From Simple to Double Pendulum: Theoretical Foundations and Numerical Simulations

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June 2025

Abstract

This document provides a self-contained and complete examination of the theoretical background and computation corresponding to the simple and double pendulum systems. The development commences by examining harmonic motion and the relationship with oscillating dynamics. The simple pendulum is next investigated under Newtonian, Lagrangian, and Hamiltonian mechanics for the purpose of demonstrating the convergence of various formalisms onto a single equation of motion. The examination continues for the double pendulum in the Lagrangian and Hamiltonian pictures but concentrates mainly upon the nonlinear and chaotic character of the system. The equations of motion are then expressed in short-hand matrix formalism amenable for theoretical examination and numerical computation. Finally, various numerical integration algorithms are presented, consisting of Euler and Runge–Kutta algorithms and symplectic algorithms such as Symplectic Euler and Verlet, and a comparative survey in terms of accuracy, stability and energy conservation of their efficacies.

1 Introduction

From the vibrations of atoms to the motion of celestial bodies, oscillatory systems connect phenomena across scales. Pendula are a classic example in which simple mechanical systems are capable of presenting a linearly determinable and a highly complicated behavior and hence relating linear approximations and nonlinear dynamics. Chaotic systems, represented by the double pendulum, constitute a rather fascinating case in point: slight variations in the initial conditions lead easily to very different outcomes.

The double pendulum exhibits nonlinear dynamics defined by many degrees of freedom and hence exhibits unpredictable yet mesmerizing trajectories. Besides having scientific value, the double pendulum also carries a distinct aesthetic. The double pendulum's motion forms complicated patterns and mesmerizes physicists, mathematicians, and enthusiasts.

Research in pendulum dynamics goes beyond classic mechanics. Oscillating systems manifest at several scales ranging from planetary motions and atomic vibrations and the double pendulum provides a rich framework for their exploration. In the domain of computer science, the double pendulum serves as a benchmark for testing numerical algorithms, machine learning strategies, and chaos-detection techniques. In mathematics, its phase-space configuration provides a prime example of nonlinear dynamics, strange attractors, and dynamical system theories. Furthermore, in quantum mechanics, the Hamiltonian formulation provides a bridge relating the classical trajectories and quantum chaos and shedding light upon relationships between deterministic classic motion and probabilistic quantum and complex behavior phenomena manifesting in nature.

The current work attempts the development of simple and double pendula from the perspectives of classic mechanics and numerical techniques, focusing on their mathematical formalism and interdisciplinary applications, ranging from harmonic motion to deterministic chaos.

Contents

2 Simple Harmonic Motion

Simple harmonic motion is a type of oscillatory motion in which the force acting on a body is proportional and opposite to its displacement from the equilibrium position. For an ideal spring, Hooke's Law is a classic example:

$$\vec{F} = -kx$$

where:
- k is a positive constant known as the spring constant;
- x also denoted as Δl, represents the difference between the spring's length under a load and its length at rest.

Simple harmonic motion describes the behavior of an ideal system known as a harmonic oscillator. Classic examples of simple harmonic oscillators in mechanics include the ideal spring with an attached mass, as well as the simple pendulum (for small oscillation angles: $\theta \ll 1\,\text{rad}$). Beyond mechanics, other examples include the electrical oscillations in RLC circuits.

2.1 Undamped Simple Harmonic Motion

The **undamped simple harmonic motion** is also known as natural harmonic motion. It is a sinusoidal oscillation with a constant angular frequency, ω. This motion is periodic with a constant oscillation because it doesn't account for energy-dissipating forces like friction or air resistance.

The position of a body oscillating with simple harmonic motion can be described by a sinusoidal function with constant amplitude and phase, assuming the origin of the reference system is at the central point of oscillation.

$$y(t) = A\,\sin\left(\omega\,t + \phi\right)$$

A **Damped simple harmonic motion** is a motion in which the oscillation decreases over time due to a damping force (or friction) that dissipates energy. This is in contrast to undamped simple harmonic motion, in which the oscillations continue indefinitely with a constant amplitude.

It is an extension of simple harmonic motion (without damping) and corresponds to many real physical phenomena, such as a spring oscillating with friction or air resistance.
Depending on the strength of the damping force, three cases can be distinguished:

- **Underdamped**: the damping force is relatively low, so the body continues to oscillate, but the amplitude slowly decreases over time until it eventually stops.
- **Critically damped**: The damping force has a precise value that reduces the oscillation to the minimum necessary for the system to reach equilibrium in the shortest possible time.
- **Overdamped**: The damping force is very strong and no oscillation occurs. The system returns to the equilibrium point, but very slowly.

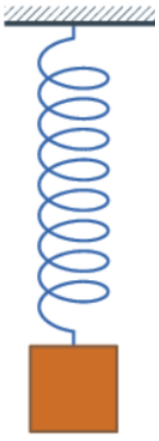


Figure 1: A damped harmonic oscillator

2.2 Forced Harmonic Motion

In forced harmonic motion, an oscillating system is subjected to an external force that varies over time, this causes an oscillation that no longer depends solely on the system's natural properties but also on the frequency and intensity of the applied force.

In the ideal, undamped case, the system can show a significant increase in oscillation amplitude when the frequency of the external force approaches the system's natural frequency, a phenomenon known as resonance.

When energy losses like friction or resistance are considered, the motion is called **damped forced harmonic motion**. In this situation, the amplitude of the oscillations reaches a finite value, and the system responds with a phase that is slightly delayed relative to the applied force. This model is more realistic and closely matches what is observed in nature.

3 Simple Pendulum

The **simple pendulum** (or mathematical pendulum) is a physical system, or more precisely a harmonic oscillator, consisting of an inextensible string of negligible mass, with a point mass (m) attached to its end. The system is subject to a gravitational pull (which we assume is uniform in space and constant over time). The pendulum is a practical application of harmonic motion.

Because the string is inextensible, the mathematical pendulum will constantly follow circular trajectories, therefore, it is helpful to use a polar coordinate system rather than a Cartesian one.

3.1 Polar Coordinate System

In mathematics, a **Polar Coordinate System** is a two-dimensional coordinate system where each point in a plane is defined by an angle, typically θ, and its distance from the center of the plane, called the pole.

The two polar coordinates, r and θ, can be converted to Cartesian coordinates x and y using the formulas for the trigonometric functions sine and cosine.

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Conversely, the two Cartesian coordinates x and y can be converted to the polar coordinate r by applying the Pythagorean theorem.

$$r = \sqrt{x^2 + y^2}$$

At the equilibrium position, the point mass is at rest, and the string is perfectly vertical. However, if the mass is displaced from this equilibrium, the pendulum will begin to oscillate under the effect of the gravitational force, which will try to pull it back toward the equilibrium position. As already described, it will trace a circular arc with a radius equal to the length l of the string. If friction is not considered, the oscillations will continue indefinitely.

4 Formulations of Mechanics

To study the motion of a simple and, later, double pendulum, it is necessary to choose the most suitable formulation of mechanics for the task. The various formulations of mechanics are different, yet equivalent, ways to describe the motion of mechanical systems, each with its own advantages and disadvantages depending on the case. They are primarily divided into Newtonian mechanics (by Newton) and rational mechanics (developed by figures like Lagrange and Hamilton).

4.1 Newtonian Mechanics

Newtonian mechanics is the most traditional formulation, with its starting point being Newton's Second Law, which establishes a direct link between force and acceleration. Particles are treated as point masses moving in three-dimensional space, and differential equations are written using Cartesian coordinates.

$$\sum \vec{F} = m\vec{a} \longleftrightarrow \vec{F} = \frac{d\vec{p}}{dt} \tag{1}$$

This approach is particularly direct: if all the forces acting on a body are known, its motion can be determined. A key disadvantage of the Newtonian approach is the need to express constraints as external forces, which can complicate the equations.

Within Newtonian mechanics, three sub-disciplines can be distinguished: **kinematics**, the description of motion through the concepts of space and time; **dynamics**, the study of the causes of motion; and **statics**, the study of system equilibrium through forces and torques.

Kinematics studies motion in an abstract way, determining the equation of motion that describes an object's position as a function of time. Dynamics and statics, on the other hand, use the fundamental equations of motion and conservation laws (such as those for momentum, angular momentum, and mechanical energy) to describe motion and equilibrium conditions.

4.2 Lagrangian Mechanics

Lagrangian mechanics introduces a paradigm shift. Instead of studying forces, it focuses on the system's energies: kinetic energy T and potential energy V. The Lagrangian function, $\mathcal{L}$, is defined as the difference between these two energies:

$$\mathcal{L} = T - V$$

$$\mathcal{L}(\dot{q}, q, t) = T(\dot{q}, q, t) - V(q, t) \tag{2}$$

where $q \in \mathbb{R}^n$ represents the generalized coordinates, $\dot{q}$ their respective velocities, and t is time.

The laws of motion are formulated through a variational principle, the **Principle of Stationary Action** (or Least Action). This principle states that a physical system, between two fixed states in time t_1 and t_2, evolves along the path that makes a quantity called action, stationary (or balanced).

Action, S, is the time integral of the difference between the kinetic and potential energies (i.e., the Lagrangian). It is a scalar quantity defined as:

$$S = \int_{t_1}^{t_2} \mathcal{L}\left[\dot{q}(t), q(t), t\right] dt$$

The Principle of Hamilton, or the principle of stationary action, states that a physical system follows the path that makes the action stationary:

$$\delta \int_{t_1}^{t_2} \left[T(\dot{q}, q, t) - U(q, t)\right] dt = 0$$

$$\delta S = 0 \tag{3}$$

Here, S is the action, which depends on the path taken by a system over time and measures the "energy cost" of a given movement. δS denotes an infinitesimal variation of the action. The equality to zero means that nature follows the most balanced path over time between kinetic and potential energy, choosing the trajectory that optimizes this balance during its evolution.

Traditional spatial coordinates are replaced by **generalized coordinates**. These are independent variables chosen to completely describe the state of a physical system while already accounting for its constraints. Unlike Cartesian coordinates (like x,y,z), they can be angles, distances along curved paths, or any other variable suited to the problem.

The goal is to reduce the number of variables to the bare minimum, choosing those that reflect the system's true degrees of freedom. This way, constraints no longer need to be handled with additional forces or separate equations; they are directly incorporated into the geometry of the coordinates themselves.

This results in the **Euler-Lagrange equations**, a system of differential equations that describe motion without the need for explicit constraint forces, making the Lagrangian approach particularly powerful for complex systems and those with constraints:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{4}$$

As a simple example, consider a system moving along the x-axis. The Lagrangian is:

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - mgx$$

Applying the Euler-Lagrange equation, we have, for the power rule:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - mgx\right) = m\dot{x}$$

and for the standard derivative of a linear function:

$$\frac{\partial \mathcal{L}}{\partial q_i} \rightarrow \frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 - mgx\right) = -mg$$

Consequently, we can conclude:

$$\frac{d}{dt}(m\dot{x}) - (-mg) = 0 \;\rightarrow\; m\ddot{x} = -mg \;\rightarrow\; \ddot{x} = -g$$

As expected, we find the constant acceleration due to a uniform gravitational field, which is the equation of motion for free fall.

4.3 Hamiltonian Mechanics

Hamiltonian mechanics is a reformulation of Lagrangian mechanics, obtained through a mathematical transformation known as the **Legendre transformation** (6). The fundamental idea consists of replacing the generalized velocities $\dot{q}_i$ with the conjugate momenta p_i. The latter are therefore defined as the partial derivative of the Lagrangian with respect to velocity:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{5}$$

This substitution allows the pairs (q_i, p_i) to be considered as independent variables, thus constituting what is called the phase space of the system. **Phase space** is a mathematical set in which each state of the system is represented by a coordinate $(q_1, ..., q_n)$ and the corresponding conjugate momentum $(p_1, ..., p_n)$, making it possible to represent the system's dynamics completely in terms of points in a $2n$-dimensional space, where n is the number of degrees of freedom.

The Hamiltonian $\mathcal{H}(q, p, t)$ is obtained from the following relationship with the Lagrangian:

$$\mathcal{H}(q, p, t) = \sum_i p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t) \tag{6}$$

In many common physical cases, when the potential depends only on the coordinates and the kinetic energy is quadratic in the velocities, the Hamiltonian coincides with the total energy of the system, namely:

$$\mathcal{H} = T + V \tag{7}$$

In the very definition of the Hamiltonian, via the Legendre transformation, however, the velocities $\dot{q}_i$ appear, which, as previously described, should not be included in this formulation, which only takes coordinates q_i and momenta p_i into account.

To eliminate the velocities from the Hamiltonian's expression, one starts from the definition of the conjugate momenta (5), which uniquely links each momentum to its corresponding velocity. This relationship, viewed as a function $p_i(q, \dot{q})$, can be inverted to obtain the velocities in terms of coordinates and momenta, i.e., $\dot{q}_i(q, p)$.

Formally, to perform this inversion, a **Hessian matrix** is required, but to make the process more intuitive, let's consider a simple example: a free particle of mass m moving along a generalized coordinate q. The Lagrangian will be defined as:

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 - V(q)$$

The conjugate momentum will therefore be defined as per (5):

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}} \left(\frac{1}{2} m \dot{q}^2 \right) = m \dot{q}$$

Which can be inverted, as a function of velocity, as:

$$\dot{q} = \frac{p}{m}$$

Finally, by substituting into the Legendre transform (6) to obtain the Hamiltonian:

$$\mathcal{H}(q, p) = p\, \dot{q} - \mathcal{L}$$

$$= p\, \frac{p}{m} - \left(\frac{1}{2} m \left(\frac{p}{m} \right)^2 - V(q) \right) = \frac{p^2}{2m} + V(q)$$

In this way, it is clearly seen how the transition from $(q, \dot{q})$ to (q, p) occurs precisely through the inversion of the definition of the momenta, which is immediate in the simple case of the free particle, while in general it is justified by the invertibility of the Hessian.

In conclusion, the velocities are replaced in the Hamiltonian by their corresponding expressions as functions of the momenta and everything is expressed only as a function of (q, p, t), meaning that $\mathcal{H}$ depends exclusively on the coordinates, momenta, and time.

Having defined the Hamiltonian in terms of coordinates and momenta, it remains to understand how it describes the temporal evolution of the system. As the principle of least action leads to the Euler-Lagrange equations in the Lagrangian formalism, the Hamiltonian formalism also has a set of fundamental equations. These are **Hamilton's equations**, which establish how coordinates and momenta evolve over time:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{8}$$

They form a system of differential equations of the first order, in contrast to the Euler-Lagrange equations, which are of the second order. The first equation describes the development of the generalised coordinates as a function of the momentum, while the second equation describes the development of the momentum as a function of the coordinates. This symmetry between coordinates and momentum makes the Hamiltonian formalism particularly suitable for the investigation of complex systems and for the mathematical formulation of statistical and quantum mechanics.

Having introduced the Hamilton equations, we can see that they provide the evolution law only for the fundamental variables of the system, namely the generalised coordinates q_i and the conjugate momenta p_i. In phase space, however, we are not only interested in these variables: we

often want to describe the temporal behaviour of a more general quantity such as the potential energy, the angular momentum or, more generally, any function $f(q, p, t)$.

To calculate the temporal evolution of any function, we can derive it with respect to time and use Hamilton's equations to substitute $\dot{q}_i$ and $\dot{p}_i$ using the chain rule:

$$\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t}.$$

To see how the derivative of any function $f(q, p, t)$ can be expressed compactly, we introduce the **Poisson brackets**, that describe how any quantity in phase space changes over time, defined, for two functions f and g that influence each other, as:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \tag{9}$$

The Poisson brackets (9) allow the equations of motion (8) to be written in a compact form. In fact, for any function $f(q, p, t)$ described using Hamiltonian mechanics, its total change over time is:

$$\dot{f} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t} \tag{10}$$

In Hamiltonian mechanics, $\dot{f}$ represents the total change in f over time. The Poisson brackets are not just a compact way of writing the equations of motion, they reflect the fact that phase space has a particular geometric structure called a **symplectic manifold**. It is a mathematical space that guarantees the conservation of fundamental characteristics of dynamics, e.g. the volume in phase space, i.e. the region of all possible states of the system, remains unchanged during evolution (Liouville's theorem) and certain physical quantities, such as energy, remain constant over time.

This formalism is also useful quantum mechanics, as it provides the fundamental mathematical structure on which the theory is based. In particular, in quantum theory the quantities, coordinates q_i and momentum p_i, become operators on a Hilbert space.

5 Application to the Single Pendulum

A theoretical-physical analysis of the simple pendulum is therefore proposed, first using Newtonian mechanics and then the Lagrangian formalism. To provide an accurate physical description, the following parameters must be considered, excluding friction: the acceleration of gravity $g = 9.81\ m/s^2$; the length of the inextensible string l in $meters$; the mass m of the point mass in kg; and the initial angle θ from the vertical in rad.

5.1 Newtonian Approach

In the case of the simple pendulum, a point mass m is constrained to move along a circular arc of radius l. It is suspended by an inextensible, massless string and is subject only to the force of gravity. The motion occurs on a vertical plane under the influence of a uniform gravitational field.

To analyze this system using Newton's Second Law, it is convenient to use a polar coordinate system with two main directions:
- **Radial**: Oriented along the string and pointing towards the center of rotation (the pivot point).
- **Tangential**: Orthogonal to the radial direction and tangent to the circular path. This is the direction along which the body moves.

The gravitational force $\vec{F_g}$, which is directed vertically downwards, is decomposed into these two directions. The radial component is balanced by the tension in the string and does not contribute to the motion along the path. Conversely, the tangential component is the one that actually acts along the direction of motion.

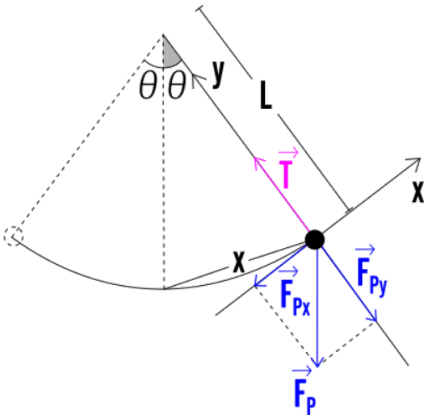


Figure 2: Force diagram of the simple pendulum

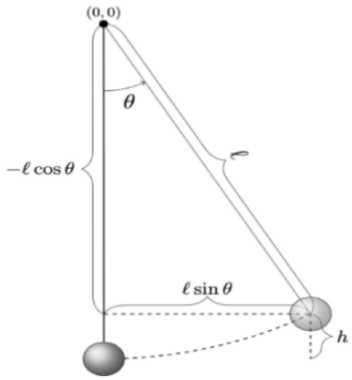


Figure 3: Coordinates expressed in terms of angle θ

If the string forms an angle θ with the vertical, then the angle between the string and the direction of the gravitational force is also θ. Consequently, we can use trigonometry to project the components of the gravitational force. The radial component, $\vec{F}_{rad}$, is balanced by the tension in the string and does not contribute to the motion. It is projected along the string (the side adjacent to the angle) using the cosine function:

$$\vec{F}_{rad} = -m\vec{g}\cos(\theta)$$

The negative sign in the tangential force formula indicates a restoring force. If θ is positive (the pendulum moves to the right), the tangential force pushes it to the left, in the negative direction. The force always acts to bring the pendulum back to its vertical equilibrium position

at $\theta = 0$, so it is always in the opposite direction of the angle θ. The tangential component, $\vec{F}_{tan}$, which acts along the direction of motion, is:

$$\vec{F}_{tan} = -m\vec{g}\,\sin(\theta)$$

In this case, where the only force causing motion is the tangential one, we can apply Newton's Second Law (1):

$$\vec{F}_{tan} = m\vec{a}_{tan}$$

$$-mg\,\sin(\theta) = m\vec{a}_{tan}$$

In the case of a pendulum, the quantity that changes over time is the angle of inclination from the vertical, $\theta(t)$. We call the first derivative of the angle the angular velocity, $\dot{\theta}$, which indicates how quickly the angle is changing. The second derivative of the angle, $\ddot{\theta}$, is the angular acceleration, which tells us how quickly the angular velocity is changing. It is written as:

$$\ddot{\theta} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

The angular velocity $\dot{\theta}$ is the ratio between the angle of the circular arc traveled, $d\theta$, and the elapsed time dt. The change of this velocity, i.e. time derivative, is the angular acceleration $\ddot{\theta}$:

$$\vec{v} = \frac{d\theta}{dt} \qquad \vec{a} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right)$$

In the case of the simple pendulum, both the tangential velocity and acceleration are particular forms of angular velocity and acceleration along a circular arc of radius l:

$$\vec{v}_{tan} = l\,\frac{d\theta}{dt} = l\,\dot{\theta} \tag{11}$$

$$\vec{a}_{tan} = l\,\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = l\,\ddot{\theta} \tag{12}$$

We can now return to the Newtonian equation from the previous section and expand it further:

$$-mg\,\sin(\theta) = m\,l\,\ddot{\theta}$$

By simplifying both sides by m and dividing by l:

$$-\frac{g}{l}\,\sin(\theta) = \ddot{\theta}$$

The final equation for $\ddot{\theta}$, is, also, the general equation of motion for a pendulum:

$$\ddot{\theta} = -\frac{g}{l}\,\sin(\theta)$$

$$\ddot{\theta} + \frac{g}{l}\,\sin(\theta) = 0 \tag{13}$$

5.2 Lagrangian Approach

We now aim to apply the Lagrangian approach to derive the equation of motion and thus verify the previous solution. This formulation is based on the Principle of Least Action (or Hamilton's Principle), which allows us to reach the same result as the Newtonian approach using an equivalent, alternative path. In this case, and even more so for the double pendulum, it is advantageous to use the Lagrangian approach because it allows for the use of generalized coordinates.

The first step is to identify a generalized coordinate that best describes the system's degree of freedom. For the simple pendulum, the motion is constrained to a circular path, which reduces the degrees of freedom to just one: the angle $\theta(t)$, measured with respect to the vertical. If we place the origin of the coordinate system at the pivot point and the y-axis is oriented upward, we get the following:

$$x = l\,\sin(\theta) \qquad y = -l\cos(\theta) \tag{14}$$

To continue with the Lagrangian formulation, the Lagrangian of the system, defined as $\mathcal{L} = T - V$, is set up using the generalized coordinates. For the simple pendulum, the most natural generalized coordinate is the angle $\theta(t)$ and its angular velocity $\dot{\theta}$.

$$\mathcal{L}(\dot{\theta}, \theta, t) = T(\dot{\theta}, \theta, t) - V(\theta, t) \tag{15}$$

We now aim to show the reader two equivalent approaches to obtain the kinetic energy T, defined as:

$$T = \frac{1}{2}mv^2$$

In the polar coordinate system, the velocity is a function of the angular velocity and the radius (11). Substituting this into the kinetic energy equation gives:

$$T = \frac{1}{2}m(l\,\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2 \tag{16}$$

In a Cartesian system, the kinetic energy is given by the sum of the squared velocities in the x and y directions, thus:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

We take the time derivatives of x and y, already proposed (14), by applying the chain rule:

$$\dot{x} = \frac{d}{dt}(l\,\sin\theta) = l\cos\theta\,\dot{\theta} \tag{17}$$

$$\dot{y} = \frac{d}{dt}(-l\cos\theta) = l\sin\theta\,\dot{\theta} \tag{18}$$

Substituting these into the kinetic energy equation:

$$T = \frac{1}{2}m((l\,\cos\theta\,\dot{\theta})^2 + (l\,\sin\theta\,\dot{\theta})^2)$$

$$T = \frac{1}{2}m(l^2\,\cos^2\theta\,\dot{\theta}^2 + l^2\,\sin^2\theta\,\dot{\theta}^2)$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta)$$

Using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, we arrive at the same result as the polar coordinate approach (16):

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

Once the kinetic energy, T, of the system has been determined, we can now look at its gravitational potential energy. In this case, we assume the zero reference point is at the lowest point of the pendulum's trajectory, i.e., at $\theta = 0$. By setting $y = -l\cos(\theta)$, the potential energy, V, is:

$$V = mgy = -mgl\cos(\theta) \tag{19}$$

Once the kinetic energy, T, and the gravitational potential energy, V, have been determined, we can proceed with the calculation of the Lagrangian (15):

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos(\theta) \tag{20}$$

We now proceed by applying the Euler-Lagrange equation, as seen above (4), adapted in this case to the chosen generalized coordinate, θ:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

Starting with the first term, the partial derivative with respect to $\dot{\theta}$, per la regola di potenza:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta\right) = ml^2\dot{\theta} \tag{21}$$

Taking the derivative with respect to time:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}(ml^2\dot{\theta}) = ml^2\ddot{\theta} \tag{22}$$

Moving on to the second term, the partial derivative with respect to θ, deriving the trigonometric function:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta\right) = -mgl\ \sin\theta \tag{23}$$

Finally, we apply the Euler-Lagrange equation, which is the sum of the first (22) and second terms (23), rewritten as:

$$ml^2\ddot{\theta} - (-mgl\sin\theta) = 0$$
$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

To obtain the equation of motion in its classical form, we divide the terms by ml^2:

$$\frac{ml^2\ddot{\theta}}{ml^2} + \frac{mgl\sin\theta}{ml^2} = 0$$

By simplifying the terms, we get the same result (13):

$$\ddot{\theta} + \frac{g}{l}\ \sin(\theta) = 0$$

Next, we can introduce a further simplification to the simple pendulum's equation of motion.

The equation is non-linear because of the trigonometric term $\sin(\theta)$ To obtain a simpler solution that describes the pendulum's motion under common conditions, we use the **small-angle approximation**. This approximation is valid when the pendulum oscillates only slightly from its vertical equilibrium position.

Mathematically, for very small values of θ in radians, the sine function can be approximated by the first term of its **Maclaurin series** expansion, a powerful technique for approximating polynomial functions:

$$\sin(\theta) \approx \theta \qquad \text{per}\,\theta \ll 1\,\text{rad}$$

By substituting this approximation into the pendulum's equation of motion, we obtain its linearized form:

$$\ddot{\theta} + \frac{g}{l}\,\theta = 0 \tag{24}$$

This equation has the same form as the differential equation for a simple harmonic oscillator, which makes the problem much easier to solve.

5.3 Hamiltonian Approach

After studying the simple pendulum using two different formalisms, the Newtonian and the Lagrangian, we now, to conclude, introduce the third and final alternative, which is an elegant and powerful method for the study of classical mechanics. The Hamiltonian formalism allows the system to be described in terms of generalized coordinates and conjugate momenta, transforming the equations of motion from second-order differential equations into first-order equations.

We begin, therefore, by considering the angle θ as the generalized coordinate q, as was also done in the previous formulations. The conjugate momentum to this coordinate (5) will therefore be defined as follows, as already calculated in the previous approach (21):

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta} \tag{25}$$

Physically, p_θ represents the angular momentum of the pendulum with respect to the point of suspension. After defining the generalized coordinate and its conjugate momentum, and consequently the system's phase space, the next step is to determine the Hamiltonian $\mathcal{H}$, using the Legendre transformation (6):

$$\mathcal{H}\ (\theta, p_\theta) = p_\theta\dot{\theta} - \mathcal{L}(\theta, \dot{\theta}) \tag{26}$$

Where the velocity $\dot{\theta}$ is easily derived as a function of p_θ, yielding:

$$\dot{\theta} = \frac{p_\theta}{ml^2} \tag{27}$$

And the Lagrangian, already calculated above (15), is defined as:

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos(\theta)$$

Substituting everything into the definition of the Hamiltonian (26):

$$\begin{aligned}
\mathcal{H}\ (\theta, p_\theta) &= p_\theta\ \frac{p_\theta}{ml^2} - \left(\frac{1}{2}ml^2\left(\frac{p_\theta}{ml^2}\right)^2 + mgl\ \cos\theta\right) \\
&= \frac{p_\theta^2}{ml^2} - \left(\frac{1}{2}ml^2\ \frac{p_\theta^2}{ml^4} + mgl\ \cos\theta\right) \\
&= \frac{p_\theta^2}{2ml^2} - mgl\ \cos\theta
\end{aligned} \tag{28}$$

Note that the Hamiltonian coincides with the total energy of the pendulum ($\mathcal{H} = T + V$) (7). The first term represents the kinetic energy expressed via the angular momentum, while the second term represents the gravitational potential energy.

From this Hamiltonian, we can derive Hamilton's equations (8), which constitute a first-order system equivalent to the Euler-Lagrange equation. For the generalized velocity $\dot{q}$, according to the power rule:

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{\partial}{\partial p_\theta}\left(\frac{p_\theta^2}{2ml^2} - mgl\ \cos\theta\right) = \frac{p_\theta}{ml^2} \tag{29}$$

While for the first derivative of the conjugate momentum $\dot{p}$:

$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = -\frac{\partial}{\partial \theta}\left(\frac{p_\theta^2}{2ml^2} - mgl\ \cos\theta\right) = -mgl\ \sin\theta \tag{30}$$

Finally, Hamilton's equations are:

$$\dot{\theta} = \frac{p_\theta}{ml^2} \qquad \dot{p}_\theta = -mgl\ \sin\theta$$

The first equation describes the evolution of the generalized coordinate θ as a function of the angular momentum p_θ, while the second describes the evolution of the angular momentum as a function of the angle θ. Alternatively, by using the previously introduced Poisson brackets, we can write, by considering $f = \theta$:

$$\{\theta, \mathcal{H}\} = \frac{\partial \theta}{\partial \theta}\frac{\partial \mathcal{H}}{\partial p_\theta} - \frac{\partial \theta}{\partial p_\theta}\frac{\partial \mathcal{H}}{\partial \theta} = \frac{\partial \mathcal{H}}{\partial p_\theta}$$

Therefore, its temporal evolution can be rewritten in a compact form as:

$$\dot{\theta} = \{\theta, \mathcal{H}\} = \frac{p_\theta}{ml^2}$$

This corresponds exactly to the first Hamilton's equation, as already determined. The approach is analogous with $f = p_\theta$:

$$\dot{p}_\theta = \{p_\theta, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial \theta} = -mgl\,\cos\theta$$

Through the Poisson brackets, it was thus possible to rewrite the evolution of the canonical variables in a compact form, obtaining the pendulum's equations of motion once again. Note that the second equation is equivalent to the pendulum's equation of motion already obtained with the Newtonian and Lagrangian approaches. From the first equation (29), we can derive the angular momentum:

$$p_\theta = ml^2\dot{\theta}$$

Differentiated with respect to time as:

$$\dot{p}_\theta = ml^2\ddot{\theta}$$

We have thus arrived at the same expression provided by the second of Hamilton's equations (30). By equating the latter and the expression just calculated, we obtain:

$$ml^2\ddot{\theta} = -mgl\,\sin\theta$$

Dividing by ml^2:

$$\ddot{\theta} + \frac{g}{l}\,\sin\theta = 0 \tag{31}$$

Finally, for the formulation obtained from the Hamiltonian, we can also apply the linearization of the equation of motion for very small angles. In particular, through the Taylor expansions of the trigonometric functions, we have:

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \qquad \text{per } \theta \ll 1 \text{ rad}$$

Therefore, the Hamiltonian (28) can be written in its linearized form:

$$\begin{aligned}\mathcal{H}(\theta, p_\theta) &\approx \frac{p_\theta^2}{2ml^2} - mgl\left(1 - \frac{\theta^2}{2}\right) \\ &\approx \frac{p_\theta^2}{2ml^2} - mgl + \frac{1}{2}mgl\,\theta^2\end{aligned} \tag{32}$$

The Hamiltonian formalism offers the advantage of observing the symmetry between coordinates and momenta, which becomes evident in phase space (θ, p_θ). Each point in phase space represents a possible state of the pendulum, and the system's motion can be interpreted as a trajectory in this space. This geometric representation facilitates the study of conserved quantities and the understanding of the system's overall dynamics.

6 Application to the Double Pendulum

The dynamics of the double pendulum, also known as the chaotic pendulum, will now be the subject of study. This is a physical system consisting of two pendulums attached one to the end of the other, each free to oscillate with respect to its pivot point. This forms a complex system that exhibits dynamic behavior with a strong sensitivity to initial conditions. The motion of a double pendulum is governed by a pair of ordinary differential equations.

Compared to the simple pendulum, the double pendulum has non-linear equations of motion and the system can exhibit chaotic dynamics for certain initial conditions.

In addition to excluding any friction and the gravitational constant $g = 9.81\ m/s^2$, the double pendulum requires considering twice as many parameters as the single pendulum: the two lengths in meters of the two inextensible and massless rods, l_1 and l_2; the two point masses, m_1 and m_2 in kg; and finally, the two initial angles, θ_1 and θ_2 with respect to the vertical.

The analysis requires using the Lagrangian formalism, which allows for a systematic derivation of the equations of motion even with multiple degrees of freedom.

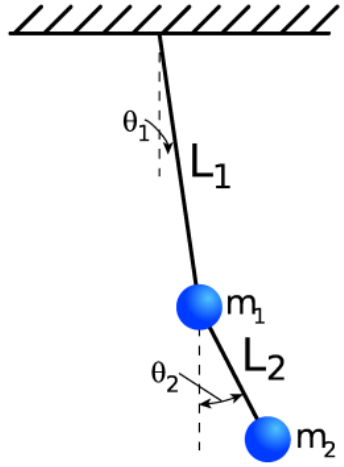


Figure 4: Idealized Double Pendulum

6.1 Lagrangian Approach

In the double pendulum, the Lagrangian formalism is used to avoid the direct calculation of tensions and the resolution of non-trivial geometric constraints. By choosing the angles θ_1 and θ_2 as generalized coordinates and then applying the Euler-Lagrange equations, we obtain the system of non-linear differential equations that describe the entire dynamics, with the constraints incorporated directly into the system's geometry.

First, we need to establish the Cartesian coordinates of the two masses, m_1 and m_2, as (x_1, y_1) and (x_2, y_2) respectively. We will use a Cartesian coordinate system with the origin at the pivot point and the y-axis oriented upwards. The coordinates are defined as follows, as was done for the simple pendulum and as described in the figure 3:

$$x_1 = l_1 \sin(\theta_1) \qquad y_1 = -l_1 \cos(\theta_1) \tag{33}$$

For the second mass, it is useful to specify that if we consider the two separate vertical lines, as shown in figure 4, the coordinates will simply be the coordinates of the first pendulum, vectorially summed with those of the pendulum attached to its end. In other words, x_2 and y_2 are derived from the position of the first pendulum plus the relative position of the second one:

$$x_2 = x_1 + l_2 \sin(\theta_2) = l_1 \sin(\theta_1) + l_2 \sin(\theta_2) \tag{34}$$

$$y_2 = y_1 - l_2 \cos(\theta_2) = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2) \tag{35}$$

The first mass m_1 moves along a circular arc of radius l and its coordinates depend only on θ_1. In contrast, the position of the second mass, m_2 is influenced by the motion of the first as well as its own angle θ_2. After calculating the generalized coordinates as functions of θ_1 and θ_2, it is useful to proceed by taking the time derivatives of the Cartesian coordinates to obtain the

respective velocities. These velocities will be part of the calculation for the subsequent kinetic energies, T_1 and T_2. By applying the chain rule:

$$\dot{x}_1 = \frac{d}{dt}\left(l_1 \sin\theta_1\right) = l_1 \cos\theta_1 \, \dot{\theta}_1 \tag{36}$$

$$\dot{y}_1 = \frac{d}{dt}\left(-l_1 \cos\theta_1\right) = l_1 \sin\theta_1 \, \dot{\theta}_1 \tag{37}$$

The procedure is analogous for the second mass:

$$\dot{x}_2 = \frac{d}{dt}\left(l_1 \sin\theta_1 + l_2 \sin\theta_2\right) = l_1 \cos\theta_1 \, \dot{\theta}_1 + l_2 \cos\theta_2 \, \dot{\theta}_2 \tag{38}$$

$$\dot{y}_2 = \frac{d}{dt}\left(-l_1 \cos\theta_1 - l_2 \cos\theta_2\right) = l_1 \sin\theta_1 \, \dot{\theta}_1 + l_2 \sin\theta_2 \, \dot{\theta}_2 \tag{39}$$

We can now proceed with the formulation of the Lagrangian:

$$\mathcal{L}(\dot{\theta}_1, \dot{\theta}_2, \theta_1, \theta_2, t) = T(\dot{\theta}_1, \dot{\theta}_2, \theta_1, \theta_2, t) - V(\theta_1, \theta_2, t) \tag{40}$$

Therefore, we consider the calculation of the total kinetic energy, T, of the system as the algebraic sum of the kinetic energies of the individual masses T_1 e T_2:

$$T = T_1 + T_2$$

$$T_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) \qquad T_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

We consider the calculation of the first mass's kinetic energy. Given the velocities in generalized coordinates, which were already calculated previously (36) (37), we can expand on them as was done for the single pendulum:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \left((l_1 \cos\theta_1 \, \dot{\theta}_1)^2 + (l_1 \sin\theta_1 \, \dot{\theta}_1)^2 \right) \\ &= \frac{1}{2} m_1 (l_1^2 \cos^2\theta_1 \, \dot{\theta}_1^2 + l_1^2 \sin^2\theta_1 \, \dot{\theta}_1^2) \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 (\cos^2\theta_1 + \sin^2\theta_1) \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 \end{aligned} \tag{41}$$

As expected, we find the kinetic energy equation for the simple pendulum. In this case, we must also define T_2, using the velocities of x_2 e y_2 (38)(39):

$$T_2 = \frac{1}{2} m_2 \left((l_1 \cos\theta_1 \, \dot{\theta}_1 + l_2 \cos\theta_2 \, \dot{\theta}_2)^2 + (l_1 \sin\theta_1 \, \dot{\theta}_1 + l_2 \sin\theta_2 \, \dot{\theta}_2)^2 \right)$$

First, we expand the squared terms:

$$\begin{aligned} T_2 = \frac{1}{2} m_2 (&l_1^2 \cos^2\theta_1 \dot{\theta}_1^2 + l_2^2 \cos^2\theta_2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 \\ &+ l_1^2 \sin^2\theta_1 \dot{\theta}_1^2 + l_2^2 \sin^2\theta_2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2) \end{aligned}$$

We'll now open a brief aside to demonstrate the summation of similar terms (placed one above the other for clarity) from the expression above. We'll begin with the first two.[1]:

$$l_1^2 \cos^2\theta_1 \dot{\theta}_1^2 + l_1^2 \sin^2\theta_1 \dot{\theta}_1^2 = l_1^2 \dot{\theta}_1^2 (\cos^2\theta_1 + \sin^2\theta_1) = l_1^2 \dot{\theta}_1^2$$

[1] Trigonometric identity: $\cos^2\alpha + \sin^2\alpha = 1$

$$l_2^2 \cos^2\theta_2\dot{\theta}_2^2 + l_2^2 \sin^2\theta_2\dot{\theta}_2^2 = l_2^2\dot{\theta}_2^2(\cos^2\theta_2 + \sin^2\theta_2) = l_2^2\dot{\theta}_2^2$$

Next, we sum the cross-terms:[2]:

$$2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2$$

$$2l_1l_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = 2l_1l_2\dot{\theta}_1\dot{\theta}_2\ \cos(\theta_1 - \theta_2)$$

With this brief demonstration closed, we can now write the kinetic energy of the second mass explicitly, by summing the results of the three previous sums:

$$\begin{aligned} T_2 &= \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\ \cos(\theta_1 - \theta_2)\right) \\ &= \frac{1}{2}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\ \cos(\theta_1 - \theta_2) \end{aligned} \tag{42}$$

Finally, by summing T_1 (41) and T_2 (42), we obtain the total kinetic energy:

$$T = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\ \cos(\theta_1 - \theta_2)$$

By collecting similar terms, such as $l_1^2\dot{\theta}_1^2$, we obtain the final form of kinetic energy T:

$$T = \frac{1}{2}l_1^2\dot{\theta}_1^2(m_1 + m_2) + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\ \cos(\theta_1 - \theta_2) \tag{43}$$

We notice three distinct terms. The first represents the kinetic energy corresponding to the motion of the first mass (where the motion and position of m_2 depend on m_1). The second is the contribution of the second mass's own motion. Finally, the third is an interaction term that links the two movements and depends on the relative angle, $\theta_1 - \theta_2$. This last dependency is at the core of the double pendulum's non-linear dynamics, which gives rise to complex and chaotic behaviors. Once the calculation of the total kinetic energy is complete, we now move on to the system's gravitational potential energy. This is also written as the algebraic sum of the energies of the individual masses, V_1 and V_2. Given y_1 (33), we define V_1 as:

$$V_1 = m_1gy_1 = -m_1gl_1\ \cos(\theta_1) \tag{44}$$

Analogously for y_2 (35), and V_2:

$$V_2 = m_2gy_2 = m_2g(-l_1\cos(\theta_1) - l_2\cos(\theta_2)) = -m_2gl_1\cos(\theta_1) - m_2gl_2\cos(\theta_2) \tag{45}$$

We define the total potential energy V as the sum of V_1 (44) and V_2(45), which gives us:

$$V = -m_1gl_1\cos\theta_1 - m_2gl_1\cos\theta_1 - m_2gl_2\cos\theta_2$$

We factor out $gl_1\ \cos\theta_1$:

$$V = -gl_1\cos\theta_1(m_1 + m_2) - m_2gl_2\cos\theta_2 \tag{46}$$

Potential energy is defined such that its value is more negative the lower the masses are. Both masses' potential energies depend on θ_1, as m_2 also changes height with the first arm. The second term, however, is due to the second mass's own motion.

Once both total energies, T (43) and V (46), have been calculated, the Lagrangian of the system can be easily determined (40):

$$\begin{aligned} \mathcal{L} =& \frac{1}{2}l_1^2\dot{\theta}_1^2(m_1 + m_2) + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\ \cos(\theta_1 - \theta_2) \\ &+ gl_1\cos\theta_1(m_1 + m_2) + m_2gl_2\cos\theta_2 \end{aligned} \tag{47}$$

[2]Trigonometric identity: $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

6.2 Euler-Lagrange Equation: First Degree of Freedom

Based on the Lagrangian, we will now proceed with writing the two Euler-Lagrange equations (4), in terms of θ_1 and θ_2 respectively. The first equation is as follows:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial \mathcal{L}}{\partial \theta_1}\right) = 0 \tag{48}$$

For simplicity of notation, the term $(\theta_1 - \theta_2)$, will be denoted as Δ. We proceed in order, first determining the contribution relative to $\dot{\theta}_1$:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = \frac{\partial}{\partial \dot{\theta}_1}\Big(&\frac{1}{2}l_1^2\dot{\theta}_1^2(m_1+m_2) + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos\Delta \\ &+ g l_1 \cos\theta_1(m_1+m_2) + m_2 g l_2 \cos\theta_2\Big)\end{aligned}$$

We begin by taking the partial derivative of the Lagrangian with respect to the angular velocity $\dot{\theta}_1$. From the Lagrangian, only the first and third terms contain $\dot{\theta}_1$ and therefore provide non-zero contributions:

$$\frac{\partial}{\partial \dot{\theta}_1}\Big(\frac{1}{2}l_1^2\dot{\theta}_1^2(m_1+m_2) + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos\Delta\Big)$$

To derive the first term, we apply the power rule. For the second, we treat it as a linear function of $\dot{\theta}_1$:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = l_1^2\dot{\theta}_1(m_1+m_2) + m_2 l_1 l_2 \dot{\theta}_2\cos\Delta \tag{49}$$

We now proceed by completely determining the first term of the first Euler-Lagrange equation, taking the time derivative of the result above (48):

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = \frac{d}{dt}\Big(l_1^2\dot{\theta}_1(m_1+m_2) + m_2 l_1 l_2 \dot{\theta}_2\cos\Delta\Big)$$

The derivative of the first term is straightforward. Since $\dot{\theta}_1$ is linear, its time derivative is simply the acceleration $\ddot{\theta}_1$:

$$\frac{d}{dt}\Big(l_1^2\dot{\theta}_1(m_1+m_2)\Big) = l_1^2\left[\frac{d}{dt}(\dot{\theta}_1)\right](m_1+m_2) = l_1^2\ddot{\theta}_1(m_1+m_2) \tag{50}$$

The derivative of the second term, $m_2 l_1 l_2 \dot{\theta}_2\cos\Delta$ contains both $\dot{\theta}_2$ and $\cos\Delta$, which are time-dependent. We must therefore apply the product rule. Furthermore, since $\cos\Delta$ is a function of θ_1 and θ_2, which are themselves functions of time, we must also apply the chain rule:

$$\frac{d}{dt}\Big(m_2 l_1 l_2 \dot{\theta}_2\cos\Delta\Big) = m_2 l_1 l_2 \frac{d}{dt}\Big[\dot{\theta}_2\cos\Delta\Big] \tag{51}$$

In a more compact way, the term to be derived (51) is determined as (in this case, the second line is a further decomposition equivalent to the first:

$$\begin{aligned}\frac{d}{dt}(\dot{\theta}_2\cos\Delta) &= \ddot{\theta}_2\cos\Delta - \dot{\theta}_2(\dot{\theta}_1-\dot{\theta}_2)\sin\Delta \\ &= \ddot{\theta}_2\cos\Delta - \dot{\theta}_1\dot{\theta}_2\sin\Delta + \dot{\theta}_2^2\sin\Delta\end{aligned} \tag{52}$$

We will now open a brief aside for the full derivation of the time-dependent term. We begin by applying the product rule to the term $\dot{\theta}_2\cos\Delta$, where $\Delta = \theta_1 - \theta_2$:

$$\frac{d}{dt}(\dot{\theta}_2)\cos\Delta + \dot{\theta}_2\frac{d}{dt}(\cos\Delta)$$

For the term on the left, the procedure is analogous to the one seen previously (50), where $\dot{\theta}_2$ is derived as $\ddot{\theta}_2$. For the term on the right, it is necessary to apply the chain rule:

$$\begin{aligned}\ddot{\theta}_2 \cos\Delta &+ \dot{\theta}_2(-\sin\Delta\dot{\Delta}) \\ &+ \dot{\theta}_2(-\sin\Delta(\dot{\theta}_1 - \dot{\theta}_2)) \\ &+ \dot{\theta}_2(-\dot{\theta}_1\sin\Delta + \dot{\theta}_2\sin\Delta) \\ &- \dot{\theta}_1\dot{\theta}_2\sin\Delta + \dot{\theta}_2^2\sin\Delta\end{aligned}$$

We have concluded the brief aside where we explicitly demonstrated each step to arrive at the same solution (52), we now continue with the calculation (51), substituting the term just obtained, into the expression $\frac{d}{dt}[\dot{\theta}_2\cos\Delta]$ to complete the second term:

$$\begin{aligned}\frac{d}{dt}\Big(m_2 l_1 l_2 \dot{\theta}_2\cos\Delta\Big) &= m_2 l_1 l_2(\ddot{\theta}_2\cos\Delta - \dot{\theta}_1\dot{\theta}_2\sin\Delta + \dot{\theta}_2^2\sin\Delta) \\ &= m_2 l_1 l_2\ddot{\theta}_2\cos\Delta - m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta + m_2 l_1 l_2\dot{\theta}_2^2\sin\Delta \qquad (53)\end{aligned}$$

We will now completely rewrite the first term of the first Euler-Lagrange equation (48), by algebraically summing the previously calculated components (50) and (53):

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) = l_1^2\ddot{\theta}_1(m_1+m_2) + m_2 l_1 l_2\ddot{\theta}_2\cos\Delta - m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta + m_2 l_1 l_2\dot{\theta}_2^2\sin\Delta \qquad (54)$$

To complete the first Euler-Lagrange equation (48), it is necessary to determine the second term:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\theta_1} = \frac{\partial}{\partial\theta_1}\Big(&\frac{1}{2}l_1^2\dot{\theta}_1^2(m_1+m_2) + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos\Delta \\ &+ g l_1\cos\theta_1(m_1+m_2) + m_2 g l_2\cos\theta_2\Big)\end{aligned}$$

In the partial derivation with respect to θ_1, only the third and fourth terms of the Lagrangian contribute. Therefore, we will have:

$$\frac{\partial}{\partial\theta_1}\Big(m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos\Delta + g l_1\cos\theta_1(m_1+m_2)\Big)$$

The term $m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos\Delta$ depends on θ_1 through $\Delta = \theta_1 - \theta_2$:

$$\frac{\partial}{\partial\theta_1}(m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos\Delta) = m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2(-\sin\Delta) = -m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) \qquad (55)$$

The gravitational term $g l_1\cos\theta_1(m_1+m_2)$ is simply derived as:

$$\frac{\partial}{\partial\theta_1}(g l_1\cos\theta_1(m_1+m_2)) = -g l_1(m_1+m_2)\sin\theta_1 \qquad (56)$$

Given the equations just calculated (55) (56), the second term of the first Euler-Lagrange equation will be:

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = -m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - g l_1(m_1+m_2)\sin\theta_1 \qquad (57)$$

Having concluded the calculation of all terms (54) (57) of the first Euler-Lagrange equation (48), we can write it explicitly as:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) - \frac{\partial\mathcal{L}}{\partial\theta_1} = 0$$

$$\begin{aligned}&l_1^2\ddot{\theta}_1(m_1+m_2) + m_2 l_1 l_2\ddot{\theta}_2\cos\Delta - m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta + m_2 l_1 l_2\dot{\theta}_2^2\sin\Delta \\ &+ m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta + g l_1(m_1+m_2)\sin\theta_1 = 0\end{aligned}$$

By simplifying the opposing terms $m_1 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta$, we obtain the final form:

$$l_1^2\ddot{\theta}_1(m_1+m_2) + m_2 l_1 l_2\ddot{\theta}_2\cos\Delta + m_2 l_1 l_2\dot{\theta}_2^2\sin\Delta + g l_1(m_1+m_2)\sin\theta_1 = 0 \qquad (58)$$

6.3 Euler-Lagrange Equation: Second Degree of Freedom

To completely determine the motion of the double pendulum, we must apply the procedure just completed in an analogous way for the second degree of freedom, θ_2, and then finally determine the second Euler-Lagrange equation, which we can already define as:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0 \tag{59}$$

We'll start, as with the previous Euler-Lagrange equation, by determining the first term in terms of $\dot{\theta}_2$. Given the double pendulum's Lagrangian (47) we can define:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = \frac{\partial}{\partial \dot{\theta}_2}\Big(&\frac{1}{2}l_1^2\dot{\theta}_1^2(m_1+m_2) + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos\Delta \\ &+ g l_1\cos\theta_1(m_1+m_2) + m_2 g l_2\cos\theta_2\Big)\end{aligned}$$

Based on the Lagrangian, only the second and third terms provide non-zero contributions when taking the partial derivative with respect to $\dot{\theta}_2$. We can rewrite this as:

$$\frac{\partial}{\partial \dot{\theta}_2}\Big(\frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos\Delta\Big)$$

Therefore, apply the power rule for the first term, and a simple linear derivation for the second:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos\Delta \tag{60}$$

The following process is analogous to the one already shown for the first Euler-Lagrange equation. We now derive the term we just calculated (60) with respect to time to complete the first of the two terms of this second Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = \frac{d}{dt}\Big(m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos\Delta\Big)$$

The derivative of the first term depends only on $\dot{\theta}_2$, which, when derived with respect to time, corresponds to $\ddot{\theta}_2$:

$$\frac{d}{dt}\Big(m_2 l_2^2\dot{\theta}_2\Big) = m_2 l_2^2\left[\frac{d}{dt}(\dot{\theta}_2)\right] = m_2 l_2^2\ddot{\theta}_2 \tag{61}$$

As previously noted, the approach for the second term is a bit more complex, as the following time-dependent terms need to be derived:

$$\frac{d}{dt}\Big(m_2 l_1 l_2\dot{\theta}_1\cos\Delta\Big) = m_2 l_1 l_2\frac{d}{dt}\Big[\dot{\theta}_1\cos\Delta\Big] \tag{62}$$

The derivative of the second term contains $\dot{\theta}_1$ and also $\cos\Delta$, where $\Delta = \theta_1 - \theta_2$ So, the derivation first requires the use of the product rule, and subsequently the chain rule. Si avrà perciò un risultato del genere (the second line is a further breakdown equivalent to the first):

$$\begin{aligned}\frac{d}{dt}(\dot{\theta}_1\cos\Delta) &= \ddot{\theta}_1\cos\Delta - \dot{\theta}_1(\dot{\theta}_1 - \dot{\theta}_2)\sin\Delta \\ &= \ddot{\theta}_1\cos\Delta - \dot{\theta}_1^2\sin\Delta + \dot{\theta}_1\dot{\theta}_2\sin\Delta\end{aligned}$$

By inserting this last result in place of the time derivative of the second term $\frac{d}{dt}[\dot{\theta}_1\cos\Delta]$, we get the full first term of the Euler-Lagrange equation:

$$\begin{aligned}\frac{d}{dt}\Big(m_2 l_1 l_2\dot{\theta}_1\cos\Delta\Big) &= m_2 l_1 l_2\left(\ddot{\theta}_1\cos\Delta - \dot{\theta}_1^2\sin\Delta + \dot{\theta}_1\dot{\theta}_2\sin\Delta\right) \\ &= m_2 l_1 l_2\ddot{\theta}_1\cos\Delta - m_2 l_1 l_2\dot{\theta}_1^2\sin\Delta + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta\end{aligned} \tag{63}$$

Completing the first term of the second Euler-Lagrange equation (59), we combine the terms we've calculated (61) (63):

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos\Delta - m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\Delta \tag{64}$$

With the calculation of the first term complete, we'll now continue with the next term, which is a function of θ_2:

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{\partial}{\partial \theta_2}\Big(\frac{1}{2} l_1^2 \dot{\theta}_1^2 (m_1 + m_2) + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\Delta + g l_1 \cos\theta_1 (m_1 + m_2) + m_2 g l_2 \cos\theta_2\Big)$$

For the terms contributing to the derivative:

$$\frac{\partial}{\partial \theta_2}\Big(m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\Delta + m_2 g l_2 \cos\theta_2\Big)$$

In calculating the latter derivative, attention must be paid to the signs of the terms derived from the product $\dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$. In fact, the function $\cos(\theta_1 - \theta_2)$ depends on both θ_1 and θ_2, and the derivative with respect to each variable produces a different sign.

When deriving with respect to θ_1, the difference $\theta_1 - \theta_2$ leads to an internal positive factor, generating the term with negative sign $-\sin(\theta_1 - \theta_2)$. Conversely, deriving with respect to θ_2, the difference produces a negative factor, which when multiplied by the $-\sin$ of the chain rule, reverses the sign, leading to a positive contribution $+\sin(\theta_1 - \theta_2)$.

Therefore, we derive the first term:

$$\frac{\partial}{\partial \theta_2}\Big(m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\Big) = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \tag{65}$$

While the gravitational term, as is usually done:

$$\frac{\partial}{\partial \theta_2}\Big(m_2 g l_2 \cos\theta_2\Big) = -m_2 g l_2 \sin\theta_2 \tag{66}$$

Finally, combining the respective terms obtained in (65) and (66), we determine the second term of the Euler-Lagrange equation (59):

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\Delta - m_2 g l_2 \sin\theta_2 \tag{67}$$

Finally, we can finally define the Euler-Lagrange equation (59) for the second degree of freedom, combining (64) and (67), arriving at:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos\Delta - m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\Delta - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\Delta + m_2 g l_2 \sin\theta_2 = 0$$

Simplifying opposite terms $m_1 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\Delta$, we get the final form of the equation:

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos\Delta - m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta + m_2 g l_2 \sin\theta_2 = 0 \tag{68}$$

6.4 Matrix Formalism

Considering a dynamic physical system like the double pendulum, which consists of two masses m_1 and m_2, constrained by two rigid rods of length l_1 and l_2, and denoting the angles of the first and second rods with their respective vertical lines as $\theta_1(t)$ and $\theta_2(t)$, we obtain a system of two coupled, non-linear, second-order ordinary differential equations (ODEs):

$$\begin{cases} l_1^2\ddot{\theta}_1(m_1+m_2) + m_2 l_1 l_2 \ddot{\theta}_2 \cos\Delta + m_2 l_1 l_2 \dot{\theta}_2^2 \sin\Delta + g l_1 (m_1+m_2)\sin\theta_1 = 0 \\ m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos\Delta - m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta + m_2 g l_2 \sin\theta_2 = 0 \end{cases} \tag{69}$$

A differential equation, such as the two just presented, is called second-order if the second derivative of the unknown variable appears in an essential way. In this case, the derivatives $\ddot{\theta}_1$ and $\ddot{\theta}_2$ are involved in the equations, as the system's dynamics depend on the acceleration of the masses.

They are called coupled because the motion of each mass depends on both its own angle and velocity as well as those of the other mass (e.g., the cross terms like $m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)$). The motion of the two pendulums is therefore interdependent and inseparable. Formally:

$$\ddot{\theta}_1 = f_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \qquad \ddot{\theta}_2 = f_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$$

They are finally non-linear due to the trigonometric terms and products between variables like $\dot{\theta}_i \dot{\theta}_j$. This implies that the principle of superposition does not apply.

In summary, the system is described by two coupled, non-linear, second-order ordinary differential equations. This structure suggests introducing the matrix formalism, where the system will no longer be described by a pair of second-order ODEs but by matrices and vectors, formally defined as:

$$\mathbf{M}(q_i)\,\ddot{\mathbf{q}} + (\mathbf{C}(q_i, \dot{q}_i)\,\dot{\mathbf{q}} + \mathbf{G}(q_i)) = 0 \tag{70}$$

Assuming, for the double pendulum, the generalized coordinates $q_1 = \theta_1$ e $q_2 = \theta_2$:

$$\mathbf{M}(\theta_1, \theta_2)\,\ddot{\mathbf{\Theta}} + \mathbf{C}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)\,\dot{\mathbf{\Theta}} + \mathbf{G}(\theta_1, \theta_2) = 0 \tag{71}$$

The matrix formalism (71) highlights several important aspects of the system. The **M**matrix collects the acceleration coefficients, clearly showing the coupling between the two masses through the off-diagonal terms. The **C** matrix groups the velocity coefficients related to Coriolis and centripetal effects, while the **G** vector contains the contributions from gravity. The two vectors $\ddot{\mathbf{\Theta}}$ and $\dot{\mathbf{\Theta}}$ contain the second and first derivatives of the generalized coordinates (the accelerations and velocities), respectively:

$$\ddot{\mathbf{\Theta}} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \qquad \dot{\mathbf{\Theta}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{72}$$

This rewriting makes the system easier to read and numerically handle, as it isolates the accelerations as functions of positions and velocities. This simplifies the implementation of numerical integration algorithms like the fourth-order Runge-Kutta. It is also extremely useful for generalizing the analysis to systems with more degrees of freedom, as all information can be contained in matrices and vectors, clarifying the system's structure and the relationships between variables.

In summary, the matrix formalism doesn't change the equations; it organizes them to highlight the linearity with respect to accelerations, the coupling between the masses, and the non-linearity of the known terms. This provides an elegant and practical tool for both theoretical analysis and numerical simulation.

Therefore, we rewrite the system of ordinary differential equations for the double pendulum in matrix form. We start with the first term in the matrix equation (71), **M**, which groups the

coefficients of the accelerations, specifically the first two terms of each of the two differential equations already written in system form (69). The $\mathbf{M}$ matrix contains only the coefficients of the accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$:

$$\mathbf{M} = \begin{bmatrix} l_1^2(m_1+m_2) & m_2 l_1 l_2 \cos\Delta \\ m_2 l_1 l_2 \cos\Delta & m_2 l_2^2 \end{bmatrix} \tag{73}$$

It is worth noting that unlike in the system of equations (69), the coefficients of the accelerations in the $\mathbf{M}$ matrix are organized into rows and columns according to the order of the generalized coordinates, (θ_1, θ_2). The first row and column correspond to θ_1, while the second row and column correspond to θ_2. Specifically, M_{11} is the coefficient of $\ddot{\theta}_1$ whereas M_{12} is the coefficient of $\ddot{\theta}_2$ in the same equation. The same logic applies to the second row and column, which correspond to the second equation of the system.

The second matrix, the $\mathbf{C}$ Coriolis matrix, groups the coefficients of the velocities. In the case of the double pendulum, it groups the terms that describe the effects of one mass's velocity on the motion of the other. Even without additional external forces, the presence of a moving mass influences the dynamics of the other. For simplicity of notation, in this case, the matrix will become a vector that groups the entire velocity term and not just its coefficients (consequently, we will have $\mathbf{C}$ and not $\mathbf{C}\dot{\mathbf{\Theta}}$):

$$\mathbf{C} = \begin{bmatrix} m_2 l_1 l_2 \dot{\theta}_2^2 \sin\Delta \\ -m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta \end{bmatrix} \tag{74}$$

The gravitational vector $\mathbf{G}$, represents the potential energy terms in the system. It corresponds to the last term of the first and second equations in the system (69):

$$\mathbf{G} = \begin{bmatrix} g l_1 (m_1+m_2) \sin\theta_1 \\ m_2 g l_2 \sin\theta_2 \end{bmatrix} \tag{75}$$

We can now complete the matrix formalism for the double pendulum (71):

$$\mathbf{M}\,\ddot{\mathbf{\Theta}} + (\mathbf{C}+\mathbf{G}) = \mathbf{0}$$

$$\begin{bmatrix} l_1^2(m_1+m_2) & m_2 l_1 l_2 \cos\Delta \\ m_2 l_1 l_2 \cos\Delta & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} m_2 l_1 l_2 \dot{\theta}_2^2 \sin\Delta \\ -m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta \end{bmatrix} + \begin{bmatrix} g l_1 (m_1+m_2) \sin\theta_1 \\ m_2 g l_2 \sin\theta_2 \end{bmatrix} = 0$$

For numerical integration, especially with a system like the double pendulum, it is useful to isolate the accelerations $\ddot{\mathbf{\Theta}}$, and the matrix formalism (71) allows us to do this easily:

$$\ddot{\mathbf{\Theta}} = -\mathbf{M}^{-1}(\mathbf{C}+\mathbf{G}) \tag{76}$$

For a given 2x2 matrix $\mathbf{M}$, its inverse is defined as:

$$\mathbf{M}^{-1} = \frac{1}{\det\mathbf{M}} \begin{bmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{bmatrix}, \qquad \det\mathbf{M} = M_{11}M_{22} - M_{12}M_{21}$$

By determining and substituting the coefficients:

$$\det\mathbf{M} = \Big[(l_1^2(m_1+m_2))\,(m_2 l_2^2)\Big] - \Big[(m_2 l_1 l_2 \cos\Delta)(m_2 l_1 l_2 \cos\Delta)\Big] = l_1 l_2 m_2 (m_1 + m_2 \sin^2\Delta)$$

$$\mathbf{M^{-1}} = \frac{1}{l_1 l_2 m_2 (m_1 + m_2 \sin^2\Delta} \begin{bmatrix} m_2 l_2^2 & -m_2 l_1 l_2 \cos\Delta \\ -m_2 l_1 l_2 \cos\Delta & l_1^2(m_1+m_2) \end{bmatrix} \tag{77}$$

The term in parentheses (76), can also be written in a more compact form:

$$\mathbf{C}+\mathbf{G} = \begin{bmatrix} m_2 l_1 l_2 \dot{\theta}_2^2 \sin\Delta + g l_1 (m_1+m_2) \sin\theta_1 \\ -m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta + m_2 g l_2 \sin\theta_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{78}$$

We have now arrived at the following matrix structure:

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \qquad \ddot{\mathbf{\Theta}} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}, \qquad \mathbf{C} + \mathbf{G} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Given the matrix structure (76), we can solve for $\ddot{\theta}_1$:

$$\ddot{\theta}_1 = -\frac{1}{\det \mathbf{M}} \left(M_{22} X_1 \; - \; M_{12} X_2 \right) = -\frac{M_{22} X_1 - M_{12} X_2}{\det \mathbf{M}} \tag{79}$$

Similarly, we can solve for $\ddot{\theta}_2$:

$$\ddot{\theta}_2 = -\frac{1}{\det \mathbf{M}} \left(-M_{21} X_1 \; + \; M_{11} X_2 \right) = -\frac{-M_{21} X_1 + M_{11} X_2}{\det \mathbf{M}} \tag{80}$$

To conclude, we can substitute the values of M_{ij} and X_1, X_2, to obtain the final formulas in terms of $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$:

$$X_1 = m_2 l_1 l_2 \dot{\theta}_2^2 \sin \Delta + g l_1 (m_1 + m_2) \sin \theta_1$$
$$X_2 = -m_2 l_1 l_2 \dot{\theta}_1^2 \sin \Delta + m_2 g l_2 \sin \theta_2$$

$$M_{11} = l_1^2 (m_1 + m_2), \qquad M_{12} = m_2 l_1 l_2 \cos \Delta$$
$$M_{21} = m_2 l_1 l_2 \cos \Delta, \qquad M_{22} = m_2 l_2^2$$

Given the terms just listed, we can make the accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$ explicit using the formulas derived above(79) (80), simplifying $(\theta_1 - \theta_2)$ to Δ:

$$\ddot{\theta}_1 = \frac{-m_2 l_2^2 \Big[m_2 l_1 l_2 \dot{\theta}_2^2 \sin \Delta + g l_1 (m_1 + m_2) \sin \theta_1 \Big] + m_2 l_1 l_2 \cos \Delta \Big[-m_2 l_1 l_2 \dot{\theta}_1^2 \sin \Delta + m_2 g l_2 \sin \theta_2 \Big]}{l_1^2 l_2^2 m_2 (m_1 + m_2 \sin^2 \Delta)} \tag{81}$$

$$\ddot{\theta}_2 = \frac{m_2 l_1 l_2 \cos \Delta \Big[m_2 l_1 l_2 \dot{\theta}_2^2 \sin \Delta + g l_1 (m_1 + m_2) \sin \theta_1 \Big] - l_1^2 (m_1 + m_2) \Big[-m_2 l_1 l_2 \dot{\theta}_1^2 \sin \Delta + m_2 g l_2 \sin \theta_2 \Big]}{l_1^2 l_2^2 m_2 (m_1 + m_2 \sin^2 \Delta)} \tag{82}$$

6.4.1 Remarks on Notation

It is important to note that the explicit form of the accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$ may look different in some textbooks or academic treatises. This is because the terms are often rewritten using algebraic simplifications, grouped in different ways or normalised in terms of lengths, masses or numerical constants to obtain more compact or didactic denominators.

However, the representation presented here is formally correct. Crucially, it can be derived directly from the matrix formalism and the previous Euler-Lagrange equations and is therefore fully verifiable. Therefore, it strictly respects the structure of the dynamic system of the double pendulum.

6.5 Hamiltonian Approach

For the double pendulum, the Hamiltonian formulation is introduced to systematically handle the equations of motion, transforming them from second-order differential equations into a first-order system in phase space. We define the conjugate momenta p_1 and p_2 associated with the generalized angles θ_1 and θ_2, leveraging the relationship between momenta and velocities derived from the system's Lagrangian. From this definition, the Hamiltonian is constructed via the Legendre transform or as the sum of kinetic energy T and potential energy V.

The subsequently obtained Hamilton's equations describe the temporal evolution of the angles and conjugate momenta, allowing for a clear study of conserved quantities and the analysis of complex phenomena, such as the chaotic behavior characteristic of the double pendulum.

We therefore begin by choosing the angles as the generalized coordinates: $q_1 = \theta_1$ and $q_2 = \theta_2$. Next, we define their respective conjugate momenta p_1 and p_2. Given the Lagrangian already determined (47):

$$\begin{aligned}\mathcal{L} =& \frac{1}{2} l_1^2 \dot{\theta}_1^2 (m_1 + m_2) + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ &+ g l_1 \cos\theta_1 (m_1 + m_2) + m_2 g l_2 \cos\theta_2 \end{aligned} \tag{83}$$

Note that the term $\theta_1 - \theta_2$ is also expressed as Δ, as has been previously used. The first conjugate momentum is defined according to the general rule (5), and was also derived in the previous formulation (49):

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = \frac{\partial}{\partial \dot{\theta}_1} \Big(\frac{1}{2} l_1^2 \dot{\theta}_1^2 (m_1 + m_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\Delta \Big) = l_1^2 \dot{\theta}_1 (m_1 + m_2) + m_2 l_1 l_2 \dot{\theta}_2 \cos\Delta$$

The same applies to the second conjugate momentum, already calculated (60):

$$p_2 = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = \frac{\partial}{\partial \dot{\theta}_2} \Big(\frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\Delta \Big) = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos\Delta$$

After determining the conjugate momenta for the generalized coordinates, it is necessary to isolate the velocities $\dot{\theta}_1$ and $\dot{\theta}_2$, which will be used along with the momenta in the Hamiltonian formulation. Note a substantial difference now compared to the simple pendulum, within the Hamiltonian $\mathcal{H}$. In the case of the simple pendulum, the conjugate momentum depends on only one velocity, and the inversion to obtain $\dot{\theta}$ as a function of p is immediate (27). For the double pendulum, however, the equations are coupled. To determine the velocities, it will therefore be necessary to write the momenta in matrix form, in order to elegantly separate the velocities from their respective coefficients:

$$\mathbf{p} = \mathbf{M}\,(\theta_1, \theta_2)\,\dot{\boldsymbol{\theta}} \tag{84}$$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_1 \end{bmatrix}$$

Where the two vectors represent, respectively, the momenta p_1 and p_2 and the velocities. The mass matrix $\mathbf{M}$, i.e., the coefficients of the velocities themselves, is composed of:

$$a = l_1^2 (m_1 + m_2) \qquad b = m_2 l_1 l_2 \cos\Delta \qquad c = m_2 l_2^2$$

To obtain the velocities as a function of the momenta, we must invert this relationship. And since the coefficient matrix is a symmetric 2×2 matrix, its inverse is immediate and is obtained as:

$$\dot{\boldsymbol{\theta}} = \mathbf{M}^{-1}(\theta_1, \theta_2)\,\mathbf{p} \tag{85}$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \frac{1}{\det \mathbf{M}} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \qquad \det \mathbf{M} = ac - b^2 \tag{86}$$

Calculating the determinant with the previous definitions[3]:

$$\begin{aligned}\det \mathbf{M} = ac - b^2 &= l_1^2(m_1 + m_2)\ \ m_2 l_2^2 - (m_2 l_1 l_2 \cos\Delta)^2 \\ &= m_2 l_1^2 l_2^2 (m_1 + m_2) - m_2^2 l_1^2 l_2^2 \cos^2\Delta \\ &= m_2 l_1^2 l_2^2 (m_1 + m_2 \sin^2\Delta)\end{aligned}$$

From the matrix formalism, specifically from the vector $\dot{\boldsymbol{\theta}}$ defined in (86), we obtain the individual explicit expressions for the velocities:

$$\dot{\theta}_1 = \frac{c\,p_1 - b\,p_2}{ac - b^2} = \frac{m_2 l_2^2 p_1 - m_2 l_1 l_2 \cos\Delta\ p_2}{m_2 l_1^2 l_2^2 (m_1 + m_2 \sin^2\Delta)} \tag{87}$$

$$\dot{\theta}_2 = \frac{-b\,p_1 + a\,p_2}{ac - b^2} = \frac{-m_2 l_1 l_2\ \cos\Delta\ p_1 + l_1^2(m_1 + m_2)\ p_2}{m_2 l_1^2 l_2^2 (m_1 + m_2 \sin^2\Delta)} \tag{88}$$

6.6 Hamiltonian of the Double Pendulum

With the conjugate momenta determined and the expressions for the velocities as a function of the phase space variables derived, we are now able to construct the system's Hamiltonian. At this point, two formally equivalent paths exist: the first is to slavishly follow the definition via the Legendre transform, substituting the velocities expressed in terms of the momenta; the second, more compact and physically transparent, consists of recognizing that the Hamiltonian coincides with the sum of the kinetic and potential energy, $\mathcal{H} = T + V$, provided these are now expressed in terms of (q_i, p_i). To be clear, we will open a brief parenthesis, where the Hamiltonian $\mathcal{H}$, via the Legendre transformation, is defined as:

$$\mathcal{H}(\theta_1, \theta_2, p_1, p_2) = \sum_{i=1}^{2} p_i \dot{\theta}_i - \mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$$

$$\mathcal{H} = p_1\dot{\theta}_1 + p_2\dot{\theta}_2 - \mathcal{L} \tag{89}$$

Where, given the velocities $\dot{\theta}_1$ and $\dot{\theta}_2$, which now depend only on the phase space (q, p), and the momenta, we can perform the summation:

$$p_1\dot{\theta}_1 + p_2\dot{\theta}_2 = p_1 \frac{c\,p_1 - b\,p_2}{ac - b^2} + p_2 \frac{-b\,p_1 + a\,p_2}{ac - b^2} = \frac{c\,p_1^2 - 2b\ p_1 p_2 + a p_2^2}{ac - b^2}$$

In explicit terms:

$$p_1\dot{\theta}_1 + p_2\dot{\theta}_2 = \frac{m_2 l_2^2\ p_1^2 - 2m_2 l_1 l_2\ \cos\Delta\ p_1 p_2 + l_1^2(m_1 + m_2)\ p_2^2}{m_2 l_1^2 l_2^2 (m_1 + m_2\ \sin^2\Delta)} \tag{90}$$

To which, to complete the Legendre transformation, the Lagrangian $\mathcal{L}$ (47) will then be subtracted. The part that follows makes the solution complex without adding anything but purely mechanical-algebraic manipulations.
Closing this brief parenthesis, in the following discussion, we will adopt the second of the two described approaches, which is equivalent to the latter but algebraically less complex. The second method is found by defining the Hamiltonian as the total energy of the system, expressing the energies as a function of (q_i, p_i), thus keeping the construction clearer without burdening it with overly cumbersome algebraic steps:

$$\mathcal{H}(q_i, p_i, t) = T(q_i, p_i, t) + V(q_i, t)$$

[3]Trigonometric identity: $\sin^2\alpha + \cos^2\alpha = 1 \longrightarrow 1 - \cos^2\alpha = \sin^2\alpha$

Recall then that the kinetic energy T of the system, already expressed as a function of the velocities, and the potential energy V are:

$$T = \frac{1}{2} l_1^2 \dot{\theta}_1^2 (m_1 + m_2) + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\Delta$$

$$V = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

The kinetic energy, however, still contains the velocities as variables. To convert it to phase space, it must now be expressed exclusively as a function of $(\theta_1, \theta_2, p_1, p_2)$. Starting from the quadratic form, which is the same for all classical mechanical systems in which the kinetic energy depends only on velocities and not on magnetic terms or non-holonomic constraints:

$$T = \frac{1}{2} \dot{\boldsymbol{\theta}}^T \mathbf{M}(\theta) \dot{\boldsymbol{\theta}} \qquad \dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{91}$$

Recalling, therefore, the matrix relationship between momenta and velocities, already expressed above (84)(85):

$$\mathbf{p} = \mathbf{M}(\theta) \dot{\boldsymbol{\theta}} \;\longrightarrow\; \dot{\boldsymbol{\theta}} = \mathbf{M}^{-1}(\theta) \mathbf{p}$$

And by substituting the aforementioned relationship into the definition of T (91) yields:

$$T(\theta, p) \;=\; \frac{1}{2} \big(\mathbf{M}^{-1}(\theta)\mathbf{p}\big)^T \mathbf{M}(\theta) \big(\mathbf{M}^{-1}(\theta)\mathbf{p}\big) \;=\; \frac{1}{2} \mathbf{p}^T \mathbf{M}^{-1}(\theta) \mathbf{p} \tag{92}$$

This identity is general for Lagrangians quadratic in velocities and avoids explicitly developing unnecessary products. Recall that the inverse of a given symmetric 2x2 matrix, already expressed (86), is defined as:

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \qquad \det \mathbf{M} = ac - b^2.$$

So, knowing the latter, we can explicitly write the terms of the kinetic energy T (92):

$$\begin{aligned} T(\theta, p) = \frac{1}{2} \mathbf{p}^T \mathbf{M}^{-1} \mathbf{p} &= \frac{1}{2 \det \mathbf{M}} \big(c\, p_1^2 - 2b\, p_1 p_2 + a\, p_2^2\big) \\ &= \frac{c\, p_1^2 - 2b\, p_1 p_2 + a\, p_2^2}{2(ac - b^2)}, \end{aligned}$$

$$a = (m_1 + m_2) l_1^2 \qquad b = m_2 l_1 l_2 \cos\Delta \qquad c = m_2 l_2^2 \qquad ac - b^2 = m_2 l_1^2 l_2^2 \big(m_1 + m_2 \sin^2\Delta\big)$$

In doing so, the kinetic energy T has been completely reduced to the phase variables. The potential energy V does not change from the Lagrangian to the Hamiltonian formulation, as it depends only on the generalized coordinates and not on the velocity terms. We will therefore have a Hamiltonian:

$$\mathcal{H}(\theta, p) = T(\theta, p) + V(\theta) = \frac{c\, p_1^2 - 2b\, p_1 p_2 + a\, p_2^2}{2 \det \mathbf{M}} + V(\theta)$$

$$\mathcal{H}(\theta_1, \theta_2, p_1, p_2) = \frac{c\, p_1^2 - 2b\, p_1 p_2 + a\, p_2^2}{2(ac - b^2)} - (m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \tag{93}$$

Note how the fractional term derives from the kinetic part of the system, appropriately rewritten as a function of the momenta, while the potential part remains formally unchanged. In expanded form, the Hamiltonian, with $\Delta = \theta_1 - \theta_2$, is defined as:

$$\mathcal{H}(\theta_1, \theta_2, p_1, p_2) = \frac{m_2 l_2^2\, p_1^2 - 2 m_2 l_1 l_2\, \cos\Delta\, p_1 p_2 + (m_1 + m_2) l_1^2\, p_2^2}{2 m_2 l_1^2 l_2^2 (m_1 + m_2\, \sin^2\Delta)} - (m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \tag{94}$$

Having completed the calculation of the Hamiltonian, the next step is to define Hamilton's equations. For systems like the double pendulum, where the conjugate momenta are coupled and the velocities are not easily isolated, it is possible to avoid the classical, term-by-term derivation by, once again, exploiting the matrix formalism, in this case of the Hamiltonian. We will therefore use the matrix formalism to determine Hamilton's equations, thus obtaining an equivalent but algebraically more compact approach.

From the Hamiltonian, expressed in matrix form as the total energy of the system:

$$\mathcal{H}(\theta, \mathbf{p}) = \frac{1}{2}\, \mathbf{p}^T\, \mathbf{M}^{-1}(\theta)\, \mathbf{p} + V(\theta)$$

The corresponding Hamilton's equations are derived, also written in matrix form:

$$\dot{\boldsymbol{\theta}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \qquad \dot{\boldsymbol{p}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{\theta}}$$

6.7 First Pair of Hamilton's Equations

For the first pair of first-order differential equations, which describes the generalized coordinates as a function of their respective momenta, we have:

$$\dot{\boldsymbol{\theta}} = \mathbf{M}^{-1}(\theta)\, \mathbf{p}$$

This coincides exactly with the relationship already found in (85). This expression shows how the angular velocities are linear functions of the conjugate momenta through the inverse mass matrix. The equations are, in fact, the same as those calculated in that (87) (88):

$$\dot{\theta}_1 = \frac{m_2 l_2^2 p_1 - m_2 l_1 l_2 \cos\Delta\, p_2}{m_2 l_1^2 l_2^2 (m_1 + m_2 \sin^2\Delta)} \qquad \dot{\theta}_2 = \frac{-m_2 l_1 l_2\, \cos\Delta\, p_1 + l_1^2 (m_1 + m_2)\, p_2}{m_2 l_1^2 l_2^2 (m_1 + m_2 \sin^2\Delta)}$$

6.8 Second Pair of Hamilton's Equations

For the second pair of Hamilton's equations for the double pendulum, the calculation is less straightforward. We begin by differentiating the Hamiltonian with respect to the generalized coordinate q:

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = -\frac{1}{2}\, \mathbf{p}^T\, \frac{\partial \mathbf{M}^{-1}}{\partial q_i} \mathbf{p} - \frac{\partial V}{\partial q_i} \tag{95}$$

Using the matrix identity:

$$\frac{\partial \mathbf{M}^{-1}}{\partial q_i} = -\mathbf{M}^{-1} \left(\frac{\partial \mathbf{M}}{\partial q_i}\right) \mathbf{M}^{-1}$$

Substituting this into (95), we find the final form for determining the second pair of Hamilton's equations:

$$\dot{p}_i = \frac{1}{2}\, \mathbf{p}^T\, \mathbf{M}^{-1} \Big(\frac{\partial \mathbf{M}}{\partial q_i}\Big) \mathbf{M}^{-1}\, \mathbf{p} - \frac{\partial V}{\partial q_i} \tag{96}$$

Once the final expression for the second pair of Hamilton's equations has been established, that is, the pair describing the evolution of the momenta p_i as functions of their respective generalized coordinates, it is useful to recall some relevant concepts below. The mass matrix $\mathbf{M}$ is, as usual, expressed as:

$$\mathbf{M} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad a = l_1^2 (m_1 + m_2) \qquad b = m_2 l_1 l_2 \cos\Delta \qquad c = m_2 l_2^2$$

$$\det \mathbf{M} = ac - b^2 = m_2 l_1^2 l_2^2 (m_1 + m_2\, \sin^2\Delta)$$

While its inverse is:

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} = \frac{1}{\det \mathbf{M}} N \qquad\qquad N = \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$

Next, we analyze the term in parentheses within the second Hamilton's equation (96), which is the derivative of $\mathbf{M}$ with respect to the generalized coordinates q_i, i.e., with respect to the two angles:

$$\frac{\partial \mathbf{M}}{\partial \theta_1} = \begin{bmatrix} 0 & -m_2 l_1 l_2 \sin\Delta \\ -m_2 l_1 l_2 \sin\Delta & 0 \end{bmatrix} = \begin{bmatrix} 0 & d_1 \\ d_1 & 0 \end{bmatrix} \qquad d_1 = -m_2 l_1 l_2 \sin\Delta \tag{97}$$

$$\frac{\partial \mathbf{M}}{\partial \theta_2} = \begin{bmatrix} 0 & +m_2 l_1 l_2 \sin\Delta \\ +m_2 l_1 l_2 \sin\Delta & 0 \end{bmatrix} = \begin{bmatrix} 0 & d_2 \\ d_2 & 0 \end{bmatrix} \qquad d_2 = +m_2 l_1 l_2 \sin\Delta \tag{98}$$

Now, knowing the partial derivatives with respect to each generalized coordinate (97) (98) as well as the inverse of the mass matrix $\mathbf{M}^{-1}$, swe can rewrite a term of the second Hamilton's equation (96) as:

$$\mathbf{p}^T\, \mathbf{M}^{-1} \Big(\frac{\partial \mathbf{M}}{\partial q_i} \Big) \mathbf{M}^{-1}\, \mathbf{p} \;=\; \frac{1}{(\det \mathbf{M})^2}\, \mathbf{p}^T\, N\, S_i\, N\, \mathbf{p} \tag{99}$$

Where S_i corresponds to:

$$S_i = \frac{\partial \mathbf{M}}{\partial q_i} = \begin{bmatrix} 0 & d_i \\ d_i & 0 \end{bmatrix}$$

Thus, we calculate the product $N\, S_i\, N$, contained within the term just written (99):

$$N\, S_i = d_i \begin{bmatrix} -b & c \\ a & -b \end{bmatrix}$$

$$N\, S_i\, N = d_i \begin{bmatrix} -2bc & b^2 + ac \\ b^2 + ac & -2ab \end{bmatrix}$$

Therefore, integrating this last product $N\, S_i\, N$ into (99), we obtain:

$$\mathbf{p}^T\, \mathbf{M}^{-1} \Big(\frac{\partial \mathbf{M}}{\partial q_i} \Big) \mathbf{M}^{-1}\, \mathbf{p} \;=\; \frac{d_i}{(\det \mathbf{M})^2} \Big(-2bc\, p_1^2 + 2(b^2 + ac)\, p_1 p_2 - 2ab\, p_2^2 \Big) \tag{100}$$

Finally, integrating this last result into the second Hamilton's equation (96):

$$\dot{p}_i = \frac{d_i}{(\det \mathbf{M})^2} \Big(-bc\, p_1^2 + (b^2 + ac)\, p_1 p_2 - ab\, p_2^2 \Big) - \frac{\partial V}{\partial q_i} \tag{101}$$

We have thus arrived at the final form of the second Hamilton's equation, which defines the evolution of the conjugate momenta $\dot{p}_i$. Next, we will complete the writing by making the various terms explicit. Starting, initially, with the derivatives of the potential with respect to the generalized coordinates:

$$\frac{\partial V}{\partial \theta_1} = (m_1 + m_2) g l_1 \sin\theta_1 \qquad\qquad \frac{\partial V}{\partial \theta_2} = m_2 g l_2 \sin\theta_2$$

Finally, knowing all the necessary terms derived and calculated in this discussion, we can write the equation (101), expanded with $\Delta = \theta_1 - \theta_2$, for $\dot{p}_1$:

$$\dot{p}_1 = \frac{-\, m_2 l_1 l_2 \sin\Delta}{\Big(m_2 l_1^2 l_2^2 (m_1 + m_2 \sin^2\Delta) \Big)^2} \Big[-\, m_2 l_2^2\, p_1^2 + \big((m_2 l_1 l_2 \cos\Delta)^2 + m_2 l_1^2 l_2^2 (m_1 + m_2) \big) p_1 p_2 \\ - l_1^2 (m_1 + m_2)\, m_2 l_1 l_2 \cos\Delta\, p_2^2 \Big] \; - (m_1 + m_2) g l_1 \sin\theta_1 \tag{102}$$

And for the evolution of the second conjugate momentum $\dot{p}_2$:

$$\dot{p}_2 = \frac{m_2 l_1 l_2 \, \sin\Delta}{\Big(m_2 l_1^2 l_2^2 (m_1 + m_2 \, \sin^2\Delta)\Big)^2} \Big[- m_2 l_2^2 \, p_1^2 + \big((m_2 l_1 l_2 \, \cos\Delta)^2 + m_2 l_1^2 l_2^2 (m_1 + m_2)\big) p_1 p_2 \\ - l_1^2 (m_1 + m_2) \, m_2 l_1 l_2 \, \cos\Delta \, p_2^2 \Big] - m_2 g l_2 \, \sin\theta_2 \tag{103}$$

Having completed the derivation of the second pair of first-order differential equations, which describe the evolution of the system's two conjugate momenta, we now want to introduce a simplification for these last equations. First, it is very useful to rewrite some terms in a compact form:

$$\Delta = \theta_1 - \theta_2 \qquad D = m_1 + m_2 \, \sin^2\Delta$$

Having defined these, we can write the second Hamilton's equation already determined (101) in a compact form, temporarily excluding the potential term V:

$$\frac{d_i}{(\det \mathbf{M})^2} K \;=\; \frac{\pm m_2 l_1 l_2 \, \sin\Delta}{\big(m_2 l_1^2 l_2^2 D\big)^2} K \;=\; \pm \frac{\sin\Delta}{m_2 \, l_1^3 l_2^3 \, D^2} K \tag{104}$$

Where the negative sign holds for $i = 1$ and the positive sign for $i = 2$. Whereas K is, as usual:

$$K = \Big(-bc \, p_1^2 + (b^2 + ac) \, p_1 p_2 - ab \, p_2^2\Big)$$

We therefore determine the coefficients of K, as has already been done:

$$-bc \;=\; -\big(m_2 l_1 l_2 \cos\Delta\big)\big(m_2 l_2^2\big) \;=\; -m_2^2 l_1 l_2^3 \cos\Delta \tag{105}$$

$$b^2 + ac \;=\; (m_2 l_1 l_2 \cos\Delta)^2 + (m_1 + m_2) l_1^2 \cdot m_2 l_2^2 \;=\; m_2 l_1^2 l_2^2 \big(m_2 \cos^2\Delta + (m_1 + m_2)\big) \tag{106}$$

$$-ab \;=\; -(m_1 + m_2) l_1^2 \cdot (m_2 l_1 l_2 \cos\Delta) \;=\; -(m_1 + m_2) m_2 l_1^3 l_2 \cos\Delta \tag{107}$$

Once the fractional pre-factor (104) has been determined and the coefficients of the polynomial K have been identified (105)(106)(107), we proceed by multiplying each term of K by this pre-factor.

For the first term, containing p_1^2, we get:

$$\pm \frac{\sin\Delta}{m_2 l_1^3 l_2^3 D^2} \Big(- m_2^2 l_1 l_2^3 \cos\Delta \Big) \, p_1^2 \;=\; \pm \frac{\sin\Delta}{D^2} \, \frac{m_2 \cos\Delta}{l_1^2} \, p_1^2$$

For the second term with $p_1 p_2$:

$$\pm \frac{\sin\Delta}{m_2 l_1^3 l_2^3 D^2} \Big(m_2 l_1^2 l_2^2 (m_2 \cos^2\Delta + (m_1 + m_2)) \Big) \, p_1 p_2 \;=\; \pm \frac{\sin\Delta}{D^2} \, \frac{m_2 \cos^2\Delta + (m_1 + m_2)}{l_1 l_2} \, p_1 p_2$$

And finally, for the third term having p_2^2:

$$\pm \frac{\sin\Delta}{m_2 l_1^3 l_2^3 D^2} \Big(- (m_1 + m_2) m_2 l_1^3 l_2 \cos\Delta \Big) \, p_2^2 \;=\; \pm \frac{\sin\Delta}{D^2} \, \frac{(m_1 + m_2) \cos\Delta}{l_2^2} \, p_2^2$$

Ultimately, by combining the three terms, we obtain an expression that is completely equivalent to the one already derived in (102) and (103), with the only difference being that in this case, the alternation of signs is indicated in a compact form for both p_1 and p_2 using a single symbol $\pm$ and the potential term V has been omitted, as it is constant. Starting from this last equivalent form, however, we can further simplify the expression by factoring out:

$$\dot{p}_1 = \frac{\sin\Delta}{D^2} \left[\frac{m_2 \cos\Delta}{l_1^2} \, p_1^2 - \frac{m_2 \cos^2\Delta + (m_1 + m_2)}{l_1 l_2} \, p_1 p_2 + \frac{(m_1 + m_2) \cos\Delta}{l_2^2} \, p_2^2 \right] - (m_1 + m_2) g l_1 \sin\theta_1 \tag{108}$$

$$\dot{p}_2 = \frac{\sin\Delta}{D^2} \left[- \frac{m_2 \cos\Delta}{l_1^2} \, p_1^2 + \frac{m_2 \cos^2\Delta + (m_1 + m_2)}{l_1 l_2} \, p_1 p_2 - \frac{(m_1 + m_2) \cos\Delta}{l_2^2} \, p_2^2 \right] - m_2 g l_2 \sin\theta_2 \tag{109}$$

In conclusion, the Hamiltonian formulation of the double pendulum was developed, starting with the generalized coordinates chosen as the angles θ_1 and θ_2. The conjugate momenta p_1 and p_2 were defined from the system's Lagrangian. It was shown that, unlike the simple pendulum, the angular velocities are coupled, requiring the use of a matrix formalism to isolate $\dot{\theta}_1$ and $\dot{\theta}_2$ as a function of the momenta.

Subsequently, the Hamiltonian was constructed, expressed as the total energy $\mathcal{H} = T + V$, by rewriting the kinetic energy exclusively in terms of the phase space variables (θ_i, p_i).

The Hamiltonian equations were derived from this Hamiltonian. The first pair, which describes the temporal evolution of the generalised coordinates, agrees with the matrix relations already defined. The second pair, which is algebraically more complex and describes the evolution of the moments, was also obtained using the matrix formalism, resulting in a more compact form. To summarise, we have obtained a system of four first-order differential equations that fully describes the motion of the double pendulum in phase space and is suitable for analytical or numerical investigations.

Both the Lagrangian and Hamiltonian formalisms can be used to introduce the small-angle approximation.

In the Lagrangian approach, a series expansion of the trigonometric terms results in a Lagrangian that is quadratically dependent on θ_i e $\dot{\theta}_i$. This leads to linearized equations of motion.

In the Hamiltonian approach, the same substitution yields a quadratic Hamiltonian in the phase space variables (θ_i, p_i). This, in turn, makes Hamilton's equations also linear.

6.9 Symplectic Geometry of the Double Pendulum

Geometrically, the double pendulum develops on a **4-dimensional symplectic manifold**, where two dimensions stand for positions and two for moments. A symplectic manifold is not just a simple three-dimensional space, but a mathematical structure in which Hamilton's laws determine the motion. Every system trajectory must adhere to this structure, in which coordinates and moments are linked by specific mathematical relationships derived from Hamilton's equations, which impose strict limits on temporal evolution.

The complexity and chaotic behaviour of the double pendulum is reflected in the intricate shape of these trajectories, which can spiral and intertwine without repeating, and which are highly sensitive to initial conditions. In this sense, symplectic geometry provides a powerful language to describe and understand why a deterministic system with two degrees of freedom can generate such rich and complex dynamics.

7 Numerical Integration

A continuous dynamic system can be formally described as a system that changes continuously over time. Examples include the velocity of a pendulum, the position of a particle, or the population of animals in an ecosystem. This is expressed by the following equation:

$$\dot{y}(t) = f(t, y(t)), \qquad y(t_0) = y_0$$

The equation indicates that, starting from a state y_0 at time t_0, the system evolves according to a certain function, f, that depends on the system's current state.

In many cases, we can't find an exact formula for $y(t)$. For instance, in the double pendulum, the equations become too complex. Numerical integration allows us to approximate the solution $y(t)$ in discrete steps $t_n = t_0 + nh$, separated by a time step h. Therefore, we get:

$$y_n \approx y(t_n)$$

An explicit numerical integration method calculates the next state at t_{n+1} using only information already known at t_n. The two most well-known methods are the Euler method and the fourth-order Runge-Kutta (RK4) method.

7.1 Euler Method

The explicit Euler method is the most basic first-order, one-step integrator. Given an equation of the type $\dot{y} = f(t, y)$, we set a time step h and iteratively build the solution as:

$$y_{n+1} = y_n + h\, f\,(t_n, y_n) \tag{110}$$

Geometrically, this means we approximate the solution curve with the tangent line at the point (t_n, y_n). The explicit Euler method is said to be consistent, which means the error committed in a single step tends to zero as the time step h decreases. Specifically, the local (or truncation) error in each step is on the order of $O(h^2)$, while the accumulated global error up to a certain time is on the order of $O(h)$, this implies that halving the time step approximately halves the final error. In conclusion, the time step h must be sufficiently small to prevent numerical errors from growing and causing the solution to diverge.

In stiff or chaotic systems, using the Euler method can require very small steps, which often makes more precise and stable methods, like the Runge-Kutta 4, preferable.

Python:

```
def euler(f, t0, t_max, y0, h):
    th = np.arange(t0, t_max + h, h)
    uh = np.zeros(len(th))
    uh[0] = y0

    for i in range(len(th)-1):
        uh[i+1] = uh[i] + h*f(th[i], uh[i])
    return th, uh
```

The provided Python code implements the explicit Euler method to solve the equation $\dot{y} = f(t, y)$. The `euler` function takes f, the initial and final times, the initial condition, and the step size h, it creates arrays for time and the solution and updates them step by step according to $y_{n+1} = y_n + h\, f(t_n, y_n)$. Finally, it returns the time and solution arrays.

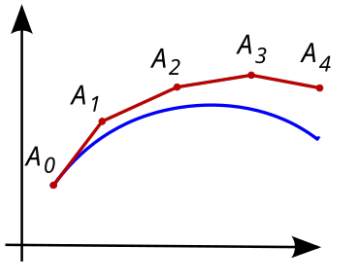


Figure 5: Illustration of Euler method. The curve to be approximated is in blue, and its polynomial approximation is in red.

7.2 Runge-Kutta 4 Method

The RK4 is another explicit integration method, but it is a four-stage method that combines multiple function evaluations within each step to achieve higher accuracy. In each interval, four increments (or slopes) are calculated:

$$
\begin{aligned}
k_1 &= f\ (t_n, y_n)\,, \\
k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \\
k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), \\
k_4 &= f\ (t_n + h,\ y_n + \ k_3)\,.
\end{aligned}
$$

The approximation for the next step is a weighted average of these values:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{111}$$

The RK4 method is a fourth-order method: the local error is $O(h^5)$, and the global error is $O(h^4)$. This means that for sufficiently small step sizes h, the error is reduced by a factor of h^4. This makes RK4 much more accurate than the Euler method for the same step size h. AFor example, by reducing h by a factor of 10, , the global error of Euler decreases by a factor of about 10, while that of RK4 decreases by a factor of approximately $\sim 10^4$ volte.

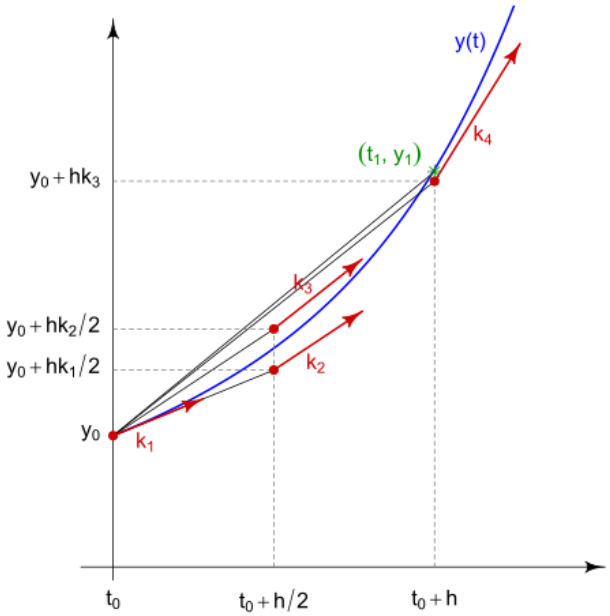
In practical terms, RK4 achieves high accuracy with a much smaller number of steps compared to Euler. The drawback is that it requires four times as many evaluations of the function f per step (four stages instead of one), but this additional cost is generally offset by the ability to use significantly larger step sizes. The RK4 method is thus considered an excellent compromise between accuracy, computational cost, and stability, even though it is more computationally expensive per step. The following is a snippet of Python code:

```
def rk4(f, t0, t_max, y0, h):
    th = np.arange(t0, t_max + h, h)
    y0 = np.array(y0, dtype=float)
    uh = np.zeros((len(th), len(y0)))
    uh[0] = y0

    for i in range(len(th)-1):
        t = th[i]
        y = uh[i]

        k1 = f(t, y)
        k2 = f(t + h/2, y + h/2 * k1)
        k3 = f(t + h/2, y + h/2 * k2)
        k4 = f(t + h,   y + h * k3)
        uh[i+1] =
        = y + (h/6) * (k1 + 2*k2 + 2*k3

    return th, uh
```



The provided Python code implements the fourth-order Runge-Kutta method to solve $\dot{y} = f(t, y)$. The `rk4` function takes f, the initial and final times, the initial condition, and the step size h, it creates arrays for time and the solution, and for each step, it calculates four intermediate slopes k_1, k_2, k_3, k_4. The solution is the weighted combination of these values.

7.3 Comparison between Euler and RK4

In the field of numerical integration of differential equations, there are many algorithms, including multi-step methods (e.g., Adams-Bashforth, Adams-Moulton), implicit methods for rigid problems, and geometric schemes that aim to preserve quantities such as the energy in Hamiltonian systems. For now, we will focus on comparing the two methods discussed previously: Euler and RK4.

Accuracy: The Euler method is a first-order method, with a global error of $O(h)$. This means that halving the step size h halves the error. In contrast, the RK4 method is a fourth-order method, with an error of $O(h^4)$. Therefore, with the same step size, RK4 is much more accurate, while Euler requires very small steps to achieve comparable results.

Computational Cost: With Euler's method, f is calculated once per step, while RK4, as the name suggests, is calculated four times. Each RK4 step therefore costs about four times as much, but in return, far fewer steps are required to achieve the same accuracy. The overall balance is generally in favour of RK4.

Stability: Both methods are explicitly and conditionally stable. RK4 has a wider stability range and tolerates slightly larger steps than Euler. In this case, and generally for chaotic systems such as the double pendulum, you must also choose a sufficiently small step size for RK4.

To summarise, the RK4 integration method better preserves invariant quantities such as energy and reduces errors. Therefore, it is generally preferable to the Euler method for simulations.

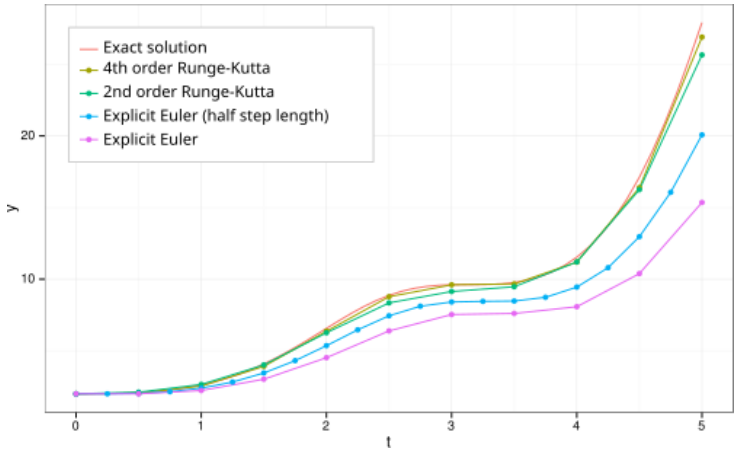


Figure 6: Comparison of different integration methods

8 Symplectic Methods

Symplectic methods are a specific class of numerical integrators designed for the study of dynamic systems with a Hamiltonian structure, meaning systems where the temporal evolution is determined by a total energy, the system's Hamiltonian:

$$\mathcal{H}(q_i, p_i, t) = T + V$$

In such systems, the generalized coordinates q_i and conjugate momenta p_i, which define the phase space, evolve over time according to Hamilton's equations, which describe the dynamic evolution of the system consistent with its total energy:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}, \qquad i = 1, \ldots, n$$

In Hamiltonian systems, the Hamiltonian function is a conserved quantity if it does not depend explicitly on time. In many physical cases, this coincides with the total energy.

A key concept in Hamiltonian systems is the **volume in phase space**. Phase space is a mathematical space where each point represents a complete state of the system. The volume in phase space measures a region of possible states.

According to Liouville's theorem, as these volumes evolve over time, they do not change: if we take a set of initial states, the volume in phase space deforms but its total remains constant.

Symplectic methods preserve this property numerically and ensure that the simulation does not artificially expand or contract the volume of the states, which helps to keep the energy invariant. Standard numerical integrators such as Euler or Runge-Kutta methods tend to accumulate errors in the energy when applied to Hamiltonian systems over the long term, leading to progressive changes, a phenomenon known as **energy drift**.

Symplectic methods, on the other hand, preserve the geometric structure of the system. Although the numerically calculated energy is not exactly constant, it oscillates around the true value without deviating with time. This property makes these methods particularly suitable for long-term simulations, such as those carried out in molecular dynamics, celestial mechanics and the investigation of planetary orbits and chaotic systems, where compliance with physical invariants is crucial.

8.1 Symplectic Euler Method

The explicit Euler method isn't symplectic, so it's not suitable for integrating Hamiltonian systems over long periods. A variant of this method, known as **symplectic Euler** (or semi-implicit, Euler–Cromer), is a first-order symplectic integrator.

Consider a dynamic system described by a second-order differential equation:

$$\ddot{x} = f(x),$$

where $x(t)$ represents position and $f(x)$ is the function that determines acceleration. To apply a numerical integrator like the symplectic Euler method, it's convenient to rewrite this as a system of first-order equations by introducing the velocity $v = \dot{x}$:

$$\dot{x} = v, \qquad \dot{v} = f(x).$$

In this form, the system is coupled: the change in x depends on the velocity v, while the change in v depends on the position x, just like in a phase space (q_i, p_i).

The symplectic Euler method applied to this system is expressed by the discrete relations, where h is the time step:

$$v_{n+1} = v_n + h\, f(x_n)$$
$$x_{n+1} = x_n + h\, v_{n+1}$$

The crucial difference to the classical Euler method is that we use the updated velocity v_{n+1} instead of the previous velocity v_n when updating the position x. This change guarantees that the symplectic structure of the system is preserved.

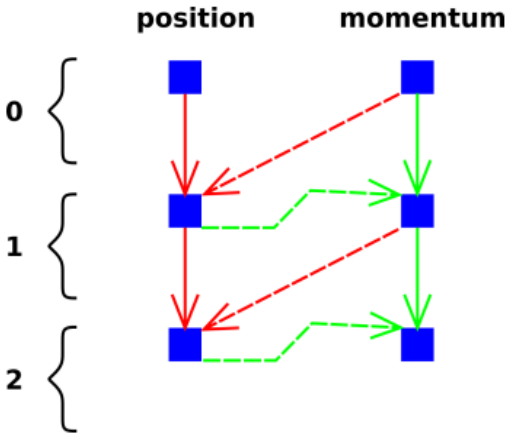
The method has a global accuracy order of $O(h)$, so it's not very precise for large time steps. However, it has the crucial advantage of avoiding a systematic energy drift, instead keeping the energy value bounded with small oscillations around the correct value.

The code below implements the symplectic Euler method for systems like $\ddot{x} = f(x)$, the velocity is updated first, followed by the updated position using v_{i+1}. Despite its low order of accuracy, the method's qualitative stability in conservative systems makes it significantly more effective than the explicit Euler method:

Python:

```
def symplectic_euler(f, t0, t_max, x0, v0, h):
    th = np.arange(t0, t_max + h, h)
    x = np.zeros(len(th))
    v = np.zeros(len(th))
    x[0] = x0
    v[0] = v0

    for i in range(len(th)-1):
        v[i+1] = v[i] + h * f(th[i], x[i])
        x[i+1] = x[i] + h * v[i+1]
    return th, x, v
```



The function `symplectic_euler` takes as input the acceleration function $f(t, x)$, the initial and final times, the initial conditions x_0 and v_0 and the time step h. It calculates the numerical evolution of the position and velocity according to the symplectic Euler method. It returns the corresponding times, positions, and velocities.

8.2 Störmer-Verlet Method (Leapfrog)

A widely used second-order symplectic method is **Verlet** (or leapfrog). Its popularity comes from its simplicity, temporal symmetry, and high accuracy in energy conservation. A common variant, called **velocity Verlet**, for systems of the form $\ddot{x} = f(x)$, is expressed as:

$$
\begin{aligned}
v_{n+\frac{1}{2}} &= v_n + \frac{h}{2} f(x_n) \\
x_{n+1} &= x_n + h\, v_{n+\frac{1}{2}} \\
v_{n+1} &= v_{n+\frac{1}{2}} + \frac{h}{2} f(x_{n+1})
\end{aligned}
$$

The method requires two evaluations of $f(x)$ per step but achieves order 2: the local error is $O(h^3)$ while the global error is $O(h^2)$. Since it's time-reversible, it avoids systematic energy drift. Numerical errors remain confined and oscillatory.

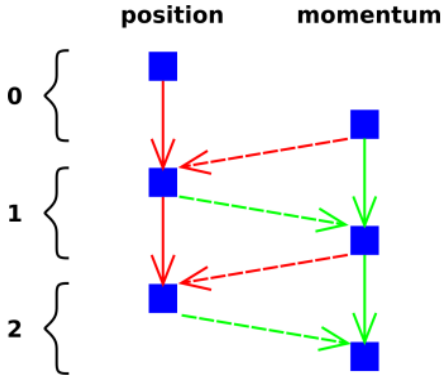
The function implements the velocity Verlet method: the velocity is updated at the half-step, the position is updated using this intermediate velocity, and the final velocity is completed with a second update. This method is now among the most used in physical simulations, from molecular dynamics to celestial mechanics.

Python:

```
def verlet(f, t0, t_max, x0, v0, h):
    th = np.arange(t0, t_max + h, h)
    x = np.zeros(len(th))
    v = np.zeros(len(th))
    x[0] = x0
    v[0] = v0

    for i in range(len(th)-1):
        v_half = v[i] + 0.5*h * f(th[i], x[i])
        x[i+1] = x[i] + h * v_half
        v[i+1] = v_half + 0.5*h * f(th[i+1], x[i+1])

    return th, x, v
```



8.3 Comparison between Symplectic Euler and Verlet

A comparison of the newly introduced symplectic methods is proposed, analysing their characteristics with respect to different parameters of interest:

Accuracy: Symplectic Euler is a first-order method ($O(h)$), while Verlet is second-order ($O(h^2)$). For the same step size, Verlet provides significantly more precise results.

Energy Conservation: Non-symplectic methods (such as Euler and RK4) accumulate an energy drift, especially in long simulations. In contrast, symplectic methods keep the energy limited, not exactly constant, but oscillating around the actual value.

Long-term Stability: In chaotic or rigid systems, symplectic methods preserve geometric invariants such as phase space volume, so that the results are physically reliable even after many iterations.

Computational Cost: The symplectic Euler method costs about as much as the classical Euler method (one evaluation per step), whereas Verlet requires two. However, thanks to its higher order and stability, Verlet often allows you to use a much larger time step, which more than makes up for the additional cost.

To summarise, symplectic methods are essential when studying Hamiltonian systems over long periods of time.
In contrast to generic high-precision methods such as RK4, which guarantee local precision but do not take geometric invariants into account, symplectic integrators ensure that the essential physical properties of the system are preserved.
For this reason, symplectic Euler and especially Verlet methods are the first choice for the numerical simulation of conservative systems.

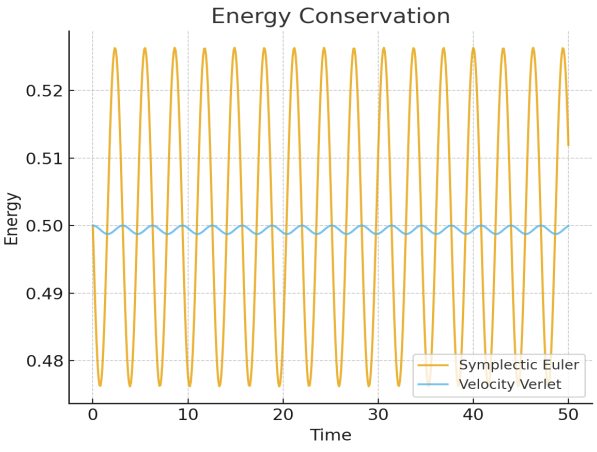


Figure 7: Symplectic Euler vs Velocity Verlet

9 Conclusions

Studies of the simple and double pendulum have revealed the fascinating coexistence of predictability and chaos and how linear approximations and full nonlinear dynamics coexist. The application of Newtonian, Lagrangian, and Hamiltonian formalisms alongside numerical simulations opens a way of presenting the richness of the classical mechanics and the necessity of the computational methods within the constraints of their respective analytical solutions.

In addition to their instructive value, pendula are general models in classical mechanics, applied maths, and numerical computing. The double pendulum, by the nature of its complicated motions and drastic sensitivity to the initial conditions, is a remarkable example of a chaotic system and a perfect testbed for numerical algorithms.

The research is restricted by idealized approximations, neglecting damping, external driving forces, and gravitational inhomogeneities, along with a lack of long-time numerical studies. Future research could address these limitations, using adaptive or improved-order integrators, and by considering the quantum analogs, in which pendula create a natural context for studies of semiclassical approximations and quantum chaos. In any of these directions, pendula continue to bridge theory, computation, and experiment by providing an elegant demonstration of the classical physics and a rich context for modern research.

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