Statistical Methods for Machine Learning Report

Image Classification with Artificial Neural Networks

Maurina Gabriele
Università degli Studi di Milano

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Abstract

This document contains a report on the project created by Gabriele Maurina for the exam Statistical Methods for Machine Learning. The project focuses on using artificial neural networks(ANNs) to perform image classification on a dataset from Keggle.com. The dataset contains 90380 images of fruits and vegetables. The project is divided into two experiments. The first experiment trains and evaluates 150 ANNs with the objective to classify images based on 10 base types, namely "apple", "banana", "cherry", "grape", "peach", "pear", "pepper", "plum", "potato" and "tomato". The second experiment trains and evaluates 150 ANNs with the objective to classify images based on all 131 labels contained in the dataset. Both experiment use Deep Neural Networks (DNNs), of which half are Convolutional Neural Networks (CNNs). The experimental results show that on average CNNs achieve a lower zero-one loss on the testing set than plain DNNs. The best networks for experiment 1 and experiment 2 are both CNNs and score a zero-one loss on the testing set of 1% and 5% respectively.

Contents

1   Introduction

This sections provides a very brief introduction to the topics discussed in the experiments.

Artificial neural networks (ANNs) are machine learning models loosely inspired by biological neural networks found in human and animal brains. ANNs consist of a set of nodes, i.e., neurons, connected to each other by edges, i.e., synapses. Nodes can send and receive signals, i.e. real numbers, through their edges. Edges typically have a weight that modulates the strength of the signal. Weights are adjusted during the training phase of a network. For each node, the sum of all weighted input signals is given to an activation function that can

forward the signal to the output edges. This project uses Deep Neural Networks (DNNs) and their subclass Convolutional Neural Networks (CNNs).

DNNs. DNNs aggregate their nodes into layers. Usually nodes from the first layer forward signals to nodes in the second layer, which, in turn, forward signals to the next layer and the forwarding is repeated until the last layer is reached. In the case of multi-class classification, the last layer has a node for each label and the signal emitted represents the probability of that label. This project uses DNNs with non-linear activation functions and dense layers. A dense layer is a layer where each node is connected to all the nodes in the next layer. Such networks are also called Multi Layer Perceptrons (MLPs).

CNNs. CNNs, are a subclass of DNNs and are most commonly used in applications such as image recognition and computer vision. CNNs reduce the number of free parameters by only connecting neurons to other neurons withing a certain range. This allows the networks to be deeper, since fewer connections are necessary. Intuitively, in an image classification task, CNNs analyze the relation between adjacent pixels and discard the relation between pixels that are far away from each other in order to extrapolate the broader meaning of the image.

2 Experiment 1

Experiment 1 trains and evaluates ANNs on the task to classify images of fruits and vegetables of 10 base types. The goal of the experiment is to try different network types in order to find the best configuration, i.e., the configuration with the lowest zero-one loss.

2.1 Setup

This experiment is carried out on a 64 bit machine running Fedora 32, with an Intel® Core™ i7-2670QM CPU @ 2.20GHz and 6GByte of RAM. THe programming language used is Python 3.8[4]. Tensorflow2[1], with the Keras[2] interface, is used to create, train and evaluate models. Numpy[3] is used to store and manipulate data easily.

Dataset. The dataset used is available on kaggle.com/moltean/fruits. It contains 90380 images of fruits and vegetables. Experiment 1 classifies images in 10 base types, namely "apple", "banana", "cherry", "grape", "peach", "pear", "pepper", "plum", "potato" and "tomato". Only the 43513 images that represents these fruit and vegetable types are considered. The images are 100x100px photos, already cropped to fit the subject perfectly.

The script `dataset_1.py` is used to manage the dataset. This script performs 5 tasks. 1)It loads images from the dataset. 2)It labels images according to the folder they are in. If the name of the folder starts with one of the 10 labels, the label is attached to the image. Otherwise the image is discarded. 3)It resizes images according to a preferred size. 4)It generates Numpy's n-dimensional arrays containing image data and labels such that they can be easily fed to a

machine learning model. The ndarray for image data is of type unsigned 8-bit integer and its shape is (`ni,is,is,3`), where `ni` is the number of images, `is` is the image side length in pixel, and 3 is the number of channels (r,g,b). The ndarray for labels is of type unsigned 8-bit integer and its shape is (`ni,`), where `ni` is the number of images. Arrays are created both for training and testing sets. 5)Finally, the script stores said arrays in binary format, so that it will load them faster at the next iteration, without having to go through the whole process again.

The script offers a simple interface, consisting of a single function `dataset(size)`, which takes care of all the aforementioned tasks in the background and returns the dataset, with images of the preferred size, ready to be used by a machine learning model. The `dataset(size)` function is declared as follows:

```python
def dataset(size):
    '''Return dataset with images of preferred size.
    If dataset already exists, it is loaded from disk,
    otherwise it is created from the image folder.'''
    files = (
        join(dataset_folder,f'x_train_{size}.npy'),
        join(dataset_folder,f'y_train_{size}.npy'),
        join(dataset_folder,f'x_test_{size}.npy'),
        join(dataset_folder,f'y_test_{size}.npy'))
    for f in files:
        if not isfile(f):
            ds = create_dataset(size)
            save_dataset(*ds,size)
            return ds
    return load_dataset(size)
```

Models. The models used differ in size, shape and type. The script `models.py` handles the creation of models according to certain parameters. The parameters supported are depth of the network, i.e., how many layers it has, width of the network, i.e., how many nodes are in each layer, type of network, i.e., DNN or CNN. Furthermore the script can handle different input and output sizes according to the needs.

The script offers a simple interface, consisting of a generator that yields models ready to be tested. The generator is called using the function `models(input_size, output_size)`. For this experiment, the generator is tuned to return 30 different networks. The number 30 is the result of the combination of 2 types of networks (DNN and CNN), 3 depths (1,2 or 3 layers), 5 widths (multiples of 32 in the case od DNN and multiples of 8 in the case of CNN).The `models(input_size, output_size)` function is coded as follows:

```python
def models(input_size,output_size):
    '''This generator returns models to testing in the experiment.'''
    #dense layers, different sizes
    for i in range(1,4):
```

```
        for j in range(1,6):
            yield dense(input_size,output_size,i,j*32),i,j*32,'dense'
    #cnn, different sizes
    for i in range(1,4):
        for j in range(1,6):
            yield conv(input_size,output_size,i,j*8),i,j*8,'conv'
```

The generator uses two more support functions to create the networks. These functions are `dense(input_size, output_size,depth,size)` and `conv(input_size, output_size,depth,size)`, and they are coded as follows:

```
def dense(input_size,output_size,depth,size):
    '''Create a dense model with specific input_size,
    output_size,depth and number of neuros.'''
    layers = [tf.keras.layers.Flatten(input_shape=(input_size,
        input_size,3))]
    for i in range(depth):
        layers.append(tf.keras.layers.Dense(size,activation='relu'))
    layers.append(tf.keras.layers.Dense(output_size))
    return tf.keras.Sequential(layers)
def conv(input_size,output_size,depth,size):
    '''Create a conv model with specific input_size,
    output_size,depth and number of neuros.'''
    layers = [tf.keras.layers.Conv2D(size,(3, 3),activation='relu',
        input_shape=(input_size,input_size,3))]
    for i in range(depth-1):
        layers += [
            tf.keras.layers.MaxPooling2D((2, 2)),
            tf.keras.layers.Conv2D(size,(3, 3),activation='relu',
                padding='same')]
    layers += [
        tf.keras.layers.Flatten(),
        tf.keras.layers.Dense(size,activation='relu'),
        tf.keras.layers.Dense(output_size)]
    return tf.keras.Sequential(layers)
```

CNNs are build alternating a `Conv2D` layer with a `MaxPooling2D` layer and finishing with a `Dense` layer, this was suggested on Tensorflow's website. The activation function used is ReLU, i.e., Rectified Linear Unit, which is one of the most commonly used with ANN. In the future, the experiment could be expanded by trying different activation functions.

Execution. The exectution of the experiment is managed by the `experiment_1.py` script. The script loads training and testing datasets of different sizes using the API of `dataset_1.py`. The script then loads models from `models.py`. Finally the script trains the networks on the training sets and evaluates them on the testing sets. For each network the training history is saved in JSON format and the zero-one loss wrt. the testing set is logged in a CSV file. Zero-one loss is
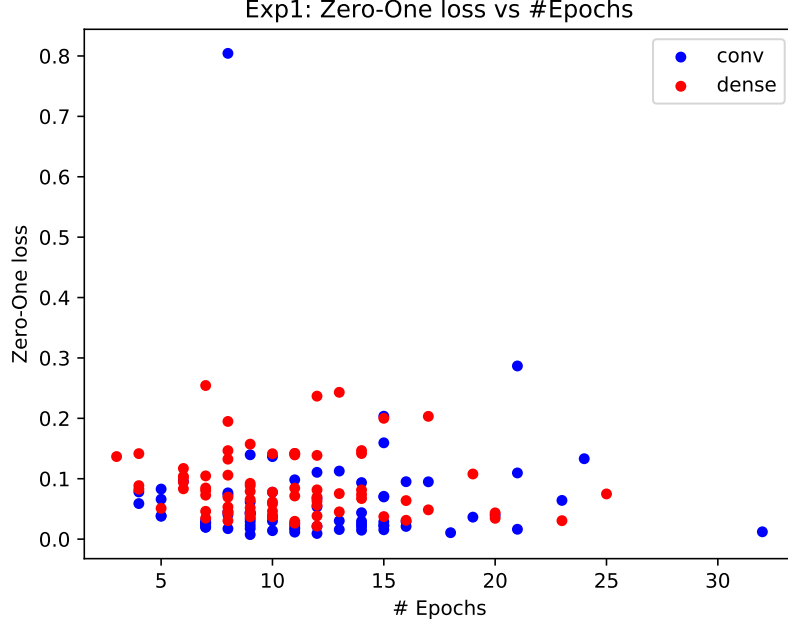
Figure 1: Experiment 1: zero-one loss vs epochs

computed by counting how many predictions on the testing set are wrong and dividing the total by the number of images in the testing set. Zero-one loss is a real number between 0 and 1, where 0 means a perfect set of predictions and 1 means a completely wrong set of predictions.

2.2 Results

The results collected are summarized in table1. The table contains the list of 150 ANNs trained and evaluated, with their parameters, i.e., depth, width and type, the number of epochs they were trained for and the zero-one loss achieved on the testing set.

Figure1 shows the zero-one loss and the number of epochs of each ANNs, differentiating the DNNs from the CNNs. On average the CNNs achieve a lower loss.

Figure2 shows the zero-one loss achieved on images of size 40x40 pixels by ANNs of different depth and width, of type DNN and CNN respectively. The figures show that deeper and wider networks achieve a lower zero-one loss. Note that there is no overfitting in that data because the zero-one loss is computed on the testing set.

Results can be reproduced by following the steps highlighted in section4.

Table 1: Experiment 1

IS	T	D	W	ZOL	#E	IS	T	D	W	ZOL	#E	IS	T	D	W	ZOL	#E
10	dense	1	32	0.07	25	20	conv	2	8	0.07	15	40	dense	3	32	0.24	13
10	dense	1	64	0.03	20	20	conv	2	16	0.03	15	40	dense	3	64	0.10	7
10	dense	1	96	0.03	23	20	conv	2	24	0.02	15	40	dense	3	96	0.09	6
10	dense	1	128	0.05	13	20	conv	2	32	0.01	10	40	dense	3	128	0.10	6
10	dense	1	160	0.08	12	20	conv	2	40	0.02	11	40	dense	3	160	0.03	8
10	dense	2	32	0.04	20	20	conv	3	8	0.11	12	40	conv	1	8	0.08	8
10	dense	2	64	0.07	8	20	conv	3	16	0.02	12	40	conv	1	16	0.11	13
10	dense	2	96	0.09	9	20	conv	3	24	0.01	9	40	conv	1	24	0.07	5
10	dense	2	128	0.06	10	20	conv	3	32	0.04	5	40	conv	1	32	0.07	12
10	dense	2	160	0.04	10	20	conv	3	40	0.02	8	40	conv	1	40	0.04	14
10	dense	3	32	0.08	11	30	dense	1	32	0.20	15	40	conv	2	8	0.16	15
10	dense	3	64	0.05	17	30	dense	1	64	0.11	19	40	conv	2	16	0.03	9
10	dense	3	96	0.03	11	30	dense	1	96	0.07	14	40	conv	2	24	0.03	9
10	dense	3	128	0.02	12	30	dense	1	128	0.08	14	40	conv	2	32	0.03	7
10	dense	3	160	0.07	9	30	dense	1	160	0.06	10	40	conv	2	40	0.02	7
10	conv	1	8	0.10	16	30	dense	2	32	0.14	12	40	conv	3	8	0.10	11
10	conv	1	16	0.04	20	30	dense	2	64	0.11	8	40	conv	3	16	0.04	9
10	conv	1	24	0.14	9	30	dense	2	96	0.05	8	40	conv	3	24	0.01	14
10	conv	1	32	0.02	16	30	dense	2	128	0.06	12	40	conv	3	32	0.02	21
10	conv	1	40	0.02	14	30	dense	2	160	0.08	4	40	conv	3	40	0.01	18
10	conv	2	8	0.06	23	30	dense	3	32	0.07	14	50	dense	1	32	0.25	7
10	conv	2	16	0.07	8	30	dense	3	64	0.10	6	50	dense	1	64	0.14	11
10	conv	2	24	0.03	7	30	dense	3	96	0.07	7	50	dense	1	96	0.14	11
10	conv	2	32	0.02	9	30	dense	3	128	0.05	9	50	dense	1	128	0.15	8
10	conv	2	40	0.02	11	30	dense	3	160	0.09	4	50	dense	1	160	0.08	13
10	conv	3	8	0.11	21	30	conv	1	8	0.29	21	50	dense	2	32	0.19	8
10	conv	3	16	0.06	9	30	conv	1	16	0.04	9	50	dense	2	64	0.12	6
10	conv	3	24	0.10	6	30	conv	1	24	0.08	4	50	dense	2	96	0.04	9
10	conv	3	32	0.02	13	30	conv	1	32	0.03	10	50	dense	2	128	0.05	8
10	conv	3	40	0.01	11	30	conv	1	40	0.06	4	50	dense	2	160	0.14	3
20	dense	1	32	0.20	17	30	conv	2	8	0.14	10	50	dense	3	32	0.08	7
20	dense	1	64	0.07	11	30	conv	2	16	0.04	9	50	dense	3	64	0.07	12
20	dense	1	96	0.03	16	30	conv	2	24	0.02	7	50	dense	3	96	0.09	9
20	dense	1	128	0.04	12	30	conv	2	32	0.03	14	50	dense	3	128	0.05	5
20	dense	1	160	0.04	15	30	conv	2	40	0.02	7	50	dense	3	160	0.03	7
20	dense	2	32	0.08	9	30	conv	3	8	0.20	15	50	conv	1	8	0.80	8
20	dense	2	64	0.08	7	30	conv	3	16	0.04	8	50	conv	1	16	0.09	14
20	dense	2	96	0.06	12	30	conv	3	24	0.03	7	50	conv	1	24	0.08	5
20	dense	2	128	0.05	10	30	conv	3	32	0.02	9	50	conv	1	32	0.04	10
20	dense	2	160	0.03	11	30	conv	3	40	0.01	12	50	conv	1	40	0.04	19
20	dense	3	32	0.06	16	40	dense	1	32	0.14	11	50	conv	2	8	0.07	15
20	dense	3	64	0.04	10	40	dense	1	64	0.13	8	50	conv	2	16	0.05	12
20	dense	3	96	0.08	10	40	dense	1	96	0.14	14	50	conv	2	24	0.03	7
20	dense	3	128	0.04	8	40	dense	1	128	0.15	14	50	conv	2	32	0.02	15
20	dense	3	160	0.05	7	40	dense	1	160	0.16	9	50	conv	2	40	0.04	5
20	conv	1	8	0.13	24	40	dense	2	32	0.24	12	50	conv	3	8	0.08	10
20	conv	1	16	0.09	17	40	dense	2	64	0.14	10	50	conv	3	16	0.03	13
20	conv	1	24	0.05	8	40	dense	2	96	0.08	6	50	conv	3	24	0.03	16
20	conv	1	32	0.03	14	40	dense	2	128	0.14	4	50	conv	3	32	0.01	32
20	conv	1	40	0.02	11	40	dense	2	160	0.04	9	50	conv	3	40	0.02	15

IS is image size in pixel;

T is the type of model, can be "dense" or "conv";

D is the depth of the model;

W is the width of the model;

ZOL is the Zero-One loss achieved on the test set;

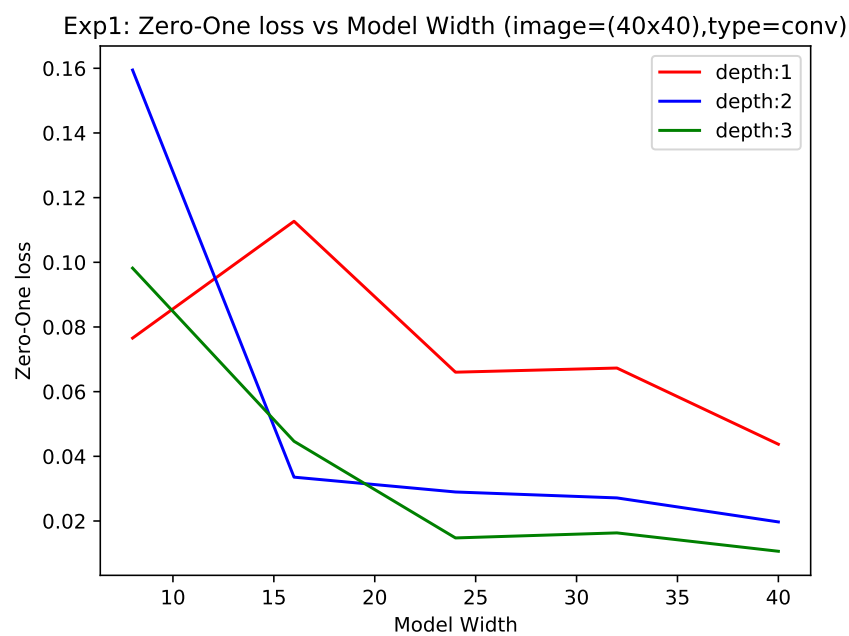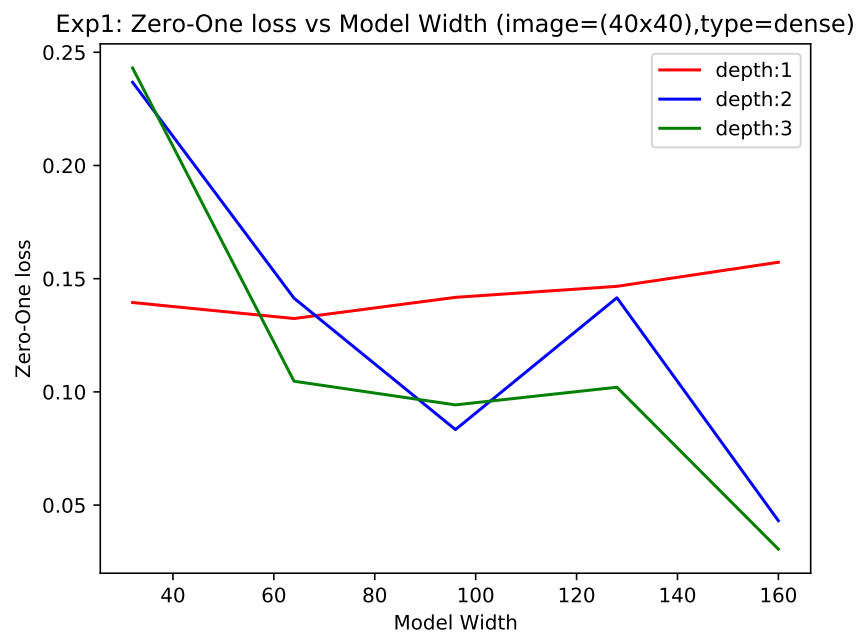**#E** is the number of epochs the model was trained for.

Figure 2: Experiment 1: zero-one loss vs ANN size

3 Experiment 2

Experiment 2, similarly to experiment 1, trains and evaluates ANNs on the task to classify images of fruits and vegetables. Experiment 2, however, considers 131 labels, making the classification task more challenging. The goal of the experiment, as in experiment 1, is to try different network types in order to find the best configuration, i.e., the configuration with the lowest Zero-One loss, and to see if the best configuration for experiment 1, still holds for experiment 2.

3.1 Setup

The setup is very similar to experiment 1. The experiment is run on the same machine, using networks build by the `models.py` script.

Dataset. The same dataset of images is used, however all the 90380 images and all the 131 labels are considered this time. For this reason, datasets are managed by a different script, `dataset_2.py`, and they are stored in a different folder.

3.2 Results

The results collected are summarized in table2.

Figure3 shows the zero-one loss and the number of epochs of each ANNs, differentiating the DNNs from the CNNs. The lowest losses are achieved by CNNs, however the chart is very scattered and also many DNNs achieve a very low loss.

Figure4 shows the zero-one loss achieved on images of size 50x50 pixels by ANNs of different depth and width, of type DNN and CNN respectively. The figures show that deeper and wider networks achieve a lower zero-one loss. Note that there is no overfitting in that data because the zero-one loss is computed on the testing set.

Results can be reproduced by following the steps highlighted in section4.

4 Reproducing the experiments

The code to reproduce the both experiments is available at github.com/GabrieleMaurina/statistical_methods_for_machine_learning_project. Note that the experiments can take up to several hours to complete, depending on the hardware used. To reproduce the experiment, follow the steps:

- clone the repository;

- install Tensorflow, Numpy and Matplotlib;

- download the image dataset;

- unzip the dataset;

Table 2: Experiment 2

IS	T	D	W	ZOL	#E	IS	T	D	W	ZOL	#E	IS	T	D	W	ZOL	#E
10	dense	1	32	0.07	28	20	conv	2	8	0.12	17	40	dense	3	32	0.38	10
10	dense	1	64	0.08	22	20	conv	2	16	0.09	11	40	dense	3	64	0.18	6
10	dense	1	96	0.08	17	20	conv	2	24	0.14	7	40	dense	3	96	0.10	8
10	dense	1	128	0.08	14	20	conv	2	32	0.05	12	40	dense	3	128	0.12	5
10	dense	1	160	0.07	11	20	conv	2	40	0.09	5	40	dense	3	160	0.16	7
10	dense	2	32	0.09	16	20	conv	3	8	0.24	12	40	conv	1	8	0.44	12
10	dense	2	64	0.09	12	20	conv	3	16	0.08	11	40	conv	1	16	0.07	10
10	dense	2	96	0.07	9	20	conv	3	24	0.10	7	40	conv	1	24	0.99	7
10	dense	2	128	0.09	10	20	conv	3	32	0.12	5	40	conv	1	32	0.08	10
10	dense	2	160	0.07	9	20	conv	3	40	0.06	7	40	conv	1	40	0.14	12
10	dense	3	32	0.10	16	30	dense	1	32	0.95	22	40	conv	2	8	0.13	13
10	dense	3	64	0.11	8	30	dense	1	64	0.37	26	40	conv	2	16	0.19	14
10	dense	3	96	0.08	11	30	dense	1	96	0.30	25	40	conv	2	24	0.12	8
10	dense	3	128	0.12	7	30	dense	1	128	0.13	17	40	conv	2	32	0.08	5
10	dense	3	160	0.13	7	30	dense	1	160	0.13	12	40	conv	2	40	0.04	8
10	conv	1	8	0.30	19	30	dense	2	32	0.34	14	40	conv	3	8	0.13	13
10	conv	1	16	0.11	20	30	dense	2	64	0.16	10	40	conv	3	16	0.22	13
10	conv	1	24	0.11	10	30	dense	2	96	0.08	8	40	conv	3	24	0.07	11
10	conv	1	32	0.06	17	30	dense	2	128	0.11	6	40	conv	3	32	0.07	8
10	conv	1	40	0.07	10	30	dense	2	160	0.11	5	40	conv	3	40	0.05	5
10	conv	2	8	0.19	16	30	dense	3	32	0.55	17	50	dense	1	32	0.99	6
10	conv	2	16	0.12	16	30	dense	3	64	0.14	11	50	dense	1	64	0.99	4
10	conv	2	24	0.09	8	30	dense	3	96	0.12	7	50	dense	1	96	0.53	18
10	conv	2	32	0.10	8	30	dense	3	128	0.08	13	50	dense	1	128	0.99	4
10	conv	2	40	0.10	6	30	dense	3	160	0.13	5	50	dense	1	160	0.53	16
10	conv	3	8	0.21	21	30	conv	1	8	0.17	15	50	dense	2	32	0.99	4
10	conv	3	16	0.11	12	30	conv	1	16	0.68	18	50	dense	2	64	0.91	14
10	conv	3	24	0.09	15	30	conv	1	24	0.67	18	50	dense	2	96	0.89	13
10	conv	3	32	0.14	7	30	conv	1	32	0.27	21	50	dense	2	128	0.31	15
10	conv	3	40	0.05	12	30	conv	1	40	0.14	14	50	dense	2	160	0.16	5
20	dense	1	32	0.52	23	30	conv	2	8	0.21	14	50	dense	3	32	0.90	14
20	dense	1	64	0.31	28	30	conv	2	16	0.10	9	50	dense	3	64	0.56	12
20	dense	1	96	0.12	22	30	conv	2	24	0.04	9	50	dense	3	96	0.16	7
20	dense	1	128	0.09	14	30	conv	2	32	0.07	6	50	dense	3	128	0.14	5
20	dense	1	160	0.09	12	30	conv	2	40	0.07	8	50	dense	3	160	0.15	4
20	dense	2	32	0.31	13	30	conv	3	8	0.40	30	50	conv	1	8	0.99	6
20	dense	2	64	0.08	8	30	conv	3	16	0.13	8	50	conv	1	16	0.39	13
20	dense	2	96	0.09	11	30	conv	3	24	0.08	7	50	conv	1	24	0.41	12
20	dense	2	128	0.10	10	30	conv	3	32	0.09	5	50	conv	1	32	0.21	17
20	dense	2	160	0.14	6	30	conv	3	40	0.06	9	50	conv	1	40	0.15	8
20	dense	3	32	0.13	16	40	dense	1	32	0.94	31	50	conv	2	8	0.39	23
20	dense	3	64	0.11	9	40	dense	1	64	0.81	20	50	conv	2	16	0.27	15
20	dense	3	96	0.12	4	40	dense	1	96	0.30	27	50	conv	2	24	0.22	17
20	dense	3	128	0.09	8	40	dense	1	128	0.41	13	50	conv	2	32	0.11	6
20	dense	3	160	0.11	4	40	dense	1	160	0.24	13	50	conv	2	40	0.06	9
20	conv	1	8	0.80	28	40	dense	2	32	0.99	10	50	conv	3	8	0.49	14
20	conv	1	16	0.37	28	40	dense	2	64	0.27	14	50	conv	3	16	0.10	6
20	conv	1	24	0.10	10	40	dense	2	96	0.11	9	50	conv	3	24	0.07	5
20	conv	1	32	0.11	7	40	dense	2	128	0.11	12	50	conv	3	32	0.07	5
20	conv	1	40	0.07	6	40	dense	2	160	0.18	4	50	conv	3	40	0.05	6

IS is the image size in pixel;

T is the type of model, can be "dense" or "conv";

D is the depth of the model;

W is the width of the model;

ZOL is the Zero-One loss achieved on the test set;

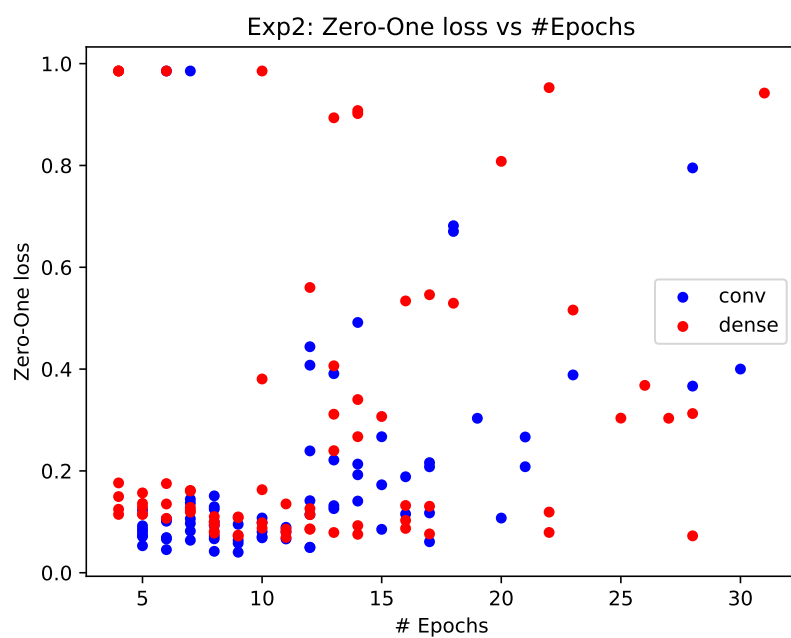**#E** is the number of epochs the model was trained for.
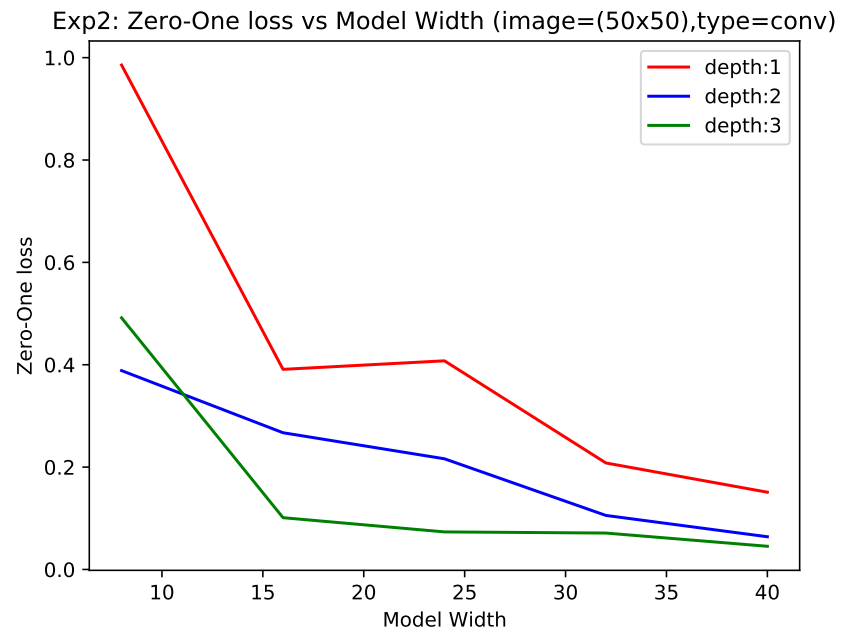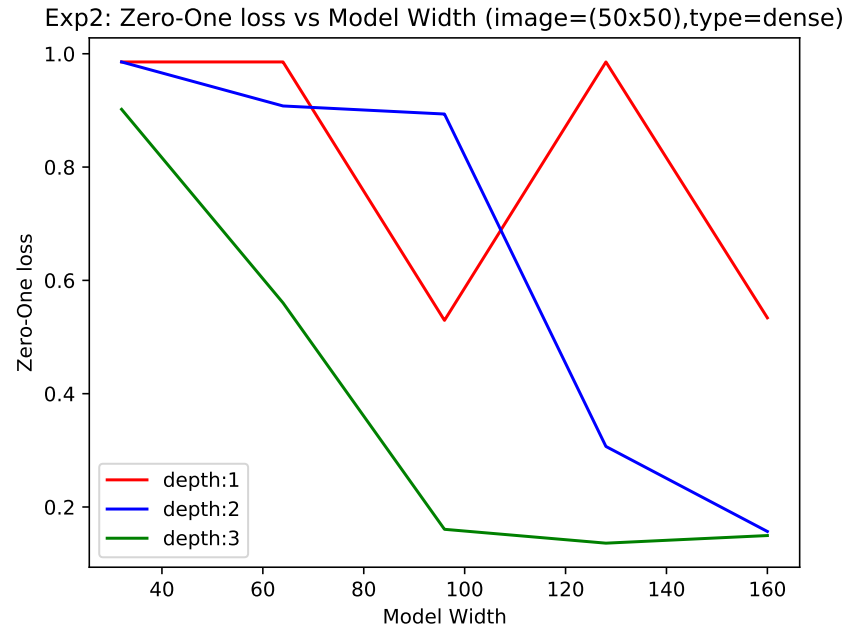
Figure 3: Experiment 2: zero-one loss vs epochs

Figure 4: Experiment 2: zero-one loss vs ANN size

- move the folder fruits-360 at the root of the cloned repository;

- run experiment 1 with `python experiment_1.py`;

- manually overwrite the `results_exp_1` folder with the newly created `results` folder;

- run experiment 2 with `python experiment_2.py`;

- manually overwrite the `results_exp_2` folder with the newly created `results` folder;

- cd into `plots/`;

- generate plots with `python plots.py`;

Both experiment 1 and experiment 2 save the results into the `results` folder. It is important to avoid mixing the results and to move them to the corresponding folders, as indicated in the steps above. This manual step is necessary because the folders `results_exp_1` and `results_exp_2` are versioned and it is undesirable to accidentally overwrite them when testing the experiment scripts. Once one is satisfied with the results, then the folder `results` should be renamed `results_exp_<number>`. Furthermore, the `plots.py` script takes the results from the folders `results_exp_1` and `results_exp_2`.

Demo. The repository also contains a demonstration of a model predicting images. To run the demonstration first run the script `demo_model.py`, which will create, train and save a model. Then run `demo.py`, which will load the model, iteratively load images from the testing dataset and for each image, show it on screen, make a prediction and show the true label.

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