EQUIVALENCE PRINCIPLE

Gedanken experiment of the elevator.

- Why does it work? Universality of gravitational interaction, i.e. $m_{inertial} = m_{gravitational}$ implies all matter falls in the same way (up to spin effect and quadrupole effects)

  $m_{in} = m_{gr}$ proven experimentally for small velocities and weak fields

  $\Rightarrow$ Hp: it is true for all conditions and type of matter (e.g. light)

- Hence

  we cannot LOCALLY distinguish between free falling and non rotating observers and inertial ones

  or we can gauge gravitation away.

  What means locally! $[R] = L^{-2} \Rightarrow$ distance $\lesssim 1/\sqrt{R}$

  Globally we can!

  Riemann normal coords $g_{\mu\nu} = \delta_{\mu\nu} + \# R_{\mu\rho|\nu\sigma} x^\rho x^\sigma + O(x^3)$ [illegible]

- In GR free obs.s are those falling (and not rotating), i.e. moving in a system where locally there is not gravitation.

The point of view is "opposite" w.r.t Newtonian mechanics.
We on earth are (almost) inertial for Newton but accelerating for Einstein.
Viceversa for a falling object!

- Charges radiate even if they fall freely since radiation is not a local process; the particle can interact with its own e.m. field!
  Locally $g \sim \eta$ hence no radiation but also in flat space radiation is seen from far away.
  Can we distinguish from self radiation coming back and radiation from a source?

- The previous formulation is heuristic.
  The mathematical formulation is
  All eq.s are generally covariant w.r.t. the C^∞ change of coordinates

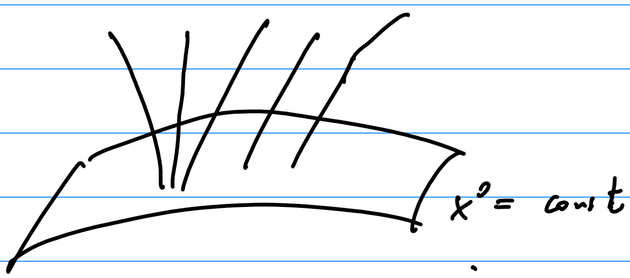
- $g_{\mu\nu}$ is not observable! Like A_μ.
  We can always choose coords such that at $x=0$ (where I am)
  $$g_{\mu\nu} = \eta_{\mu\nu} + O(x^2).$$

  Like QED there are gauge transformations and they are diffeomorphisms in GR

  (also $SO(1,d-1)$ transformations in spin connection formalism are gauge transformations whose gauge field is $\omega^a{}_\nu$)

MEANING OF COORDINATES

Coordinates have no meaning in GR, but we can think of them as labels attached to worldlines.



Fix x^0 then x^i can be used to label the different worldlines which are <u>not</u> necessarily geodesic.

Generally observers experience acceleration.

GEODETICS

Two ways to get geodetics as free particle trajectory

- In a local inertial frame at x_0

$$\ddot{x}^\mu\big|_{x_0} = 0 \qquad \Gamma^\mu_{\rho\sigma}\big|_{x_0} = 0$$

but we need tensorial quantities, hence

$$\underset{\substack{\uparrow\\ \text{acceleration}}}{a^\mu} = \ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0 \qquad \dot{x}^2 = \underset{\substack{\uparrow\\ \text{4-velocity}}}{u^2} = -1$$

In particular we can identify the affine parameter $\lambda = c\tau$ since
$ds^2 = -c^2 d\tau^2 = g_{\mu\nu}\,dx^\mu dx^\nu$

The previous eq. can be written (cheating a little) as

$$a = \nabla_u u = 0 \neq \nabla_{\dot{\gamma}}\dot{\gamma} \quad \text{since } u^2 = -1 \text{ while } \dot{\gamma}^2 = ?$$

This can be verified starting from a generic vector field Y

$$\nabla_u Y = (\underset{\substack{\uparrow\\ \frac{d}{d\lambda}Y^\nu(x(\lambda))}}{\dot{Y}^\nu} + \Gamma^\nu_{\rho\sigma}u^\rho Y^\sigma)e_\nu$$

- $S = -mc\int ds = -mc^2\int dt\sqrt{1-\vec{v}^2/c^2}$ is the free particle action in Minkowski.

ds is a good covariant, actually invariant, object so

$S = -mc\int ds$ is the action for a "free" particle.

Let us see an example.

Consider Minkowski, take uniformly rotating observers.

$\mathbb{R}^{1,2}$ $\quad ds^2 = -c^2dt^2 + dr^2 + r^2 d\vartheta^2$

insert $\quad \theta = \alpha + \omega t$

$\uparrow$ new coord!

tipical of rotation

BUT $\downarrow$ $\partial_{[i} g_{j]0} \neq 0$ is needed

get $\quad ds^2 = -(c^2 - r^2\omega^2)\,dt^2 + 2r^2\omega\, dt\, d\alpha + dr^2 + r^2 d\alpha^2$

Obs are labelled by (r, α).

The path followed by one is $\gamma = (t, r_0, \alpha_0 + \omega t)$ in Mink coords

$= (t, r_0, \alpha_0)$ in other coords

Are they geodetic?

- Minkowski coords

$$\dot\gamma = (1, 0, \omega) = \partial_t + \omega\partial_\theta$$

$$\Rightarrow \quad \|\dot\gamma\|^2 = -c^2 + r_0^2\omega^2 < 0 \Rightarrow r_0^2 < \frac{\omega^2}{c^2}$$

hence not all coords can be associated to physical obs.

trivelocity. $\quad u = \frac{\dot\gamma}{\|\gamma\|} = \left.\frac{\partial_t + \omega\partial_\theta}{\sqrt{c^2 - r_0^2\omega^2}}\right|_\gamma \qquad u^2 = -1$

triaccel: $\quad a = \nabla_u u \quad$ or $\quad a = \nabla_{\dot\gamma}\dot\gamma$?

Remember $\quad \nabla\partial_t = 0 \quad \nabla\partial_\theta = \frac{1}{r}\partial_\theta \otimes dr - r\,\partial_r \otimes d\vartheta \quad \nabla\partial_r = \frac{1}{r}\partial_\theta \otimes d\theta$

Minkowski

Chk $\quad S = \int d\lambda \, -\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \int d\lambda \, \frac{1}{2} \left(c^2 \dot{t}^2 - \dot{r}^2 - r^2 \dot{\theta}^2 \right)$

$\Rightarrow \quad \delta t \quad \ddot{t} = 0 \quad \Rightarrow \quad \Gamma^t_{\mu\nu} = 0$

$\delta\theta \quad (r^2\dot{\theta})^{\cdot} = r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \quad \Rightarrow \quad \Gamma^\theta_{r\theta} = \frac{1}{r} = \Gamma^\theta_{\theta r}$

$\delta r \quad \ddot{r} - r\dot{\theta}^2 = 0 \quad \Rightarrow \quad \Gamma^r_{\theta\theta} = -r$

$\Rightarrow \quad \nabla \partial_r = \partial_\mu \otimes dx^\rho \, \Gamma^\mu_{\rho r} = \partial_\theta \otimes d\theta \, \Gamma^\theta_{\theta r} = \frac{1}{r} \partial_\theta \otimes d\theta$

$\nabla \partial_\theta = \partial_\theta \otimes dr \, \frac{1}{r} + \partial_r \otimes d\theta \, (-r)$

Compute

$\nabla_{\dot{\gamma}} \dot{\gamma} = \left(\nabla_{\partial_t} \dot{\gamma} + \omega \nabla_{\partial_\theta} \dot{\gamma} \right) = \nabla_{\partial_t} (\partial_t + \omega \partial_\theta) + \omega \nabla_{\partial_\theta} (\partial_t + \omega \partial_\theta)$

$= \omega^2 \nabla_{\partial_\theta} \partial_\theta \big|_\gamma = -\omega^2 r_0 \, \partial_r \big|_\gamma$

and

$\nabla_u u = \frac{1}{\sqrt{c^2 - r_0^2 \omega^2}} \left(\nabla_{\partial_t} u + \omega \nabla_{\partial_\theta} u \right)$ use r_0 is constant

$= \frac{1}{(\sqrt{\quad})^2} \left(\nabla_{\partial_t} (\partial_t + \omega \partial_\theta) + \omega \nabla_{\partial_\theta} (\partial_t + \omega \partial_\theta) \right)$

$= \frac{1}{(\sqrt{\quad})^2} \omega^2 \nabla_{\partial_\theta} \partial_\theta \big|_\gamma = -\frac{\omega^2}{c^2 - r_0^2 \omega^2} r_0 \, \partial_r \big|_\gamma \sim -r_0 \omega^2 \partial_r \big|_r$

$\left| \frac{r_0 \omega}{c} \right| \ll 1$

Which one? Remember geodetics in affine parametrization

$a^\mu = \ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma$

only if $\dot{x}^2 = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1$

hence we need using u since $u^2 = -1$

Notice we could compute a^μ directly from geodetic eq.s!

$a^t = 0$

$a^\theta = 0$

$$a^r = (0 - r_0 \omega^2) \left(\frac{1}{\sqrt{c^2 - r_0^2 \omega^2}} \right)^2$$

due to the need of using the affine param $\lambda = c\tau$ so that

$$\frac{d}{d\lambda} = \frac{dt}{dc\tau} \frac{d}{dt} = \frac{1}{\sqrt{c^2 - r_0^2 \omega^2}} \frac{d}{dt}$$

- Rotating coords

The accel. comes from the metric!

$\dot{\gamma} = \hat{\partial}_t|_\gamma$, $\|\dot{\gamma}\|^2 = -(c^2 - r_0^2\omega^2)$, $u = \dfrac{\hat{\partial}_t|_\gamma}{\sqrt{c^2 - r_0^2\omega^2}}$

Now we need the new Christoffel symbols.

$$S = -\int d\lambda \, \frac{1}{2}\left[(c^2 - r^2\omega^2)\dot{t}^2 - \dot{r}^2 - r^2\dot{\alpha}^2 - 2r^2\omega \, \dot{t}\dot{\alpha} \right]$$

δt $\quad [\, (c^2 - r^2\omega^2)\dot{t} - r^2\omega\dot{\alpha}\,]^{\cdot} =$

$$= (c^2 - r^2\omega^2)\ddot{t} - 2r\dot{r}\omega^2\dot{t} - 2\omega r \, \dot{r}\dot{\alpha} - r^2\omega\ddot{\alpha} = 0$$

$\delta\alpha$ $\quad [\, r^2\dot{\alpha} + \omega r^2\dot{t}\,]^{\cdot} =$

$$= r^2\ddot{\alpha} + \omega r^2\ddot{t} + 2r\dot{r}\dot{\alpha} + 2\omega r\dot{r}\dot{t} = 0$$

$$\Rightarrow \quad \ddot{\alpha} = -\omega\ddot{t} - 2\frac{\dot{r}\dot{\alpha}}{r} - 2\frac{\omega}{r}\dot{r}\dot{t} \qquad \times$$

δr $\quad + \ddot{r} - r\omega^2\dot{t}^2 - r\dot{\alpha}^2 - 2\omega r\dot{t}\dot{\alpha} = 0$

hence

$$(c^2 - r^2\omega^2)\ddot{t} - 2r\omega^2\dot{r}\dot{t} + \overset{\times}{r^2\omega^2\ddot{t}} + \overset{\times}{2r\omega^2\dot{r}\dot{t}}$$
$$-2\omega r\dot{r}\dot{\alpha} + \underset{\times}{2r\omega\dot{r}\dot{\alpha}} = 0$$

or $c^2 \ddot{t} = 0$

then $\ddot{t} = 0$

and $\ddot{\alpha} + \frac{2}{r} \dot{r}\dot{\alpha} + 2\frac{\omega}{r} \dot{r}\dot{t} = 0$

$$\Rightarrow \quad \hat{\Gamma}^t_{\mu\nu} = 0 \qquad \hat{\Gamma}^\alpha_{r\alpha} = \frac{1}{r} \qquad \hat{\Gamma}^\alpha_{rt} = \frac{\omega}{r}$$

$$\hat{\Gamma}^r_{tt} = -r^2\omega$$

then we get

$$a = \nabla_u u = \frac{1}{\sqrt{c^2 - r_0^2\omega^2}} \nabla_{\hat{\partial}_t} u = \frac{1}{c^2 - r_0^2\omega^2} \nabla_{\hat{\partial}_t} \hat{\partial}_t$$

$$= \frac{1}{c^2 - r_0^2\omega^2} e_\mu \Gamma^\mu_{tt}$$

$$= -\frac{r_0\omega^2}{(c^2 - r_0^2\omega^2)} \hat{\partial}_r \Big|_\gamma$$

is the same ∂_r as before

chk

$$\nabla_{\hat{\partial}_t} \hat{\partial}_t = \nabla_{\partial_t + \omega\partial_\theta} (\partial_t + \omega\partial_\theta) = \omega^2 \nabla_{\partial_\theta} \partial_\theta = -\omega r^2 \partial_r$$