Abcd +

Gabriele Giudici

Universitá degli studi di Milano-Bicocca

18 September 2018

Index

Relations before crisis

Before the financial crisis, independently from the considered tenor, the below reported relation was valid:

$$1 + F_x(t)\tau_x(t) = \frac{D_x(t)}{D_x(t+x)} \tag{1}$$

that can be written as:

$$F_x(t) = \frac{1}{\tau_x(t)} \left(e^{\int_t^{t+x}[f(u)]\mathrm{d}u} \right) \tag{2}$$

Relations after crisis

Later on, the above reported equation ceased to be valid because the markets started to consider maturity and credit premia among different tenors and therefore multiple curves world arose. In order to compensate them, given a tenor x, for retrieving a forward rate between t and t+x, starting from a different curve, in continuous time, it is necessary to add a basis on an instantaneous forward rate.

$$F_x(t) = \frac{1}{\tau_x(t)} \left(e^{\int_t^{t+x} [s_x(u) + f(u)] \mathrm{d}u} \right) \tag{3}$$

If this is the overnight one, then all the curves are modelled as a basis on the ON and therefore are called "absolute":

$$s_x(t) = f_x(t) - f_{ON}(t), \tag{4}$$

Relations after crisis

However, it is possible to model a curve as a basis on another one that differs from the ON and therefore it is possible to talk about relative basis:

$$s_{x,y}(t) = f_x(t) - f_y(t) \tag{5}$$

Obtaining:

$$F_x(t) = \frac{1}{\tau_x(t)} \left(e^{\int_t^{t+x} [s_{x,y}(u) + f_y(u)] \mathrm{d}u} \right) \tag{6}$$

In both cases the basis can be empirically modelled as an abcd function:

$$s_x(t), s_{x,y}(t) = (a + bt)e^{-ct} + d \tag{7}$$

Aim

Given the previous research findings, the aim is to stress the following points:

1. model inputs form;
2. fixing of parameter c and its effect on the model;
3. fixing of t_{max} and its effect on the model;
4. fixing of both t_{max} and c evaluating effects on the model.

New model inputs form

Given the abcd functional form of the basis:

$$s(t) = (a + bt)e^{-ct} + d; \tag{8}$$

there are two main matters:

1. model has too many parameters;
2. only a and d have a clear financial meaning: a + d represents the value of the basis in $t = 0$, while d is the long run value of the basis.

Therefore, the first idea is to rewrite the basis as function of:

$$a_x, d_x, t_{max}, s_x(t_{max}) \tag{9}$$

that still gives the user 4 parameters, but only financial meaningful ones. Indeed, t_{max} indicates when the peak of uncertainty comes, while $s_x(t_{max})$ represents the basis value in that moment.

Problematic form

The idea is to bring back the model to the abcd in order to exploit the available framework. Unfortunately, it is not possible retrieving the abcd parameters because of the problematic form:

$$y = xe^x \tag{10}$$

that makes not possible to rewrite $x = f(y)$.
Therefore, a change of plan is needed. The choice is to substitute $s_x(t_{max})$ with c. It makes sense, because c is a fundamental model parameter (later better explained) that should be chosen by the user. In the end, the new inputs form is:

$$a, c, d, t \tag{11}$$

and the previous problem ceases.

Calibration matter

In the "old" abcd framework, as already explained, it is possible to calibrate the basis with respect to the overnight rate according to not incremental method :

$$s_x(t) = f_x(t) - f_{ON}(t), \tag{12}$$

or exploiting the incremental method:

$$s_{x,y}(t) = f_x(t) - f_y(t) \tag{13}$$

these different procedures lead to model different basis and it is not possible to swap from absolute to relative basis and viceversa.

Common c model effects

However, it is valid that given two absolute basis $s_x(t)$ and $s_y(t)$, where tenor x is greater than tenor y, if $c_x = c_y = c_{x,y}$, then the relative basis is still an abcd function:

$$\begin{aligned} s_{x,y}(t) &= s_x(t) - s_y(t) \\ &= (a_x + b_x t)e^{-c_x t} + d_x - ((a_y + b_y t)e^{-c_y t} + d_y) \\ &= (a_x - a_y + (b_x - b_y)t)e^{-c_{x,y} t} + d_x - d_y \\ &= (a_{x,y} + b_{x,y} t)e^{-c_{x,y} t} + d_{x,y} \end{aligned} \tag{14}$$

Difference of absolute basis as relative basis

Moreover, it is also valid that, considering f_x for a generic tenors x, given that:

$$f_x(t) = s_x(t) + f_{ON}(t) \tag{15}$$

then:

$$\begin{aligned} s_{x,y}(t) &= f_x(t) - f_y(t) \\ &= s_x(t) + f_{ON}(t) - s_y(t) - f_{ON}(t) \\ &= s_x(t) - s_y(t) \end{aligned} \tag{16}$$

if the two basis are abcd and share the common exponential term, $s_{x,y}$ is still an abcd basis.

Consequences

Therefore, shared c allows to shift from incremental to not incremental basis leading to a global model. Given the absence of a QuantLib implementation for managing this type of calibration, the model is calibrated exploiting the excel solver capabilities and making the following choices:

1. guesses of the parameters have been chosen picking from the results of the previous paper, that exhibits the best fit as possible according to this framework, while for c have been manually tempted different values and in the end the empirical attempts led to a value of 0.7, which avoids the abortion of the output;
2. if a curve can't be calibrated, the algorithm fails;
3. the algorithm has been tried for each available excel solver method.

C fixing (NI) outputs: optimization algorithm

The results that follow come from not incremental (NI) approach:

	Working column	Output		
Solving method	GRG Non linear	GRG Non linear	Simplex LP	Evolutionary
Guessed c	0,498261576	0,7 V 0,498261576047114	0,498261576	0,498261576
Calibrated c	0,498261358	0,498261576	Linear Condition not satisfied	Error
Error function	7,41	7,41	-	-

Figure: c fixing output

As it is possible to see, the Solver works only with the "GRG non linear" algorithm, probably because the problem is highly not linear (yeah, not so smart consideration).

C fixing (NI) outputs: k correction factors

6M Continuous Basis	
k max	1,02470
k min	0,99573
Max Error (bps)	4,79
Root Mean Square Error	2,16

3M Continuous Basis	
k max	1,02118
k min	0,99691
Max Error (bps)	3,05
Root Mean Square Error	1,36

1Y Continuous Basis	
k max	1,03197
k min	0,99460
Max Error (bps)	5,76
Root Mean Square Error	3,62

1M Continuous Basis	
k max	1,00168
k min	0,99953
Max Error (bps)	0,68
Root Mean Square Error	0,27

Figure: Correction factors k from not incremental calibration with global c

C fixing (NI) outputs: parameters and statistics

6M Continuous	Calibration	
Basis Parameters	Continuous	Simple
a	0,0436%	0,0885%
b	0,0023	0,0021
c	0,498	0,498
d	0,1373%	0,1373%
a+d	0,1809%	0,2257%
Max Location	21-dic-17	23-set-17
a/b	0,186939453	0,429116034
1/c	2,006978834	2,006978834
T max th	1,820039381	1,5778628
T max from Location	1,825518833	1,583342252
Max Value	0,3263%	0,3258%

3M Continuous	Calibration	
Basis Parameters	Continuous	Simple
a	0,0097%	0,0221%
b	0,0011	0,0011
c	0,498	0,498
d	0,0980%	0,0980%
a+d	0,1077%	0,1201%
Max Location	26-gen-18	13-dic-17
a/b	0,08646261	0,20886817
1/c	2,00697883	2,00697883
T max th	1,92051622	1,79811066
T max from Locat	1,92599568	1,80359012
Max Value	0,1847%	0,1846%

1Y Continuous	Calibration	
Basis Parameters	Continuous	Simple
a	0,0616%	0,2144%
b	0,0045	0,0036
c	0,498	0,498
d	0,1670%	0,1670%
a+d	0,2286%	0,3814%
Max Location	09-gen-18	22-lug-17
a/b	0,13564709	0,60240153
1/c	2,00697883	2,00697883
T max th	1,87133175	1,40457731
T max from Locat	1,8768112	1,41005676
Max Value	0,5256%	0,5217%

1M Continuous	Calibration	
Basis Parameters	Continuous	Simple
a	0,0104%	0,0097%
b	-0,0001	-0,0001
c	0,498	0,498
d	0,0184%	0,0184%
a+d	0,0289%	0,0281%
Max Location	24-dic-18	08-dic-18
a/b	-0,82314956	-0,779033178
1/c	2,006978834	2,006978834
T max th	2,830128394	2,786012012
T max from Locat	2,835607846	2,791491464
Max Value	0,0122%	0,0122%

Figure: Parameters from not incremental calibration with global c

For 6M, 3M and 1Y the $t_{max}(x)$ values are closed to the other and this may mean that a global shared value of t_{max} is a sane idea. The only problem that arises is that with respect to the legacy curve, the new basis seems losing fitting on the legacy one. Anyway it should not be a problem in the extend that is valid the idea that legacy curve brings with itself more noise than signal with respect to an abcd basis.

C fixing (NI) outputs: 1M problem

Graphically, it appears that the matter, already encountered in the old abcd framework, about the shape of 1M lasts:

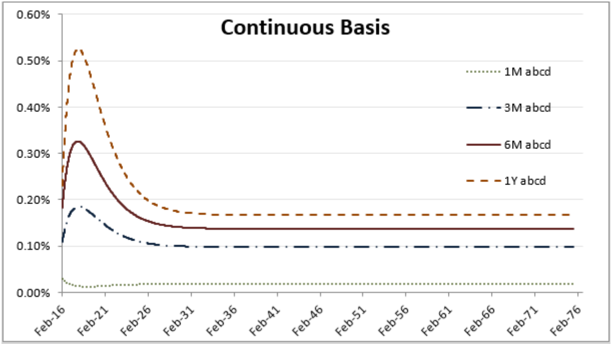

Figure: Absolute basis s_x from not incremental calibration

C fixing (I) outputs:optimization algorithm

The outputs follow:

	Working column	Output		
Solving method	GRG Non linear	GRG Non linear	Simplex LP	Evolutionary
Guessed c	0.498261576	0.498261576	0.498261576	0.498261576
Calibrated c	0.305782434	0.305782434	Linear Condition not satisfi	Error
Error function	3.40	3.40	-	-

Figure: c fixing output

Should be noted that the overall error is smaller than in the not incremental approach.

C fixing (I) outputs: k correction factors

However, considering that the distribution of the correction factor is what was looking for, the results are still excellent:

6M Continuous Basis	
k max	1.00941
k min	0.99733
Max Error (bps)	1.86
Root Mean Square Error	0.98

3M, 6M Continuous Basis	
k max	1.01074
k min	0.99769
Max Error (bps)	1.51
Root Mean Square Error	0.74

1Y, 6M Continuous Basis	
k max	1.00849
k min	0.99818
Max Error (bps)	4.36
Root Mean Square Error	1.35

1M, 6M Continuous Basis	
k max	1.00154
k min	0.99946
Max Error (bps)	0.80
Root Mean Square Error	0.33

Figure: Correction factors k from incremental calibration with global c

Range of k values is closed to 1 that is the best value for the calibration.

C fixing (I) outputs:parameters and statistics

6M Continuous	Calibration	
Basis Parameters	Simple	Continuous
A	0,1502%	0,1246%
B	0,0014	0,0015
C	0,306	0,306
D	0,0871%	0,0871%
A+D	0,2373%	0,2117%
Max Location	09-mag-18	08-ago-18
a/b	1,065708441	0,819440891
1/c	3,270299043	3,270299043
T max th	2,204590601	2,450858152
T max from Locat	2,205479452	2,454794521
Max Value	0,3219%	0,3222%

3M, 6M Continuous	Calibration	
Basis Parameters	Simple	Continuous
A	0,0852%	0,0793%
B	0,0007	0,0007
C	0,306	0,306
D	0,0146%	0,0146%
A+D	0,0998%	0,0940%
Max Location	26-mar-18	09-mag-18
a/b	1,19127169	1,067864154
1/c	3,27029904	3,270299043
T max th	2,07902735	2,202434889
T max from Locat:	2,08493151	2,205479452
Max Value	0,1385%	0,1385%

12M, 6M Continuous	Calibration	
Basis Parameters	Simple	Continuous
A	0,1155%	0,0853%
B	0,0009	0,0010
C	0,306	0,306
D	0,0174%	0,0174%
A+D	0,1330%	0,1027%
Max Location	06-feb-18	01-ago-18
a/b	1,32181911	0,83853656
1/c	3,27029904	3,27029904
T max th	1,94847994	2,43176248
T max from Locatic	1,95342466	2,43561644
Max Value	0,1749%	0,1756%

1M, 3M Continuous	Calibration	
Basis Parameters	Simple	Continuous
A	0,0403%	0,0371%
B	0,0008	0,0008
C	0,306	0,306
D	0,0540%	0,0540%
A+D	0,0943%	0,0911%
Max Location	12-dic-18	27-dic-18
a/b	0,4832735	0,43903039
1/c	3,27029904	3,27029904
T max th	2,78702554	2,83126865
T max from Locat:	2,8	2,84109589
Max Value	0,1702%	0,1702%

Figure: Parameters from incremental calibration with global c

The statistics are still good because not only the repricing error is low, but because the $t_{max}(x)$ values are closed to the other.

C fixing (I) outputs: curve shapes

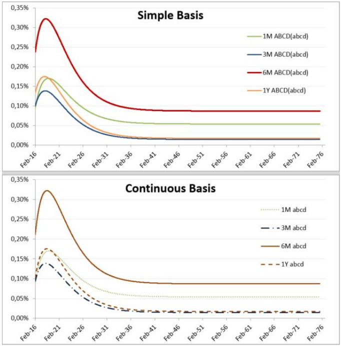

Figure: Relative basis from incremental calibration

Basis Shifting

Once tested the model fitting and ensured that it works, it makes sense to go a step ahead for evaluating the new model features, i.e. the capabilities of retrieving the relative basis in a not incremental context and the absolute in an incremental one, through a repricing.

Not incremental approach

After globally calibrated the model, where absolute basis for 3M, 12M and 1M have been built according to the abcd not incremental framework, the relative basis w.r.t. the 6M instantaneous forward rate can be retrieved thanks to:

$$s_{x,6M} = s_x - s_{6M} \tag{17}$$

where: $x \in (3M, 12M, 1M)$. The parameters of $s_{x,6M}$ have been retrieved exploiting the below reported formulas.
If $x > 6M$:

$$\begin{aligned} a_{x,6M} &= a_x - a_{6M}; \\ b_{x,6M} &= b_x - b_{6M}; \\ d_{x,6M} &= d_x - d_{6M}; \end{aligned} \tag{18}$$

otherwise the signs are inverted.

Not incremental approach: k correction factors

The results for the basis with respect to 6M are:

3M, 6M Continuous Basis	
k max	1.02116
k min	0.99689
Max Error (bps)	3.04
Root Mean Square Error	1.46

1Y, 6M Continuous Basis	
k max	1.02239
k min	0.99374
Max Error (bps)	7.56
Root Mean Square Error	5.27

1M, 6M Continuous Basis	
k max	1.00167
k min	0.99952
Max Error (bps)	0.75
Root Mean Square Error	0.28

Figure: Correction factors k from not incremental calibration with global c and relative basis

Not incremental approach:k correction factors

While for 1M from $s_{1M,3M}$ are:

1M, 3M Continuous Basis	
k max	1.00170
k min	0.99954
Max Error (bps)	1.82
Root Mean Square Error	0.68

Figure: 1M correction factors k from not incremental calibration with global c and relative basis $s_{1M,3M}$

The k are generally closed to 1, but the root mean square error and the maximum error in basis points for 12M are high. Therefore, it seems that the k factor range is not a good metric for understanding how much the repricing is good, then a better metric is needed to summarize and interpret the correction factors.

Not incremental approach: parameters and statistics

6M Continuous Basis Parameters	Calibration	
	Continuous	Simple
a	0,0436%	0,0885%
b	0,0023	0,0021
c	0,498	0,498
d	0,1373%	0,1373%
a+d	0,1809%	0,2257%
Max Location	21-dic-17	23-set-17
a/b	0,186939453	0,429116034
1/c	2,006978834	2,006978834
T max th	1,820039381	1,5778628
T max from Locatio	1,825518833	1,583342252
Max Value	0,3263%	0,3258%

3M, 6M Continuous Basis Parameters	Calibration	
	Continuous	Simple
a	0,0215%	0,0349%
b	0,0013	0,0012
c	0,498	0,498
d	0,0392%	0,0392%
a+d	0,0608%	0,0742%
Max Location	27-dic-17	13-nov-17
a/b	0,1687559	0,29116146
1/c	2,00697883	2,00697883
T max th	1,83822294	1,71581738
T max from Locatio	1,84370239	1,72129683
Max Value	0,1416%	0,1416%

1Y, 6M Continuous Basis Parameters	Calibration	
	Continuous	Simple
a	0,1708%	0,1788%
b	0,0012	0,0010
c	0,498	0,498
d	0,0297%	0,0297%
a+d	0,2005%	0,2085%
Max Location	07-ott-16	19-apr-16
a/b	1,39251694	1,85927138
1/c	2,00697883	2,00697883
T max th	0,6144619	0,14770746
T max from Locatio	0,61994135	0,15318691
Max Value	0,2110%	0,2090%

1M, 6M Continuous Basis Parameters	Calibration	
	Continuous	Simple
a	0.0339%	0.0438%
b	0.0025	0.0024
c	0.498	0.498
d	0.1189%	0.1189%
a+d	0.1528%	0.1627%
Max Location	08-Jan-18	23-Dec-17
a/b	0.138113924	0.182230306
1/c	2.006978834	2.006978834
T max th	1.86886491	1.824748528
T max from Location	1.874344362	1.83022798
Max Value	0.3132%	0.3131%

Figure: Parameters from not incremental calibration with global c and relative basis

Not incremental approach: parameters and statistics

Moreover for $s_{1M,3M}$:

1M,3M Continuous Basis Parameters	Calibration	
	Continuous	Simple
a	0,0124%	0,0172%
b	0,0012	0,0012
c	0,498	0,498
d	0,0796%	0,0796%
a+d	0,0920%	0,0969%
Max Location	04-gen-18	20-gen-18
a/b	0,105044401	0,149160783
1/c	2,006978834	2,006978834
T max th	1,901934433	1,857818051
T max from Location	1,863297503	1,907413885
Max Value	0,1715%	0,1715%

Figure: Parameters from not incremental calibration with global c and relative basis $s_{1M,3M}$

The times of maximum are closed, the only problem appears for the 12M that seems to behave in a different way w.r.t. the other.

Not incremental approach: curves shape

Also looking at graphical comparison (excluding $s_{1M,3M}$):

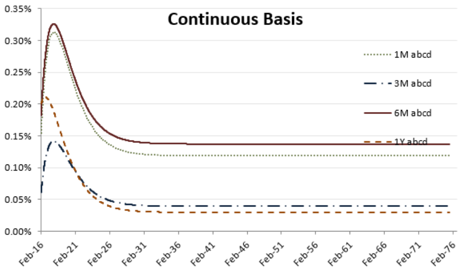

Figure: Absolute s_{6M} and its respective $s_{x,6M}$ relative basis

the basis dominance is respected and it makes sense that the $s_{1M,6M}$ is next to the s_{6M}.

Incremental approach

After globally calibrated the model, where relative basis for 3M, 12M and 1M have been built according to the abcd incremental framework, the absolute basis can be retrieved exploiting:

$$s_x = s_{x,6M} + s_{6M} \tag{19}$$

where: $x \in (3M, 12M)$.
While for 1M because of incremental principle:

$$s_x = s_{x,3M} + s_{3M} \tag{20}$$

Where: $s_{3M} = s_{6M} - s_{3M,6M}$

The parameters of s_x have been retrieved exploiting the above reported formulas. If $x > y$:

$$a_x = a_{x,y} + a_y;$$
$$b_x = b_{x,y} + b_y;$$
$$d_x = d_{x,y} + d_y;$$

Otherwise, signs are inverted.
Given the basis, for each one has been made the repricing of the market quotes, modelling the considered instantaneous forward rate as:

$$f_x = f_{ON} + s_y \pm s_{x,y} \tag{21}$$

Incremental approach: k correction factors

3M Continuous Basis	
k max	1,01049
k min	0,99748
Max Error (bps)	0,95
Root Mean Square Error	1,11

1Y Continuous Basis	
k max	1,00537
k min	0,99741
Max Error (bps)	4,01
Root Mean Square Error	2,29

1M Continuous Basis	
k max	1,00131
k min	0,99927
Max Error (bps)	1,57
Root Mean Square Error	0,94

Figure: Correction factors k from incremental calibration with global c and absolute basis

The k are generally closed to 1, but the root mean square error and the maximum error in basis points for 12M are still high. Once again, is it k a good metric? The answer should be "yes, it is", but just to evaluate the amount of information addressed by the model (research target), for evaluating the overall goodness of the model the repricing error is what matters (therefore they are the pivotal values in the calibration process).

Incremental approach: parameters and statistics

6M Continuous	Calibration	
Basis Parameters	Continuous	Simple
a	0,1246%	0,1502%
b	0,0015	0,0014
c	0,306	0,306
d	0,0871%	0,0871%
a+d	0,2117%	0,2373%
Max Location	08-ago-18	10-mag-18
a/b	0,81944089	1,0657084
1/c	3,27029904	3,270299
T max th	2,45085815	2,2045906
T max from Locat:	2,4563376	2,2100701
Max Value	0,3222%	0,3219%

3M Continuous	Calibration	
Basis Parameters	Continuous	Simple
a	0,0650%	0,0708%
b	0,0007	0,0007
c	0,306	0,306
d	0,0724%	0,0724%
a+d	0,1374%	0,1433%
Max Location	26-giu-18	12-mag-18
a/b	0,93635939	1,05976693
1/c	3,27029904	3,27029904
T max th	2,33393965	2,21053212
T max from Locat:	2,33941911	2,21601157
Max Value	0,1836%	0,1836%

1Y Continuous	Calibration	
Basis Parameters	Continuous	Simple
a	0,2657%	0,3231%
b	0,0023	0,0020
c	0,306	0,306
d	0,1045%	0,1045%
a+d	0,3702%	0,4276%
Max Location	04-apr-18	10-ott-17
a/b	1,16373285	1,6470154
1/c	3,270299043	3,27029904
T max th	2,106566192	1,62328364
T max from Locati	2,112045644	1,6287631
Max Value	0,4966%	0,4950%

1M Continuous	Calibration	
Basis Parameters	Continuous	Simple
a	0,0247%	0,0238%
b	-0,0001	-0,0001
c	0,306	0,306
d	0,0184%	0,0184%
a+d	0,0432%	0,0422%
Max Location	15-mar-21	27-feb-21
a/b	-1,7842555	-1,740012
1/c	3,27029904	3,270299
T max th	5,05455453	5,0103114
T max from Locati	5,06003398	5,0157909
Max Value	0,0087%	0,0087%

Figure: Parameters from incremental calibration with global c and absolute basis

The time of maximum are not so closed and the recursive problem with 1M appears.

Incremental approach: curves shape

Also looking at graphical comparison:

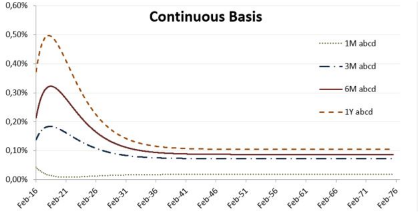

Figure: Absolute basis from relative ones

the basis dominance is respected and it is coherent that problem with 1M still remains for the absolute basis with both the approach.

Fixing of t

Given a multiple curves world with N of them, there are not financial reasons why should be forecast different uncertainty horizons for different tenors, therefore it makes sense to commonly fix, during the calibration process, the t_{max} value.
As a results from the previous abcd framework, has been shown that, in the model continuous form, it is possible to retrieve t_{max} equation as:

$$t_{max}(x) = \frac{1}{c_x} - \frac{a_x}{b_x} \tag{22}$$

From abcd to acdt

Fixing t together with the new input form means that the user only chooses the guessed t_{max}, a, c, d values and internally the model is linked to the old abcd framework. Therefore, it is necessary to retrieve the form of b, that in the new scheme is a given value:

$$b_x = \frac{a_x c_x}{1 - t_{max} c_x} \tag{23}$$

Implementation idea

While retrieving acdt from abcd is trivial, for globally fixing a parameter without turning to the excel solver is more challenging, two solution are available. While the first simply exploits a QL solver, the second creates a model of models, therefore the calibration process builds a new model with a single parameter (it makes more sense in case of c fixing because there are model implications). Anyway, both of them rely on the construction of a global error that summarizes each curve repricing errors as:

$$\sum_{j=1}^{4} \sqrt{\frac{\sum_{i=1}^{n} (\hat{q_{j,i}} - q_{j,i})^2}{n}} \tag{24}$$

where: q is the market quote and $\hat{q}$ is the model repriced quote.

Fixing of c and time of maximum

The aim of this experiment is to validate the previous intuition from both a financial and a model point of view. Indeed, in order to fix t_{max} the role of c is fundamental and the previous empirical results show that often $\frac{1}{c}$ dominates $\frac{a}{b}$ and therefore the time of maximum converge. Therefore, fixing c implies a strong effect on the fixing of t_{max} and the exactness of the equation:

$$t_{max} = \frac{1}{c} - \frac{a_x}{b_x} \tag{25}$$

depends on a particular relation that needs to be generally valid among different tenors x and y:

$$\frac{a_x}{b_x} = \frac{a_y}{b_y} \tag{26}$$