— II CONNESSIONI SENZA METRICA

Metrics and Riemann manifolds

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We have done all what we could do without being able of defining distances and angles, more precisely.

We are not able

- to compute distances
- to compute the modulus of the velocity
- to compare vectors at different points (unless they are connected by a diffeo) i.e. how to transport them from a point to another

The 2 problems are disconnected, in principle but they can be connected requiring that the length is unchanged when transported.

A pseudo Riemannian manifold is a differentiable manifold endowed by a $C^\infty(M)$ family (in any chart all components $g_{ij} \in C^\infty$) symmetric, bilinear non degenerate form on T_pM

$g_p(\ ,\ ) : T_pM \times T_pM \to \mathbb{R}$ such that for all $X, Y \in \Gamma(TM, X)$ $g_p(X_p, Y_p) \in C^\infty(M)$

In formulae
$$g(X, Y) = g(Y, X) \qquad X, Y \in \Gamma(TM, M)$$
$$g(aX + bY, Z) = a\, g(X, Z) + b\, g(Y, Z)$$
$$\forall X \ g(X, Y) = 0 \implies Y = 0$$

- A Riemannian manifold is pseudo + $\quad g(X, X) \geq 0 \quad g(X, X) = 0 \implies X = 0$
  g is positive definite

- A Minkowskian manifold is pseudo + $\quad g_p$ with signature $(-, +^n)$

We can now define the length of a vector

$\|X\|^2 = g(X,X)$

the angle between two vectors

$\|X\|\,\|Y\|\cos\theta_{XY} = g(X,Y)$ ex $\cos\theta_{\mu\nu} = \frac{g_{\mu\nu}}{\sqrt{g_{\mu\mu}\, g_{\nu\nu}}}$

and the length of a curve $\gamma : [0,1] \to M$ as

$\ell(\gamma) = \int_0^1 dt \sqrt{g(\dot\gamma(t), \dot\gamma(t))}$

since $\dot\gamma(t)$ is a vector

In a chart since $dx^\mu(\vec\partial_\nu) = \delta^\mu_\nu$ $X = X^\mu \vec\partial_\mu$

we can write

$g(X,Y) = g_{\mu\nu(\alpha)}(x_{(\alpha)})\, X^\mu_{(\alpha)} Y^\nu_{(\alpha)} \Rightarrow g = g_{\mu\nu(\alpha)}(x_{(\alpha)})\, dx^\mu_{(\alpha)} \otimes dx^\nu_{(\alpha)}$

If we write $dP = dx^\mu \vec e_\mu = dx^\mu \vec\partial_\mu$ we see

- When can we put a metric on a manifold M?

- Given a metric g we can define a distance

$d(P,Q) = \inf\{\ell(\gamma) \,/\, \gamma \in C^1(M) \quad \gamma(0)=P \quad \gamma(1)=Q\}$

. The inf and not min is necessary since on non compact spaces we are not granted to reach inf by a curve ex $R^2-\{0\}$ $P=(-1,0)$ $Q=(1,0)$

$R^2-\{0\}$

. There can be more ways of attaining inf ex S^2 and the two poles

The equation for the geodetics

We want to find the minimum of $S(c)$.

We consider the extremum. In a chart U_α we have

$$S[x^\mu] = \int_0^1 d\sigma \sqrt{g_{\mu\nu}(x(\sigma)) \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}$$

This is a lagrangian with coordinates x : $\left(\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{\ }}\right)^\cdot - \frac{1}{2\sqrt{\ }}\partial_\mu g_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0$

We can apply Euler-Lagrange eq.s directly but there is a $\sqrt{\ }$

so we resort to a trick and we introduce an einbein e

$$S_{(1)}[x^\mu, e] = \frac{1}{2}\int d\sigma \left(e_\sigma g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{e_\sigma}\right)$$

e is auxiliary and non propagating, i.e. we have not an dynamical eq of motion for e, the $\frac{\partial L}{\partial \dot{e}} = 0$ (constrained dynamics)

δe : $\dot{x}^2 - \frac{1}{e_\sigma^2} = 0 \Rightarrow \frac{1}{e_\sigma} = \sqrt{\dot{x}^2}$

$\Rightarrow S_{(1)}[x, e_\sigma^{-1} = \sqrt{\dot{x}^2}] = \frac{1}{2}\int d\sigma \sqrt{\dot{x}^2} + \sqrt{\dot{x}^2} = \int d\sigma \sqrt{\dot{x}^2} = S[x]$

we can then derive the eq. of motion

δx^μ $\frac{d}{d\sigma}(e\, g_{\mu\nu}\dot{x}^\nu) - \frac{1}{2}e\,\partial_\mu g_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0 \Rightarrow \left(\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{\dot{x}^2}}\right)^\cdot - \frac{1}{2}\frac{\partial_\mu g_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}{\sqrt{\dot{x}^2}} = 0$ ok

If we change parametrization on $\sigma = \sigma(\lambda)$ $e_\sigma = \frac{d\sigma}{d\lambda}e_\lambda \Leftrightarrow e$

$$S_{(1)} = \frac{1}{2}\int d\lambda \frac{d\sigma}{d\lambda}\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\left(\frac{d\lambda}{d\sigma}\right)^2 e_\lambda\frac{d\sigma}{d\lambda} + \frac{1}{e_\lambda}\frac{d\lambda}{d\sigma}\right)$$
$$= \frac{1}{2}\int d\lambda \left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}e_\lambda + \frac{1}{e_\lambda}\right)$$

Then we can change integration variable $\sigma \to \lambda$ s.t. $e_\lambda = 1$

So we get the eq. of motion

$$\delta\frac{1}{2}\int d\lambda\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 1\right) = 0 \Rightarrow \frac{d}{d\lambda}\left(g_{\mu\nu}\frac{dx^\nu}{d\lambda}\right) - \frac{1}{2}\partial_\mu g_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

or $g_{\mu\nu}\frac{d^2x^\nu}{d\lambda^2} + \frac{1}{2}\left(\partial_\sigma g_{\mu\rho} + \partial_\rho g_{\mu\sigma} - \partial_\mu g_{\rho\sigma}\right)\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$

or $\frac{d^2x^\mu}{d\lambda^2} + \underbrace{\frac{1}{2}g^{\mu\nu}\left(\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\nu\rho} - \partial_\nu g_{\rho\sigma}\right)}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$

$A^\mu = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$ $\Gamma^\mu_{\rho\sigma}$ Christoffel symbols not tensors!

but setting $c_\lambda = 1$ does not eliminate this com so we need to impose

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 1 = \left(\frac{d\tau}{d\lambda}\right)^2 \leftarrow \text{"proper time" } \tau$$

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notice $a^\mu = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = A^\mu \left(\frac{d\lambda}{d\tau}\right)^2 = A^\mu$

Examples

1) $M = \mathbb{R}^n$ $g_{\mu\nu} = \delta_{\mu\nu}$

$$\frac{d^2 x^\mu}{d\lambda^2} = 0 \Rightarrow x^\mu = x^\mu_0 + v^\mu_0 \lambda$$

constraint $\|\dot{x}\|^2 = \delta_{\mu\nu} v^\mu_0 v^\nu_0 = v_0^2 = 1$

We can always choose $\|v_0\| = 1$ by rescaling the λ!
But we are left with the direction

2) $M = \mathbb{R}^2$ $ds^2 = dr^2 + r^2 d\theta^2$

$$S = \frac{1}{2}\int d\lambda \; \dot{r}^2 + r^2 \dot{\theta}^2 + 1$$

$\delta\theta$ $(r^2\dot{\theta})^\cdot = 0 \to r^2\ddot{\theta} + 2\dot{r}r\dot{\theta} = 0 \to \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0$ $\quad \Gamma^\theta_{r\theta} = \frac{1}{r}$

δr $\ddot{r} - r\dot{\theta}^2 = 0$ $\quad \Gamma^r_{\theta\theta} = -r$

hence Γ depend on the metric and can be non-trivial also on a trivial space

Solving directly

$r^2\dot{\theta} = r^2_0 \dot{\theta}_0 = L$,

$$\to \ddot{r} - \frac{L^2}{r^3} = 0 \to (\dot{r}^2)^\cdot + \left(\frac{L^2}{r^2}\right)^\cdot = 0 \to \dot{r}^2 + \frac{L^2}{r^2} = E$$

$$\to \frac{dr}{\sqrt{E - \frac{L^2}{r^2}}} = \pm d\lambda \qquad \frac{\sqrt{E - \frac{L^2}{r^2}}\; r}{E} = \pm(\lambda - \lambda_0)$$

$$\to r^2 = \frac{L^2 + E^2(\lambda - \lambda_0)^2}{E}$$

$$\to d\theta = \frac{L E}{L^2 + E^2(\lambda - \lambda_0)^2} d\lambda \to \theta = \tilde{\theta}_0 + \arctg\left(\frac{E}{L}(\lambda - \lambda_0)\right)$$

Constraint $\dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + \frac{L^2}{r^2} = 1 \Rightarrow E = 1$

If $\nabla_{e_\mu} \nu = e_\kappa \Gamma^{\nu\kappa}_{\mu} \nu$ $\qquad \nabla_{\partial/\partial x^\mu} \partial/\partial x^\nu = \frac{\partial}{\partial x^\kappa} \Gamma^{\kappa}_{\mu\nu}$ (59.1)

then $\nabla_{\dot\gamma} \dot\gamma = 0$

Questions 1) $\dot\gamma^\mu \frac{\partial}{\partial x^\mu}\Big|_{\zeta} \dot\gamma \Rightarrow \frac{\partial}{\partial x^\mu} \dot\gamma(t)$ meaning?

2) $\dot\gamma^\mu = \frac{d\gamma}{d\tau}$ or $\dot\gamma = \frac{d\gamma}{d\lambda}$ or $\dot\gamma = \frac{d\gamma}{dt}$?

Notice $\partial_\tau = \dot\lambda\, \partial_\lambda \qquad \partial_\tau^2 = \ddot\lambda\, \partial_\lambda + \dot\lambda^2 \partial_\lambda^2$

$\ddot x + \Gamma \dot x \dot x = \ddot\lambda x' + \dot\lambda^2 x'' + \Gamma x' x' \dot\lambda^2 = \ddot\lambda x' + \dot\lambda^2 (x'' + \Gamma x' x') = 0$

$\Rightarrow x'' + \Gamma x' x' = -\frac{\ddot\lambda}{\dot\lambda^2} x' = \left(\frac{1}{\dot\lambda}\right)^{\cdot} x' = (t')^{\cdot} x' = \frac{t''}{t'} x' = (\ln t')' x' = f(\lambda)\, x'$

when $f(\lambda) = 0$ the parametrization is called affine

Notice that $(\dot x^2)^{\cdot} = 2 g_{\lambda\mu} \dot x^\lambda \ddot x^\mu + \dot x^\lambda \dot x^\rho \partial_\rho g_{\lambda\sigma} \dot x^\sigma$

$= 2 \dot x^\lambda \left(g_{\lambda\mu} \ddot x^\mu + \frac{1}{2} \partial_\rho g_{\lambda\sigma} \dot x^\rho \dot x^\sigma \right) = 2 \dot x^\lambda \left((g_{\lambda\mu} \dot x^\mu)' - \frac{1}{2} \partial_\rho g_{\lambda\sigma} \dot x^\rho \dot x^\sigma \right)$

$= 2 \dot x^\lambda \left(\frac{1}{2} \partial_\lambda g_{\rho\sigma} \dot x^\rho \dot x^\sigma + g_{\lambda\rho} \dot x^\rho f(\lambda) - \frac{1}{2} \partial_\rho g_{\lambda\sigma} \dot x^\rho \dot x^\sigma \right)$

$= 2 f(\lambda)\, \dot x^2 \Longrightarrow 0$ if $f(\lambda) = 0$ hence we can fix $\dot x^2 = \text{const}$

- Assume $\nabla g = e^{*\mu} \otimes e^{*\nu} \otimes (\partial g_{\mu\nu} - g_{\mu\rho} \Gamma^\rho_{\ \nu} - g_{\nu\rho} \Gamma^\rho_{\ \mu}) = 0$

$\Rightarrow \partial_\lambda g_{\mu\nu} = \Gamma_{\lambda;\nu\mu} + \Gamma_{\lambda;\mu\nu}$ where $g_{\mu\rho} \Gamma^\rho_{\lambda,\nu} = \Gamma_{\lambda;\nu\mu}$ symm part of

then
$$\left.\begin{aligned} \partial_\lambda g_{\mu\nu} &= \Gamma_{\lambda;\nu\mu} + \Gamma_{\lambda;\mu\nu} \\ -\partial_\mu g_{\lambda\nu} &= -\Gamma_{\mu;\lambda\nu} - \Gamma_{\mu;\nu\lambda} \\ \partial_\nu g_{\mu\lambda} &= \Gamma_{\nu;\mu\lambda} + \Gamma_{\nu;\lambda\mu} \end{aligned}\right\} \Rightarrow \partial_\lambda g_{\mu\nu} - \partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} = 2\Gamma_{[\lambda,\mu]\nu} + 2\Gamma_{[\nu;\mu]\lambda} + \Gamma_{\lambda;\nu\mu} + 2\Gamma_{[\lambda;\nu]\mu} +$$

$$\Gamma^\rho_{\lambda\mu} = \frac{1}{2} g^{\rho\nu} \left(\partial_\lambda g_{\nu\mu} + \partial_\nu g_{\lambda\mu} - \partial_\mu g_{\lambda\nu} + \underbrace{2\Gamma_{[\mu,\lambda]\nu} + 2\Gamma_{[\mu;\nu]\lambda} - \ldots}_{} \right)$$

symm. part ; anti-symm. mixed part

where $\Gamma^{\ \nu}_{[\mu;\lambda]} = \frac{1}{2} T^\nu_{\ \mu\lambda}$ is the torsion

Notice that $\nabla_X Y - \nabla_Y X = \left([X,Y]^\mu + 2 X^\rho Y^\sigma \Gamma^\mu_{[\rho\sigma]} \right) e_\mu$

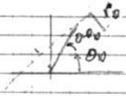
requiring $\nabla_X Y - \nabla_Y X = [X,Y]$ we get the christoffel symbol
$+ \ldots T(X,Y)$

or $$\frac{E(t-t_0)}{L} = \operatorname{tg}(\vartheta-\tilde{\vartheta}_0)$$ (60)

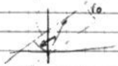
so $$r^2 = \frac{L^2}{E}\left(1+\operatorname{tg}^2(\vartheta-\tilde{\vartheta}_0)\right) = \frac{L^2}{E}\,\frac{1}{\cos^2(\vartheta-\tilde{\vartheta}_0)}$$

$$r\cos(\vartheta-\tilde{\vartheta}_0) = \pm\frac{L}{\sqrt{E}}$$

This is a line, as expected



$\vartheta \cdot \tilde{\vartheta}_0$ $r\sin(\vartheta-\vartheta_0) = r_0$



$-\vartheta_0 = \frac{\pi}{2} - \tilde{\vartheta}_0$

Notice we could save some steps by noticing that the previous action can be interpret as Lagrangian and so because of the symmetry $\vartheta \to \vartheta + \hat{\vartheta}_0$ we would find an integral of motion L and because of "time" shift we would find the energy conservation E

3) S^2 $ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$

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$S = \frac{1}{2}\int d\lambda \;\; \dot\theta^2 + \sin^2\theta \, \dot\varphi^2$

$p_\varphi = \sin^2\theta \dot\varphi \qquad \rightarrow \qquad \dot p_\varphi = 0$

$p_\theta = \dot\theta$

$\rightarrow \quad H = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} - \frac{1}{2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) = \frac{1}{2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} \right) = \frac{1}{2}\left(\dot\theta^2 + \frac{L^2}{\sin^2\theta}\right) = E = \frac{1}{2}$

used $p_\varphi = L$ conserved

$\frac{d\theta}{d\lambda} = \pm \sqrt{2E - \frac{L^2}{\sin^2\theta}} \qquad \frac{d\theta}{\sqrt{2E - \frac{L^2}{\sin^2\theta}}} = \pm d\lambda$

$\frac{\sin\theta \, d\theta}{\sqrt{2E \sin^2\theta - L^2}} = \frac{d(-\cos\theta)}{\sqrt{2E(1-\cos^2\theta) - L^2}} = \frac{-dx}{\sqrt{2E - L^2 - 2Ex^2}} = \frac{-dx}{\sqrt{2E}\sqrt{\left(1 - \frac{L^2}{2E}\right) - x^2}}$

$\Rightarrow -\frac{1}{\sqrt{2E}} \arcsin \frac{x}{\sqrt{1 - \frac{L^2}{2E}}} = \pm(\lambda - \lambda_0) \qquad$ notice $\frac{L^2}{2E} = L^2 < 1$

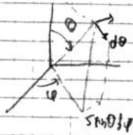
$\cos\theta = \mp\sqrt{1 - \frac{L^2}{2E}} \; \sin\left(\sqrt{2E}\,(\lambda - \lambda_0) \right)$

$\frac{d\varphi}{d\lambda} = \frac{L}{1 - \left(1 - \frac{L^2}{2E}\right)\sin^2\left(\sqrt{2E}\,(\lambda - \lambda_0)\right)}$

$\varphi - \varphi_0 = \frac{L/\sqrt{2E}}{\sqrt{L^2/(2E)}} \arctg\left[\sqrt{\frac{L^2}{2E}} \; tg\left(\sqrt{2E}\,(\lambda - \lambda_0) \right) \right] = \arctg\left[\frac{L}{\sqrt{2E}} \, tg \sqrt{2E}\,(\lambda - \lambda_0) \right]$

Why have we chosen this particular metric on S^2? (62)

Because we have induced the metric using the immersion of S^2 in R^3



$$ds^2 = d\theta^2 + (\sin\theta\, d\varphi)^2$$

We can always pull back the metric.

Given $f: M \to N$ and a metric on g

we can define f^*g the metric on M as

$$(f^*g)_p(X_p, Y_p) \qquad X, Y \in T_pM$$

$$= g_{f(p)}\big((f_*X)_{f(p)}, (f_*Y)_{f(p)}\big)$$

In components $\quad p = (x^\mu) \quad f(p) = (y^m)$

$$(f^*g)_p\big(X^\mu \partial_\mu|_p, Y^\nu \partial_\nu|_p\big) = X^\mu Y^\nu (f^*g)_{\mu\nu}(x)$$

$$= g\big(X^\mu \partial_\mu y^m \partial_m|_y, Y^\nu \partial_\nu y^n \partial_n|_y\big)$$

$$= X^\mu \partial_\mu y^m|_x \, Y^\nu \partial_\nu y^n|_x \, g_{mn}(y = f(x))$$

$$\Rightarrow (f^*g)_{\mu\nu}(x) = g_{mn}(f(x))\, \partial_\mu y^m|_x \, \partial_\nu y^n$$

In our case $\quad (x^\mu) = (\theta, \varphi) \quad (y^m) = (x, y, z) = (\vec{x})$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases}$$

$$f^*g_{\theta\theta} = \partial_\theta \vec{x} \cdot \partial_\theta \vec{x} = (\cos\theta\cos\varphi)^2 + (\cos\theta\sin\varphi)^2 + (-\sin\theta)^2 = 1$$

Notice that the length of a tangent vector to a curve can be understood as

$$(\gamma^* g)(\partial_t, \partial_t)$$

But in the picture we used a different approach..

We defined two 1-form

$\delta P_\theta = d\theta \qquad \delta P_\varphi = \sin\theta\, d\varphi$ — δ since not globally d of something

and we supposed $ds^2 = (\delta P_\theta)^2 + (\delta P_\varphi)^2$

Why does this work?

We have a metric for any point we choose an orthogonal frame

$\vec{e}_{i\,p} = V_i^{\ \mu}\, \partial_\mu \qquad g(e_i, e_j) = \delta_{ij} \qquad i, j = 1..n \quad \mu = 1..n$

This amounts to choose a $GL(n, \mathbb{R})$ matrix V_i^{μ} such that

$g_{\mu\nu}\, V_i^{\mu}\, V_j^{\nu} = \delta_{ij}$ define $V^{-1} = \| V_i^{\ \mu} \| \qquad V^{-1} g\, V^{-1\,T} = \mathbb{1}$

or $\| V^i_{\ \mu} \| = V \qquad g_{\mu\nu} = V^i_{\ \mu}\, \delta_{ij}\, V^j_{\ \nu} \Rightarrow V^T \mathbb{1}\, V = g$

This V can be chosen to vary smoothly since g varies smoothly.

→ where - $V^i_{\ \mu}$ is not unique since $V' = OV \quad O \in O(n)$ does the same job

- $\vec{e}_i$ is a moving frame and V are the vielbein

- This is what we have done before, in another way

← We can also define the dual 1-form as

$V^i = V^i_{\ \mu}\, dx^\mu$

s.t. $V^i(\vec{e}_j) = V^i_{\ \mu}\, dx^\mu(\vec{e}_j) = V^i_{\ \mu}\, V_j^{\ \nu}\, dx^\mu(\partial_\nu)$

$= V_\mu^{\ i}\, \delta^\mu_\nu\, V_j^{\ \nu} = V_j^{\ \mu}\, V_\mu^{\ i} = \delta^j_i$

Then

$ds^2 = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = \delta_{ij}\, V^i_{\ \mu}\, V^j_{\ \nu}\, dx^\mu \otimes dx^\nu =$

$= \delta_{ij}\, V^i \otimes V^j$

Using this distance the metric seems orthogonal

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Ex
$$ds^2 = (1+r^2\omega^2)\,dz^2 + dr^2 + 2r^2\omega\,dz\,d\alpha + r^2 d\alpha^2$$
$$= (dz)^2 + (dr)^2 + [r(d\alpha + \omega dz)]^2$$
$$= \left[\sqrt{1+r^2\omega^2}\left(dz + \frac{r^2\omega}{1+r^2\omega^2}d\alpha\right)\right]^2 + dr^2 + \left(r^2 - \frac{r^4\omega^2}{1+r^2\omega^2}\right)d\alpha^2$$

$$\begin{pmatrix} V'^z \\ V'^r \\ V'^\alpha \end{pmatrix} = \begin{pmatrix} \sqrt{1+r^2\omega^2}\,dz + \frac{r^2\omega}{\sqrt{}}\,d\alpha \\ dr \\ \frac{d\alpha}{\sqrt{1+r^2\omega^2}} \end{pmatrix} = O\cdot V = O\begin{pmatrix} dz \\ dr \\ r(d\alpha+\omega dz) \end{pmatrix}$$

where O is orthogonal

Consider the metric in R^3 in polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

then it is natural to define

$$\hat{V}^r = dr$$
$$\hat{V}^\theta = r d\theta \qquad \Rightarrow \quad ds^2 = \hat{V}_r \otimes \hat{V}_r + \hat{V}_\theta \otimes \hat{V}_\theta + \hat{V}_\varphi \otimes \hat{V}_\varphi$$
$$\hat{V}^\varphi = r \sin\theta \, d\varphi$$

Again we can restrict these vierbein to S^2 $r=1$, i.e. we can do the pullback on S^2 to get

$$f_* V^r = 0$$
$$V^\theta = f_* \hat{V}^\theta = d\theta$$
$$V^\varphi = f_* \hat{V}^\varphi = \sin\theta \, d\varphi$$

and then we can compute the metric on

$$ds^2 = V^\theta \otimes V^\theta + V^\varphi \otimes V^\varphi = f_*(ds^2_{\mathbb{R}^3})$$

We can write explicitly the matrix V^ν_μ as $\nu = \theta, \varphi$, $\mu = 1, 2$

$$V = \begin{array}{c|cc} & 1 & 2 \\ \hline \theta & 1 & 0 \\ \varphi & 0 & \sin\theta \end{array} \qquad V^{-1} = \begin{pmatrix} 1 & \\ & \frac{1}{\sin\theta} \end{pmatrix}$$

so that the moving frame is given by

$$e_\theta = \frac{\partial}{\partial\theta}$$

$$e_\varphi = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$$

Bianchi identities

Simplest example

$F = dA$

$dF = 0$ Bianchi ($*d*F = j$ true Maxwell equation)

We define

$T^i = dV^i + \omega^i{}_j V^j$ (suppose $\neq 0$)

$R^i{}_j = d\omega^i{}_j + \omega^i{}_k \omega^k{}_j$

and we derive

$dT^i = ..$

$(\mathcal{D}T)^i = dT^i + \omega^i{}_k \wedge T^k = \underset{\uparrow}{R^i{}_j \wedge V^j}$

therefore we introduce the curvature

Then we get

$(\mathcal{D}R)^i{}_j = dR^i{}_j + \omega^i{}_k \wedge R^k{}_j - R^i{}_k \wedge \omega^k{}_j \Rightarrow$

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Consequences

Expand

$T^i = \frac{1}{2} T^i{}_{kl} \, V^k \wedge V^l = 0$ ← assume if not use $\mathring{\omega}^i{}_j = \omega^i{}_j + \frac{1}{2} T^i{}_{jk}$

$dx \; \overset{\circ}{T}{}^i_j = T^i_j + \frac{1}{2} T^i{}_{jk} V^k) \wedge V^l$

$R^i{}_j = \frac{1}{2} R^i{}_{jkl} \, V^k \wedge V^l$ $R^i{}_{jkl}$ is the Riemann tensor!

- 1^{st} Bianchi identity

$0 = R^i{}_k \wedge V^k = \frac{1}{2} R^i{}_{[jkl]} \, V^j \wedge V^k \wedge V^l = \frac{1}{6} \left(R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} \right) V^{jkl}$

$\Rightarrow R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0$

Consequence $R_{ij|kl} = R_{kl|ij}$

$R_{ij|kl} - R_{kl|ij} = R_{ij|kl} - (-R_{ki|jl} - R_{kj|li})$

$= R_{ij|kl} + R_{ik|lj} + R_{kj|li}$

$= (\qquad\qquad) + R_{kj|li}$

$= -R_{il|jk} + R_{kj|li}$

$= R_{li|jk} - R_{jk|li}$ $\qquad ij|kl \to li|jk$

reapply same rule $li|jk \to kl|ij$

$= R_{kl|ij} - R_{ij|kl}$

$\Rightarrow R_{ij|kl} = R_{kl|ij}$

We can define the Ricci tensor

$Ric_{jk} = R^{i}{}_{j|ik}$ $\qquad$ Ric_{jk} in order not to confuse with $R^{j}{}_{i}$

contracting the previous Bianchi we get

$Ric_{jl} + R^{i}{}_{i|lj} - Ric_{lj} = 0$

if $\omega^{ij} = -\omega^{ji}$ then $R_{ji} = d\omega_{ji} + \omega_{jk}\omega^{k}{}_{i} = -d\omega_{ij} - \omega_{i}{}^{k}\omega_{kj}$
$= -R_{ij}$

so we get

$Ric_{jl} = Ric_{lj}$

Therefore we can define the curvature scalar

$R = g^{ij} Ric_{ij}$
better g^{ij} is g_{ij}

- Independent components

$R_{[ij]|[kl]}$ $\quad \left(\frac{n(n-1)}{2}\right)^2 = \frac{n^2(n-1)^2}{4}$

$R_{i[j|kl]} = T_{i|jkl}$ $\quad n \cdot \frac{n(n-1)(n-2)}{6} = \frac{n^2(n-1)(n-2)}{6}$

$\# R = \frac{n^2(n-1)}{12}\left[3(n-1) - 2(n-2)\right] = \frac{n^2(n-1)}{12}(n+1) = \frac{n^2(n^2-1)}{12}$

2. Bianchi identity

$$\textcircled{D} R^i{}_j = \frac{1}{2} \mathcal{D}_m R^i{}_{jkl} \quad V^m V^k V^l + R^i{}_{jkl} \; T^k{}_n V^l = 0$$

when $T=0$

$$3\, \mathcal{D}_{[m} R^i{}_{j|kl]} = \mathcal{D}_m R^i{}_{j|kl} + \mathcal{D}_k R^i{}_{j|lm} + \mathcal{D}_l R^i{}_{j|mk} = 0$$

Contract with δ^k_i to get

$$\mathcal{D}_m R_{kjl} + \mathcal{D}_k R^k{}_{jlm} - \mathcal{D}_l R_{kjm} = 0$$

or $\mathcal{D}_m R^k_{lj} - \mathcal{D}_l R^k_{mj} + \mathcal{D}_k R^k{}_{jlm} = 0$

Contract g^{jl} and use metric $R_{kjl} = -R_{jkl}$ to get

$$\mathcal{D}_m R - \mathcal{D}_l R^l_m - \mathcal{D}_k R^k_m = 0$$

or $\mathcal{D}_k R^k_m = \frac{1}{2} \mathcal{D}_m R$

which can be rewritten as

$$\mathcal{D}_k \left(R^k_m - \frac{1}{2} \delta^k_m R \right) = 0$$

The Einstein equation for the vacuum (no matter in the space where they apply) are

$G^k{}_m = R^k_m - \frac{1}{2} \delta^k_m R = 0$

↑ Einstein tensor

The full Einstein equations are

$$G_{km} = R_{km} - \frac{1}{2} g_{km} R = \underbrace{\text{const}}_{\frac{8\pi G}{c^4}} \; T_{km} - \Lambda g_{km}$$

↑ cosmological constant

↑ conserved tensor which represents the matter

$T_{km} = T_{mk}$ $\qquad \mathcal{D}_k T^{km} = 0$

↑ the energy-momentum tensor