2)\, F(t, u)\, du, \tag{3.8}$$

where $\widehat{w}(\cdot, \tau_1, \tau_2)$ is a weight function (often derived from discounting) connecting delivery-period contracts to pointwise forwards.

Takeaway for practice. For *storable* commodities, the cost-of-carry relationship provides a tight structural link between spot and forwards, and forwards can often be interpreted through storage/financing economics. For *electricity*, limited storability and delivery-period settlement motivate modelling the forward/swap curve directly and carefully accounting for delivery aggregation when mapping between traded contracts and an (often unobserved) instantaneous forward curve.

1.6 Schwartz-type Models for Commodity Prices

1.6.1 Context and modelling goal

A recurring empirical feature of many commodity (and energy) prices is that a pure geometric Brownian motion (GBM) is too restrictive: it fails to reproduce, among other stylised facts, *mean reversion* and *seasonality*. In particular, a key requirement for term-structure modelling is to obtain realistic *futures/forward volatility* as a function of maturity, typically declining with time-to-maturity and often showing seasonal patterns.

In what follows we summarise the family of models attributed to Schwartz in the text: a one-factor mean-reverting spot model, a two-factor model with stochastic convenience yield, a two-factor decomposition model (Schwartz–Smith), and a three-factor extension including the short rate.

1.6.2 Schwartz one-factor mean-reverting spot model (log-OU / mean-reverting log-price)

Dynamics (spot). Let $S(t)$ denote the spot commodity price. Schwartz proposes a mean-reverting specification in log-price form:

$$dS(t) = \alpha\big(\mu - \log S(t)\big)S(t)\, dt + \sigma S(t)\, dW(t), \qquad S(0) = s,$$

equivalently, for $x(t) := \log S(t)$,

$$dx(t) = \alpha\big(\mu_b - x(t)\big)\, dt + \sigma\, dW(t), \qquad \mu_b = \mu - \frac{\sigma^2}{2\alpha}.$$

A key interpretable quantity is the *half-life* of mean reversion:

$$t_{1/2} = \frac{\log 2}{\alpha}.$$

Risk-neutral adjustment. Under the risk-neutral measure, the long-run mean in the drift is shifted by the market price of risk (as presented in the text via $\mu_b \mapsto \mu_b - \lambda$).

Forward/futures implication: maturity-dependent volatility. This model yields a closed-form forward/futures price and implies a *futures volatility term structure*

$$\sigma(t,T) = \sigma e^{-\alpha(T-t)},$$

hence volatility declines when:

- the mean-reversion speed α increases;
- the contract maturity T increases (larger $T-t$).

Advantages (as stated in the text).

- Produces a realistic term structure of forward/futures volatilities, decreasing with maturity via $\sigma(t,T) = \sigma e^{-\alpha(T-t)}$.
- Captures mean reversion, a key commodity feature.

Limitations (safe statements from the text).

- Compared to richer commodity dynamics, it remains a single-factor diffusion and is not designed to capture multiple structural drivers (e.g. explicit convenience yield dynamics, stochastic rates, etc.).

1.6.3 Schwartz two-factor model with stochastic convenience yield

Motivation. Convenience yield is a central commodity concept in cost-of-carry relationships. Introducing it as a stochastic factor enriches the spot dynamics beyond a single mean-reverting component.

Dynamics (physical measure). Schwartz proposes the coupled system

$$\begin{cases} dS(t) = \big(\mu - \delta(t)\big)S(t)\,dt + \sigma_S S(t)\,dW_1(t), \\ d\delta(t) = \alpha\big(k - \delta(t)\big)\,dt + \sigma_\delta\,dW_2(t), \\ dW_1(t)\,dW_2(t) = \rho_{S\delta}\,dt, \end{cases}$$

where $\delta(t)$ is the (instantaneous) convenience yield, mean-reverting around k with speed α.

Risk-neutral adjustment. Under the risk-neutral measure the spot drift becomes r (risk-free rate) and the convenience-yield drift is shifted by the market price of risk (written in the text as $\alpha(k-\delta(t)) \mapsto \alpha(k-\delta(t)) - \lambda$).

Advantages (safe statements from the text).

- Explicitly models convenience yield as a stochastic (mean-reverting) factor, allowing a richer representation of commodity economics than pure one-factor spot models.
- Correlation $\rho_{S\delta}$ couples spot and convenience yield shocks, improving flexibility for term-structure and joint dynamics.

Limitations (safe statements from the text).

- Requires additional state variable(s) and parameters (e.g. $\alpha, k, \sigma_\delta, \rho_{S\delta}$ and risk premia adjustments), hence higher calibration burden than one-factor models.

1.6.4 Schwartz–Smith two-factor model (short-term deviation + stochastic equilibrium level)

Model structure and motivation. The model decomposes the log-spot into:

$$\log S(t) = \chi(t) + \xi(t),$$

where $\chi(t)$ is a *short-term deviation* and $\xi(t)$ is the *equilibrium price level*. The stated characteristics are:

- mean reversion in short-term prices;
- uncertainty in the equilibrium level to which prices revert.

A key remark in the text is that this model *does not explicitly model convenience yields*, although its state variables can be represented in terms of a stochastic convenience yield.

Dynamics (empirical measure).

$$\begin{cases} d\chi(t) = -k\chi(t)\,dt + \sigma_\chi\,dW_\chi(t), \\ d\xi(t) = \mu_\xi\,dt + \sigma_\xi\,dW_\xi(t), \\ dW_\chi(t)\,dW_\xi(t) = \rho_{\chi,\xi}\,dt. \end{cases}$$

Risk-neutral dynamics and forward curve. Under the risk-neutral measure, the book writes

$$d\chi(t) = -(k\chi(t) + \lambda_\chi)\,dt + \sigma_\chi\,dW_\chi^f(t), \qquad d\xi(t) = (\mu_\xi - \lambda_\xi)\,dt + \sigma_\xi\,dW_\xi^f(t),$$

with correlated Brownian motions. Under risk-neutrality $\chi(t)$ reverts to λ_χ/k (whereas under the empirical measure it reverts to 0). The forward curve admits a closed-form expression:

$$\log F(t,T) = e^{-k(T-t)}\chi(t) + \xi(t) + A(T-t),$$

where $A(\cdot)$ is an explicit function of the parameters (as given in the text), including contributions from $(\mu_\xi - \lambda_\xi)$, mean-reversion k, volatilities σ_χ, σ_ξ, and correlation $\rho_{\chi,\xi}$.

Advantages (as stated in the text).

- Separates short-term mean-reverting deviations from a stochastic long-run equilibrium, matching the two stated qualitative features (short-term mean reversion and uncertain equilibrium).
- The forward curve can be expressed in closed form; parameters are typically calibrated using futures panel data.
- The text explicitly reports that the forward-curve specification is *very successful in fitting observed futures term structures and its dynamics*, and that risk-neutral parameters are usually *precisely estimated* when calibrating to futures (since futures are expectations under the risk-neutral distribution).

Limitations (safe statements from the text).

- Convenience yield is not an explicit state variable (even if a representation is possible).
- Introduces additional parameters and latent components, implying a non-trivial calibration step (the book describes a nonlinear least-squares calibration to futures).

1.6.5 Schwartz three-factor model (spot, convenience yield, and short rate)

Motivation. To enrich the joint dynamics further, Schwartz also proposes a three-factor framework adding stochastic interest rates to the spot–convenience yield system.

Risk-neutral dynamics. Under the risk-neutral probability measure, the model is:

$$\begin{cases} dS(t) = (r(t) - \delta(t))S(t)\,dt + \sigma_S S(t)\,dW_1^f(t), \\ d\delta(t) = \alpha(\bar{\delta} - \delta(t))\,dt + \sigma_\delta\,dW_2^f(t), \\ dr(t) = \beta(\bar{r} - r(t))\,dt + \sigma_r\,dW_3^f(t), \end{cases}$$

with instantaneous correlations

$$dW_1^f\,dW_2^f = \rho_{S\delta}\,dt, \qquad dW_1^f\,dW_3^f = \rho_{Sr}\,dt, \qquad dW_2^f\,dW_3^f = \rho_{r\delta}\,dt.$$

The text notes that the volatility term structure depends on volatilities, mean-reversion speeds of the three variables, and their correlations; it also remarks that sometimes the interest rate is modelled within an HJM-type framework.

Advantages (safe statements from the text).

- Adds stochastic interest-rate risk as a third factor, enabling joint modelling of spot, convenience yield, and rates.
- Increased flexibility for term-structure dynamics via additional mean-reversion/volatility/correlation channels.

Limitations (safe statements from the text).

- Higher dimensionality implies more parameters and correlations to estimate, increasing calibration complexity relative to one- and two-factor specifications.

1.6.6 Comparison: what is clearly stated about strengths/weaknesses

A reference baseline: Black's commodity adaptation (for contrast). The text highlights a specific drawback of the Black-style spot model with constant proportional volatility: it implies *constant* futures/forward volatility equal to spot volatility, whereas in practice futures/forward volatilities *vary with maturity*, typically declining and often showing *seasonal patterns*. This contrast motivates mean-reverting and multi-factor Schwartz-type models.

Summary of the Schwartz family in the book's perspective.

- The **one-factor mean-reverting Schwartz model** addresses the key empirical fact of decreasing futures volatility with maturity through an explicit exponential term structure $\sigma(t,T) = \sigma e^{-\alpha(T-t)}$.
- The **two-factor convenience-yield Schwartz model** enriches the economics by explicitly introducing a mean-reverting stochastic convenience yield and its correlation with spot.
- The **Schwartz–Smith two-factor model** targets two qualitative traits (short-term mean reversion and uncertain equilibrium) and is reported to fit futures term structures and their dynamics very well when calibrated to futures data.
- The **three-factor Schwartz model** adds stochastic interest rates, increasing realism and flexibility at the expense of additional parameters and calibration effort.