SHORT INTRODUCTION TO TENSOR PRODUCT, EXTERNAL ALGEBRA AND FORMS IN FLAT SPACE

1) Which is the difference among
$R \times R = R^2$, $R \oplus R$, $R \otimes R$?

The easiest way we a base: $x\vec{i} \in R$ $y\vec{j} \in R$

$R^2 = \{ (x\vec{i}, y\vec{j}) \}$ 2 dim
$R \oplus R = \{ x\vec{i} + y\vec{j} \}$ 2 dim
$R \otimes R = \{ u \; \vec{i} \otimes \vec{j} \}$ 1 dim

Notice
- $\times$ is defined for all sets
- $E \oplus F$, $E \otimes F$ require the concept of sum and scalar multipl.

2) $R^5 \simeq R^3 \oplus R^2$ $R^3 \otimes R^2$

$R^5 \ni \sum_{i=1}^{3} x^i \vec{e}_i + \sum_{a=1}^{2} y^a \vec{f}_a$ 5 dim

$R^3 \otimes R^5 \ni \sum_{i=1}^{3} \sum_{a=1}^{2} u^{ia} \; \vec{e}_i \otimes \vec{f}_a$ 6 dim

Notice
- generically $u \in R^3 \otimes R^5$ can NOT be written as $(\sum x^i \vec{e}_i) \otimes (\sum y^a \vec{f}_a)$

Count d.o.f. $\# u^{ia} = 3 \cdot 2 = 6$ $\# x^i + \# y^a = 3+2=5$

- Already seen in Q.M.

  $|\psi_{tot}\rangle \Leftrightarrow \psi(x)\ \psi_{spin}$

  is actually $\quad |\psi(x)\rangle \otimes |\psi_{spin}\rangle$

  and $\quad |\psi_{tot}\rangle = \sum c_{x,s}\ |\psi_x(x)\rangle \otimes |\psi_s\rangle$

- Entangled state $\quad |1,0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\rangle_1 \otimes |-\tfrac{1}{2}\rangle_2 + |-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}\rangle_2 \right)$

- Think of density matrix

  States which can be written as $\rho = |\psi\rangle\langle\psi|$ are pure states but generic ρ can be written as

  $\rho = \sum c_i\ |\psi_i\rangle\langle\psi_i|$

TENSORS

- They are generalization of vectors, covectors and matrices.
- Take a vector space V with basis $\{\vec{e}_i\}$

A vector is $v = v^i e_i$

Why write v and not (v^i)?

v is an entity, a concept which happens to have components v^i w.r.t. a basis $\{e_i\}$

Suppose we change basis $\tilde{e}_i = \Lambda_i{}^j e_j$, the vector v is unchanged, its components no

$$v = v^i e_i = \tilde{v}^i \tilde{e}_i = \tilde{v}^i \Lambda_i{}^j e_j$$

$$\Rightarrow \quad v^i = \tilde{v}^i \Lambda_i{}^j \qquad \text{or} \qquad \tilde{v}^i = v^j (\Lambda^{-1})_j{}^i$$

HENCE it is better to work with concepts
BUT computations are done with components.

- We can construct the dual space V^* as the space of linear function on V

$$f \in V^* \iff f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)$$

In particular V^* is a vector space and has a basis.

Given a basis for V we can easily construct one for V^*, explicitly given $\{e_i\}$ we define the linear functions $\{e^i\}$ so that

$$e^i(v) = v^j e^i(e_j) = v^j \delta^i_j = v^i$$

Now for a generic f we have

$$f(v) = v^i f(e_i) = v^i f_i = f_i e^i(v)$$

hence $f = f_i e^i$

- Now $f(v)$ can be written as $v(f)$, i.e.

$$f(v) = v(f)$$

i.e. $V^{**} = V$

$$v(f) = v^i e_i(f) = v^i f_j e_i(e^j) = v^i f_j \delta_i^j$$

- $\tilde{e}_i = \Lambda_i{}^j e_j \Rightarrow \tilde{e}^l = (\Lambda^{-1})_m{}^l e^m = (\Lambda^{-T})^l{}_m e^m$

chk $\tilde{e}^l(\tilde{e}_i) = (\Lambda^{-T})^l{}_m \Lambda_i{}^j \delta_j^m = (\Lambda \Lambda^{-1})_i{}^l = \delta_i^l$

- $\tilde{f}_i = f(\tilde{e}_i) = \Lambda_i{}^j f_j$

- A matrix $\|M_{ij}\|$ can be thought as a bilinear function and we can write

$$M = M_{ij}\, e^i \otimes e^j$$

so that $M(u, v) = u^l v^m \, M(e_l, v^m)$

$$= u^l v^m \, M_{ij} \, e^i(e_l) \, e^j(e_m)$$

$$= M_{ij} \, u^i v^j$$

The entries M_{ij} can be computed as

$$M(e_i, e_j) = M_{ij}$$

Formally we write $M: V \times V \longrightarrow \mathbb{R}$

or $M \in V^* \otimes V^*$

- Tensors are generalization of matrices

$$T: \underbrace{V^* \times \quad \times V^*}_{m} \times \underbrace{V \times \quad \times V}_{n} \longrightarrow \mathbb{R}$$

components $T(e^{i_1}, \; e^{i_m}, e_{j_1}, \; e_n) = T^{i_1 \dots i_m}{}_{j_1 \dots j_n}$

transformation

$$\tilde{T}(\tilde{e}^{i_1}, \dots \tilde{e}_{j_n}) = \tilde{T}^{i_1 \dots i_m}{}_{j_1 \dots j_n} =$$

$$= (\Lambda^{-T})^{i_1}{}_{l_1} \dots \Lambda_{j_n}{}^{m_n} \; T(e^{l_1} \dots e_{m_n})$$

$$= (\Lambda^{-T})^{i_1}{}_{l_1} \dots (\Lambda^{-T})^{i_m}{}_{l_m} \; \Lambda_{j_1}{}^{m_1} \dots \Lambda_{j_n}{}^{m_n} \; T^{l_1 \dots l_m}{}_{m_1 \dots m_n}$$

- Up to now we have not required a metric which is a bilinear $g(u,v)$, i.e. a tensor $g = g_{ij} e^i \otimes e^j$
- Invertible matrices M_{ab} $\det M \neq 0$ can be used to raise and lower indices. There are two big classes $M = g$, i.e. symmetric matrix, and $M = \omega$, i.e. antisymmetric matrix, which correspond to metric and symplectic structure
- Tensor can be used to define groups.

If we take $M_{ij} = \delta_{ij}$ and we consider transformations which leave it invariant we get the group $O(n)$.

If we consider transformations which leave both δ_{ij} and $\varepsilon_{i_1 \dots i_n}$ invariant we get $SO(n)$.

The procedure can be generalized to exceptional groups!

DIFFERENTIAL FORMS

1) Example, take a function $f(x)$, define

$$df \equiv dx^\mu \partial_\mu f$$

then extract the meaning of $d \equiv dx^\mu \partial_\mu$.

2) Suppose now we have a covector $\omega_\mu(x)$, we can make a 1-form as

$$\omega = \omega_\mu(x)\, dx^\mu$$

Notice it can be integrated on a curve

$$\gamma^\mu : [0,1] \to R^n$$
$$t \mapsto \gamma^\mu(t)$$

as follows

$$\int_\gamma \omega \equiv \int_0^1 dt\, \omega_\mu(\gamma(t)) \frac{d\gamma^\mu}{dt}$$

How does d act on ω?

$$d\omega \Rightarrow \text{"} dx^\mu \partial_\mu \omega_\nu dx^\nu \text{"} \qquad \text{since}$$

$$d\omega = \partial_\mu \omega_\nu\, dx^\mu \wedge dx^\nu = -\partial_\mu \omega_\nu\, dx^\nu \wedge dx^\mu = -\partial_\nu \omega_\mu\, dx^\mu \wedge dx^\nu$$

$$= \frac{1}{2} (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)\, dx^\mu \wedge dx^\nu = \partial_{[\mu} \omega_{\nu]}\, dx^\mu \wedge dx^\nu$$

3) Generalize to 2-form

$$\omega^{(2)} = \frac{1}{2} \omega_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu = \sum_{\mu<\nu} \omega_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu$$

It can be integrated on a 2-surface

$$\gamma_{(2)} : [0,1] \times [0,1] \longrightarrow R^n$$
$$(u,t) \longmapsto \gamma^\mu(u,t)$$

as follows

$$\int_{\gamma_{(2)}} \omega^{(2)} = \int_0^1 du \int_0^1 dt \quad \omega_{\mu\nu}(\gamma(u,t)) \frac{\partial \gamma^\mu}{\partial u} \frac{\partial \gamma^\nu}{\partial t}$$

since naively evaluating $\omega_{(2)}$ on $\gamma_{(2)}$ gives

$$\omega^{(2)}\big|_{\gamma(2)} = \frac{1}{2} \omega_{\mu\nu}(\gamma(u,v)) \left[\frac{\partial \gamma^\mu}{\partial u} du + \frac{\partial \gamma^\mu}{\partial v} dv \right] \wedge \left[\frac{\partial \gamma^\nu}{\partial u} du + \frac{\partial \gamma^\nu}{\partial v} dv \right]$$

$$= \frac{1}{2} \omega_{\mu\nu} \left(\frac{\partial \gamma^\mu}{\partial u} \frac{\partial \gamma^\nu}{\partial v} du \wedge dv + \frac{\partial \gamma^\mu}{\partial v} \frac{\partial \gamma^\nu}{\partial u} dv \wedge du \right)$$

$$= \frac{1}{2} \omega_{\mu\nu} \left(\frac{\partial \gamma^\mu}{\partial u} \frac{\partial \gamma^\nu}{\partial v} - \frac{\partial \gamma^\mu}{\partial v} \frac{\partial \gamma^\nu}{\partial u} \right) du \wedge dv$$

$$= \frac{1}{2} \left(\omega_{\mu\nu} \partial_u \gamma^\mu \partial_v \gamma^\nu - \omega_{\nu\mu} \partial_v \gamma^\nu \partial_u \gamma^\mu \right) du \wedge dv$$

$$= \omega_{\mu\nu} \partial_u \gamma^\mu \partial_v \gamma^\nu \, du \wedge dv$$

The action of d is given by

$$d\omega^{(2)} = \frac{1}{2} \partial_\lambda \omega_{\mu\nu} \, dx^\lambda \wedge dx^\mu \wedge dx^\nu =$$

$$= \frac{1}{3!} 3 \partial_{[\lambda} \omega_{\mu\nu]} \, dx^\lambda \wedge dx^\mu \wedge dx^\nu$$

4) Generalize to k-form

$$\omega^{(k)} = \frac{1}{k!}\, \omega_{\mu_1 \dots \mu_k}(x)\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}$$

· Integrate over k-surface and act with d

Notice

- In R^n there are at most n-forms since

$$dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{n+1}} \equiv 0$$

because of antisymmetry

5) Duals

In R^n we can define an object

$$\varepsilon_{\mu_1 \dots \mu_n}$$

completely antisymmetric with $\varepsilon_{12\dots n} = +1$

Ex R^3 ε_{ijk}

The same is true for $\varepsilon^{\mu_1 \dots \mu_n}$ with indexes raised using $\delta^{\mu\nu}$.

Then we define

$$* \omega(k) = \frac{1}{k!} \frac{1}{(n-k)!}\, \varepsilon_{\mu_1 \dots \mu_{n-k}}{}^{\nu_1 \dots \nu_k}\, \omega_{\nu_1 \dots \nu_k}\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{n-k}}$$

6) Product of forms

Examples

- $\omega_{(1)} \wedge \psi_{(1)} = \omega_\mu dx^\mu \wedge \psi_\nu dx^\nu = \omega_\mu \psi_\nu \, dx^\mu \wedge dx^\nu$

  $= - \omega_\mu \psi_\nu \, dx^\nu \wedge dx^\mu = - \psi_{(1)} \wedge \omega_{(1)}$

- $\omega_{(1)} \wedge \psi_{(2)} = + \psi_{(2)} \wedge \omega_{(1)}$

Generically

$$\omega_{(k)} \wedge \omega_{(l)} = (-)^{kl} \omega_{(l)} \wedge \omega_{(k)}$$

7) $d^2 = 0$

$$dd = dx^\mu \wedge dx^\nu \, \partial_\mu \partial_\nu \quad \wedge \ldots$$

$$= dx^\mu \wedge dx^\nu \, \partial_{[\mu} \partial_{\nu]} \quad \wedge \ldots$$

$$= 0$$

8) Worked examples in R^3, indices $i, j, k = 1,2,3$

- $\omega_{(0)} = f$

$df = dx^i \, \partial_i f = (\vec{\nabla} f)_i \, dx^i$ gradient

$*f = \frac{1}{0!} \frac{1}{3!} f \, \varepsilon_{ijk} \, dx^i dx^j dx^k = f \, dx^1 dx^2 dx^3$

$d^2 f = \partial_i \partial_j f \, dx^i \wedge dx^j = 0$

$d*f = 0$

$*df = \frac{1}{2!1!} \varepsilon_{ij}{}^{k} \, \partial_k f \, dx^i \wedge dx^j$

- $\omega_{(1)} = v_i \, dx^i$

$d\omega_{(1)} = \partial_{[i} v_{j]} \, dx^i \wedge dx^j$

$* \omega_{(1)} = \frac{1}{2!} \frac{1}{1!} \varepsilon_{ij}{}^{k} \, v_k \, dx^i \wedge dx^j$

$* d\omega_{(1)} = \frac{1}{2!1!} \varepsilon_{i}{}^{jk} \, \partial_j v_k \, dx^i = \frac{1}{2} (\vec{\nabla} \times \vec{v})_i \, dx^i$

$d * \omega_{(1)} = \frac{1}{2!1!} \varepsilon_{ij}{}^{k} \, \partial_l v_k \, dx^l \wedge dx^i \wedge dx^j$

$= \frac{1}{2} \varepsilon_{ij}{}^{k} \, \partial_l v_k \, \varepsilon^{lij} \, dx^1 dx^2 dx^3$

$= \frac{1}{2} \, 2 \, \delta^{kl} \, \partial_l v_k \, d^3x$

$= (\vec{\nabla} \cdot \vec{v}) \, d^3x$

- $\omega_{(2)} = \frac{1}{2} B_{ij} \; dx^i \wedge dx^j = \frac{1}{2} \varepsilon_{ijk} \tilde{B}_k \; dx^i \wedge dx^j$

$\Rightarrow$ it is the same of $* \omega_{(1)}$

Green theorem in R^2

$$\int_\gamma f\,dx + g\,dy = \int_S \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right) dx\,dy$$

$$\int_{\partial S} \omega_{(1)} = \int_S d\omega_{(1)}$$

Gauss theorem in R^3

$$\int_V \vec{\nabla}\cdot\vec{f}\, d^3V = \int_{\partial V} \vec{f}\cdot\vec{n}\, d^2S$$

$$\int_V d{*}\omega_{(1)} = \int_{\partial V} {*}\omega_{(1)}$$

More generically

$$\int_{S_{(k+1)}} d\omega_{(k)} = \int_{\partial S_{(k)}} \omega_{(k)}$$

corr [13/01/15]

Maxwell equation (in rationalized MKS)

We are in $D=4$ with metric $\|\eta_{\mu\nu}\| = (-1,1,1,1)$
so we need paying attention when raising and lowering indexes. In particular

$$\varepsilon_{0123} = 1 \Rightarrow \varepsilon^{0123} = -1$$

Start from Lorentz

$$m \frac{d^2 x^\mu}{d\tau^2} = q F^\mu{}_\nu \frac{dx^\nu}{d\tau}$$

write it in components for low speed s.t. $\tau \approx t$ and $x^0 = ct$

Naively

$$\mu=0 \qquad 0 = q\,(F^0{}_i\, \dot{x}^i) \qquad \text{STRANGE ?!}$$

$$\mu=i \qquad m\,\ddot{x}^i = q\,(F^i{}_j\, \dot{x}^j + F^i{}_0\, c) = +q\, F_{i0} c + q\, F_{ij}\, \dot{x}^j$$

$$= q\, E_i + q\, \varepsilon_{ijk}\, \dot{x}^j B^k$$

hence

$$F_{i0} = \frac{1}{c} E_i \qquad F_{ij} = \varepsilon_{ijk} B^k$$

Notice

$$F^0{}_i\, \dot{x}^i = E_i \frac{v^i}{c} \sim 0 \qquad \text{ok!}$$

Homogeneous Maxwell equations

$$\begin{cases} \vec{\nabla}\cdot\vec{B}=0 \\ \vec{\nabla}\times\vec{E}+\dfrac{\partial\vec{B}}{\partial t}=0 \end{cases}$$

are nothing else but

$$\partial_{[\mu}F_{\rho\sigma]}=0 \quad \longleftrightarrow \quad dF=0$$

on introducing $F=\frac{1}{2}F_{\mu\nu}\,dx^\mu\wedge dx^\nu$

Let us check it

$$F=\frac{1}{c}E_i\,dx^i\wedge(c\,dt)+\frac{1}{2}\varepsilon_{ijk}B^i\,dx^j\wedge dx^k$$

$$dF=\partial_\mu E_i\,dx^\mu\wedge dx^i\wedge dt+\frac{1}{2}\varepsilon_{ijk}\partial_\mu B^i\,dx^\mu\wedge dx^j\wedge dx^k$$

$$=\partial_i E_j\,dx^i\wedge dx^j\wedge dt+\frac{1}{2}\varepsilon_{ijk}\,\partial_t B^i\,dt\wedge dx^j\wedge dx^k$$

$$+\frac{1}{2}\varepsilon_{ijk}\,\partial_\ell B^i\,dx^\ell\wedge dx^j\wedge dx^k$$

$$=\left(\partial_{[i}E_{j]}+\frac{1}{2}\varepsilon_{\ell ij}\,\partial_t B^\ell\right)dx^i\wedge dx^j\wedge dt$$

$$+\frac{1}{2}\varepsilon_{\ell jk}\,\partial_i B^\ell\,dx^i\wedge dx^j\wedge dx^k$$

Then we get

$$\varepsilon_{kij} \left(\partial_{[i} E_{j]} + \frac{1}{2} \varepsilon_{lij} \partial_t B^l \right)$$

$$= (\vec{\nabla} \times \vec{E})_k + \frac{1}{2} 2 \delta_{kl} \partial_t B^l =$$

$$= (\vec{\nabla} \times \vec{E})_k + \partial_t B_k \Rightarrow$$

and

$$\varepsilon_{ijk} \frac{1}{2} \varepsilon_{ljk} \partial_i B^l = \delta_{il} \partial_i B^l = \partial_i B^i \Rightarrow$$

Inhomogeneous Maxwell equations

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \\ \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} = \frac{1}{c^2\varepsilon_0}\vec{J} \end{cases}$$ since $\mu_0\varepsilon_0 = \frac{1}{c^2}$

are nothing else but

$$\partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu\left(\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right) = J^\nu$$

or $\quad d * F = *j\,\frac{1}{\varepsilon_0} \quad$ with $J = J_\mu dx^\mu = \frac{\rho}{c}\, d(ct) + J_i\, dx^i$

Let us check it

$$*F = \frac{1}{2!}\frac{1}{2!}\varepsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}\, dx^\mu\wedge dx^\nu$$

$$= \frac{1}{2}\varepsilon_{0i}{}^{jk}F_{jk}\, d(ct)\wedge dx^i + \frac{1}{2}\varepsilon_{ij}{}^{0l}F_{0l}\, dx^i\wedge dx^j$$

$$= \frac{c}{2}\varepsilon_{ijk}F_{jk}\, dt\wedge dx^i + \frac{1}{2}(-)\varepsilon_{lij}F_{0l}\, dx^i\wedge dx^j$$

$$= \frac{c}{2}\varepsilon_{ijk}\varepsilon_{jkl}B_l\, dt\wedge dx^i - \frac{1}{2}\varepsilon_{lij}\left(-\frac{1}{c}E_l\right)dx^i\wedge dx^j$$

$$= c\, B_i\, dt\wedge dx^i + \frac{1}{2c}\varepsilon_{lij}E_l\, dx^i\wedge dx^j$$

Then we get

$$d*F = c\,\partial_j B_i\, dt\wedge dx^i\wedge dx^j + \frac{1}{2c}\varepsilon_{lij}\partial_t E_l\, dt\wedge dx^i\wedge dx^j$$

$$+ \frac{1}{2c}\varepsilon_{lij}\partial_k E_l\, dx^k\wedge dx^i\wedge dx^j$$

$$= \left(-c\, \partial_{[i} B_{j]} + \frac{1}{2c} \varepsilon_{\ell i j}\, \partial_t E_\ell \right) dt \wedge dx^i \wedge dx^j$$

$$+ \left(\frac{1}{2c} \varepsilon_{\ell i j}\, \partial_k E_\ell \right) dx^k \wedge dx^i \wedge dx^j$$

Now we can exam $\star j$

$$\star j = \frac{1}{3!\,1!} \varepsilon_{\mu\nu\rho\sigma}\, j^\sigma\, dx^\mu \wedge dx^\nu \wedge dx^\rho$$

$$= \frac{1}{3!} \varepsilon_{ijk0}\, j^0\, dx^i \wedge dx^j \wedge dx^k + \frac{1}{2} \varepsilon_{0ijk}\, j^k\, dx^0 \wedge dx^i \wedge dx^j$$

$$= (-)^2 \frac{1}{6} \varepsilon_{ijk}\, j_0\, dx^i \wedge dx^j \wedge dx^k + \frac{c}{2} \varepsilon_{ijk}\, j_k\, dt \wedge dx^i \wedge dx^k$$

therefore we get

- $$\frac{1}{2c} \varepsilon_{\ell[ij}\, \partial_{k]} E_\ell = \frac{1}{6} \varepsilon_{ijk}\, j_0$$

$$\Rightarrow \quad \frac{1}{2c}\, 2\, \partial_\ell E_\ell = \frac{1}{6} \cdot 6\, j_0 / \varepsilon_0$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{E} = \frac{c}{\varepsilon_0} j_0 = \rho \frac{1}{\varepsilon_0}$$

- and

$$-c\, \partial_{[i} B_{j]} + \frac{1}{2c} \varepsilon_{k i j}\, \partial_t E_k = \frac{c}{2} \varepsilon_{ijk}\, j_k$$

$\Rightarrow \varepsilon_{ij\ell} \times$

$$-c\, (\vec{\nabla} \times \vec{B})_\ell + \frac{1}{c} \partial_t E_\ell = c\, j_\ell$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \frac{1}{c^2 \varepsilon_0} \vec{J}$$