Definitions

① Topological space

is a couple (X, τ)

where τ is a collection of subsets of X, such that

1) $\phi \in \tau$

2) $U_\alpha \in \tau \Rightarrow \bigcup_\alpha U_\alpha \in \tau$

3) $U_i \in \tau \quad i = 1..n \Rightarrow \bigcap_i U_i \in \tau$

→ $U_i \in \tau$ is an open $\Rightarrow X \setminus U_i$ is closed

→ τ is the topology

→ A neighbourhood of point P is an open U which contains P

• To give all {U} is length better to give the minimal number of U which allow to reconstruct τ, ie a basis is a set $\{V_\alpha\}$ / $\forall U \in \tau \quad U = \underset{\alpha \text{ subset}}{\bigcup} V_\alpha$

. Functions

$f: M \to N$ $\quad$ M, N topo spaces

→ f is continuous iff $\forall\ V \in \tau_N \Rightarrow f^{-1}(V) \in \tau_M$

or the inverse image of every open set is open

→ f is a homeomorphism iff

f is a bijection and both f and f^{-1} are continuous

EX

1) $M = \{0,1\}$

- discrete topology $\tau = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$
- Sierpinski space $\tau = \{\emptyset, \{\phi\}, \{0,1\}\}$ $\{\phi\}$ is open but not closed

→ it is T_0 but not T_1 (and not T_2)
(↑ Hausdorff)

T_0 $\forall P,Q \in M$ $P \neq Q$ $\exists U$ either $P \in U$ $Q \notin U$ or $P \notin U$ $Q \in U$

T_1 $\forall P,Q \in M$ $P \neq Q$ $\exists U_P, U_Q$ $P \in U_P$ $Q \notin U_P$ & $P \notin U_Q$ $Q \in U_Q$

T_2 as T_1 plus $U_P \cap U_Q = \emptyset$

→ it is connected

M is Connected iff not$(M = U \cup V$ $U \cap V = \emptyset)$ or if $\emptyset$ and M are the only sets which are both open and closed

→ it is separable

M is separable if $\exists D \subset M$ $\overline{D} = M$ D is numerable

→ it is compact

M is compact if from any open covering we can extract a finite subcovering

Covering $\{\{0\}, \{0,1\}\} \to \{\{0,1\}\}$

Come è che $\mathbb{R}$ non è compatto se posso prendere $\mathbb{R}$ stesso come ricoprimento aperto?
Perché se prendo il covering $(n, n+1)$ $(n \in \mathbb{Z}, n+3/4)$
non posso estrarre nulla di finito

Until now we can define

→ connectness
→ separation
→ continuous functions
→ separability
→ compactness
- partition of unity

(15)

It is paracompact

$\forall V_\alpha\ \exists U_\alpha\ V_\alpha \subset U_\alpha$

M is paracompact if any open covering admit a <u>locally finite</u> refinement

$\forall P\ \exists U_P \,/\, U \cap V_\alpha \neq \emptyset$ and $\alpha \in I$ in finite

② Topological space M is locally Euclidean H

- M is a topological space locally homeomorphic to R^n, i.e.

$\forall P \in M \quad \exists V \in \tau \quad P \in V \quad V$ homeomorphic to an open subset of $\mathbb{R}^n$

(*) exist at least one, not all V!

→ Topological manifold M

is a locally Euclidean Hausdorff, paracompact topological space

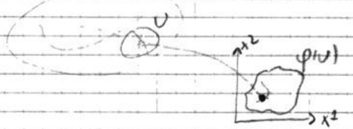
admits a partition of unity ← ↳ METRIZABLE by Urizohn's theorem

Now we can introduce charts and coordinates, atlas

For any point P we can find a neighborhood U_p which is homeomorphic to an open in $\mathbb{R}^n$, U_p is called Euclidean neighborhood

$$\varphi: U \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$$

φ is a coordinate chart on U

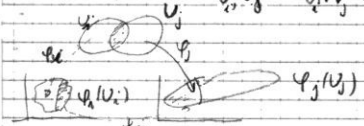


← we can use coordinates in U_p around P

M is locally Euclidean iff can be covered by Eucl. neigh

A set of Euclidean neighborhoods which cover M is an atlas

Now consider $U_i, U_j \quad U_i \cap U_j = U_{ij} \neq \emptyset$



We define a transition function

$$\varphi_j \circ \varphi_i^{-1}\Big|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij}) \qquad \psi_{ij} = \varphi_j \circ \varphi_i^{-1}$$

ψ_{ij}

→ it is a homeomorphism from $\varphi_i(U_{ij}) \subseteq \mathbb{R}^n$ to $\varphi_j(U_{ij}) \subseteq \mathbb{R}^n$

→ it describes a change in coordinates $\psi_{ij}^n(x^m_{(i)}) = x^n_{(j)}$

and $\psi_{ij} \in C^0(\quad)$ i.e. it continuous (= homeomorphism)

METRIZABLE = it exists a distance which is less than a metric
which allows to compute angles too

(*)  is not locally Euclidean
since  is not!

Now we have coordinates on different patches and a way to pass from one set to the other.

Is it enough?

No, we want to be able to take derivatives to compute i.e. velocities and so on hence we require — to do calculus

$\forall_{ij}\ \psi_{ij} \in C^K \qquad K \geq 1 \qquad \Rightarrow$ diffeomorphism!

We want a differentiable manifold

Actually we have different classes

$\psi \in C^K$ diff

$\psi \in C^\infty$ smooth

$\psi \in C^\omega$ analytic — ψ can be Taylor expanded and the series absolutely converges

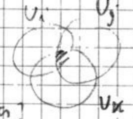
$\psi : \mathbb{C}^n \to \mathbb{C}^n$ complex — ψ is a holomorphic function

$\to$ holomorphic atlas

$\to$ Moreover when we construct a manifold by patching we cannot give ψ_{ij} arbitrarily

$\psi_{ij} \circ \psi_{ji} = \mathbb{1}$

$\psi_{ik} \circ \psi_{kj} \circ \psi_{ji} \big|_{U_{ijk}} = \mathbb{1}$



therefore there are constraints!

Are all possible atlases on M equivalent?

Suppose we have two atlases (U_i, φ_i) (V_a, ϕ_a)

both C^k $(C^\infty, C^\omega, \mathcal{O})$

- they are compatible iff $(V_i V_a, \varphi_i \phi_a)$ is still $C^k (C^\infty, C^\omega, \mathcal{O})$

then $(U, \varphi) \sim (V, \phi)$

→ This is an equivalence relation since it is reflexive, symmetric and transitive

→ We can consider the equivalence classes, and within a given class we can construct the maximal atlas as the sum of all atlases in the equivalence class

The equivalence class is a C^k differential structure (variété

Now manifolds of dimension 1,2,3 have a unique differential structure and therefore there is no difference between topo and diff manifold

But Milnor showed in 1956 that there exist exotic 7-spheres

An exotic sphere is a differentiable manifold M which is homeomorphic but not diffeomorphic to the usual S^n

(There are 28 exotic 7-spheres which form a cyclic group under the operation of connected sum)

And there are ∞ differential structures on $\mathbb{R}^4$!

(Locally they are all equal since diffeomorphic to an open)

Finally note that C^k $k>0$ implies the existence of one C^∞ s.t. C^∞ is compatible with C^k (the sum is still C^k) hence it is smoothable!

1) $C^\infty \neq C^\omega$

Consider $f(x) = \begin{cases} x>0 & e^{-\frac{1}{x^2}} \\ x\le 0 & 0 \end{cases}$

$f'(x) = \begin{cases} e^{-\frac{1}{x^2}} \frac{2}{x^3} & x>0 \\ 0 & x\le 0 \end{cases}$ $\qquad \lim_{x\to 0^-} f'(x) = 0$

Moreover $f^{(n)}(x) = e^{-\frac{1}{x^2}} \dfrac{P_{n}(x)}{x^{3n}} \underset{x\to 0^-}{\longrightarrow} 0$

Therefore $f \in C^\infty([-\varepsilon, \varepsilon])$

But let us compute the Taylor series around $x=0$

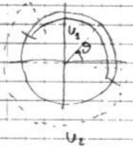
$f_T(x) = \sum \frac{f^{(n)}(0)}{n!} x^n \equiv 0$

which is absolutely convergent but not to f!

Therefore $f \notin C^\omega$

Differently an analytic function with an acc point of zeros is zero

2) The first nontrivial manifold S^1



$S^1 = \{ (U_1, \varphi_1), (U_2, \varphi_2) \}$

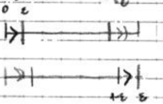
$U_1 \underset{\varphi_1}{\sim}]0,1[\qquad U_2 \underset{\varphi_2}{\sim}]0,1[$

To make a manifold we need to say how we glue U_1 and U_2

Suppose $U_{12} = \;]0,\varepsilon[\;\cup\;]1-\eta, 1[\;\subset U_1$

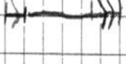
$= \;]0,\eta[\;\cup\;]1-\varepsilon, 1[\;\subset U_2$

Notice U_{12} is <u>NOT</u> connected



$\varphi_{12}(x_{(1)}) = x_{(2)} = \begin{cases} x_{(1)} + 1 - \varepsilon & \begin{cases} 0 \to 1-\varepsilon \\ \varepsilon \to 1 \end{cases} \\ x_{(1)} - 1 + \eta & \begin{cases} 1-\eta \to 0 \\ 1 \to \eta \end{cases} \end{cases}$

• Notice ?



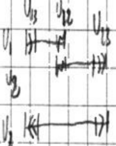
$$\varphi_{12}(x_{(1)}) = x_{(2)} = \begin{cases} -x_{(1)}+1 \\ x_{(1)} - 1 + \eta \end{cases}$$

$0 \to 1$
$\varepsilon \to 1-\varepsilon$

should not be a manifold

since $\varphi_{12}' = \begin{cases} -1 \\ 1 \end{cases}$ is not well defined

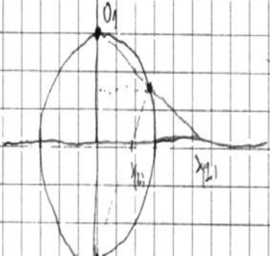
• With 3 charts

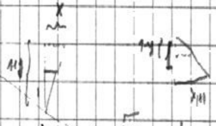


U_{ij} are connected

• It is possible to use stereographic projection

$U_{(1)} = S^1 - \{O_2\}$ $\varphi_1(U_1) = \mathbb{R}$; $U_{(2)} = S^1 - \{O_2\}$ $\varphi_2(U_2) = \mathbb{R}$

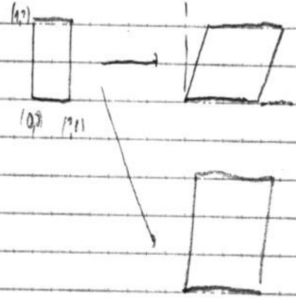




$$\frac{x_{(2)}}{1} = \frac{x}{1+y} = \sqrt{\frac{1-y}{1+y}} \quad 1/x_{(1)} = \frac{1-y}{x} \cdot \frac{1+y}{\sqrt{1-y^2}} = \sqrt{\frac{1-y}{1+y}}$$

$$x_{(2)} x_{(1)} = \frac{x}{1+y} \, \frac{x}{1-y} = \frac{x^2}{1-y^2} = 1$$

Torus and complex structure



$z = x + \tau y \qquad \tau_y = \mathrm{Im}\,\tau > 0$

$w = x + \sigma y$

$$w = z + (\sigma - \tau) y = z + \frac{\sigma - \tau}{2i\tau_y}(z - \bar{z}) \leftarrow \text{not holomorphic}$$

More subtle: $\tau \equiv \tau + 1 \qquad \tau \equiv -\frac{1}{\tau}$

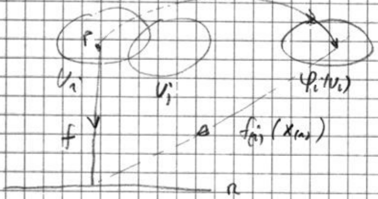
Functions on Manifolds

A scalar function on a manifold is a function

$$f: M \to \mathbb{R}$$
$$p \in M \mapsto f(p) \in \mathbb{R}$$

How can we represent it?

$p \equiv (x_{(i)}(p)) \in \mathbb{R}^m$ i.e. $(x_{(i)}^{\mu}(p))$ $\mu = 1, \dots, m$



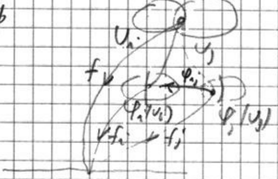
$$f_{(i)} \stackrel{df}{\equiv} f \circ \varphi_{(i)}^{-1} : \varphi_i(U_i) \longrightarrow \mathbb{R}$$

but $f_{(j)} = f_{(i)} \circ \varphi_{ij}$ $\quad p \in U_{ij} \neq \emptyset$

$= f \circ \varphi_{(j)}^{-1} = f \circ \varphi_{(i)}^{-1} \circ (\varphi_{(i)} \circ \varphi_{(j)}^{-1})$

Explicitly

$$f_{(j)}(x_{(j)}) = f_{(i)}(\varphi_{ij}(x_{(j)}))$$



Practically one gives $\{f_{(i)}\}$

Notice there are usually few globally well defined holomorphic functions!

The problem is the patching and the asymptotic behaviour.

Ex Take $S^2 = \{(U_N, z_{(N)}), (U_S, z_{(S)})\}$ $\quad \varphi_{NS}(z_{(S)}) = \frac{1}{z_{(S)}} = z_{(N)}$

$f_{(N)}(z_{(N)}) = a_0 + a_1 z_{(N)} + a_2 z_{(N)}^2 + \dots$ in $\mathbb{C} \simeq S^2 - \{S\}$

in $U_{NS} = U_N \cap U_S = \mathbb{C} - \{0\} = S^2 - \{S, N\}$

$f_{(S)}(z_{(S)}) = f_{(N)}(\varphi_{NS}(z_{(S)})) = a_0 + a_1 \frac{1}{z_{(S)}} + a_2 \frac{1}{z_{(S)}^2} + \dots$

Regularity in U_S $\Rightarrow$ $a_1 = a_2 = \dots = 0$

THE ONLY COMPLEX FUNCTIONS ARE CONSTANTS!

BUT consider the real functions

$$f_{(x_0,y_0;\varepsilon)}\binom{\xi}{\eta} = \begin{cases} \exp\left\{-\dfrac{1}{(x_{(m)}-x_0)^2+(y_{(m)}-y_0)^2-\varepsilon^2}\right\} & |x_{(m)}-x_0|^2+|y_{(m)}-y_0|^2<\varepsilon^2 \\ 0 & \end{cases}$$

and $f_{(x_0,y_0;\varepsilon)}(z) = \begin{cases} \exp\{- \dots \\ 0 \end{cases}$

They are not C^ω but C^∞ and there are infinitely many

- One can construct also the functions

$$p_{(x_0,y_0)(m)}\binom{x}{y} = \begin{cases} e^{\frac{1}{\varepsilon^2}} - f_{(x_0,y_0;\varepsilon)}\binom{x}{y} & (x_{(m)}-x_0)^2+(y_{(m)}-y_0)^2<\varepsilon \\ 1 & \end{cases}$$

with $p_{(x_0,y_0)}(x_0,y_0)=0$ and $p\neq 0$ otherwise

- It is also important to study the real, bounded (w.r.t the supremum norm $\|f\| = \sup f$) functions on a manifold M.

These functions forms a

Now by the Banach-Stone theorem we can consider this algebra instead of the manifold since there is a 1-1 correspondence between algebras and compact Hausdorff spaces.

This is the basic idea of today's algebraic geometry.

$f,g \in C_b^0(M)$ → $f\cdot g$, $f\cdot g$ is a ring
$\quad$ $0\cdot f$ $\;$ $1\cdot f = f$

→ $\lambda f + \mu g$ is a vector space

Intermezzo on algebraic structures

- Group

  $(G,+)$ usual notation if abelian

  $(G,\cdot)$

  ex $(\mathbb{Z}_p,+)$

  $(\mathbb{Z}_p^*,\cdot)$ p prime otherwise $\exists$ zero divisors ex $p=6$ $2\cdot3=6=0 \notin \mathbb{Z}_6^*$

  $SO(2)\simeq U(1)$

  $SU(2)$

- Ring $(R,+)$ is a group, $(R,\cdot)$ is " " (we miss the inverse)

  $a\cdot(b+c) = a\cdot b + a\cdot c$ distributive

  $a\cdot(b\cdot c) = (a\cdot b)\cdot c$ associative

  $\exists 1 \quad a\cdot 1 = 1\cdot a = a$

Almost a field K : if $(K^*,\cdot)$ is a group ie there exists an inverse, for all $a\in K$ but $a=0$.

ex $\mathbb{Z}$

$K[t]$ polynomials

only with a set R

with two sets

- Module $(M,+)$ is a group $(R,+,\cdot)$ is a ring

  $R\times M \to M \qquad (x,v) \mapsto xv \in M$

  s.t. $(x+y)\,v = xv + yv$

  $x(v+w) = xv + xw$

  $(xy)\,v = x\,(yv)$

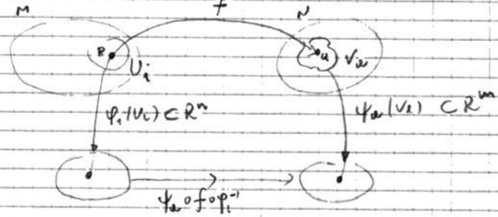
  $1\cdot v = v$

Notice it is almost a vector space, to get a vector space R must be a field

Function between manifolds

- We have seen how we define function $f: M \to \mathbb{R}\ (\mathbb{C})$

- This can be generalized to $f: M \to N$



(eventually $m \neq n$)

$$f_{(a,i)} = \psi_{(a)} \circ f \circ \varphi^{-1}_{(i)} : \varphi_i(U_i) \subset \mathbb{R}^n \longrightarrow \psi_a(V_a) \subset \mathbb{R}^m$$

Again we can give a function f by giving $\{f_{(a,i)}\}$ subject to constraints

$$f_{(b,j)} = \psi_{(ba)} \circ f_{(a,i)} \circ \varphi_{(ij)} \qquad p \in U_{ij} \quad q \in V_{ab}$$

$$\psi_{(b)} \circ f \circ \varphi^{-1}_{(j)} = (\psi_b \psi_a^{-1}) \circ f_{(a,i)} \circ (\varphi_{(i)} \circ \varphi^{-1}_{(j)})$$

- We define a diffeomorphism between M and N a function f s.t. $f_{(a,i)} \in C^k$ $(k > 0)$ $\forall a, i$

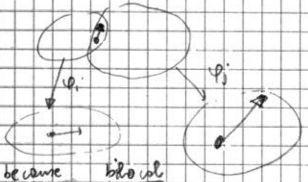
Then we write $f \in C^k(M, N)$

Tangent vectors

In $\mathbb{R}^n$ we have the concept of vector.

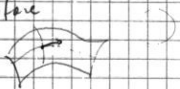
→ When $\mathbb{R}^n$ is considered as an affine space a vector is a difference of points.

This approach does not work with charts



because bilocal

→ A better starting point can be obtained by considering an immersed surface



Here we have the concept of tangent vector which are "infinitesimal" difference of points which is local and locally the manifold looks Euclidean.

– A natural vector is associated to the velocity of a path

$\gamma : [0,1] \longrightarrow M$ at a point $t = t_0$ where γ is at least C^1

– We want to be able to describe this vector without using components.

A good way is to consider the directional derivative

We take a function $f : M \to \mathbb{R}$

then we can compute

$$\frac{d}{dt} f(\gamma(t)) \Big|_{t=t_0}$$

This gives a map from functions to numbers

$$\nabla_\gamma : f \longmapsto \frac{d f(\gamma(t))}{dt}\Big|_{t=t_0}$$

which is equivalent to give a vector.

given a vector v we can compute the directional der. of f i.e. ∇_v and viceversa

Exercise 1

Exercice 1 $S^1 = \{\theta \mid \theta \in [0, 2\pi[\,\}$

$\gamma(t) : \theta = \cos(t^3 + 2t)$

$$\dot{\gamma}\big|_{t=1/2} = -\sin(t^3+2t)\cdot(3t^2+2)\,\partial_\theta\big|_{t=1/2}$$

$$= -\sin\left(1+\tfrac{1}{8}\right)\cdot\left(\tfrac{3}{4}+2\right)\partial_\theta\Big|_{\theta=\cos\left(\frac{1}{8}+1\right)}$$

$$= -\frac{11}{4}\sin\left(\frac{9}{8}\right)\partial_\theta\Big|_{\theta=\cos\frac{9}{8}}$$

Let us give a closer look to the previous approach. (27)

Consider a chart U_i

$\gamma \equiv (\gamma^\mu_{(i)}(t)) \qquad \mu = 1 \dots n$

$f \equiv f_{(i)}(x^\mu_{(i)})$

$$\Rightarrow \frac{d}{dt} f_{(i)}(\gamma^\mu_{(i)}(t))\Big|_{t=t_0} = \frac{d\gamma^\mu_{(i)}}{dt}\Big|_{t=t_0} \frac{\partial f_{(i)}}{\partial x^\mu_{(i)}}\Big|_{\gamma(t_0)}$$

We see that this result depends only on $\dot\gamma|_{t=0}$ and not on γ as a whole.

The vector at $P = \gamma(t_0)$ is therefore an equivalence class

with $\gamma \sim \gamma'$ iff $\gamma(t_0) = \gamma'(t_0) \quad \dot\gamma(t_0) = \dot\gamma'(t_0)$

and it is given by $\frac{d\gamma^\mu_{(i)}}{dt}\Big|_{t=0} = c^\mu_{(i)}$

Actually also for $f_{(i)}$ we can consider an equivalence class

The set of all vectors is the tangent space at p $\quad T_pM$

Notice that 1) $\gamma_{(\lambda)}(\lambda t)$ at $t = \frac{t_0}{\lambda}$ is $\gamma(t_0)$

but $\dot\gamma_{(\lambda)}(t_0) = \lambda\, \dot\gamma(t_0)$

2) How can we define $\dot\gamma_1 + \dot\gamma_2$?

In charts $\gamma^\mu_{1(i)} = x^\mu_0 + v^\mu_1(t-t_0) \qquad \gamma^\mu_{2(i)} = x^\mu_0 + v^\mu_2(t-t_0)$ for small $t - t_0$

we can construct

$$\frac{1}{2}(\gamma_{1(i)} + \gamma_{2(i)})(2t - t_0) = x_0 + (v_1 + v_2)t \qquad \text{at } t=0$$

Or we can use the previous two curves as representatives of the vectors (for small t)

→ What happens if we change chart?

$$c^{\mu}_{(j)} = \left.\frac{d\gamma^{\mu}_{(j)}}{dt}\right|_{t=t_0} = \left.\frac{d\,\psi^{\mu}_{ji}(\gamma_{(i)}(t))}{dt}\right|_{t=t_0} = \left.\frac{\partial \psi^{\mu}_{ji}}{\partial x^{\nu}_{(i)}}\right|_{x_{(i)}=\gamma_{(i)}(t_0)} \left.\frac{d\gamma^{\nu}_{(i)}}{dt}\right|_{t=t_0}$$

$$= \left.\frac{\partial x^{\mu}_{(j)}}{\partial x^{\nu}_{(i)}}\right|_{x_{(i)}=\gamma_{(i)}(t_0)} c^{\nu}_{(i)}$$

notice however that

$$c^{\mu}_{(i)} \frac{\partial}{\partial x^{\mu}_{(i)}} = c^{\mu}_{(i)} \frac{\partial x^{\nu}_{(j)}}{\partial x^{\mu}_{(i)}} \frac{\partial}{\partial x^{\nu}_{(j)}} = c^{\nu}_{(j)} \frac{\partial}{\partial x^{\nu}_{(j)}} = \vec{c}$$

is independent on the chart so we can write

$$\vec{c}_p = c^{\mu}\, \vec{\partial}_{\mu}\big|_p$$

↑ this is a component and it is a number

↑ this is the basis vector like the gradient $\vec{\nabla}$

and we have the action

$$\vec{c}_p(f) = c^{\mu}\partial_{\mu} f\big|_p$$

The set of all possible vectors in a point is the tangent space

$$T_pM \approx \mathbb{R}^n$$

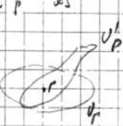
↑ n same dimension as M

→ At a closer look we realize we do not need a function to define $\vec{c}_p$ but only the behavior of the function at p.

We define therefore the germ of a C^{∞} function at p as

given $f \in C^{\infty}(U_p)$ $\quad g \in C^{\infty}(U'_p)$

$$f \sim g \iff f|_{U_p \cap U'_p} = g|_{U_p \cap U'_p}$$



and the space of germs of smooth functions at p as

$$C^{\infty}_p = \frac{\cup_{U_p} C^{\infty}(U_p)}{\sim}$$

→ Finally we define the tangent vector as the action of directional derivative on C^{∞}_p

We have constructed a bilinear function

$$\langle\langle \vec{c}_p, f \rangle\rangle = c^\mu \partial_\mu f_p$$

We cannot however take $C_p^\infty(M)$ as the dual space since we get a degenerate bilinear function since the germs $f \sim f_1$ for which all have $\partial_\mu f|_p = 0$

→ We can however define the differential

$$df_p = dx^\mu_{(\alpha)}\Big|_p \frac{\partial}{\partial x^\mu_{(\alpha)}} f_{(\alpha)}\Big|_p \quad \text{as} \quad df_p(\vec{c}_p) = \langle \vec{c}_p, f \rangle$$

and take it to be the dual

$$\langle \vec{c}_p, df_p \rangle = c^\mu \partial_\mu f_p = \vec{c}(df_p) = df_p(\vec{c})$$

– Remember that the dual of a vector space V is the set of all linear functions defined on V, i.e. $V^* = \{ \varphi \ / \ \varphi(\lambda u + \mu v) = \lambda \varphi(u) + \mu \varphi(v) \}$

→ The set of all differentials df_p is the cotangent space at p $T^*_p M$

→ Change of chart

$$df = dx^\mu_{(\alpha)}\Big|_p \frac{\partial f_{(\alpha)}}{\partial x^\mu_{(\alpha)}}\Big|_p = dx^\mu_{(\alpha)}\Big|_p \frac{\partial x^\nu_{(\beta)}}{\partial x^\mu_{(\alpha)}}\Big|_p \frac{\partial f_{(\beta)}}{\partial x^\nu_{(\beta)}}\Big|_p \qquad p \in U_{\alpha\beta}$$

in $U_{\alpha\beta}$: $f_{(\alpha)} = f_{(\beta)} \circ \psi_{(\beta\alpha)}$, i.e. $f_{(\alpha)}(x_{(\alpha)}) = f_{(\beta)}(x^{(\beta)}(x^{(\alpha)}))$

$$\Rightarrow dx^\mu_{(\beta)}\Big|_p = dx^\nu_{(\alpha)}\Big|_p \frac{\partial x^\mu_{(\beta)}}{\partial x^\nu_{(\alpha)}}\Big|_p$$

→ basis of the $T_p M$ and $T^*_p M$

$$\begin{array}{cc} \partial_\mu|_p & dx^\nu|_p \\ \updownarrow & \updownarrow \\ e_\mu & e^{*\nu} \\ \in & \in \\ V & V^* \end{array} \qquad e^{*\nu}(e_\mu) = \delta^\nu_\mu \qquad \text{but also} \quad e_\mu(e^{*\nu}) = \delta^\nu_\mu$$

$$(V^*)^* = V$$

Change of basis

$$e'_\mu = \Lambda_\mu{}^\nu e_\nu \rightarrow e'^{\mu *} = (\Lambda^{-1})^\mu{}_\nu e^{\nu *}$$

We can now define tensors

Tensors (n,m) are multilinear applications on the product space $V^{*n} \underset{\times}{\otimes} V^{m}$

I.e.

$$T : \underbrace{V^* \times \ldots \times V^*}_{n} \times \underbrace{V \times \ldots V}_{m} \to \mathbb{R}$$

Given a basis we can compute the components

$$T(e^{*i_1}, \ldots, e^{*i_n}, e_{j_1} \ldots e_{j_m}) = T^{i_1 \ldots i_n}_{\quad j_1 \ldots j_m}$$

This generalizes the idea of a matrix as a bilinear application

- What happens if we change basis

$$\tilde{e}_i = \Lambda_i^{\ j} e_j \quad \Rightarrow \quad \tilde{e}^{*i} = (\Lambda^{-1})^i_{\ j} e^{*j}$$

then

$$\tilde{T}^{i_1 \ldots i_n}_{\quad j_1 \ldots j_m} = T(\tilde{e}^{*i_1}, \ldots, \tilde{e}^{*i_n}, \tilde{e}_{j_1} \ldots \tilde{e}_{j_m}) = \Lambda^{-1\,i_1}_{\ \ k_1} \ldots (\Lambda^{-1})^{i_n}_{\ k_n} \Lambda_{j_1}^{\ l_1} \ldots \Lambda_{j_m}^{\ l_m} T^{k_1 \ldots k_n}_{\quad l_1 \ldots l_m}$$

This is the "old" way of defining a tensor.

- Which is a basis for $\mathcal{T}^n_m$?

For applications on $V \rightarrow V^*$ $\quad e^{*i}$

For applications on $V^* \rightarrow V^{**} \simeq V$ $\quad e_i$

For applications on $V \times V$?

We can define $e^{*i} \otimes e^{*j}$ as

$$(e^{*i} \otimes e^{*j})(u, v) = e^{*i}(u) \otimes e^{*j}(v) = u^i v^j$$

then $T = \sum_{i,j} T_{ij} \, e^{*i} \otimes e^{*j}$

generalize

$$T = T^{i_1 \ldots i_n}_{\quad j_1 \ldots j_m} e_{i_1} \otimes \ldots \otimes e_{i_n} \otimes e^{*j_1} \otimes \ldots \otimes e^{*j_m}$$

- Operations

0) $T_1 \in \mathcal{T}^m_n \quad T_2 \in \mathcal{T}^r_s \rightarrow T_1 \otimes T_2 \in \mathcal{T}^{m+r}_{n+s}$

1) contraction $\sum_l T^{\ldots l \ldots}_{\ldots l \ldots} \Rightarrow \sum_l T(\ldots; e^{*l}, \ldots, e_l, \ldots)$

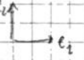
invariant for basis change

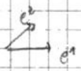
3) forms $\omega \in \Lambda^m V$ antisymmetric $\quad \omega(v_1, v_l, v_k, v_m) = -\omega(\ldots, v_k, v_l, \ldots)$

h) Check Consider $\mathbb{R}^2$ e^{x1}, e^{x2}

compute $\Omega = e^{x1} \wedge e^{x2}$ $\Omega(\vec{v},\vec{w}) = \frac{1}{2}(e^{x1}\otimes e^{x2} - e^{x2}\otimes e^{x1})(\vec{v},\vec{w})$

$= \frac{1}{2}(v^1 w^2 - v^2 w^1) = \frac{1}{2} |\vec{v}\times\vec{w}|$

It looks like the area if 

but if  it is not!

We need the concept of metric!

NOTICE

Mind that $E\otimes F$ has dimension $\dim E \cdot \dim F = n \cdot m$, in particular

$v = t^{ai} e_a \otimes f_i$

where $t^{ai} \neq v^a u^i$ generally since $\# t^{ai} = nm$ while $\# v^a u^i = n+m$

It is like the pure states in the density matrix

On the other side $E \oplus F$ has dimension $n+m$

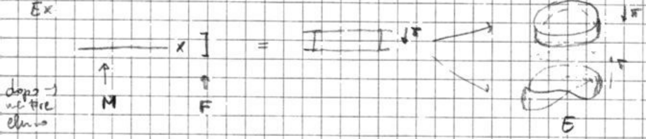
$v = v^a e_a \oplus u^i f_i = v^a (e_a \oplus 0) + u^i (0 \oplus f_i)$

We have constructed $\forall p \in M$ T_pM can we glue together all these vector spaces?

The problem is to glue together something which locally looks like $U_i \times R^n$ since $T_pM \approx R^n$

The solution is the concept of fibre bundle

Ex



A fibre bundle is the following collections of requirements

1) A topological space E (the total space)
2) A topological space M (the base)
3) A topological space F the fibre
4) A projection $\pi : E \to M$
5) a set $\{(U_\alpha, \phi_\alpha)\}$ where U_α is a covering of X

ϕ_α is a homeomorphism (local trivialisations)

$$\phi_\alpha : \pi^{-1}(U_\alpha) \subset E \to U_\alpha \times F$$

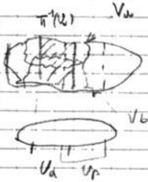
s.t. $\pi \, \phi^{-1}_\alpha (x,f) = x$ $\quad x \in U_\alpha \quad f \in F$

Can be written as $E \xrightarrow{\pi} M$

→ (6) see later

→ A topological group is a group with a topology s.t. the internal operation is continuous wrt the topology.

V_a charts of E ; U_α charts of X (before it was M)



$$V_{\alpha a} = \pi^{-1}(U_\alpha) \cap V_a \qquad V_{\alpha b} = \pi^{-1}(U_\alpha) \cap V_b$$

$$\Psi_a : V_{\alpha a} \to (y_{(a)}) \in \mathbb{R}^{n+f} \longrightarrow (y_{(b)}) \in \mathbb{R}^{n+f}, \quad \Psi_b : V_{\alpha b} \to (y_{(b)})$$

$y_{(b)} = \Psi_{ba}(y_{(a)})$ trans funct in E

ϕ_α

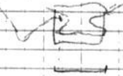
trivialization implies the existence of "good" coordinates $(y_{(a)}) \mapsto (X, f)$

$\Longrightarrow (X_{(\alpha a)}, f_{(\alpha a)})\big|_{U} = (x_{(\alpha)}, f_{(\alpha)}) = (x_{(\alpha)}, f_{(\alpha)})$ ↑ since it depends only on ϕ_α

Relation between "good" and "bad" coordinates

?→ $(y_{(a)}) = \Psi_a^{-1} \circ \phi_\alpha^{-1}(x_{(\alpha)}, f_{(\alpha)}) \qquad (x, f) \in \pi(\pi^{-1}(U_\alpha) \cap V_a)$

but V_a and $\pi(f) \notin V_a$



→ ↝ A is better

$$(x_{(\alpha)}, f_{(\alpha)}) = \phi_\alpha \circ \Psi_{(a)}^{-1}(y_{(a)})$$

if $P \in V_a \cap V_b \cap \pi_\alpha^{-1}(U_\alpha) \qquad (x_{(\alpha)}, f_{(\alpha)}) = \phi_\alpha \circ \Psi_a^{-1}(y_{(a)}) - \phi_\alpha \circ \Psi_b^{-1}(y_{(b)})$

this trivializes the y coordinates

→ now consider $U_\alpha \cap U_\beta \neq \emptyset$

if $P \in V_a \cap \pi_\alpha^{-1}(U_{\alpha\beta}) \qquad P \in V_a \cap \pi_\beta^{-1}(U_{\alpha\beta})$

$(x_{(\alpha)}, f_{(\alpha)}) = \phi_\alpha \circ \Psi_a^{-1}(y_{(a)}) \qquad (x_{(\beta)}, f_{(\beta)}) = \phi_\beta \circ \Psi_a^{-1}(y_{(a)})$

in the base we have $x_{(\beta)} = \varphi_{\beta\alpha}(x_{(\alpha)})$

and considering all V_a s.t. $\bigcup V_a \supset \pi^{-1}(U_\alpha)$ we can write

$$f_{(\beta)} = \varphi_{\beta\alpha}(x_{(\alpha)}, f_{(\alpha)})$$

↑ it depends on both x and f

We can now add requirement 6

6) A topological group G of homeomorphisms (i.e. at least C^0) on the fibre

- A G-atlas is a local trivialization $\{(U_\alpha, \phi_\alpha)\}$ s.t. $U_{\alpha\beta} \neq \emptyset$ $\quad \phi_\beta \circ \phi_\alpha^{-1} : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$

is given by

$$\phi_\beta \phi_\alpha^{-1} (x_{(\alpha)}, f_{(\alpha)}) = (\psi_{\beta\alpha}(x_{(\alpha)}), t_{\beta\alpha}(x_{(\alpha)}) f_{(\alpha)}) = (x_{(\beta)}, f_{(\beta)})$$

where $\quad t_{\beta\alpha}(x_\alpha) \in G$

and $\quad t_{\beta\alpha} : U_{\alpha\beta} \to G$ is continuous

$\longrightarrow$ Esempio!

- Two G-atlas $\{(U_\alpha, \phi_\alpha)\}$, $\{(U'_\alpha, \phi'_\alpha)\}$ are equivalent iff $\{(U_\alpha, \phi_\alpha), (U'_\alpha, \phi'_\alpha)\}$ is also a G-atlas

(as for the differential atlases)

$\rightsquigarrow$ ϕ, t can be $C^0, C^k, C^\infty, C^\omega$, holomorphic ecc

- A G-bundle is a bundle with an equivalence class of G-atlases (as for a differential variety)

- The group G is called structure group.

When G is a Lie group it is the "gauge group"

$\to$ and t are gauge transformations

- We can define the bundle by giving $(U_\alpha, \varphi_\alpha)$ to define the base M and $t_{\beta\alpha}$ to glue the fibers $U_\alpha \times F$

BUT we have again some consistency conditions

$t_{\alpha\alpha} = 1$

$t_{\alpha\beta} = t_{\beta\alpha}^{-1}$

$t_{\alpha\beta} t_{\beta\gamma} t_{\gamma\alpha} = 1 \quad \Leftrightarrow \quad t_{\gamma\alpha}(x_{(\alpha)}) = t_{\gamma\beta}(\psi_{\beta\alpha}(x_{(\alpha)}))\, t_{\beta\alpha}(x_{(\alpha)}) \qquad x_{(\alpha)} \in U_{\alpha\beta\gamma}$

cocycle condition.

A principal bundle is one such that $F = G$ (Yang Mills)

A vector bundle is one with $F = V$ = vector space and $G = \mathrm{Hom}(V,V)$

esempio $\qquad \qquad \mathbb{Z}_2 \mathbb{R}^n$

• An associated vector bundle in $F = V$ = vector space seen as representation ~~end of~~ of G

"V" is matter and G is the gauge group

Notice a representation of a group G on a vector space is a map

$$g \in G \longmapsto D(g) \in \mathrm{Hom}(V,V)$$

s.t.

$$D(g_1 g_2) = D(g_1)\, D(g_2)$$

$$D(e) = \mathbb{1}$$

so that $\quad D(g^{-1}) = D(g)^{-1}$

→ D can be represented as a matrix

→ D is irreducible iff there is not a proper invariant subspace

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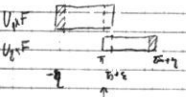
Non Abelian [illegible] fields YM [illegible] $U(1)$

Ex Moebius

$X = S^1 = \{(U_1, \varphi_1), (U_2, \varphi_2)\}$

$F = [-1, 1]$

$G = \mathbb{Z}_2 = \{-1, 1\}$ topo discrete



trivialization $t_{21} = 1$ $\qquad \phi_{21}(\theta_{(1)}, f_{(1)}) = (\theta_{(1)}, f_{(1)}) = (\theta_{(2)}, f_{(2)})$

trivialization $t_{22} = -1$

$\phi_{21}(\theta_{(1)}, x_{(1)}) = (2\pi + \eta + \theta_{(1)}, -f_{(1)})$

Klein

$X = S^1 \to \theta$

$F = S^1$ $\alpha \in [0, 2\pi[$

$G = \mathbb{Z}_2$

$U_{12} = [-\eta, 0] \cup [\pi, \pi+\varepsilon]$

$\phi_{21}(\theta_{(1)}, \alpha_{(1)}) = (2\pi + \eta + \theta_{(1)}, 2\pi - \alpha_{(1)})$ $\qquad$ the same as before with the identification $[0,1] \sim S^1$

(1) We can now define the tangent bundle TM and the cotangent T^*M

Locally we have $:- T_pM \cong \mathbb{R}^n = F$

The trivialization is given by

$(U_\alpha, \phi_\alpha) \rightsquigarrow p \in TM|_{U_\alpha} \quad p = (x_{(\alpha)}, \vec{v}_{(\alpha)})$

The transition functions are simply

$$t_{\beta\alpha}(x_{(\alpha)}) = \left\| \frac{\partial x^\nu_{(\beta)}}{\partial x^\mu_{(\alpha)}} \right\| \in GL(n, \mathbb{R})$$

as follows from the change of coordinates

$$v = v^\mu_{(\alpha)} \partial_{\mu(\alpha)} = v^\nu_{(\beta)} \partial_{\nu(\beta)} \Rightarrow v^\nu_{(\beta)} = \frac{\partial x^\nu_{(\beta)}}{\partial x^\mu_{(\alpha)}} v^\mu_{(\alpha)}$$

hence $G = GL(n, \mathbb{R})$

Obviously $t_{\beta\alpha}$ satisfy the cocycle conditions

- The coordinates on TM are therefore $(x, \vec{v})$

The Lagrangian is a scalar function defined on TM $\quad L = L(x, \vec{v})$

(2) In a similar way I can define T^*M

ex find $t_{\beta\alpha}$

(3) It is also possible to define $(TM)^p \otimes (T^*M)^q$

↳ More generally we can define a product of bundles $E_1 \otimes E_2$ on the same X as the bundle with fiber $F_1 \otimes F_2$ and transition functions $t_{\alpha\beta} = t_{\alpha\beta 1} \otimes t_{\alpha\beta 2}$ (less trivial than expected: discussed in Whitney's Spaces and ensemble a new)

(4) We can also define the Whitney sum $E_1 + E_2$ as the bundle with fiber $F_1 \oplus F_2$ and $t_{\alpha\beta} = t_{\alpha\beta 1} \oplus t_{\alpha\beta 2} = \begin{pmatrix} t_{\alpha\beta 1} & \\ & t_{\alpha\beta 2} \end{pmatrix}$

The non trivial part of these operations is due to the transition functions and the same base

(5) frame bundle $F(M)$

(5) We define the tangent frame bundle $F(M) \xrightarrow{\text{usually}} F(M)$ is an ordered basis of T_p for each point p

Sections of bundles

The idea: for any $x \in X$ we associate a point in the fibre.

- Locally it is clear. Globally one must be more careful.

Def a section is a map $s: X \to E$ s.t.

$\pi(s(x)) = x$

For a vector bundle $s(x) = s^a(x)\, \vec{f}_a(x)$ $\vec{f}_a$ is a basis as in the $F(M)$

- Again s can be $C^0, C^k, C^\infty, C^\omega, \ldots$ when $s: X \to E$ is $C^0 \ldots$
  i.e. $f = f(x)_{(x)}^{(a)}$ is $C^0 \ldots$

- The smooth sections are denoted as $\Gamma(E, X)$

Ex Holom sections of $X = S^2$ · $z_{(S)} = 1/z_{(N)}$

$E = TS^2 \Rightarrow$ vector fields

$$v = v_{(N)} \frac{\partial}{\partial z_{(N)}} = - v_{(N)} z_{(S)}^2 \frac{\partial}{\partial z_{(S)}} \Rightarrow v_{(S)} = - z_{(S)}^2 v_{(N)}\left(z_{(N)} = \frac{1}{z_{(S)}}\right)$$

$$\uparrow \psi_{(SN)} = - z_{(S)}^2$$

Regularity $\Rightarrow$ $v_{(N)}(z_{(N)}) = \sum_{n=0}^{\infty} c_n z_{(N)}^n$

$$\Rightarrow v_{(S)}(z_{(S)}) = - z_{(S)}^2 \sum_0^{\infty} \frac{c_n}{z_{(S)}^n} \Rightarrow n = 0, 1, 2$$

$E = (TS^2)^2$ $\qquad \psi_{(SN)} = z_{(S)}^4$

$$v = v_{(N)} \frac{\partial}{\partial z_{(N)}} \otimes \frac{\partial}{\partial z_{(N)}} = v_{(N)} z_{(S)}^4 \frac{\partial}{\partial z_{(S)}} \otimes \frac{\partial}{\partial z_{(S)}}$$

Regularity $\Rightarrow$ $v_{(S)}(z_{(S)}) = z_{(S)}^4 \sum_0^{\infty} \frac{c_n}{z_{(S)}^n} \Rightarrow n = 0, 1, 2, 3, 4$

$E = TS^2 \otimes T^*S^2$

$$v = v_{(N)} \frac{\partial}{\partial z_{(N)}} \otimes dz_{(N)}$$

To introduce the next argument notice that

$$v = (\alpha + \beta z + \gamma z^2) \frac{\partial}{\partial z}$$

generates infinitesimally $z' = z + \alpha + \beta z + \gamma z^2$

Now $z' = \frac{(1+a)z + b}{cz + 1 + d} \sim [(1+a)z + b][1 - d - cz] = (1+a)(1-d)z - c(1+a)z^2 + b(1-d) - bcz$

$$\sim z + b + (a - d)z - cz^2$$

$b = \alpha \qquad a - d = \beta \qquad -c = \gamma$

Vector fields, Algebra of vector fields and Lie derivative

Consider the section $\vec{v} \in \Gamma(TM, M) \equiv \mathrm{Diff}(M)$

We can consider $\vec{v}\varepsilon$ or $\vec{v}$ as the generator of an infinitesimal change of coordinates or diffeomorphism;

in a patch U_α $x^\mu_{(\alpha)}$

$$\phi^\mu_{(\alpha)}(x_{(\alpha)}, t) = x'^\mu_{(\alpha)} = x^\mu_{(\alpha)} + v^\mu_{(\alpha)}(x_{(\alpha)})\, t \qquad |t| \ll 1$$

notice $\left|\frac{\partial\phi}{\partial x}\right| = |1 + \partial v t| \sim 1$ so in 1-1!

The $\phi^\mu_{(\alpha)}$ can be also described as

$$\frac{d\phi^\mu_{(\alpha)}(x,t)}{dt} = v^\mu_{(\alpha)}(x,t) \qquad \phi^\mu_{(\alpha)}(x, t=0) = x^\mu_{(\alpha)}$$

→ Example: $\phi(x_0, t) = \gamma_{x_0}(t)$, $\frac{d\gamma_{x_0}}{dt} = v(x = \gamma_{x_0}(t), t)$

How can we write this in a coordinate independent way?

- Given any function f

$$\vec{v}(f) = v^\mu_{(\alpha)} \partial_\mu f = \frac{d\phi^\mu_{(\alpha)}}{dt} \partial_\mu f = \frac{df(\phi(x,t))}{dt}$$

i.e. $$\frac{df(\phi(x,t))}{dt} = \vec{v}(f)\Big|_{\phi(x,t)}$$

- Given a vector field $X = X^\mu_{(\alpha)} \partial_{\mu(\alpha)}$

we can ask what happens under the diffeo ϕ_t

It's probably clearer to consider a diffeo $\phi: M \to N$, i.e. $y^m_{(a)} = \phi_{(a)}(x^\mu_{(\alpha)})$

Suppose we have $X \in TM$ $\quad X = X^\mu_{(\alpha)} \partial_{\mu(\alpha)}$

then it is natural to consider $\left[X^\mu_{(\alpha)}(x) \frac{\partial\phi^m_{(a)}(x)}{\partial x^\mu_{(\alpha)}}\right] \frac{\partial}{\partial y^m_{(a)}} \in TN$

This vector field is well defined on N I can write $x = x(y) = \phi^{-1}(y)$

so if we can invert ϕ

we can define the push-forward $\phi_*: TM \to TN$ (it acts exactly in the same direction as $\phi: M \to N$) and $\phi_*(X)$ for the previous vector TM

Notice if I cannot invert ϕ we do not get a vector field on N but a vector on $\phi_*(M) \subset N$, e.g. $i: S^2 \hookrightarrow \mathbb{R}^3$ a vector field on S^2 yields a vector field on $i(S^2) \subset \mathbb{R}^3$ but not on $\mathbb{R}^3$

Ex 2) $\mathbb{R}^2 - \{0\}$ with coordinates (r,θ)

$v = 2\partial_\theta$

$$\begin{cases} \frac{d\phi^r(r,\theta,t)}{dt} = 0 \\ \frac{d\phi^\theta(r,\theta,t)}{dt} = 2 \end{cases} \qquad \text{s.t. } (\phi^r(r,\theta,t=0), \phi^\theta(r,\theta,t=0)) = (r,\theta) \quad \forall r,\theta$$

$\Rightarrow \phi^r(r,\theta,t) = r$

$\phi^\theta(r,\theta,t) = \theta + 2t \mod 2\pi$

Ex 3) $v = x\partial_y - y\partial_x = x^1\partial_2 - x^2\partial_1$

$$\begin{pmatrix} \dot\gamma^1 \\ \dot\gamma^2 \end{pmatrix} = \begin{pmatrix} -\gamma^2 \\ \gamma^1 \end{pmatrix} \qquad \gamma^\mu(0) = (x_0, y_0)$$

$\ddot\gamma^1 + \gamma^1 = 0$

$\gamma^1(x,t) = \sqrt{x^2+y^2}\cos(t + \varphi(x^i))$

$\gamma^2(x,t) = \sqrt{x^2+y^2}\sin(t + \varphi(x^i))$

Ex 1) $v = x\partial_x$

$$\frac{d\phi(x,t)}{dt} = v(x) = x$$

1st try $\phi(x,t) = x(1+t)$ $\quad \phi(x,t=0) = x$

BUT fixed $x = x_0$ the curve $C(x_0,t) = x_0(1+t)$

has $v = x_0\partial_x|_{\gamma(t)}$ $\forall t \Rightarrow \forall x \in C$

SO it is wrong!

2nd try $\gamma_{x_0}(t) \to \frac{d\gamma_{x_0}}{dt} = v|_{\gamma_{x_0}}$

$\gamma_{x_0}(t=0) = x_0$

$\dot\gamma(t) = \gamma(t) \qquad \gamma(t) = x_0 e^t$

This is a consequence of applying N times the ...

$$x \to (1+\tfrac{t}{N})x \to (1+\tfrac{t}{N})^2 x \to \quad (1+\tfrac{t}{N})^N x = e^t x$$

In our case $\phi_t : M \to M$ and we can therefore compute the push-forward

$$\phi_{t*} X(y) = X^\mu_{(x)}(x) \frac{\partial \phi^\nu_{t(x)}}{\partial x^\mu_{(x)}}\Big|_{x=\phi_t^{-1}(y)} \frac{\partial}{\partial x^\nu}\Big|_{y} \sim X^\mu_{(x)}(y_{(x)} + v_{(x)} t)(\delta^\nu_\mu - \partial_\mu v^\nu_{(x)} t)\partial_\nu \quad \text{Here } y = x + vt$$

Then if we take the $\frac{d}{dt}\Big|_{t=0}$ we get

$$\frac{d}{dt}\phi_{-t*} X(y)\Big|_{t=0} = \frac{d}{dt}\Big|_{t=0}\Big[X^\mu_{(x)}(y_{(x)} + v_{(x)} t)\,(\delta^\nu_\mu - t\partial_\mu v^\nu_{(x)})\Big] \frac{\partial}{\partial y^\nu_{(x)}}\Big|_y$$

$$= \left[+ v^\lambda_{(x)} \partial_\lambda X^\mu_{(x)} \delta^\nu_\mu - X^\mu_{(x)} \partial_\mu v^\nu_{(x)}\right] \partial_\nu|_y$$

$$= \left[-X^\lambda \partial_\lambda v^\nu + v^\lambda \partial_\lambda X^\nu\right] \partial_\nu|_y$$

$$= [v, X]^\nu \partial_\nu|_y$$

→ Therefore the vector fields form a Lie algebra where the "product" is given by the commutator $[X, v]$

The same result can be obtained in a simpler way as

$[\partial_X, \partial_v] = \partial_X \partial_v - \partial_v \partial_X = \partial_{[X,v]}$ $\Rightarrow$ $\partial_X \partial_v - \partial_v \partial_X - \partial_{[X,v]} = 0$ "curvature"!

- Consider now a differential one-form $\omega = \omega_{\mu(x)} dx^\mu_{(x)}$

We can proceed as before and start from $\phi : M \to N$

If we want to use ϕ and not ϕ^{-1} we need to start with the $\omega \in T^*N$ $\quad \omega = \omega_{m(y)} dy^m_{(y)}$

so that $\left[\omega_{m(x)}(y=\phi(x)) \frac{\partial \phi^m_{(x)}}{\partial x^\mu_{(x)}}\right] dx^\mu_{(x)} =$

Now we can avoid asking ϕ admits an inverse!

and the map we have got the pull-back goes in the opposite direction $\quad \phi_* : T^*N \to T^*M \quad (\phi : M \to N)$

As before

$$\phi_t \omega = \omega_{\mu(x)}(\phi_t(x)) \frac{\partial \phi^\mu_t}{\partial x^\nu_{(x)}} dx^\nu_{(x)} \sim \omega_{\mu(x)}(x_{(x)} + v_{(x)} t)(\delta^\mu_\nu + \partial_\nu v^\mu_{(x)} t) dx^\nu_{(x)}$$

hence

$$\frac{d}{dt}\phi_t \omega\Big|_{t=0} = v^\nu \partial_\nu \omega_\mu \, dx^\mu + (\partial_\nu v^\mu)\omega_\mu \, dx^\nu$$

$$= \left[\partial_\nu(v^\lambda \omega_\lambda) + v^\lambda(\partial_\lambda \omega_\nu - \partial_\nu \omega_\lambda)\right] dx^\nu$$

$$= d(i_v \omega) + i_v d\omega \quad = \mathcal{L}_{\vec v}\, \omega =$$

We have defined the Lie derivative $\mathcal{L}_{\vec v}$

$f : M \to N$
$x \mapsto y = f(x)$

$\phi_t : M \to M$
$x \mapsto y = x + vt$

(42.1)

$f^* : T^*N \to T^*M$
$\omega(y) \mapsto \omega(y(x))$

ϕ_t^*
$\omega(y) \mapsto \omega(x+vt)$

$f_* : TM \to TN$
$V(x) \mapsto V(x = x^{-1}(y))$

ϕ_{t*}
$V(x) \mapsto V(x - vt)$

hence we use

ϕ_t^* but ϕ_{-t*}

other way

$(\phi_t^* \omega)(\phi_{-t}(x)) \in T^*_{\phi_{-t}(x)} M$

$(\phi_t^* \omega)(y) \in T_y^* M$

$(\phi_{-t*} V)(y) \in T_y M$

Here we have introduced the new symbols/operator $\wedge$,

Given two 1-forms $\omega, \chi \in \Omega^1$ we define

$$\omega_{(1)} \wedge \chi_{(1)} = \omega_{\mu(1)} \chi_{\nu(1)} \, dx^{\mu}_{(1)} \wedge dx^{\nu}_{(1)} =$$ where $dx^\mu \wedge dx^\nu = - dx^\nu \wedge dx^\mu$

$$= \frac{1}{2} \left(\omega_\mu \chi_\nu - \omega_\nu \chi_\mu \right) \, dx^{\mu}_{(1)} \wedge dx^{\nu}_{(1)}$$

$$= \frac{1}{2} \omega_{\mu\nu\,(2)} \, dx^{\mu}_{(1)} \wedge dx^{\nu}_{(1)}$$ $\omega_{\mu\nu} = - \omega_{\nu\mu}$

which is a 2-form

More generally we can start from a K-form $\omega_{(K)} \in \Omega^K$ and a l-form $\chi_{(l)}$

$$\omega_{(K)} = \frac{1}{K!} \omega_{\mu_1 \dots \mu_K} \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_K}$$ $\omega_{\mu_1 \dots \mu_K} = \omega_{[\mu_1 \dots \mu_K]}$ totally antisymmetric

$$\chi_{(l)} = \frac{1}{l!} \chi_{\nu_1 \dots \nu_l} \, dx^{\nu_1} \wedge \dots \wedge dx^{\nu_l}$$

and define

$$\omega_{(K)} \wedge \chi_{(l)} = (-)^{K \cdot l} \chi_{(l)} \wedge \omega_{(K)}$$

$$= \frac{1}{K!\, l!} \omega_{\mu_1 \dots \mu_K} \chi_{\nu_1 \dots \nu_l} \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_K} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_l}$$

$$= \frac{1}{(K+l)!} \sum_{\substack{\text{partitions} \\ \lambda_1 \dots \lambda_{K+l} \\ \text{into } K+l}} \omega_{\lambda_1 \dots \lambda_K} \chi_{\lambda_{K+1} \dots \lambda_{K+l}} \, dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_{K+l}}$$

$\leftarrow$ there are $\frac{(K+l)!}{K!\, l!}$ partitions

Notice that if $K+l > \dim M$ the result is zero!

- We are now ready to define $d : \Omega^p \to \Omega^{p+1}$ Notice $\Omega^0 \equiv$ functions globally defined in a patch as

$$d\, \omega_{(K)} = \frac{1}{K!} dx^\nu \wedge \partial_\nu \omega_{\mu_1 \dots \mu_K} \, dx^{\mu_1} \dots dx^{\mu_K} =$$

$$= \frac{1}{K!} \partial_{\mu_1} \omega_{\mu_2 \dots \mu_{K+1}} \, dx^{\mu_1} \dots dx^{\mu_{K+1}}$$ $\leftarrow$ totally antisym in $\mu_1 \dots \mu_{K+1}$

$$= \frac{1}{(K+1)!} \left(\partial_{\mu_1} \omega_{\mu_2 \dots \mu_{K+1}} + (-1)^K \partial_{\mu_2} \omega_{\mu_3 \dots \mu_{K+1} \mu_1} + \dots + (-)^{K-1} \partial_{\mu_{K+1}} \omega_{\mu_1 \dots \mu_K} \right) (dx)^{K+1}$$

$$= \dots$$

dx^μ is a section of $\Gamma(T^*M)$

(43.1)

$p \in M \to dx^\mu|_p$

more precisely

$(x^\mu) \in M \longmapsto (x^\mu, v^\nu_\mu = \delta^{(\mu\nu)}_\mu)$

$\hookrightarrow v^\nu_\mu \in C^\infty(M)$

Notice the fundamental property

$$d^2=0$$

since $\quad d^2 = dx^\mu \partial_\mu \, dx^\lambda \partial_\lambda = dx^\mu dx^\lambda \, \partial_\mu \partial_\lambda = dx^\mu dx^\lambda \partial_{[\mu}\partial_{\lambda]} = 0$

Because of this we can ask whether

$$d\omega_{(n)}=0 \Rightarrow \omega_{(n)} = d\chi_{(n-1)}$$

Locally it is true but not globally

Ex1 $\quad S^1 = \{ \vartheta \,/\, 0 \le \vartheta < 2\pi \}$

Consider $d\vartheta \qquad d^2\vartheta \equiv 0$

but ϑ is not a globally well defined $\omega_{(0)}$

on the contrary $\omega_{(1)} = \sin\vartheta \, d\vartheta = d(-\cos\vartheta)$ and $-\cos\vartheta$ is globally well defined on S^1!

Ex2 vedi sopra

Ex3 Consider $R^2 - \{0\}$ with coordinates (r, ϑ)

I substract $\rho = 0$ since at $\rho = 0$ ϑ is not well defined

Still $d\vartheta \neq d\omega_{(0)}$ globally and $d^2\vartheta = 0$ on $R^2 - \{0\}$ BUT

If we write $z = re^{i\vartheta}$ then $dz = e^{i\vartheta} dr + i\, r e^{i\vartheta} d\vartheta = z\left(\frac{dr}{r} + i\, d\vartheta\right)$

$$d\bar z = e^{-i\vartheta} dr - i\, r e^{-i\vartheta} d\vartheta$$

$$\Rightarrow dz \wedge d\bar z = e^{i\vartheta} dr \wedge (-i r e^{-i\vartheta} d\vartheta) + (i\, r e^{i\vartheta} d\vartheta) \wedge (e^{-i\vartheta} dr)$$

$$= -2i\, r\, dr \wedge d\vartheta$$

$$\Rightarrow \frac{dz}{z} - \frac{d\bar z}{\bar z} = 2i\, d\vartheta \qquad \Rightarrow \qquad \frac{d\vartheta}{2\pi} = \frac{1}{2}\left(\frac{dz}{2\pi i z} - \frac{d\bar z}{2\pi i \bar z} \right)$$

also defined on $R^2 - \{0\}$

This is compatible with residues

$$\int_{C_\delta} \frac{d\vartheta}{2\pi} = 1 = \frac{1}{2} \oint_C \frac{dz}{2\pi i z} - \frac{1}{2} \oint \frac{d\bar z}{2\pi i \bar z}$$

Apply Green

$$\int_{C_\delta = \partial D_\delta} \frac{d\vartheta}{2\pi} = \int_{D_\delta} \frac{d^2\vartheta}{2\pi} \Rightarrow \frac{d^2\vartheta}{2\pi} = \delta^2(x)\, dx dy \text{ in } R^2$$

Ex2

(54.1)

- $H \in \Gamma(T^*M)$ $\qquad H(q,p)$

  $dH \in T^*(T^*M)$ $\qquad \left(q, p, \frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right) \in T^*(T^*M)$

Given ω / $d\omega = 0$ $\qquad \omega$ non degenerate

- We define the Hamiltonian vector field as

$\forall Y \; dH(Y) = \omega(X_H, Y)$ $\qquad d^2 H = d\omega = 0$

$$\frac{\partial H}{\partial x^i} = -\omega_{ij} X^j_H \qquad \Rightarrow \qquad (\omega^{-1})^{ij} \frac{\partial H}{\partial x^j} = -X^i_H$$

- Hamilton eq.s

$$\dot{x} = X_H \qquad \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \omega^{-1} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

- Because $d\omega = 0$ then

  $\mathcal{L}_{X_H} \omega = d\, i_{X_H}\omega + i_{X_H} d\omega = d(dH) + 0 = 0$

  ω is conserved along the Hamiltonian flow

In complex coordinates

$d\left(\frac{dz}{z}\right) = \frac{d^2 z}{z} - \frac{1}{z^2} dz \wedge dz = 0$ naively but the singularity

at $z=0$ matters so $d\frac{dz}{z} = d\, d\ln z = d^2 \ln z = 2\pi i\, \delta^2(x)\, dx\, dy$

as follows from $\oint \frac{dz}{z} = 2\pi i$

or $d\frac{dz}{z} = \pi\, dz\, d\bar{z}\ \delta^2(x)$

The number of k-forms which are closed, i.e. $d\omega_{(k)} = 0$

but not exact $\omega_{(k)} \neq d\omega_{(k-1)}$ is finite and depends

on the manifold

Actually we are dealing of the equivalence classes $\omega_{(k)} \sim \omega'_{(k)}$ iff

$\omega'_{(k)} = \omega_{(k)} + d\chi_{(k-1)}$

and of the dimension of a real vector space $H^k(M,\mathbb{R})$

$\lambda\, \omega_{(k)} + \mu\, \psi_{(k)}$

$H^k(M)$ is the k-cohomology group on M with coefficient in $\mathbb{R}$

Integration: if $\omega = \omega^{(m)}$ where $m = \dim M$ and there is only one chart U_{α_0}

where $\omega_{(\alpha_0)} \neq 0$ then $\int_M \omega^{(m)} = \int_{\phi_{\alpha_0}(U_{\alpha_0})} \omega^{(m)}_{(\alpha_0)}$

If not use the existence of a partition of unity $1 = \sum_a \rho_a$ where

only a finite number is locally $\neq 0$ to set $\int_M \omega^{(m)} = \sum \int_{\phi_a(U_a)} \rho_a\, \omega^{(m)}_a$

- How we define $\int \omega^{(k)}$ we need a $\Sigma^{(k)}$ a subvariety defined by $i: S^{(k)} \to \Sigma^{(k)} \subset M$

then
$$\int_{\Sigma^{(k)}} \omega^{(k)} = \int_{S^{(k)}} i^* \omega^{(k)}$$

i.e.
$$\omega^{(2)} = \frac{1}{2} \omega_{\mu\nu}^{(2)} dx^\mu \wedge dx^\nu$$
$$\Sigma^{(2)} = (x^\mu (u^1, u^2))$$ simplest case with $S^{(2)} = [0,1]^2$
$$i^* \omega^{(2)} = \frac{1}{2} \omega_{\mu\nu} (x(u)) \left(\frac{\partial x^\mu}{\partial u^1} \frac{\partial x^\nu}{\partial u^2} - \frac{\partial x^\mu}{\partial u^2} \frac{\partial x^\nu}{\partial u^1} \right) du^1 \wedge du^2$$
$$\int_{\Sigma^{(2)}} \omega^{(2)} = \int_0^1 du^1 du^2 \; \frac{1}{2} \omega_{\mu\nu} (x(u)) \frac{\partial (x^\mu)}{\partial (u^\nu)}$$

- General Stokes th
$$\int_{\Sigma^{(k)}} d\omega^{(k-1)} = \int_{\partial \Sigma^{(k-1)}} \omega^{(k-1)}$$
↑ boundary

When we have a metric we can define a Vol form at a Hodge dual $\omega \wedge * \omega = \mathrm{Vol} \cdot |\omega|^2$

- We have shown that given $X, Y \in \Gamma(TM, M)$

$\mathcal{L}_X Y = [X,Y] = -\mathcal{L}_Y X$

In particular we find also

$\mathcal{L}_{fX} Y = [fX, Y] = [(f X^\mu)\partial_\mu Y^\nu - Y^\mu \partial_\mu (f X^\nu)]\partial_\nu =$

$= f[X,Y] - X\, Y(f)$

and

$\mathcal{L}_X (fY) = f[X,Y] + Y X(f)$

Now we can ask whether $\mathcal{L}_X$ can be used to compare two very close points, or better the vectors defined on the tangent space of two very close points

But it does not work properly since we can imagine that the comparison of $Y(x)$ and $Y(x + \overset{\delta x}{X\varepsilon})$ be proportional to $\delta x = X\varepsilon$, even if ε is x dependent independently on how I write $\delta x = X\varepsilon = f Y_\varepsilon$ since what matters is δx

So we are led to introduce a new object called connection s.t.

$\nabla_{fX} Y = f \nabla_X Y$ ← linear!

$\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$ $\quad\nabla_{X_1 + X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$

$\nabla_X (fY) = \nabla_X Y f + Y X(f)$

The previous definition is valid also for a generic vector bundle E where we take $\sigma \in \Gamma(E, M)$ and $X \in \Gamma(TM, M)$

The ~~main~~ difference ~~between $\mathcal{L}_X$ and ∇~~ is now due to the fact that we want to compare the vector $\sigma(x) \in F_x$ to the vector $\sigma(x+\delta x) \in F_{x+\delta x}$ with $\delta x = X\varepsilon$

$\nabla_{fX} \sigma = f \nabla_X \sigma$

$\nabla_X (f\sigma) = \nabla_X \sigma f + \sigma X(f)$

The basic object we need is therefore

$$\nabla_{e_\mu} f_a = f_b \Gamma^b_{\mu a}$$ How many Γ? $\dim X \cdot (\dim F)^2$

where $e_\mu = \partial/\partial x^\mu$

so that

$$\nabla_X \sigma = \nabla_{X^\mu e_\mu}(\sigma^a f_a) = X^\mu \nabla_{e_\mu}(\sigma^a f_a) = X^\mu (\nabla_{e_\mu}\sigma^a f_a + \sigma^a \nabla_{e_\mu} f_a)$$

$$= X^\mu (\partial_\mu \sigma^a f_a + \sigma^b f_a \Gamma^a_{\mu b}) =$$

$$= X^\mu (\partial_\mu \sigma^a + \Gamma^a_{\mu b}\sigma^b) f_a$$

$$= X^\mu (\mathcal{D}_\mu \sigma)^a f_a$$

↑ covariant derivative of Y^a

MEANING: intrinsic does not change under coord's changes

Why covariant derivative?

Suppose we change basis in F explicitly $\tilde{f}_a = \Lambda_a{}^b f_b$ } then since $\nabla_X Y$ must be invariant $(\mathcal{D}_\mu \tilde{\sigma}^a) = (\mathcal{D}_\mu \sigma)^b (\Lambda^{-1})_b{}^a$

$$\nabla \tilde{f}_a = \nabla(\Lambda_a{}^b f_b) = \Lambda_a{}^b \nabla f_b + \nabla \Lambda_a{}^b f_b$$

$$= \Lambda_a{}^b f_c \Gamma^c_{\mu b} + \partial_\mu \Lambda_a{}^b f_b$$

$$= \Lambda_a{}^d \Gamma^c_{d\mu} (\Lambda^{-1})_c{}^b \tilde{f}_b + \partial_\mu \Lambda_a{}^c (\Lambda^{-1})_c{}^b \tilde{f}_b$$

$$= (\Lambda_a{}^d \Gamma^c_{\mu d} (\Lambda^{-1})_c{}^b + \partial_\mu \Lambda_a{}^c (\Lambda^{-1})_c{}^b) \tilde{f}_b$$

$$= \tilde{f}_b (\Lambda^{-T}\Gamma_\mu \Lambda^T + \Lambda^{-T}\partial_\mu \Lambda^T)^b{}_a$$

$$\Rightarrow \tilde{\Gamma}_\mu = \Lambda^{-T}\Gamma_\mu \Lambda^T + \Lambda^{-T}\partial_\mu \Lambda^T$$ $\tilde{\Gamma}$ is <u>NOT</u> a tensor

but now

$$Y = \sigma^a f_a = \tilde{\sigma}^a \tilde{f}_a \Rightarrow \tilde{\sigma}^a = \sigma^b (\Lambda^{-1})_b{}^a$$

so

$$(\mathcal{D}_\mu \tilde{\sigma})^a = \partial_\mu \tilde{\sigma}^a + \tilde{\Gamma}^a_{\mu b}\tilde{\sigma}^b = \partial_\mu(\Lambda^{-T}\sigma) + (\Lambda^{-T}\Gamma_\mu \Lambda^T + \Lambda^{-T}\partial_\mu \Lambda^T)\Lambda^{-T}\sigma$$

$$= \partial_\mu \Lambda^{-T}\sigma + \Lambda^{-T}\partial_\mu \sigma + \Lambda^{-T}\Gamma_\mu \sigma + \underbrace{\Lambda^{-T}\partial_\mu \Lambda^T \Lambda^{-T}}_{-\partial_\mu \Lambda^{-T}}\sigma$$

$$= \Lambda^{-T}(\partial_\mu \sigma + \Gamma_\mu \sigma)$$

$$= (\partial_\mu Y^b + \Gamma^b_{\mu c} Y^c) (\Lambda^{-1})_b{}^a$$

i.e. $(\mathcal{D}_\mu \tilde{\sigma})$ transforms "well"

Ex 1. $E = X \times F \qquad X = S^1 \quad F = \mathbb{R}^2$

$\nabla_{\partial/\partial\theta} f_a = 0$

$$\tilde{f}_a = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & 1 \end{pmatrix} f = \Lambda f \rightarrow \det \Lambda = 1 \qquad \Lambda^{-1} = \begin{pmatrix} 1 & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\nabla \tilde{f}_a = \tilde{f}_b \tilde{\Gamma}^b{}_a \qquad \tilde{\Gamma} = \Lambda^{-1} d\Lambda^T = \begin{pmatrix} 1 & s \\ -s & c \end{pmatrix} \begin{pmatrix} -2cs & -c \\ c & 0 \end{pmatrix} d\theta$$

$$= \begin{pmatrix} -cs & -c \\ 2cs^2 + c^2 & sc \end{pmatrix} d\theta$$

+we globally?

Ex 2 $\quad E = \mathbb{R}^2 \times \mathbb{R}^2 \qquad X = \mathbb{R}^2 \qquad F = \mathbb{R}^2$ (50)

$$\begin{cases} \nabla f_1 = F(r)\, f_2 \otimes d\vartheta \\ \nabla f_2 = -F(r)\, f_1 \otimes d\vartheta \end{cases} \rightarrow \quad \Gamma^2{}_1 = F(r)\, d\vartheta \qquad \Gamma^1{}_2 = -F(r)\, d\vartheta \qquad \Gamma = \begin{pmatrix} & \Gamma^1_2 \\ \Gamma^2_1 & \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} F\, d\vartheta$$

a) $\gamma(t) = (r_0, \omega t)$

$\dot\gamma = \omega\, \partial_\theta|_\gamma$

b) $0 = \nabla_{\dot\gamma}\sigma = \omega\, \nabla_{\partial/\partial\theta}(\sigma^1 f_1 + \sigma^2 f_2)|_\gamma$ $\quad \sigma^1(r,\vartheta)$

$= \omega\,[\, F(r_0)\, f_2\, \sigma^1(r_0,\vartheta) + \partial_\theta \sigma^1|_{\theta=\omega t} f_1 - F(r_0)\, f_1\, \sigma^2(r,\vartheta) + f_2\, \partial_\theta \sigma^2(r_0,\vartheta)|_{\theta=}$

$= \omega\,[\,(\partial_\theta \sigma^1|_{\theta=\omega t} - F(r_0)\sigma^2)\, f_1 + (\partial_\theta \sigma^2|_{\theta=\omega t} + F(r_0)\sigma^1)\, f_2\,]$

$$\begin{cases} \frac{d}{dt}\sigma^1(r_0,\omega t) - \omega F(r_0)\,\sigma^2 = 0 \\ \frac{d}{dt}\sigma^2(r_0,\omega t) + \omega F(r_0)\,\sigma^1 = 0 \end{cases} \rightarrow \quad \frac{d^2}{dt^2}\sigma^1 - \omega^2 F^2(r_0)\,\sigma^1 = 0$$

$\sigma^1(r_0, \omega t) = \sigma^1(r_0, \omega t_0)\cos(F(r_0)\,\omega(t-t_0))$

$t_0 = 0 \quad \sigma^1(r_0, 2\pi) = \sigma^1(r_0, 0)\cos(2\pi F(r_0))$

$\sigma^1 \ldots = \sigma^1(r_0,0)\cos(2\pi F(r_0)) + \sigma^1(r_0,0)\ldots$

c) $\gamma(t) = (r_0 \sin\omega t, \omega t)$

$\dot\gamma = r_0\omega\cos\omega t\, \partial_r|_\gamma + \omega\, \partial_\theta|_\gamma$

d) $0 = \nabla_{\dot\gamma}\sigma = \frac{d}{dt}\sigma^1 f_1 + \frac{d}{dt}\sigma^2 f_2 + \sigma^1 \nabla_{r_0\omega\cos\omega t\,\partial_r + \omega\partial_\theta} f_1|_\gamma +$

$+ \sigma^2\, \omega\, (-F(r_0\sin\omega t)\, f_1)$

$= f_1[\,\dot\sigma^1_\gamma - \omega F(r_0\sin\omega t)\,\sigma^2_\gamma\,] + f_2[\,\dot\sigma^2 + \omega F(r_0\sin\omega t)\,\sigma^1\,]$

$$\begin{cases} \dot\sigma^1 = \omega F(r_0\sin\omega t)\,\sigma^2 \\ \dot\sigma^2 = -\omega F(r_0\sin\omega t)\,\sigma^1 \end{cases} \qquad \frac{1}{F(r_0\sin\omega t)}\frac{d}{dt} = \frac{d}{du} \Rightarrow \frac{dt}{du} = \frac{1}{F(r_0\sin\omega t)}$$

$du = F(r_0\sin\omega t)\, dt$

$$\begin{cases} \frac{d\sigma^1}{du} = \omega\sigma^2 \\ \frac{d\sigma^2}{du} = -\omega\sigma^1 \end{cases} \rightarrow \quad \sigma^1(u) = \sigma^1(u_0)\cos(\omega(u-u_0)) \qquad \sigma^2(u) = -\sigma^1(u_0)\sin(\omega(u-u_0))$$

e) $R = d\Gamma + \Gamma\wedge\Gamma = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \frac{dF}{dr}\, dr\wedge d\vartheta$

Why $F = \text{const} \Rightarrow R = 0$ while $R\sigma \neq \sigma$?

We can use $\Lambda = \begin{pmatrix} \cos F_0\theta & \sin F_0\theta \\ -\sin F_0\theta & \cos F_0\theta \end{pmatrix}$ to set $\tilde\Gamma = 0$ with $F_0 = F = \text{const}$

but Λ is not well defined on R^2 so we cannot set $\tilde\Gamma$ globally

Like $A = a_0$ on S^1 $\quad F = 0$ but $\phi(\theta + 2\pi) = e^{i\ldots a_0}\phi(\theta)$

Notice that $F_0\, d\vartheta$ is not well-defined at $r = 0$!

Notice that we can reassemble all ∇_{e_μ} into $\nabla = \nabla_{e_\mu} \otimes dx^\mu$, i.e.

$$\nabla f_a = f_b \Gamma^b{}_a = f_b \Gamma^b{}_{a\mu} \otimes dx^\mu$$

in this meaning

$$\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M) = \Omega^1(E)$$

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Can we interpret the connection as

$$f_a(x+\delta x) = f_a(x) + f_b(x)\, \Gamma^b{}_{aj}\, \delta x^j$$

$\hat{T}_{x+\delta x}M$ even if $\hat{T}_x M$?

What do we mean?

It is better to interpret ∇ as a way to parallel transport σ along a curve which locally can be taken $\gamma^\mu_{(t)} = x^\mu + X^\mu t$ (Here)

We say that the section σ is parallel transported along γ if

$$\nabla_{\dot\gamma} \sigma = 0$$

or $\dot\gamma^\mu (\partial_\mu \sigma)^a_{|\gamma(t)} f_a |_{\gamma(t)} = 0 \Rightarrow \dot\gamma^\mu (D_\mu \sigma)^a = 0$

at $t=0$

$$X^\mu(x) (\partial_\mu \sigma)^a = 0$$

Deli-ato sopra ... qui $X(x)$ quando deriverei ... $X(t)$ "ipunk"

or $X^\mu(x)\, \partial_\mu \sigma^a_{(x)} = - \Gamma^a{}_{b\mu}(x)\, \sigma^b(x)\, X^\mu(x)$ (like $\partial_t \psi^a = H^a_{tb} \psi^b$)

so we can set $\delta x^\mu = X^\mu \varepsilon$ and try with

$$(P_{x+dx,x}\, \sigma)^a = Y^a(x) + \varepsilon X^\mu \partial_\mu Y^a = Y^a(x) - \delta x^\mu \Gamma^a{}_{b\mu} Y^b(x) =$$

$$= (\delta^a_b - \delta x \cdot \Gamma^a{}_b(x))\, \sigma^b(x)$$

$$= [(\mathbb{1} - \delta x \cdot \Gamma(x))\, \sigma]^a$$

BUT with the previous definition γ_0 $x \to x+\delta x \to x$ (52)

$$P\gamma_0\, \sigma = P_{x,x+\delta x}\, P_{x+\delta x,x}\, \sigma$$
$$= (\mathbb{1} + \delta x\cdot\Gamma(x+\delta x))\,(\mathbb{1} - \delta x\cdot\Gamma(x))\,\sigma$$
$$= (\mathbb{1} + \delta x\cdot\partial(\delta x\cdot\Gamma) + \delta x\cdot\Gamma(x))\,(\mathbb{1} - \delta x\cdot\Gamma(x))\,\sigma$$
$$= (\mathbb{1} - (\delta x\cdot\Gamma(x))^2 + \delta x\cdot\partial(\delta x\cdot\Gamma))\,\sigma$$
$$\uparrow \text{ error } = O(\delta x^2)$$

Can we do better? Yes we can get a $O(\delta x^3)$ error

$$P_{x+\delta x,x}\,\sigma = \Big(\mathbb{1} - \delta x\cdot\Gamma(x) + \underbrace{\tfrac{1}{2}(\delta x\cdot\Gamma(x))^2 - \tfrac{1}{2}\delta x\,\partial(\delta x\cdot\Gamma)}_{\Delta\Gamma^{(2)}_{\delta x}}\Big)\,\sigma$$

is such that

$$P_{\gamma_0}\,\sigma = [\mathbb{1} + O(\delta x^3)]\,\sigma$$

.

.

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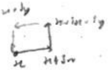
Now we can ask what happens if we compute along 

$$P_\gamma\,\sigma = (\mathbb{1} + \delta y\cdot\Gamma(x+\delta y) + \Delta\Gamma^{(2)}_{\delta y})\,(\mathbb{1} + \delta x\cdot\Gamma(x+\delta x+\delta y) + \Delta\Gamma^{(2)}_{\delta x})$$
$$(\mathbb{1} - \delta y\cdot\Gamma(x+\delta x) + \Delta\Gamma^{(2)}_{\delta y})\,(\mathbb{1} - \delta x\cdot\Gamma(x) + \Delta\Gamma^{(2)}_{\delta x})\,\sigma$$
$$= (\mathbb{1} + \delta y\cdot\Gamma(x) + \delta y\cdot\partial(\delta y\cdot\Gamma) + \Delta\Gamma_{\delta y})\,(\mathbb{1} + \delta x\cdot\Gamma + \delta x\cdot\partial(\delta x\cdot\Gamma) + \delta y\cdot\partial(\delta x\cdot\Gamma) + \Delta\Gamma_{\delta x})$$
$$(\mathbb{1} - \delta y\cdot\Gamma - \delta x\cdot\partial(\delta y\,\Gamma) + \Delta\Gamma_{\delta y})\,(\mathbb{1} - \delta x\cdot\Gamma + \Delta\Gamma_{\delta x})\,\sigma$$
$$= [\ \mathbb{1}$$
$$+ \cancel{\delta y\cdot\Gamma} + \delta x\cdot\Gamma - \cancel{\delta y\cdot\Gamma} - \delta x\cdot\Gamma$$
$$+ \delta y\cdot\Gamma\,\delta x\cdot\Gamma - \underline{(\delta y\,\Gamma)^2} - \delta y\,\Gamma\,\delta x\cdot\Gamma - \delta x\cdot\Gamma\,\delta y\,\Gamma - \underline{(\delta x\cdot\Gamma)^2} + \delta y\,\Gamma\,\delta x\cdot\Gamma$$
$$+ \underline{\delta y\cdot\partial(\delta y\cdot\Gamma)} + \underline{\Delta\Gamma_{\delta y}} + \underline{\delta x\cdot\partial(\delta x\cdot\Gamma)} + \delta y\cdot\partial(\delta x\cdot\Gamma) + \underline{\Gamma_{\delta x}}$$
$$- \delta x\cdot\partial(\delta y\cdot\Gamma) + \underline{\Delta\Gamma_{\delta y}} + \underline{\Delta\Gamma_{\delta x}}\ ]\,\sigma$$
$$= [\mathbb{1} + \delta x^\mu\,\delta y^\nu\,(-\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu + \partial_\nu\Gamma_\mu - \partial_\mu\Gamma_\nu)]\,\sigma$$
$$= [\mathbb{1} - \delta x^\mu\delta y^\nu R_{\mu\nu}]\,\sigma$$

where $R^a{}_{b\mu\nu} = (\partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu + \Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu)^a{}_b$

is the curvature <u>tensor</u> (prove it!)

Since $R_{\mu\nu} = - R_{\nu\mu}$

we can interpret the previous result as a two forms

$R^a{}_b = \frac{1}{2} R^a{}_{b\,\mu\nu}\, dx^\mu \wedge dx^\nu = (d\Gamma + \Gamma\wedge\Gamma)^a{}_b$

which takes a vector and gives a vector back, i.e.

$R \in \Gamma(E^* \otimes E \otimes \Lambda^2 TM)$

$R = R^a{}_b\, f_a \otimes f^{b*} = R^a{}_{b\mu\nu}\, f_a \otimes f^{b*} \otimes dx^\mu \wedge dx^\nu$

- Moreover the previous expression for a closed

$P_\gamma\, \sigma = (1 + \delta x^\mu \delta y^\nu R_{\mu\nu})\, \sigma$

is an infinitesimal $End(E)$, i.e. $P_{\gamma(x)}: F_x \to F_x$

Collecting all P_γ we get a subgroup of End E

$(P_{\gamma_1} P_{\gamma_2} = P_{\gamma_1 \gamma_2} \ ; \quad P^{-1}_\gamma = P_{-\gamma})$

This subgroup is the holonomy group

- Which is the connection between ∇_X and R?

$R(X,Y)(\sigma) = \nabla_X \nabla_Y(\sigma) - \nabla_Y \nabla_X(\sigma) - \nabla_{[X,Y]}(\sigma)$

Ex $X = e_\mu$ $Y = e_\nu$

this is similar to the result of the Lie derivative

$$\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]} \quad \Rightarrow \quad \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X - \mathcal{L}_{[X,Y]} = 0$$

Ex $\mathcal{L}_X \mathcal{L}_Y Z = \mathcal{L}_X [Y,Z] = [X,[Y,Z]]$

$$[X[Y,Z]] - [Y,[X,Z]] = [X,[Y,Z]] + [Y[Z,X]] = - [Z[X,Y]] = [[X,Y]Z]$$

which acts on $\Gamma(TM)$

$\nabla_X \sigma = X^\mu \Gamma_\mu \sigma$, $\nabla_X f^T = f^T X\cdot\Gamma$ $\nabla_X f_a = f_b\, X\cdot\Gamma^b{}_a$

$\nabla_Y(\nabla_X f^T) = \nabla_Y(f^T X\cdot\Gamma) = f^T Y\cdot\Gamma\, X\cdot\Gamma + f^T\, Y\cdot\partial(X\cdot\Gamma)$

$$R(X,Y)(f^T) = f^T \tfrac{1}{2} X^\mu Y^\nu \left(\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu \right)$$

$$= \tfrac{1}{2} f^T \left(Y^\nu X\cdot\partial\Gamma_\nu - X^\nu Y\cdot\partial\Gamma_\nu + X\cdot\Gamma\, Y\cdot\Gamma - Y\cdot\Gamma\, X\cdot\Gamma \right)$$

$$= \tfrac{1}{2} \left(\nabla_X \nabla_Y f^T - f^T X\cdot\partial(Y\cdot\Gamma) - \nabla_Y \nabla_X f^T + f^T Y\cdot\partial(X\cdot\Gamma) + f^T \left[\underbrace{Y^\nu X\cdot\partial\Gamma_\nu} - X^\nu Y\cdot\partial\Gamma_\nu \right] \right) =$$

$$= \tfrac{1}{2} \left(\nabla_X \nabla_Y f^T - \nabla_Y \nabla_X f^T - f^T X\cdot\partial Y^i\, \Gamma_i + f^T Y\cdot\partial X^i\, \Gamma_i \right)$$

$$= \tfrac{1}{2} \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \right)(f^T)$$

• Is a connection unique? no

Given 2 connections ∇, ∇' $\sigma \in \Gamma(E,M)$ $f \in \Gamma(M)$

$(\nabla-\nabla')(f\sigma) = f\nabla\sigma + \sigma\otimes df - f\nabla'\sigma - \sigma\otimes df =$

$= f(\nabla-\nabla')\sigma$

so it is linear also in the section, i.e. $\nabla-\nabla'$ is a 1-form with values in $End(E) = E\otimes E^*$

$\nabla-\nabla' \in \Gamma(End(E)\otimes T^*M)$

$(\nabla-\nabla^*)_{e_\mu} f_a = f_b A^b_{\mu a}$

or $\nabla' = \nabla + A$

• Finally note that given $\nabla : \Gamma(E) \to \Gamma(E\otimes T^*M) = \Omega^1(E)$

we can induce $\nabla^* : \Gamma(E^*) \to \Gamma(E^*\otimes T^*M)$

How $\sigma \in \Gamma(E)$ $\tau^* \in \Gamma(E^*)$

$d\langle\tau^*,\sigma\rangle = \tau_a \sigma^a$

then

introduce covariant derivatives for having nice objects

$d\langle\tau^*,\sigma\rangle = d(\tau_a\sigma^a) = (\nabla^*\tau_a)\sigma^a + \tau_a\nabla\sigma^a$

$= \langle\nabla^*\tau^*,\sigma\rangle + \langle\tau^*,\nabla\sigma\rangle$

then

$\nabla^* f^{*a} = -f^{*b}\Gamma^a{}_b$

due $0 = \langle\nabla^* f^{*a}, f_b\rangle + \langle f^{*a}, \nabla f_b\rangle$

$= -\Gamma^a{}_b + \Gamma^a{}_b = 0$

or $\nabla^*\tau_a = d\tau_a - \Gamma^b{}_a\tau_b$

due $\nabla^*\tau_a\sigma^a + \tau_a\nabla\sigma^a \to (-\Gamma^b{}_a\tau_b)\sigma^a + \tau_a(\Gamma^a{}_b\sigma^b) = 0$

We can generalize to $\Gamma(E^{*r} \otimes E^s)$

(55.1)

$T = T^{a_1 \ldots a_s}_{b_1 \ldots b_r} f^{*b_1} \otimes \; \otimes f^{*b_r} \otimes f_{a_1} \otimes \; \otimes f_{a_s}$

with $\mathcal{D} T = \mathcal{D} T^{a_1 \ldots a_s}_{b_1 \ldots b_r} f^{*b_1} \otimes \ldots \otimes f_{a_s}$

$$(\mathcal{D} T)^{a_1 \ldots a_s}_{b_1 \ldots b_r} = dT^{a_1 \ldots a_s}_{b_1 \ldots b_r} + \Gamma^{a_1}{}_c T^{c a_2 \ldots a_s}_{b_1 \ldots b_r} + \ldots + \Gamma^{a_s}{}_c T^{a_1 \ldots a_{s-1} c}_{b_1 \ldots b_r}$$
$$- \Gamma^c{}_{b_1} T^{a_1 \ldots a_s}_{c b_2 \ldots b_r} - \quad - \Gamma^c{}_{b_r} T^{a_1 \ldots a_s}_{b_1 \ldots b_{r-1} c}$$

What about Lie derivative? And forms?