VIELBEIN AND LEVI-CIVITA

We have introduced the vielbein V^i and the spin connection ω^{ij} $(i,j = 1..n)$ as

$$\begin{cases} T^i = dV^i + \omega^i{}_j \wedge V^j \\ R^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j \end{cases}$$

We require (metric postulate)

$$\omega_{ij} = -\omega_{ji}$$

We can write

$$T^i = \frac{1}{2} T^i{}_{\ell m} V^\ell \wedge V^m \qquad \text{with} \qquad T^i{}_{\ell m} = -T^i{}_{m\ell}$$

then if we set

$$\mathring{\omega}^i{}_j = \omega^i{}_j + \frac{1}{2} T^i{}_{jm} V^m$$

we get

$$\mathring{T}^i = dV^i + \mathring{\omega}^i{}_j \wedge V^j = 0$$

Chk $0 = \mathring{T}^i = T^i + \frac{1}{2} T^i{}_{jm} V^m \wedge V^j = T^i - \frac{1}{2} T^i{}_{\ell m} V^\ell \wedge V^m$

Notice: A priori $T_{i|\ell m} V^m \neq - T_{\ell|im} V^m$

then $\overset{\circ}{\omega}_{ij} \neq - \overset{\circ}{\omega}_{ji}$.

But if $T_{i|\ell m} = - T_{\ell|im}$ then $T_{i|\ell m} = T_{[i\ell m]}$

since also $T_{m|\ell i} = - T_{m|i\ell} = T_{i|m\ell} = - T_{i|\ell m}$

Is it compatible with Bianchi $(\overset{\circ}{\mathcal{D}} T)^i = 0$?

Yes because Bianchi follows from $d^2 = 0$.

Notice however that 1st Bianchi identity is modified

since $\overset{\circ}{R}{}^i{}_j = d\overset{\circ}{\omega}{}^i{}_j + \overset{\circ}{\omega}{}^i{}_k \wedge \overset{\circ}{\omega}{}^k{}_j$

is such that

$$\overset{\circ}{R}_{ij|k\ell} \neq - \overset{\circ}{R}_{ji|k\ell}$$

and $$\overset{\circ}{R}_{ij|k\ell} = - \overset{\circ}{R}_{ij|\ell k}$$

hence $$\overset{\circ}{R}_{ij|k\ell} \neq \overset{\circ}{R}_{k\ell|ij}$$

Main difference between Cartan (vielbein) and connection

Cartan uses forms $\quad dV^i = -\omega^i{}_j \wedge V^j$

Connection uses tensor products $\quad (\nabla^* e)^\mu{}_{\cdot} = -\Gamma^\mu{}_\nu \otimes e^\nu$

• From $\quad \nabla^*_\mu dx^\rho = \nabla^*_\mu e^\rho = -\Gamma_\mu{}^\rho{}_\sigma e^\sigma$

and from $\quad V^i = V^i{}_\mu e^\mu$

we get

$$\nabla^*_\mu V^i = \nabla^*_\mu (V^i{}_\rho e^\rho) = (\partial_\mu V^i{}_\rho - \Gamma_\mu{}^\sigma{}_\rho V^i{}_\sigma) e^\rho$$

$$= (\partial_\mu V^i{}_\rho - \Gamma_\mu{}^\sigma{}_\rho V^i{}_\sigma) V_j{}^\rho V^j$$

$$= -(V_j{}^\rho \Gamma_\mu{}^\sigma{}_\rho V^i{}_\sigma - V_j{}^\rho \partial_\mu V^i{}_\sigma) V^j$$

$$= -\Gamma_\mu{}^i{}_j V^j$$

This is nothing else but a change of basis

In general vector bundle

$$(\nabla^* f)^a = -\Gamma^a{}_b \otimes f^b \Rightarrow (\nabla^* f')^a = -\Gamma'^a{}_b \otimes f^{b\prime}$$

with $\quad f'(x)^a = \Lambda^a{}_b(x) f^b(x)$

implies $\quad \Gamma' = \Lambda^{-1} \Gamma \Lambda - \Lambda \, d\Lambda^{-1}$

From this expression we read the covariant derivative

$$(\nabla^*_\mu V^i)_\rho$$

It follows

$$\nabla^*_{[\mu} V^i_{\rho]} = \partial_{[\mu} V^i_{\rho]} - \Gamma^\sigma_{[\mu\rho]} V^i_\sigma = - \Gamma_\mu{}^i{}_j V^j_\rho$$

or

$$\underset{\shortparallel}{d V^i} = - \Gamma^i{}_j \wedge V^j + \Gamma^i_{[\mu\rho]} dx^\mu \wedge dx^\rho$$

$$\partial_\mu V^i_\rho \, dx^\mu \wedge dx^\rho$$

$\odot$ Let us compare the Cartan formulation

$$dV^i = -\omega^i{}_j \wedge V^j + \frac{1}{2} T^i{}_{kl} V^k \wedge V^l$$

and the Riemann

$$dV^i = -\Gamma^i{}_j \wedge V^j + \Gamma^i{}_{[\mu\rho]} dx^\mu \wedge dx^\rho$$

We get

$$\omega^i{}_j = \Gamma^i{}_j$$

$$\frac{1}{2} T^i{}_{kl} V^k_\mu V^l_\nu = \Gamma^i{}_{[\mu\rho]}$$

It follows also that

$$R^i{}_j = V^i{}_\mu V^\nu{}_j R^\mu{}_\nu$$

with $\quad R^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu$

where $\quad \Gamma^\mu{}_\nu = dx^\rho \, \Gamma^\mu_{\rho\nu}$

- How many components?

Levi-Civita $\quad \# \Gamma^\mu_{\rho\nu} = n \cdot \frac{n(n+1)}{2}$

spin connection $\quad \# \Gamma_{ij|\rho} = n \cdot \frac{n(n-1)}{2}$

Computationally Γ_{ij} is "easier"

- Let us require the metric be covariantly costant, i.e. scalar product does not change under parallel transport

- From $\nabla^* g = 0$ then we get

$$0 = V^\mu_i V^\nu_j (\nabla^* g)_{\mu\nu} = [d g_{\mu\nu} - g_{\mu\rho} \Gamma^\rho{}_\nu - g_{\nu\rho} \Gamma^\rho{}_\mu] V^\mu_i V^\nu_j$$

then from

$$g_{\mu\nu} = \eta_{ij} V^i_\mu V^j_\nu$$

we get

$$V^\mu_i d g_{\mu\nu} V^\nu_j = V^\mu_i V^\nu_j d(\eta_{em} V^e_\mu V^m_\nu) = V^\mu_i d V_{\mu j} + V^\mu_j d V_{\mu i}$$

it follows

$$0 = V^\mu_i d V_{\mu j} - V_{\rho i} \Gamma^\rho{}_\nu V^\nu_j + (i \leftrightarrow j)$$

$$= - \Gamma_{i|j} - \Gamma_{j|i}$$

i.e. $\nabla^* g = 0 \Rightarrow \Gamma_{ij} = \Gamma_{[ij]}$

Hence if $\Gamma_{i|j} \neq \Gamma_{[i|j]} \Rightarrow \nabla^* g \neq 0$

⊙ How do we compute ω?

Expand $\omega_{ij} = C_{ij|k} \, V^k$

Then $dV^i = \partial_\mu V^i_\nu \, dx^\mu \wedge dx^\nu = \partial_{[\mu} V_{\nu]}^i \, dx^\mu \wedge dx^\nu$

$= \partial_{[\mu} V_{\nu]}^i \, V_j^\mu V_k^\nu \, V^j \wedge V^k$

Therefore from $dV_i = - \omega_{ij} \wedge V^j$ we get

$$\partial_\mu V_{\nu i} \, V^\mu_{[j} V^\nu_{k]} \, V^j \wedge V^k = - C_{ij|k} \, V^k \wedge V^j$$

$$\Rightarrow \quad \partial_\mu V_{\nu i} \, V^\mu_{[j} V^\nu_{k]} = C_{i[j|k]}$$

As done for the Levi-Civita connection

sum

$$+ \quad C_{ij|k} - C_{ik|j} = \left(\partial_\mu V_{\nu i} \, V^\mu_{[j} V^\nu_{k]} \right)$$

$$- \quad C_{ki|j} - C_{kj|i} =$$

$$+ \quad C_{jk|i} - C_{ji|k} =$$

$$2 C_{ij|k} = \Big(+ \partial_\mu V_{\nu i} \, V^\mu_{[j} V^\nu_{k]}$$

$$- \partial_\mu V_{\nu k} \, V^\mu_{[i} V^\nu_{j]}$$

$$+ \partial_\mu V_{\nu j} \, V^\mu_{[k} V^\nu_{i]} \Big)$$

⊙ Example of S^2

metric $\qquad ds^2 = L^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

zweibein $\qquad V^\theta = L\, d\theta \qquad V^\varphi = L \sin\theta\, d\varphi$

$$\Rightarrow \quad ds^2 = (V^\theta)^2 + (V^\varphi)^2$$

Spin connection $\qquad \omega^\theta{}_\varphi = \omega_{\theta\varphi} = -\omega_{\varphi\theta} = -\omega^\varphi{}_\theta$

$dV^\theta = 0 = -\omega^\theta{}_\varphi \wedge V^\varphi \qquad \Rightarrow \qquad \omega^\theta{}_\varphi = A\, V^\varphi$

since $\quad V^\theta \wedge V^\varphi \neq 0 \quad$ and $\quad V^\varphi \wedge V^\varphi = 0$

$$dV^\varphi = L \cos\theta\, d\theta \wedge d\varphi = \frac{1}{L}\frac{\cos\theta}{\sin\theta}(L\, d\theta) \wedge (L \sin\theta\, d\varphi)$$

$$= \frac{\operatorname{ctg}\theta}{L} V^\theta \wedge V^\varphi = \omega^\varphi{}_\theta \wedge V^\theta = A\, V^\varphi \wedge V^\theta$$

$$\Rightarrow \quad A = -\frac{\operatorname{ctg}\theta}{L}$$

$$\Rightarrow \quad \omega^\theta{}_\varphi = -\frac{\operatorname{ctg}\theta}{L} V^\varphi = -\cos\theta\, d\varphi$$

Curvature

$$R^\theta{}_\varphi = d\omega^\theta{}_\varphi - \omega^\theta{}_i \wedge \omega^i{}_\varphi = d\omega^\theta{}_\varphi =$$

$$= \sin\theta\, d\theta \wedge d\varphi = \frac{1}{L^2} V^\theta \wedge V^\varphi = \frac{1}{2} \cdot 2\, R^\theta{}_{\varphi|\theta\varphi} V^\theta \wedge V^\varphi$$

$$\Rightarrow \quad R^\theta{}_{\varphi|\theta\varphi} = -R^\theta{}_{\varphi|\varphi\theta} = \frac{1}{L^2}$$

- $Ric_{\theta\theta} = R^{i}{}_{\theta|i\theta} = R^{\varphi}{}_{\theta|\varphi\theta} = R_{\theta\varphi|\theta\varphi} = \frac{1}{L^2}$

  $Ric_{\varphi\varphi} = R^{\theta}{}_{\varphi|\theta\varphi} = \frac{1}{L^2}$

- $R = g^{ij} Ric_{ij} = \frac{2}{L^2}$