From Simple to Double Pendulum: Theoretical Foundations and Numerical Simulations

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Abstract

This paper presents a coherent and self-contained framework of the main theoretical materials related to the simple pendulum and the double pendulum, redistributed and integrated into a single coherent framework. First, the harmonic motion is briefly introduced to illustrate the connection with the oscillators. Then the simple pendulum is analysed using both Newtonian and Lagrangian mechanics, and the two approaches are compared to find the same solution. The double pendulum is then studied using Lagrangian mechanics, emphasising the chaotic nature of the pendulum, which makes a simulation using a computational approach even more complex. Finally, the ordinary differential equations of the double pendulum are rewritten using a matrix formalism. In the end, numerical integration methods such as Euler and Runge-Kutta 4 are presented, comparing precision, stability and energy conservation.

This approach provides a complete overview of both elementary and chaotic oscillatory systems, their mechanics, and the numerical methods required for their approximation.

1 Simple Harmonic Motion

Simple harmonic motion is a type of oscillatory motion in which the force acting on a body is proportional and opposite to its displacement from the equilibrium position. For an ideal spring, Hooke's Law is a classic example:

$$\vec{F} = -kx$$

where:
- k is a positive constant known as the spring constant;
- x also denoted as Δl, represents the difference between the spring's length under a load and its length at rest.

Simple harmonic motion describes the behavior of an ideal system known as a harmonic oscillator. Classic examples of simple harmonic oscillators in mechanics include the ideal spring with an attached mass, as well as the simple pendulum (for small oscillation angles: $\theta \ll 1\,\text{rad}$). Beyond mechanics, other examples include the electrical oscillations in RLC circuits.

1.1 Undamped Simple Harmonic Motion

The **undamped simple harmonic motion** is also known as natural harmonic motion. It is a sinusoidal oscillation with a constant angular frequency, ω. This motion is periodic with a constant oscillation because it doesn't account for energy-dissipating forces like friction or air resistance.

The position of a body oscillating with simple harmonic motion can be described by a sinusoidal function with constant amplitude and phase, assuming the origin of the reference system is at the central point of oscillation.

$$y(t) = A\,\sin\left(\omega\,t + \phi\right)$$

A **Damped simple harmonic motion** is a motion in which the oscillation decreases over time due to a damping force (or friction) that dissipates energy. This is in contrast to undamped simple harmonic motion, in which the oscillations continue indefinitely with a constant amplitude.

It is an extension of simple harmonic motion (without damping) and corresponds to many real physical phenomena, such as a spring oscillating with friction or air resistance.
Depending on the strength of the damping force, three cases can be distinguished:

- **Underdamped**: the damping force is relatively low, so the body continues to oscillate, but the amplitude slowly decreases over time until it eventually stops.
- **Critically damped**: The damping force has a precise value that reduces the oscillation to the minimum necessary for the system to reach equilibrium in the shortest possible time.
- **Overdamped**: The damping force is very strong and no oscillation occurs. The system returns to the equilibrium point, but very slowly.

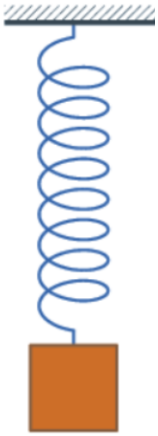


Figure 1: A damped harmonic oscillator

1.2 Forced Harmonic Motion

In forced harmonic motion, an oscillating system is subjected to an external force that varies over time, this causes an oscillation that no longer depends solely on the system's natural properties but also on the frequency and intensity of the applied force.

In the ideal, undamped case, the system can show a significant increase in oscillation amplitude when the frequency of the external force approaches the system's natural frequency, a phenomenon known as resonance.

When energy losses like friction or resistance are considered, the motion is called **damped forced harmonic motion**. In this situation, the amplitude of the oscillations reaches a finite value, and the system responds with a phase that is slightly delayed relative to the applied force. This model is more realistic and closely matches what is observed in nature.

2 Simple Pendulum

The **simple pendulum** (or mathematical pendulum) is a physical system, or more precisely a harmonic oscillator, consisting of an inextensible string of negligible mass, with a point mass (m) attached to its end. The system is subject to a gravitational pull (which we assume is uniform in space and constant over time). The pendulum is a practical application of harmonic motion.

Because the string is inextensible, the mathematical pendulum will constantly follow circular trajectories, therefore, it is helpful to use a polar coordinate system rather than a Cartesian one.

2.1 Polar Coordinate System

In mathematics, a **Polar Coordinate System** is a two-dimensional coordinate system where each point in a plane is defined by an angle, typically θ, and its distance from the center of the plane, called the pole.

The two polar coordinates, r and θ, can be converted to Cartesian coordinates x and y using the formulas for the trigonometric functions sine and cosine.

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Conversely, the two Cartesian coordinates x and y can be converted to the polar coordinate r by applying the Pythagorean theorem.

$$r = \sqrt{x^2 + y^2}$$

At the equilibrium position, the point mass is at rest, and the string is perfectly vertical. However, if the mass is displaced from this equilibrium, the pendulum will begin to oscillate under the effect of the gravitational force, which will try to pull it back toward the equilibrium position. As already described, it will trace a circular arc with a radius equal to the length l of the string. If friction is not considered, the oscillations will continue indefinitely.

3 Formulations of Mechanics

To study the motion of a simple pendulum, it is necessary to choose the most suitable formulation of mechanics for the task. The various formulations of mechanics are different, yet equivalent, ways to describe the motion of mechanical systems, each with its own advantages and disadvantages depending on the case. They are primarily divided into Newtonian mechanics (by Newton) and rational mechanics (developed by figures like Lagrange and Hamilton).

3.1 Newtonian Mechanics

Newtonian mechanics is the most traditional formulation, with its starting point being Newton's Second Law, which establishes a direct link between force and acceleration. Particles are treated as point masses moving in three-dimensional space, and differential equations are written using Cartesian coordinates.

$$\vec{F} = m\vec{a} \longleftrightarrow \vec{F} = \frac{d\vec{p}}{dt} \tag{1}$$

This approach is particularly direct: if all the forces acting on a body are known, its motion can be determined. A key disadvantage of the Newtonian approach is the need to express constraints as external forces, which can complicate the equations.

Within Newtonian mechanics, three sub-disciplines can be distinguished: **kinematics**, the description of motion through the concepts of space and time; **dynamics**, the study of the causes of motion; and **statics**, the study of system equilibrium through forces and torques.

Kinematics studies motion in an abstract way, determining the equation of motion that describes an object's position as a function of time. Dynamics and statics, on the other hand, use the fundamental equations of motion and conservation laws (such as those for momentum, angular momentum, and mechanical energy) to describe motion and equilibrium conditions.

3.2 Lagrangian Mechanics

Lagrangian mechanics introduces a paradigm shift. Instead of studying forces, it focuses on the system's energies: kinetic energy T and potential energy V. The Lagrangian function, $\mathcal{L}$, is defined as the difference between these two energies:

$$\mathcal{L} = T - V$$

$$\mathcal{L}(\dot{q}, q, t) = T(\dot{q}, q, t) - V(q, t) \tag{2}$$

where $q \in \mathbb{R}^n$ represents the generalized coordinates, $\dot{q}$ their respective velocities, and t is time.

The laws of motion are formulated through a variational principle, the **Principle of Stationary Action** (or Least Action). This principle states that a physical system, between two fixed states in time t_1 and t_2, evolves along the path that makes a quantity called action, stationary (or balanced).

Action, S, is the time integral of the difference between the kinetic and potential energies (i.e., the Lagrangian). It is a scalar quantity defined as:

$$S = \int_{t_1}^{t_2} \mathcal{L}\left[\dot{q}(t), q(t), t\right] dt$$

The Principle of Hamilton, or the principle of stationary action, states that a physical system follows the path that makes the action stationary:

$$\delta \int_{t_1}^{t_2} \left[T(\dot{q}, q, t) - U(q, t)\right] dt = 0$$

$$\delta S = 0 \tag{3}$$

Here, S is the action, which depends on the path taken by a system over time and measures the "energy cost" of a given movement. δS denotes an infinitesimal variation of the action. The equality to zero means that nature follows the most balanced path over time between kinetic and potential energy, choosing the trajectory that optimizes this balance during its evolution.

Traditional spatial coordinates are replaced by **generalized coordinates**. These are independent variables chosen to completely describe the state of a physical system while already accounting for its constraints. Unlike Cartesian coordinates (like x,y,z), they can be angles, distances along curved paths, or any other variable suited to the problem.

The goal is to reduce the number of variables to the bare minimum, choosing those that reflect the system's true degrees of freedom. This way, constraints no longer need to be handled with additional forces or separate equations; they are directly incorporated into the geometry of the coordinates themselves.

This results in the **Euler-Lagrange equations**, a system of differential equations that describe motion without the need for explicit constraint forces, making the Lagrangian approach particularly powerful for complex systems and those with constraints:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{4}$$

As a simple example, consider a system moving along the x-axis. The Lagrangian is:

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 \; - mgx$$

Applying the Euler-Lagrange equation, we have, for the power rule:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m\dot{x}^2 - mgx\right) = m\dot{x}$$

and for the standard derivative of a linear function:

$$\frac{\partial \mathcal{L}}{\partial q_i} \rightarrow \frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2} m\dot{x}^2 - mgx\right) = -mg$$

Consequently, we can conclude:

$$\frac{d}{dt}(m\dot{x}) - (-mg) = 0 \;\rightarrow\; m\ddot{x} = -mg \;\rightarrow\; \ddot{x} = -g$$

As expected, we find the constant acceleration due to a uniform gravitational field, which is the equation of motion for free fall.

3.3 Brief introduction on Hamiltonian Mechanics

Hamiltonian mechanics is a further reformulation that emerges from the Lagrangian via a **Legendre transformation**:

$$\mathcal{H}(q, p, t) = \sum_i p_i \dot{q}_i - \mathcal{L}(q, \dot{q}, t) \tag{5}$$

Instead of describing motion in terms of generalized coordinates q_i and velocities $\dot{q}_i$, the Hamiltonian formulation uses coordinates q_i and their conjugate momenta, p_i. By defining the conjugate momentum, we can reformulate the system in terms of these pairs of variables, coordinates q and momenta p:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad p_i = -\frac{\partial \mathcal{H}}{\partial \dot{q}_i}$$

4 Application to the Single Pendulum

A theoretical-physical analysis of the simple pendulum is therefore proposed, first using Newtonian mechanics and then the Lagrangian formalism. To provide an accurate physical description, the following parameters must be considered, excluding friction: the acceleration of gravity $g = 9.81\ m/s^2$; the length of the inextensible string l in $meters$; the mass m of the point mass in kg; and the initial angle θ from the vertical in rad.

4.1 Newtonian Approach

In the case of the simple pendulum, a point mass m is constrained to move along a circular arc of radius l. It is suspended by an inextensible, massless string and is subject only to the force of gravity. The motion occurs on a vertical plane under the influence of a uniform gravitational field.

To analyze this system using Newton's Second Law, it is convenient to use a polar coordinate system with two main directions:
- **Radial**: Oriented along the string and pointing towards the center of rotation (the pivot point).
- **Tangential**: Orthogonal to the radial direction and tangent to the circular path. This is the direction along which the body moves.

The gravitational force $\vec{F_g}$, which is directed vertically downwards, is decomposed into these two directions. The radial component is balanced by the tension in the string and does not contribute to the motion along the path. Conversely, the tangential component is the one that actually acts along the direction of motion.

If the string forms an angle θ with the vertical, then the angle between the string and the direction of the gravitational force is also θ. Consequently, we can use trigonometry to project the components of the gravitational force. The radial component, $\vec{F}_{rad}$, is balanced by the tension in the string and does not contribute to the motion. It is projected along the string (the side adjacent to the angle) using the cosine function:

$$\vec{F}_{rad} = -m\vec{g}\cos(\theta)$$

The negative sign in the tangential force formula indicates a restoring force. If θ is positive (the pendulum moves to the right), the tangential force pushes it to the left, in the negative direction. The force always acts to bring the pendulum back to its vertical equilibrium position at $\theta = 0$, so it is always in the opposite direction of the angle θ. The tangential component, $\vec{F}_{tan}$, which acts along the direction of motion, is:

$$\vec{F}_{tan} = -m\vec{g}\sin(\theta)$$

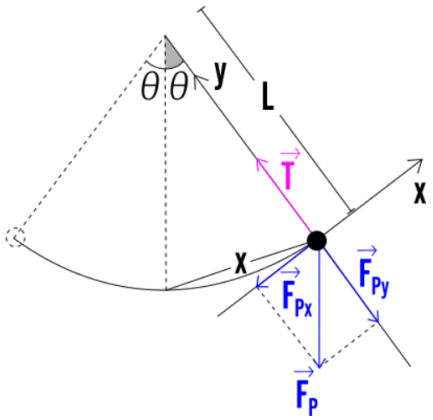


Figure 2: Force diagram of the simple pendulum

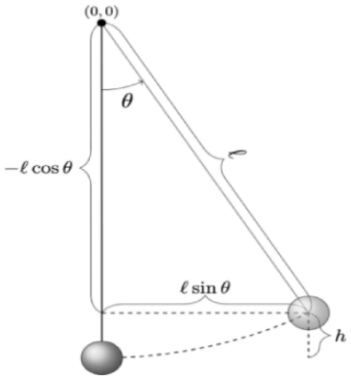


Figure 3: Coordinates expressed in terms of angle θ

In this case, where the only force causing motion is the tangential one, we can apply Newton's Second Law (1):

$$\vec{F}_{tan} = m\vec{a}_{tan}$$

$$-mg\,\sin(\theta) = m\vec{a}_{tan}$$

In the case of a pendulum, the quantity that changes over time is the angle of inclination from the vertical, $\theta(t)$. We call the first derivative of the angle the angular velocity, $\dot{\theta}$, which indicates how quickly the angle is changing. The second derivative of the angle, $\ddot{\theta}$, is the angular acceleration, which tells us how quickly the angular velocity is changing. It is written as:

$$\ddot{\theta} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

The angular velocity $\dot{\theta}$ is the ratio between the angle of the circular arc traveled, $d\theta$, and the elapsed time dt. The change of this velocity, i.e. time derivative, is the angular acceleration $\ddot{\theta}$:

$$\vec{v} = \frac{d\theta}{dt} \qquad \vec{a} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right)$$

In the case of the simple pendulum, both the tangential velocity and acceleration are particular forms of angular velocity and acceleration along a circular arc of radius l:

$$\vec{v}_{tan} = l\,\frac{d\theta}{dt} = l\,\dot{\theta} \tag{6}$$

$$\vec{a}_{tan} = l\,\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = l\,\ddot{\theta} \tag{7}$$

We can now return to the Newtonian equation from the previous section and expand it further:

$$-mg\,\sin(\theta) = m\,l\,\ddot{\theta}$$

By simplifying both sides by m and dividing by l:

$$-\frac{g}{l}\,\sin(\theta) = \ddot{\theta}$$

The final equation for $\ddot{\theta}$, is, also, the general equation of motion for a pendulum:

$$\ddot{\theta} = -\frac{g}{l}\,\sin(\theta)$$

$$\ddot{\theta} + \frac{g}{l}\,\sin(\theta) = 0 \tag{8}$$

4.2 Lagrangian Approach

We now aim to apply the Lagrangian approach to derive the equation of motion and thus verify the previous solution. This formulation is based on the Principle of Least Action (or Hamilton's Principle), which allows us to reach the same result as the Newtonian approach using an equivalent, alternative path. In this case, and even more so for the double pendulum, it is advantageous to use the Lagrangian approach because it allows for the use of generalized coordinates.

The first step is to identify a generalized coordinate that best describes the system's degree of freedom. For the simple pendulum, the motion is constrained to a circular path, which reduces the degrees of freedom to just one: the angle $\theta(t)$, measured with respect to the vertical. If we place the origin of the coordinate system at the pivot point and the y-axis is oriented upward, we get the following:

$$x = l\,\sin(\theta) \qquad y = -l\cos(\theta) \tag{9}$$

To continue with the Lagrangian formulation, the Lagrangian of the system, defined as $\mathcal{L} = T - V$, is set up using the generalized coordinates. For the simple pendulum, the most natural generalized coordinate is the angle $\theta(t)$ and its angular velocity $\dot{\theta}$.

$$\mathcal{L}(\dot{\theta}, \theta, t) = T(\dot{\theta}, \theta, t) - V(\theta, t) \tag{10}$$

We now aim to show the reader two equivalent approaches to obtain the kinetic energy T, defined as:

$$T = \frac{1}{2}mv^2$$

In the polar coordinate system, the velocity is a function of the angular velocity and the radius (6). Substituting this into the kinetic energy equation gives:

$$T = \frac{1}{2}m(l\,\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2 \tag{11}$$

In a Cartesian system, the kinetic energy is given by the sum of the squared velocities in the x and y directions, thus:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

We take the time derivatives of x and y, already proposed (9), by applying the chain rule:

$$\dot{x} = \frac{d}{dt}(l\,\sin\theta) = l\cos\theta\,\dot{\theta} \tag{12}$$

$$\dot{y} = \frac{d}{dt}(-l\cos\theta) = l\sin\theta\,\dot{\theta} \tag{13}$$

Substituting these into the kinetic energy equation::

$$T = \frac{1}{2}m((l\,\cos\theta\,\dot{\theta})^2 + (l\,\sin\theta\,\dot{\theta})^2)$$

$$T = \frac{1}{2}m(l^2\,\cos^2\theta\,\dot{\theta}^2 + l^2\,\sin^2\theta\,\dot{\theta}^2)$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta)$$

using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, we arrive at the same result as the polar coordinate approach: (11):

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

Once the kinetic energy, T, of the system has been determined, we can now look at its gravitational potential energy. In this case, we assume the zero reference point is at the lowest point of the pendulum's trajectory, i.e., at $\theta = 0$. By setting $y = -l\cos(\theta)$, the potential energy, V, is:

$$V = mgy = -mgl\cos(\theta) \tag{14}$$

Once the kinetic energy, T, and the gravitational potential energy, V, have been determined, we can proceed with the calculation of the Lagrangian:(10):

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos(\theta) \tag{15}$$

We now proceed by applying the Euler-Lagrange equation, as seen above (4), adapted in this case to the chosen generalized coordinate, θ:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

Starting with the first term, the partial derivative with respect to $\dot{\theta}$, per la regola di potenza:

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta\right) = ml^2\dot{\theta} \tag{16}$$

Taking the derivative with respect to time:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{d}{dt}(ml^2\dot{\theta}) = ml^2\ddot{\theta} \tag{17}$$

Moving on to the second term, the partial derivative with respect to θ, deriving the trigonometric function:

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta\right) = -mgl\ \sin\theta \tag{18}$$

Finally, we apply the Euler-Lagrange equation, which is the sum of the first (17) and second terms (18), rewritten as:

$$ml^2\ddot{\theta} - (-mgl\sin\theta) = 0$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

To obtain the equation of motion in its classical form, we divide the terms by ml^2:

$$\frac{ml^2\ddot{\theta}}{ml^2} + \frac{mgl\sin\theta}{ml^2} = 0$$

By simplifying the terms, we get the same result (8):

$$\ddot{\theta} + \frac{g}{l}\ \sin(\theta) = 0$$

Next, we can introduce a further simplification to the simple pendulum's equation of motion.

The equation is non-linear because of the trigonometric term $\sin(\theta)$ To obtain a simpler solution that describes the pendulum's motion under common conditions, we use the **small-angle approximation**. This approximation is valid when the pendulum oscillates only slightly from its vertical equilibrium position.

Mathematically, for very small values of θ in radians, the sine function can be approximated by the first term of its **Maclaurin series** expansion, a powerful technique for approximating polynomial functions:

$$\sin(\theta) \approx \theta \qquad \text{per}\,\theta \ll 1\,\text{rad}$$

By substituting this approximation into the pendulum's equation of motion, we obtain its linearized form:

$$\ddot{\theta} + \frac{g}{l}\,\theta = 0 \tag{19}$$

This equation has the same form as the differential equation for a simple harmonic oscillator, which makes the problem much easier to solve.

5 Application to the Double Pendulum

The dynamics of the double pendulum, also known as the chaotic pendulum, will now be the subject of study. This is a physical system consisting of two pendulums attached one to the end of the other, each free to oscillate with respect to its pivot point. This forms a complex system that exhibits dynamic behavior with a strong sensitivity to initial conditions. The motion of a double pendulum is governed by a pair of ordinary differential equations.

Compared to the simple pendulum, the double pendulum has non-linear equations of motion and the system can exhibit chaotic dynamics for certain initial conditions.

In addition to excluding any friction and the gravitational constant $g = 9.81\ m/s^2$, the double pendulum requires considering twice as many parameters as the single pendulum: the two lengths in meters of the two inextensible and massless rods, l_1 and l_2; the two point masses, m_1 and m_2 in kg; and finally, the two initial angles, θ_1 and θ_2 with respect to the vertical.

The analysis requires using the Lagrangian formalism, which allows for a systematic derivation of the equations of motion even with multiple degrees of freedom.

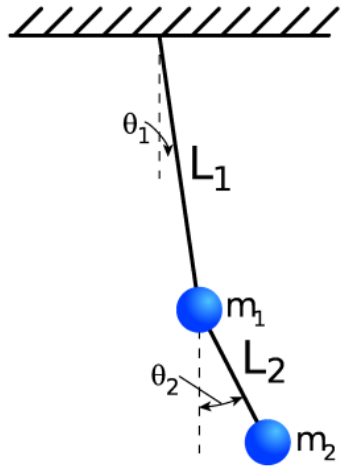


Figure 4: Idealized Double Pendulum

5.1 Lagrangian Approach

In the double pendulum, the Lagrangian formalism is used to avoid the direct calculation of tensions and the resolution of non-trivial geometric constraints. By choosing the angles θ_1 and θ_2 as generalized coordinates and then applying the Euler-Lagrange equations, we obtain the system of non-linear differential equations that describe the entire dynamics, with the constraints incorporated directly into the system's geometry.

First, we need to establish the Cartesian coordinates of the two masses, m_1 and m_2, as (x_1, y_1) and (x_2, y_2) respectively. We will use a Cartesian coordinate system with the origin at the pivot point and the y-axis oriented upwards. The coordinates are defined as follows, as was done for the simple pendulum and as described in the figure 3:

$$x_1 = l_1\,\sin(\theta_1) \qquad y_1 = -l_1\cos(\theta_1) \tag{20}$$

For the second mass, it is useful to specify that if we consider the two separate vertical lines, as shown in figure 4, the coordinates will simply be the coordinates of the first pendulum, vectorially summed with those of the pendulum attached to its end. In other words, x_2 and y_2 are derived from the position of the first pendulum plus the relative position of the second one:

$$x_2 = x_1 + l_2\,\sin(\theta_2) = l_1\,\sin(\theta_1) + l_2\,\sin(\theta_2) \tag{21}$$

$$y_2 = y_1 - l_2\,\cos(\theta_2) = -l_1\,\cos(\theta_1) - l_2\,\cos(\theta_2) \tag{22}$$

The first mass m_1 moves along a circular arc of radius l and its coordinates depend only on θ_1. In contrast, the position of the second mass, m_2 is influenced by the motion of the first as well as its own angle θ_2. After calculating the generalized coordinates as functions of θ_1 and θ_2, it is useful to proceed by taking the time derivatives of the Cartesian coordinates to obtain the respective velocities. These velocities will be part of the calculation for the subsequent kinetic energies, T_1 and T_2. By applying the chain rule:

$$\dot{x}_1 = \frac{d}{dt}\left(l_1 \sin\theta_1\right) = l_1 \cos\theta_1\,\dot{\theta}_1 \tag{23}$$

$$\dot{y}_1 = \frac{d}{dt}\left(-l_1 \cos\theta_1\right) = l_1 \sin\theta_1\,\dot{\theta}_1 \tag{24}$$

The procedure is analogous for the second mass:

$$\dot{x}_2 = \frac{d}{dt}\left(l_1 \sin\theta_1 + l_2 \sin\theta_2\right) = l_1 \cos\theta_1\,\dot{\theta}_1 + l_2 \cos\theta_2\,\dot{\theta}_2 \tag{25}$$

$$\dot{y}_2 = \frac{d}{dt}\left(-l_1 \cos\theta_1 - l_2 \cos\theta_2\right) = l_1 \sin\theta_1\,\dot{\theta}_1 + l_2 \sin\theta_2\,\dot{\theta}_2 \tag{26}$$

We can now proceed with the formulation of the Lagrangian:

$$\mathcal{L}(\dot{\theta}_1, \dot{\theta}_2, \theta_1, \theta_2, t) = T(\dot{\theta}_1, \dot{\theta}_2, \theta_1, \theta_2, t) - V(\theta_1, \theta_2, t) \tag{27}$$

Therefore, we consider the calculation of the total kinetic energy, T, of the system as the algebraic sum of the kinetic energies of the individual masses T_1 e T_2:

$$T = T_1 + T_2$$

$$T_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) \qquad T_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

We consider the calculation of the first mass's kinetic energy. Given the velocities in generalized coordinates, which were already calculated previously (23) (24), we can expand on them as was done for the single pendulum:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \left((l_1 \cos\theta_1\,\dot{\theta}_1)^2 + (l_1 \sin\theta_1\,\dot{\theta}_1)^2 \right) \\ &= \frac{1}{2} m_1 (l_1^2 \cos^2\theta_1\,\dot{\theta}_1^2 + l_1^2 \sin^2\theta_1\,\dot{\theta}_1^2) \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 (\cos^2\theta_1 + \sin^2\theta_1) \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 \end{aligned} \tag{28}$$

As expected, we find the kinetic energy equation for the simple pendulum. In this case, we must also define T_2, using the velocities of x_2 e y_2 (25)(26):

$$T_2 = \frac{1}{2} m_2 \left((l_1 \cos\theta_1\,\dot{\theta}_1 + l_2 \cos\theta_2\,\dot{\theta}_2)^2 + (l_1 \sin\theta_1\,\dot{\theta}_1 + l_2 \sin\theta_2\,\dot{\theta}_2)^2 \right)$$

First, we expand the squared terms:

$$\begin{aligned} T_2 = \frac{1}{2} m_2 (&l_1^2 \cos^2\theta_1 \dot{\theta}_1^2 + l_2^2 \cos^2\theta_2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 \\ &+ l_1^2 \sin^2\theta_1 \dot{\theta}_1^2 + l_2^2 \sin^2\theta_2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2) \end{aligned}$$

We'll now open a brief aside to demonstrate the summation of similar terms (placed one above the other for clarity) from the expression above. We'll begin with the first two.[1]:

$$l_1^2 \cos^2\theta_1 \dot{\theta}_1^2 + l_1^2 \sin^2\theta_1 \dot{\theta}_1^2 = l_1^2 \dot{\theta}_1^2 (\cos^2\theta_1 + \sin^2\theta_1) = l_1^2 \dot{\theta}_1^2$$
$$l_2^2 \cos^2\theta_2 \dot{\theta}_2^2 + l_2^2 \sin^2\theta_2 \dot{\theta}_2^2 = l_2^2 \dot{\theta}_2^2 (\cos^2\theta_2 + \sin^2\theta_2) = l_2^2 \dot{\theta}_2^2$$

Next, we sum the cross-terms:[2]:

$$2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2$$
$$2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) = 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

With this brief demonstration closed, we can now write the kinetic energy of the second mass explicitly, by summing the results of the three previous sums:

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right) \\ &= \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \end{aligned} \tag{29}$$

Finally, by summing T_1 (28) and T_2 (29), we obtain the total kinetic energy:

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

By collecting similar terms, such as $l_1^2 \dot{\theta}_1^2$, we obtain the final form of kinetic energy T:

$$T = \frac{1}{2} l_1^2 \dot{\theta}_1^2 (m_1 + m_2) + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \tag{30}$$

We notice three distinct terms. The first represents the kinetic energy corresponding to the motion of the first mass (where the motion and position of m_2 depend on m_1). The second is the contribution of the second mass's own motion. Finally, the third is an interaction term that links the two movements and depends on the relative angle, $\theta_1 - \theta_2$. This last dependency is at the core of the double pendulum's non-linear dynamics, which gives rise to complex and chaotic behaviors. Once the calculation of the total kinetic energy is complete, we now move on to the system's gravitational potential energy. This is also written as the algebraic sum of the energies of the individual masses, V_1 and V_2. Given y_1 (20), we define V_1 as:

$$V_1 = m_1 g y_1 = -m_1 g l_1 \cos(\theta_1) \tag{31}$$

Analogously for y_2 (22), and V_2:

$$V_2 = m_2 g y_2 = m_2 g(-l_1 \cos(\theta_1) - l_2 \cos(\theta_2)) = -m_2 g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2) \tag{32}$$

We define the total potential energy V as the sum of V_1 (31) and V_2(32), which gives us:

$$V = -m_1 g l_1 \cos\theta_1 - m_2 g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

We factor out $g l_1 \cos\theta_1$:

$$V = -g l_1 \cos\theta_1 (m_1 + m_2) - m_2 g l_2 \cos\theta_2 \tag{33}$$

Potential energy is defined such that its value is more negative the lower the masses are. Both masses' potential energies depend on θ_1, as m_2 also changes height with the first arm. The second term, however, is due to the second mass's own motion.

Once both total energies, T (30) and V (33), have been calculated, the Lagrangian of the system can be easily determined (27):

$$\begin{aligned} \mathcal{L} = &\frac{1}{2} l_1^2 \dot{\theta}_1^2 (m_1 + m_2) + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ &+ g l_1 \cos\theta_1 (m_1 + m_2) + m_2 g l_2 \cos\theta_2 \end{aligned} \tag{34}$$

[1] Trigonometric identity: $\cos^2\alpha + \sin^2\alpha = 1$

[2] Trigonometric identity: $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

5.2 Euler-Lagrange Equation: First Degree of Freedom

Based on the Lagrangian, we will now proceed with writing the two Euler-Lagrange equations (4), in terms of θ_1 and θ_2 respectively. The first equation is as follows:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial \mathcal{L}}{\partial \theta_1}\right) = 0 \tag{35}$$

For simplicity of notation, the term $(\theta_1 - \theta_2)$, will be denoted as Δ. We proceed in order, first determining the contribution relative to $\dot{\theta}_1$:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = \frac{\partial}{\partial \dot{\theta}_1}\Big(&\frac{1}{2}l_1^2\dot{\theta}_1^2(m_1+m_2) + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos\Delta \\ &+ g l_1 \cos\theta_1(m_1+m_2) + m_2 g l_2 \cos\theta_2\Big)\end{aligned}$$

We begin by taking the partial derivative of the Lagrangian with respect to the angular velocity $\dot{\theta}_1$. From the Lagrangian, only the first and third terms contain $\dot{\theta}_1$ and therefore provide non-zero contributions:

$$\frac{\partial}{\partial \dot{\theta}_1}\Big(\frac{1}{2}l_1^2\dot{\theta}_1^2(m_1+m_2) + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos\Delta\Big)$$

To derive the first term, we apply the power rule. For the second, we treat it as a linear function of $\dot{\theta}_1$:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = l_1^2\dot{\theta}_1(m_1+m_2) + m_2 l_1 l_2 \dot{\theta}_2\cos\Delta \tag{36}$$

We now proceed by completely determining the first term of the first Euler-Lagrange equation, taking the time derivative of the result above (35):

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = \frac{d}{dt}\Big(l_1^2\dot{\theta}_1(m_1+m_2) + m_2 l_1 l_2 \dot{\theta}_2\cos\Delta\Big)$$

The derivative of the first term is straightforward. Since $\dot{\theta}_1$ is linear, its time derivative is simply the acceleration $\ddot{\theta}_1$:

$$\frac{d}{dt}\Big(l_1^2\dot{\theta}_1(m_1+m_2)\Big) = l_1^2\left[\frac{d}{dt}(\dot{\theta}_1)\right](m_1+m_2) = l_1^2\ddot{\theta}_1(m_1+m_2) \tag{37}$$

The derivative of the second term, $m_2 l_1 l_2 \dot{\theta}_2\cos\Delta$ contains both $\dot{\theta}_2$ and $\cos\Delta$, which are time-dependent. We must therefore apply the product rule. Furthermore, since $\cos\Delta$ is a function of θ_1 and θ_2, which are themselves functions of time, we must also apply the chain rule:

$$\frac{d}{dt}\Big(m_2 l_1 l_2 \dot{\theta}_2\cos\Delta\Big) = m_2 l_1 l_2 \frac{d}{dt}\Big[\dot{\theta}_2\cos\Delta\Big] \tag{38}$$

In a more compact way, the term to be derived (38) is determined as (in this case, the second line is a further decomposition equivalent to the first:

$$\begin{aligned}\frac{d}{dt}(\dot{\theta}_2\cos\Delta) &= \ddot{\theta}_2\cos\Delta - \dot{\theta}_2(\dot{\theta}_1-\dot{\theta}_2)\sin\Delta \\ &= \ddot{\theta}_2\cos\Delta - \dot{\theta}_1\dot{\theta}_2\sin\Delta + \dot{\theta}_2^2\sin\Delta\end{aligned} \tag{39}$$

We will now open a brief aside for the full derivation of the time-dependent term. We begin by applying the product rule to the term $\dot{\theta}_2\cos\Delta$, where $\Delta = \theta_1 - \theta_2$:

$$\frac{d}{dt}(\dot{\theta}_2)\cos\Delta + \dot{\theta}_2\frac{d}{dt}(\cos\Delta)$$

For the term on the left, the procedure is analogous to the one seen previously (37), where $\dot{\theta}_2$ is derived as $\ddot{\theta}_2$. For the term on the right, it is necessary to apply the chain rule:

$$\begin{aligned}\ddot{\theta}_2 \cos\Delta + \dot{\theta}_2(-\sin\Delta\dot{\Delta}) \\ + \dot{\theta}_2(-\sin\Delta(\dot{\theta}_1 - \dot{\theta}_2)) \\ + \dot{\theta}_2(-\dot{\theta}_1\sin\Delta + \dot{\theta}_2\sin\Delta) \\ - \dot{\theta}_1\dot{\theta}_2\sin\Delta + \dot{\theta}_2^2\sin\Delta\end{aligned}$$

We have concluded the brief aside where we explicitly demonstrated each step to arrive at the same solution (39), we now continue with the calculation (38), substituting the term just obtained, into the expression $\frac{d}{dt}[\dot{\theta}_2\cos\Delta]$ to complete the second term:

$$\begin{aligned}\frac{d}{dt}\Big(m_2 l_1 l_2 \dot{\theta}_2\cos\Delta\Big) &= m_2 l_1 l_2(\ddot{\theta}_2\cos\Delta - \dot{\theta}_1\dot{\theta}_2\sin\Delta + \dot{\theta}_2^2\sin\Delta) \\ &= m_2 l_1 l_2\ddot{\theta}_2\cos\Delta - m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta + m_2 l_1 l_2\dot{\theta}_2^2\sin\Delta \qquad (40)\end{aligned}$$

We will now completely rewrite the first term of the first Euler-Lagrange equation (35), by algebraically summing the previously calculated components (37) and (40):

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) = l_1^2\ddot{\theta}_1(m_1+m_2) + m_2 l_1 l_2\ddot{\theta}_2\cos\Delta - m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta + m_2 l_1 l_2\dot{\theta}_2^2\sin\Delta \qquad (41)$$

To complete the first Euler-Lagrange equation (35), it is necessary to determine the second term:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\theta_1} = \frac{\partial}{\partial\theta_1}\Big(&\frac{1}{2}l_1^2\dot{\theta}_1^2(m_1+m_2) + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos\Delta \\ &+ g l_1\cos\theta_1(m_1+m_2) + m_2 g l_2\cos\theta_2\Big)\end{aligned}$$

In the partial derivation with respect to θ_1, only the third and fourth terms of the Lagrangian contribute. Therefore, we will have:

$$\frac{\partial}{\partial\theta_1}\Big(m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos\Delta + g l_1\cos\theta_1(m_1+m_2)\Big)$$

The term $m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos\Delta$ depends on θ_1 through $\Delta = \theta_1 - \theta_2$:

$$\frac{\partial}{\partial\theta_1}(m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos\Delta) = m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2(-\sin\Delta) = -m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) \qquad (42)$$

The gravitational term $g l_1\cos\theta_1(m_1+m_2)$ is simply derived as:

$$\frac{\partial}{\partial\theta_1}(g l_1\cos\theta_1(m_1+m_2)) = -g l_1(m_1+m_2)\sin\theta_1 \qquad (43)$$

Given the equations just calculated (42) (43), the second term of the first Euler-Lagrange equation will be:

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = -m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - g l_1(m_1+m_2)\sin\theta_1 \qquad (44)$$

Having concluded the calculation of all terms (41) (44) of the first Euler-Lagrange equation (35), we can write it explicitly as:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) - \frac{\partial\mathcal{L}}{\partial\theta_1} = 0$$

$$\begin{aligned}&l_1^2\ddot{\theta}_1(m_1+m_2) + m_2 l_1 l_2\ddot{\theta}_2\cos\Delta - m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta + m_2 l_1 l_2\dot{\theta}_2^2\sin\Delta \\ &+ m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta + g l_1(m_1+m_2)\sin\theta_1 = 0\end{aligned}$$

By simplifying the opposing terms $m_1 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta$, we obtain the final form:

$$l_1^2\ddot{\theta}_1(m_1+m_2) + m_2 l_1 l_2\ddot{\theta}_2\cos\Delta + m_2 l_1 l_2\dot{\theta}_2^2\sin\Delta + g l_1(m_1+m_2)\sin\theta_1 = 0 \qquad (45)$$

5.3 Euler-Lagrange Equation: Second Degree of Freedom

To completely determine the motion of the double pendulum, we must apply the procedure just completed in an analogous way for the second degree of freedom, θ_2, and then finally determine the second Euler-Lagrange equation, which we can already define as:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0 \tag{46}$$

We'll start, as with the previous Euler-Lagrange equation, by determining the first term in terms of $\dot{\theta}_2$. Given the double pendulum's Lagrangian (34) we can define:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = \frac{\partial}{\partial \dot{\theta}_2}\Big(&\frac{1}{2}l_1^2\dot{\theta}_1^2(m_1+m_2) + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos\Delta \\ &+ g l_1\cos\theta_1(m_1+m_2) + m_2 g l_2\cos\theta_2\Big)\end{aligned}$$

Based on the Lagrangian, only the second and third terms provide non-zero contributions when taking the partial derivative with respect to $\dot{\theta}_2$. We can rewrite this as:

$$\frac{\partial}{\partial \dot{\theta}_2}\Big(\frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos\Delta\Big)$$

Therefore, apply the power rule for the first term, and a simple linear derivation for the second:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos\Delta \tag{47}$$

The following process is analogous to the one already shown for the first Euler-Lagrange equation. We now derive the term we just calculated (47) with respect to time to complete the first of the two terms of this second Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = \frac{d}{dt}\Big(m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos\Delta\Big)$$

The derivative of the first term depends only on $\dot{\theta}_2$, which, when derived with respect to time, corresponds to $\ddot{\theta}_2$:

$$\frac{d}{dt}\Big(m_2 l_2^2\dot{\theta}_2\Big) = m_2 l_2^2\left[\frac{d}{dt}(\dot{\theta}_2)\right] = m_2 l_2^2\ddot{\theta}_2 \tag{48}$$

As previously noted, the approach for the second term is a bit more complex, as the following time-dependent terms need to be derived:

$$\frac{d}{dt}\Big(m_2 l_1 l_2\dot{\theta}_1\cos\Delta\Big) = m_2 l_1 l_2\frac{d}{dt}\Big[\dot{\theta}_1\cos\Delta\Big] \tag{49}$$

The derivative of the second term contains $\dot{\theta}_1$ and also $\cos\Delta$, where $\Delta = \theta_1 - \theta_2$ So, the derivation first requires the use of the product rule, and subsequently the chain rule. Si avrà perciò un risultato del genere (the second line is a further breakdown equivalent to the first):

$$\begin{aligned}\frac{d}{dt}(\dot{\theta}_1\cos\Delta) &= \ddot{\theta}_1\cos\Delta - \dot{\theta}_1(\dot{\theta}_1 - \dot{\theta}_2)\sin\Delta \\ &= \ddot{\theta}_1\cos\Delta - \dot{\theta}_1^2\sin\Delta + \dot{\theta}_1\dot{\theta}_2\sin\Delta\end{aligned}$$

By inserting this last result in place of the time derivative of the second term $\frac{d}{dt}[\dot{\theta}_1\cos\Delta]$, we get the full first term of the Euler-Lagrange equation:

$$\begin{aligned}\frac{d}{dt}\Big(m_2 l_1 l_2\dot{\theta}_1\cos\Delta\Big) &= m_2 l_1 l_2\left(\ddot{\theta}_1\cos\Delta - \dot{\theta}_1^2\sin\Delta + \dot{\theta}_1\dot{\theta}_2\sin\Delta\right) \\ &= m_2 l_1 l_2\ddot{\theta}_1\cos\Delta - m_2 l_1 l_2\dot{\theta}_1^2\sin\Delta + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\Delta\end{aligned} \tag{50}$$

Completing the first term of the second Euler-Lagrange equation (46), we combine the terms we've calculated (48) (50):

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos\Delta - m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\Delta \tag{51}$$

With the calculation of the first term complete, we'll now continue with the next term, which is a function of θ_2:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{\partial}{\partial \theta_2}\Big(&\frac{1}{2} l_1^2 \dot{\theta}_1^2 (m_1 + m_2) + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\Delta \\ &+ g l_1 \cos\theta_1 (m_1 + m_2) + m_2 g l_2 \cos\theta_2\Big)\end{aligned}$$

For the terms contributing to the derivative:

$$\frac{\partial}{\partial \theta_2}\Big(m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\Delta + m_2 g l_2 \cos\theta_2\Big)$$

In calculating the latter derivative, attention must be paid to the signs of the terms derived from the product $\dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$. In fact, the function $\cos(\theta_1 - \theta_2)$ depends on both θ_1 and θ_2, and the derivative with respect to each variable produces a different sign.

When deriving with respect to θ_1, the difference $\theta_1 - \theta_2$ leads to an internal positive factor, generating the term with negative sign $-\sin(\theta_1 - \theta_2)$. Conversely, deriving with respect to θ_2, the difference produces a negative factor, which when multiplied by the $-\sin$ of the chain rule, reverses the sign, leading to a positive contribution $+\sin(\theta_1 - \theta_2)$.

Therefore, we derive the first term:

$$\frac{\partial}{\partial \theta_2}\Big(m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\Big) = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \tag{52}$$

While the gravitational term, as is usually done:

$$\frac{\partial}{\partial \theta_2}\Big(m_2 g l_2 \cos\theta_2\Big) = -m_2 g l_2 \sin\theta_2 \tag{53}$$

Finally, combining the respective terms obtained in (52) and (53), we determine the second term of the Euler-Lagrange equation (46):

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\Delta - m_2 g l_2 \sin\theta_2 \tag{54}$$

Finally, we can finally define the Euler-Lagrange equation (46) for the second degree of freedom, combining (51) and (54), arriving at:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0$$

$$\begin{aligned}&m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos\Delta - m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\Delta \\ &- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\Delta + m_2 g l_2 \sin\theta_2 = 0\end{aligned}$$

Simplifying opposite terms $m_1 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\Delta$, we get the final form of the equation:

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos\Delta - m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta + m_2 g l_2 \sin\theta_2 = 0 \tag{55}$$

5.4 Matrix Formalism

Considering a dynamic physical system like the double pendulum, which consists of two masses m_1 and m_2, constrained by two rigid rods of length l_1 and l_2, and denoting the angles of the first and second rods with their respective vertical lines as $\theta_1(t)$ and $\theta_2(t)$, we obtain a system of two coupled, non-linear, second-order ordinary differential equations (ODEs):

$$\begin{cases} l_1^2\ddot{\theta}_1(m_1+m_2) + m_2 l_1 l_2 \ddot{\theta}_2 \cos\Delta + m_2 l_1 l_2 \dot{\theta}_2^2 \sin\Delta + g l_1 (m_1+m_2)\sin\theta_1 = 0 \\ m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos\Delta - m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta + m_2 g l_2 \sin\theta_2 = 0 \end{cases} \tag{56}$$

A differential equation, such as the two just presented, is called second-order if the second derivative of the unknown variable appears in an essential way. In this case, the derivatives $\ddot{\theta}_1$ and $\ddot{\theta}_2$ are involved in the equations, as the system's dynamics depend on the acceleration of the masses.

They are called coupled because the motion of each mass depends on both its own angle and velocity as well as those of the other mass (e.g., the cross terms like $m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)$). The motion of the two pendulums is therefore interdependent and inseparable. Formally:

$$\ddot{\theta}_1 = f_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \qquad \ddot{\theta}_2 = f_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$$

They are finally non-linear due to the trigonometric terms and products between variables like $\dot{\theta}_i \dot{\theta}_j$. This implies that the principle of superposition does not apply.

In summary, the system is described by two coupled, non-linear, second-order ordinary differential equations. This structure suggests introducing the matrix formalism, where the system will no longer be described by a pair of second-order ODEs but by matrices and vectors, formally defined as:

$$\mathbf{M}(q_i)\,\ddot{\mathbf{q}} + (\mathbf{C}(q_i, \dot{q}_i)\,\dot{\mathbf{q}} + \mathbf{G}(q_i)) = 0 \tag{57}$$

Assuming, for the double pendulum, the generalized coordinates $q_1 = \theta_1$ e $q_2 = \theta_2$:

$$\mathbf{M}(\theta_1, \theta_2)\,\ddot{\mathbf{\Theta}} + \mathbf{C}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)\,\dot{\mathbf{\Theta}} + \mathbf{G}(\theta_1, \theta_2) = 0 \tag{58}$$

The matrix formalism (58) highlights several important aspects of the system. The **M**matrix collects the acceleration coefficients, clearly showing the coupling between the two masses through the off-diagonal terms. The **C** matrix groups the velocity coefficients related to Coriolis and centripetal effects, while the **G** vector contains the contributions from gravity. The two vectors $\ddot{\mathbf{\Theta}}$ and $\dot{\mathbf{\Theta}}$ contain the second and first derivatives of the generalized coordinates (the accelerations and velocities), respectively:

$$\ddot{\mathbf{\Theta}} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \qquad \dot{\mathbf{\Theta}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{59}$$

This rewriting makes the system easier to read and numerically handle, as it isolates the accelerations as functions of positions and velocities. This simplifies the implementation of numerical integration algorithms like the fourth-order Runge-Kutta. It is also extremely useful for generalizing the analysis to systems with more degrees of freedom, as all information can be contained in matrices and vectors, clarifying the system's structure and the relationships between variables.

In summary, the matrix formalism doesn't change the equations; it organizes them to highlight the linearity with respect to accelerations, the coupling between the masses, and the non-linearity of the known terms. This provides an elegant and practical tool for both theoretical analysis and numerical simulation.

Therefore, we rewrite the system of ordinary differential equations for the double pendulum in matrix form. We start with the first term in the matrix equation (58), **M**, which groups the

coefficients of the accelerations, specifically the first two terms of each of the two differential equations already written in system form (56). The $\mathbf{M}$ matrix contains only the coefficients of the accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$:

$$\mathbf{M} = \begin{bmatrix} l_1^2(m_1+m_2) & m_2 l_1 l_2 \cos\Delta \\ m_2 l_1 l_2 \cos\Delta & m_2 l_2^2 \end{bmatrix} \tag{60}$$

It is worth noting that unlike in the system of equations (56), the coefficients of the accelerations in the $\mathbf{M}$ matrix are organized into rows and columns according to the order of the generalized coordinates, (θ_1, θ_2). The first row and column correspond to θ_1, while the second row and column correspond to θ_2. Specifically, M_{11} is the coefficient of $\ddot{\theta}_1$ whereas M_{12} is the coefficient of $\ddot{\theta}_2$ in the same equation. The same logic applies to the second row and column, which correspond to the second equation of the system.

The second matrix, the $\mathbf{C}$ Coriolis matrix, groups the coefficients of the velocities. In the case of the double pendulum, it groups the terms that describe the effects of one mass's velocity on the motion of the other. Even without additional external forces, the presence of a moving mass influences the dynamics of the other. For simplicity of notation, in this case, the matrix will become a vector that groups the entire velocity term and not just its coefficients (consequently, we will have $\mathbf{C}$ and not $\mathbf{C}\dot{\mathbf{\Theta}}$):

$$\mathbf{C} = \begin{bmatrix} m_2 l_1 l_2 \dot{\theta}_2^2 \sin\Delta \\ -m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta \end{bmatrix} \tag{61}$$

The gravitational vector $\mathbf{G}$, represents the potential energy terms in the system. It corresponds to the last term of the first and second equations in the system (56):

$$\mathbf{G} = \begin{bmatrix} g l_1 (m_1+m_2) \sin\theta_1 \\ m_2 g l_2 \sin\theta_2 \end{bmatrix} \tag{62}$$

We can now complete the matrix formalism for the double pendulum (58):

$$\mathbf{M}\,\ddot{\mathbf{\Theta}} + (\mathbf{C}+\mathbf{G}) = \mathbf{0}$$

$$\begin{bmatrix} l_1^2(m_1+m_2) & m_2 l_1 l_2 \cos\Delta \\ m_2 l_1 l_2 \cos\Delta & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} m_2 l_1 l_2 \dot{\theta}_2^2 \sin\Delta \\ -m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta \end{bmatrix} + \begin{bmatrix} g l_1 (m_1+m_2) \sin\theta_1 \\ m_2 g l_2 \sin\theta_2 \end{bmatrix} = 0$$

For numerical integration, especially with a system like the double pendulum, it is useful to isolate the accelerations $\ddot{\mathbf{\Theta}}$, and the matrix formalism (58) allows us to do this easily:

$$\ddot{\mathbf{\Theta}} = -\mathbf{M}^{-1}(\mathbf{C}+\mathbf{G}) \tag{63}$$

For a given 2x2 matrix $\mathbf{M}$, its inverse is defined as:

$$\mathbf{M}^{-1} = \frac{1}{\det\mathbf{M}} \begin{bmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{bmatrix}, \qquad \det\mathbf{M} = M_{11}M_{22} - M_{12}M_{21}$$

By determining and substituting the coefficients:

$$\det\mathbf{M} = \Big[(l_1^2(m_1+m_2))\,(m_2 l_2^2)\Big] - \Big[(m_2 l_1 l_2 \cos\Delta)(m_2 l_1 l_2 \cos\Delta)\Big] = l_1 l_2 m_2 (m_1 + m_2 \sin^2\Delta)$$

$$\mathbf{M^{-1}} = \frac{1}{l_1 l_2 m_2 (m_1 + m_2 \sin^2\Delta} \begin{bmatrix} m_2 l_2^2 & -m_2 l_1 l_2 \cos\Delta \\ -m_2 l_1 l_2 \cos\Delta & l_1^2(m_1+m_2) \end{bmatrix} \tag{64}$$

The term in parentheses (63), can also be written in a more compact form:

$$\mathbf{C}+\mathbf{G} = \begin{bmatrix} m_2 l_1 l_2 \dot{\theta}_2^2 \sin\Delta + g l_1 (m_1+m_2) \sin\theta_1 \\ -m_2 l_1 l_2 \dot{\theta}_1^2 \sin\Delta + m_2 g l_2 \sin\theta_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{65}$$

We have now arrived at the following matrix structure:

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \qquad \ddot{\mathbf{\Theta}} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}, \qquad \mathbf{C} + \mathbf{G} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Given the matrix structure (63), we can solve for $\ddot{\theta}_1$:

$$\ddot{\theta}_1 = -\frac{1}{\det \mathbf{M}} \left(M_{22} X_1 \; - \; M_{12} X_2 \right) = -\frac{M_{22} X_1 - M_{12} X_2}{\det \mathbf{M}} \tag{66}$$

Similarly, we can solve for $\ddot{\theta}_2$:

$$\ddot{\theta}_2 = -\frac{1}{\det \mathbf{M}} \left(-M_{21} X_1 \; + \; M_{11} X_2 \right) = -\frac{-M_{21} X_1 + M_{11} X_2}{\det \mathbf{M}} \tag{67}$$

To conclude, we can substitute the values of M_{ij} and X_1, X_2, to obtain the final formulas in terms of $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$:

$$X_1 = m_2 l_1 l_2 \dot{\theta}_2^2 \sin \Delta + g l_1 (m_1 + m_2) \sin \theta_1$$
$$X_2 = -m_2 l_1 l_2 \dot{\theta}_1^2 \sin \Delta + m_2 g l_2 \sin \theta_2$$

$$M_{11} = l_1^2 (m_1 + m_2), \qquad M_{12} = m_2 l_1 l_2 \cos \Delta$$
$$M_{21} = m_2 l_1 l_2 \cos \Delta, \qquad M_{22} = m_2 l_2^2$$

Given the terms just listed, we can make the accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$ explicit using the formulas derived above(66) (67), simplifying $(\theta_1 - \theta_2)$ to Δ:

$$\ddot{\theta}_1 = \frac{-m_2 l_2^2 \Big[m_2 l_1 l_2 \dot{\theta}_2^2 \sin \Delta + g l_1 (m_1 + m_2) \sin \theta_1 \Big] + m_2 l_1 l_2 \cos \Delta \Big[-m_2 l_1 l_2 \dot{\theta}_1^2 \sin \Delta + m_2 g l_2 \sin \theta_2 \Big]}{l_1^2 l_2^2 m_2 (m_1 + m_2 \sin^2 \Delta)} \tag{68}$$

$$\ddot{\theta}_2 = \frac{m_2 l_1 l_2 \cos \Delta \Big[m_2 l_1 l_2 \dot{\theta}_2^2 \sin \Delta + g l_1 (m_1 + m_2) \sin \theta_1 \Big] - l_1^2 (m_1 + m_2) \Big[-m_2 l_1 l_2 \dot{\theta}_1^2 \sin \Delta + m_2 g l_2 \sin \theta_2 \Big]}{l_1^2 l_2^2 m_2 (m_1 + m_2 \sin^2 \Delta)} \tag{69}$$

5.4.1 Remarks on Notation

It is important to note that the explicit form of the accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$ may look different in some textbooks or academic treatises. This is because the terms are often rewritten using algebraic simplifications, grouped in different ways or normalised in terms of lengths, masses or numerical constants to obtain more compact or didactic denominators.

However, the representation presented here is formally correct. Crucially, it can be derived directly from the matrix formalism and the previous Euler-Lagrange equations and is therefore fully verifiable. Therefore, it strictly respects the structure of the dynamic system of the double pendulum.

6 Numerical Integration

A continuous dynamic system can be formally described as a system that changes continuously over time. Examples include the velocity of a pendulum, the position of a particle, or the population of animals in an ecosystem. This is expressed by the following equation:

$$\dot{y}(t) = f(t, y(t)), \qquad y(t_0) = y_0$$

The equation indicates that, starting from a state y_0 at time t_0, the system evolves according to a certain function, f, that depends on the system's current state

In many cases, we can't find an exact formula for $y(t)$. For instance, in the double pendulum, the equations become too complex. Numerical integration allows us to approximate the solution $y(t)$ in discrete steps $t_n = t_0 + nh$, separated by a time step h. Therefore, we get:

$$y_n \approx y(t_n)$$

An explicit numerical integration method calculates the next state at t_{n+1} using only information already known at t_n. The two most well-known methods are the Euler method and the fourth-order Runge-Kutta (RK4) method.

6.1 Euler Method

The explicit Euler method is the most basic first-order, one-step integrator. Given an equation of the type $\dot{y} = f(t, y)$, we set a time step h and iteratively build the solution as:

$$y_{n+1} = y_n + h\, f\,(t_n, y_n) \tag{70}$$

Geometrically, this means we approximate the solution curve with the tangent line at the point (t_n, y_n). The explicit Euler method is said to be consistent, which means the error committed in a single step tends to zero as the time step h decreases. Specifically, the local (or truncation) error in each step is on the order of $O(h^2)$, while the accumulated global error up to a certain time is on the order of $O(h)$, this implies that halving the time step approximately halves the final error. In conclusion, the time step h must be sufficiently small to prevent numerical errors from growing and causing the solution to diverge.

In stiff or chaotic systems, using the Euler method can require very small steps, which often makes more precise and stable methods, like the Runge-Kutta 4, preferable.

Python:

```
def euler(f, t0, t_max, y0, h):
    th = np.arange(t0, t_max + h, h)
    uh = np.zeros(len(th))
    uh[0] = y0

    for i in range(len(th)-1):
        uh[i+1] = uh[i] + h*f(th[i], uh[i])
    return th, uh
```

The provided Python code implements the explicit Euler method to solve the equation $\dot{y} = f(t, y)$. The `euler` function takes f, the initial and final times, the initial condition, and the step size h, it creates arrays for time and the solution and updates them step by step according to $y_{n+1} = y_n + h\, f(t_n, y_n)$. Finally, it returns the time and solution arrays.

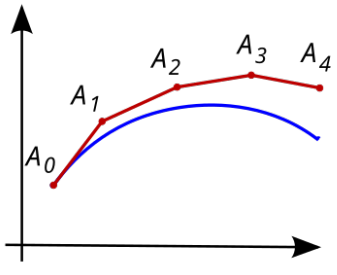


Figure 5: Illustration of Euler method. The curve to be approximated is in blue, and its polynomial approximation is in red.

6.2 Runge-Kutta 4 Method

The RK4 is another explicit integration method, but it is a four-stage method that combines multiple function evaluations within each step to achieve higher accuracy. In each interval, four increments (or slopes) are calculated:

$$
\begin{aligned}
k_1 &= f\ (t_n, y_n)\,, \\
k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \\
k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), \\
k_4 &= f\ (t_n + h,\ y_n + \ k_3)\,.
\end{aligned}
$$

The approximation for the next step is a weighted average of these values:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{71}$$

The RK4 method is a fourth-order method: the local error is $O(h^5)$, and the global error is $O(h^4)$. This means that for sufficiently small step sizes h, the error is reduced by a factor of h^4. This makes RK4 much more accurate than the Euler method for the same step size h. AFor example, by reducing h by a factor of 10, , the global error of Euler decreases by a factor of about 10, while that of RK4 decreases by a factor of approximately $\sim 10^4$ volte.

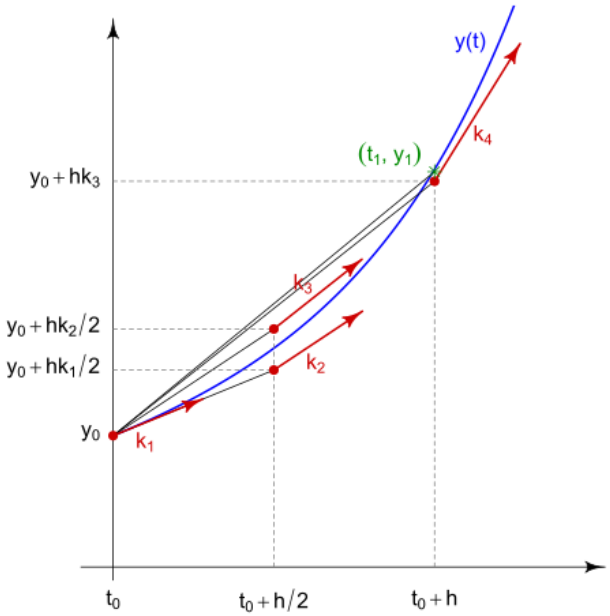
In practical terms, RK4 achieves high accuracy with a much smaller number of steps compared to Euler. The drawback is that it requires four times as many evaluations of the function f per step (four stages instead of one), but this additional cost is generally offset by the ability to use significantly larger step sizes. The RK4 method is thus considered an excellent compromise between accuracy, computational cost, and stability, even though it is more computationally expensive per step. The following is a snippet of Python code:

```
def rk4(f, t0, t_max, y0, h):
    th = np.arange(t0, t_max + h, h)
    y0 = np.array(y0, dtype=float)
    uh = np.zeros((len(th), len(y0)))
    uh[0] = y0

    for i in range(len(th)-1):
        t = th[i]
        y = uh[i]

        k1 = f(t, y)
        k2 = f(t + h/2, y + h/2 * k1)
        k3 = f(t + h/2, y + h/2 * k2)
        k4 = f(t + h,   y + h * k3)
        uh[i+1] =
        = y + (h/6) * (k1 + 2*k2 + 2*k3

    return th, uh
```



The provided Python code implements the fourth-order Runge-Kutta method to solve $\dot{y} = f(t, y)$. The `rk4` function takes f, the initial and final times, the initial condition, and the step size h, it creates arrays for time and the solution, and for each step, it calculates four intermediate slopes k_1, k_2, k_3, k_4. The solution is the weighted combination of these values.

6.3 Comparison of Integration Methods

In the field of numerical integration of differential equations, there are many algorithms, including multi-step methods (e.g., Adams-Bashforth, Adams-Moulton), implicit methods for rigid problems, and geometric schemes that aim to preserve quantities such as the energy in Hamiltonian systems. For now, we will focus on comparing the two methods discussed previously: Euler and RK4.

Accuracy: The Euler method is a first-order method, with a global error of $O(h)$. This means that halving the step size h halves the error. In contrast, the RK4 method is a fourth-order method, with an error of $O(h^4)$. Therefore, with the same step size, RK4 is much more accurate, while Euler requires very small steps to achieve comparable results.

Computational Cost: With Euler's method, f is calculated once per step, while RK4, as the name suggests, is calculated four times. Each RK4 step therefore costs about four times as much, but in return, far fewer steps are required to achieve the same accuracy. The overall balance is generally in favour of RK4.

Stability: Both methods are explicitly and conditionally stable. RK4 has a wider stability range and tolerates slightly larger steps than Euler. In this case, and generally for chaotic systems such as the double pendulum, you must also choose a sufficiently small step size for RK4.

To summarise, the RK4 integration method better preserves invariant quantities such as energy and reduces errors. Therefore, it is generally preferable to the Euler method for simulations.

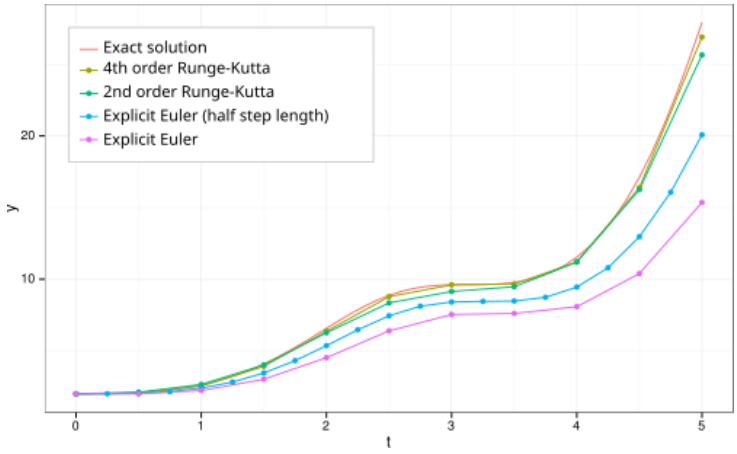


Figure 6: Cmparison of different methods

7 Discussion and Conclusions

This study has presented a unified framework for analysing oscillating systems, ranging from the simple pendulum to the complex dynamics of the double pendulum. By combining Newtonian and Lagrangian formulations with numerical integration techniques, the work illustrates the equivalence of theoretical approaches in classical mechanics while emphasising the need for computational methods for nonlinear and chaotic dynamics.

The simple pendulum provided a basis for understanding the transition from linear approximations at small angles to the exact nonlinear equations of motion. The comparison between

Newtonian and Lagrangian mechanics showed how different conceptual frameworks lead to equivalent physical descriptions and underlined the usefulness of generalised coordinates in managing constraints.

The analysis of the double pendulum illustrated the complexity of non-linear systems in which the sensitivity to the initial conditions leads to chaotic and unpredictable long-term behaviour. This motivated the application of numerical methods, including the Euler method and the fourth-order Runge-Kutta method, to approximate solutions. While classical methods such as Runge-Kutta 4 can exhibit energy drift in long simulations, symplectic integrators preserve the structure of the phase space and thus ensure physically meaningful results in long-term studies of conservative systems.

Beyond its educational value, the double pendulum serves as a paradigmatic model for complex systems in all disciplines: Engineering and applied physics for testing bed for control strategies in unstable mechanical systems and analogues in coupled electrical circuits, but also in statistical physics and chemistry to illustrate transitions from ordered to chaotic dynamics, analogous to non-equilibrium phenomena and oscillatory chemical reactions, and finally in Quantum Mechanics for quantum double pendulums shed light on the interplay between classical chaos and quantum behaviour.

Certain limitations are acknowledged, such as dissipative effects, like friction have been neglected and advanced topics (the quantisation of pendulums and the use of symplectic integrators for Hamiltonian invariants) have not been addressed.

By integrating analytical theory and computational practise, this study shows how classical mechanics continues to offer deep insights into the nature of order, disorder, and unpredictability that are relevant to physics, engineering, biology, and computer science.

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