GPS and curved space

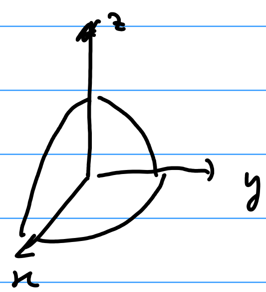
1) The GPS system is made of

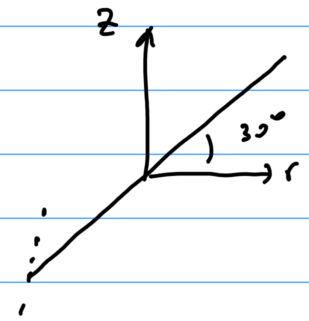
- 6×4 satellites at $R_S = 26600$ Km, i.e. 20200 Km from earth surface ($R_\oplus = 6400$ Km, $g = 9.8\ m/s^2$)

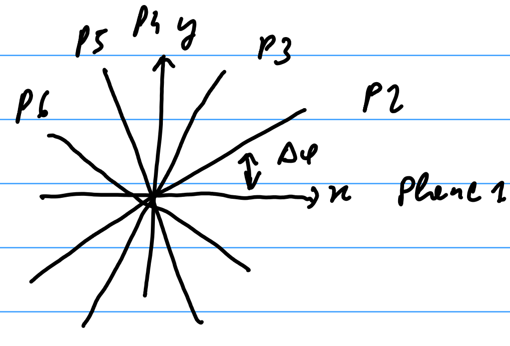
- 4 satellites on each of the 6 planes

1 plane splits the angles into	2 pieces
2 planes	4
3 planes	6
⋮	
6	12 $\Rightarrow \quad \Delta\varphi = \frac{360^\circ}{12} = 30^\circ$

- each plane is at 30° w.r.t. the previous (this does not mean that all planes are obtained from one plane by a rotation along the same axis)







- almost always at least 4 satellites are visible

2) Why 4 ?
We need to solve the eq.

$$(*) \quad |\vec{r}_{GPS} - \vec{r}_s| = c\,(t_{GPS} - t_s) \qquad s = 1, .. 4$$

where $(ct_s, \vec{r}_s)$ are the coords of satellites
and $(ct_{GPS}, \vec{r}_{GPS})$ are the coords of receiver
$\Rightarrow$ 4 unknowns !

Precision
If we want $|\Delta r_{GPS}| \sim 1\,m$ $\Rightarrow$ $|c\Delta t_s| \sim 1\,m$
$\Rightarrow$ $|\Delta t_s| \sim 3 \cdot 10^{-9}\,s$

so time on satellites must be precise with such an error
and they must be synchronized with such an error, i.e.
$|t_r - t_s| \leq 3 \cdot 10^{-9}\,s$
This must happen also after $1\,d = 86400\,s$ hence the error
must be $\left|\frac{\Delta t}{d}\right| \sim 3.47 \cdot 10^{-14}$

3) Fact

The synchronized GPS clocks have frequency

$\nu_{earth} = 10.23$ MHz on earth measured with earth proper time

$\nu_{satellite} = 10.2299\,9999\,543$ MHz on satellites in their proper time

so $\left|\frac{\Delta\nu}{\nu_\oplus}\right| = \frac{\nu_\oplus - \nu_S}{\nu_\oplus} = 4.47 \cdot 10^{-10}$

or $\left|\frac{\Delta P}{P_\oplus}\right| = \frac{P_S - P_\oplus}{P_\oplus} = \frac{\nu_\oplus - \nu_S}{\nu_S} \sim \frac{\nu_\oplus - \nu_S}{\nu_\oplus} \sim 4.47 \cdot 10^{-10}$

N.B. $\oplus$ Earth ; $\odot$ Sun ; ♀ venus ; ♂ Mars

In other words.
If we use the maximum of $\vec{E}$ in an e.m. wave to measure time intervals, an e.m. wave emitted with frequency $\nu_{satellite}$ when measured on satellite with its proper time is measured with frequency $\nu_\oplus$ on earth in earth proper time.

4) Why?

Can we understand this fact using special relativity?

- 1^{st} try

Satellites move "fast"

$$v_s = 3.87 \text{ km s}^{-1} \qquad \beta = 1.29 \cdot 10^{-5}$$

$$\gamma - 1 = \frac{1}{\sqrt{1-\beta^2}} - 1 \qquad \sim \frac{1}{2}\beta^2 \sim 0.83 \cdot 10^{-10}$$

Easy to derive $\quad g = \frac{GM_\oplus}{R_\oplus^2} \qquad \frac{v^2}{R_s} = \frac{GM_\oplus}{R_s^2}$

$$\Rightarrow \frac{v^2}{R_s g} = \frac{R_\oplus^2}{R_s^2} \qquad \Rightarrow v^2 = g \frac{R_\oplus^2}{R_s}$$

$$v^2 \sim 10 \frac{m}{s^2} \frac{(6 \cdot 10^6 m)^2}{27 \cdot 10^6 m} = \frac{1}{2.7} \cdot 36 \cdot 10^6 \frac{m^2}{s^2} \sim 12 \cdot 10^6 \frac{m^2}{s^2}$$

It follows that

$P_s = \gamma P_\oplus$ for a signal emitted from earth since $\Delta x_\oplus = 0$

hence $$\frac{\Delta P}{P} = \frac{P_s - P_\oplus}{P_\oplus} = \gamma - 1 = +0.83 \cdot 10^{-10}$$

or

$P_\oplus = \gamma P_s$ for a signal emitted from satellite since $\Delta x_s = 0$ and measured on earth

hence $$\frac{\Delta P}{P} = \frac{P_s - P_\oplus}{P_\oplus} = \frac{1-\gamma}{\gamma} \sim +1-\gamma \sim -0.83 \cdot 10^{-10}$$

Which is correct?

- We must ask what we want measure in which system.

  We measure 1) a time interval

  2) in earth system

  $\Rightarrow$ The 2nd is right

- Here we are actually comparing frequencies as measured in satellite proper time with those measured by an observer which is static wrt the $\oplus$ observer and syncronized with the $\oplus$ obs

• 2^{nd} try

Suppose light feels gravitation, or better energy feels gravitation and not only mass.

Hence write

$$V = -\frac{G M_\oplus m}{r} \Rightarrow -\frac{G M_\oplus}{c^2 r} E$$

Then given photon of frequency $\nu \Rightarrow E = h\nu$ and we can compute what happens when it falls from the satellite.

$$E_S + V_S = \left(1 - \frac{G M_\oplus}{c^2 R_S}\right) E_S = E_\oplus + V_\oplus = \left(1 - \frac{G M_\oplus}{c^2 R_\oplus}\right) E_\oplus$$

$$\Rightarrow E_\oplus \sim E_S \left(1 + \frac{G M_\oplus}{c^2}\left(\frac{1}{R_\oplus} - \frac{1}{R_S}\right)\right) > E_S$$

i.e. photons acquire energy when falling

Equivalently

$$\left|\frac{\Delta\nu}{\nu_\oplus}\right| = \frac{E_\oplus - E_S}{E_S} = \frac{G M_\oplus}{c^2}\left(\frac{1}{R_\oplus} - \frac{1}{R_S}\right) =$$

$$= \frac{g R_\oplus^2}{c^2}\left(\frac{1}{R_\oplus} - \frac{1}{R_S}\right) \sim 5.3 \cdot 10^{-10}$$

Combining 1^{st} and 2^{nd} we get

$$\frac{\Delta\nu}{\nu_\oplus} \sim 5.3 \cdot 10^{-10} - 0.83 \cdot 10^{-10} \sim 4.47 \cdot 10^{-10}$$

which is the right result

How do we interpret this?

We compare a static satellite in orbit with a static obs on earth so we miss time dilatation for the satellite and Doppler for the $\oplus$ obs

$\Rightarrow$ We added time dilatation to gravity effects

Earth observer sees in her system and with her proper time

$$\overset{\text{proper time}}{\nu_\oplus} = 23\ \text{MHz} = \overset{\text{blue shifted}}{\nu_S} > \overset{\text{proper time}}{\nu_S}$$

Satellite observer sees

$$\overset{\text{red shifted}}{\nu_\oplus} = \overset{\text{proper time}}{\nu_S} < \overset{\text{proper time}}{\nu_\oplus}$$

5) Principle of equivalence

All matter behaves in the same way in gravitational fields (up to spin dependent coupling)

The previous is an application of it.

NOTICE

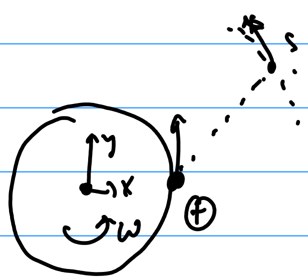
1) Eq (*) is strictly speaking valid in ECI system (Earth Centered Inertial system) since on earth light geodetic is not a straight line because of earth rotation

2) We have to consider that receiver is moving so

$$-ds^2 = c^2 dt^2_{GPS} = c^2 \gamma^2_{rec} dt^2_{receiver}$$

3) Because receiver is moving signals from satellites are subject to Doppler effect (both transverse which we have already considered and radial which we consider now which is bigger!)

$$\frac{\nu_{rec}}{\nu_{GPS}} = \frac{\sqrt{1-\beta^2}}{1+\beta_r} \sim 1 - \frac{v_r}{c} + \frac{1}{2}\left(\frac{v_r}{c}\right)^2 - \frac{1}{2}\left(\frac{v}{c}\right)^2 + O(c^{-3})$$



Now $v_r \sim O(100\ m/s)$

as an obs on equator is moving at $v_{eq} = \frac{4\cdot 10^4 km}{1d} = 462 \frac{m}{s}$

4) To keep satellites synchronized they must be updated frequently.

Clocks on earth must be synchronized but earth is rotating and with different speeds at different latitudes

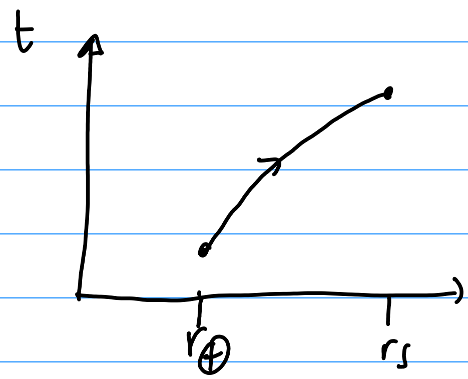
$\Rightarrow$ Sagnac effect

not possible to synchronize all clocks

Schild's argument

Can we explain blue/red shift in flat space?

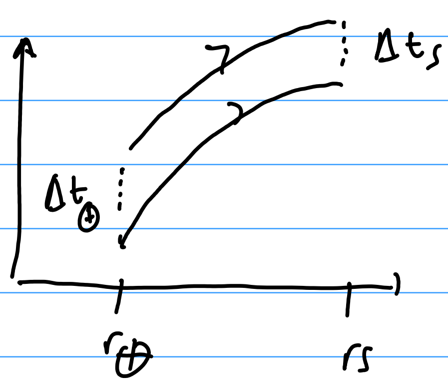
Suppose space is Minkowski.
Forget earth rotation.



A γ climbing out the gravitational well.

The path followed may be at non constant speed

Consider a 2nd photon.

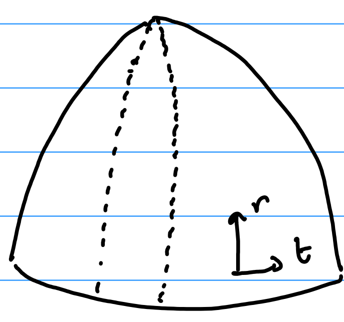


It must follow an equal path since the system is static.

Hence $\Delta t_{\oplus} = \Delta t_s$

Hence the space cannot be flat

Ex.



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