# An Introduction to AdS/CFT Correspondence

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# Introduction

- Holography
- Anti-de Sitter spacetime
- Conformal invariance
- Large N expansion
- Strings, D-branes, and all that
- Statement of duality
- Applications

# Statement of Duality

*String theory on* $AdS_5 \times S^5 \simeq \mathcal{N} = 4, \mathrm{SU}(N)$ *gauge theory in 4D*

# Statement of Duality

Resources:

- Gauge/Gravity Duality (Ammon and Erdmenger)
- Introduction to the AdS/CFT Correspondence (Natase)
- String Theory and Holographic Duality (Liu)

Arxiv:

- The Large N Limit of Superconformal Field Theories and Supergravity (Maldacena 1997)
- Anti De Sitter Space And Holography (Witten)
- The AdS/CFT Correspondence (Hubeny)
- TASI Lectures on the Emergence of the Bulk in AdS/CFT (Harlow)

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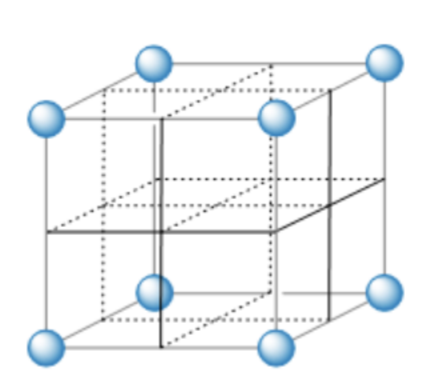
# Holographic Principle I

- Spin system on lattice (size $a$) has $V/a^3$ total spins or $N = 2^{V/a^3}$ dof
- $\implies S_{spin} = \frac{V}{a^3} \log 2$



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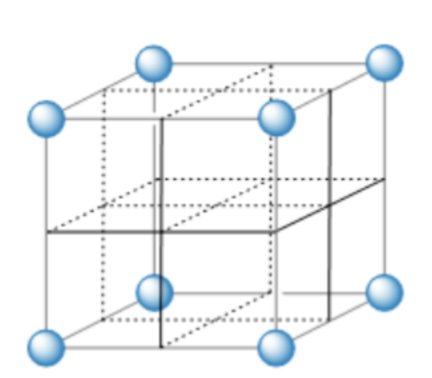
- Hawking showed $S_{BH} = \frac{A}{4\hbar G_N} = \frac{A}{4\ell_p^2}$
- $S_{spin} \geq S_{BH}$



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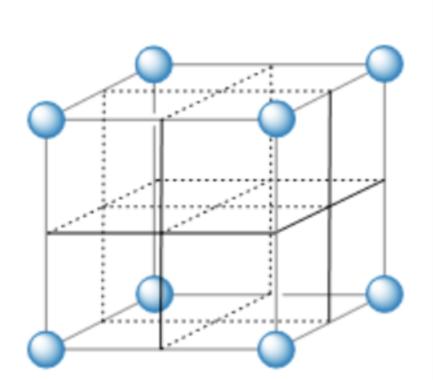
- Including gravity drastically reduces number of dof



# Holographic Principle II

**Theorem (Holographic Principle)**
*In QG, a region $R$ with codimension-1 surface $B = \partial R$ can be described by the dynamics in $B$.*

i.e. by no more than $A/4\ell_p^2$ dof.

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# Anti-de Sitter space I

- Spacetime with $\Lambda < 0$
- $-X_0^2+\sum_{i=1}^{d-1} X_i^2-X_{d+1}^2 = -R^2$
  and has $\mathrm{SO}(2, d-1)$
  isometry

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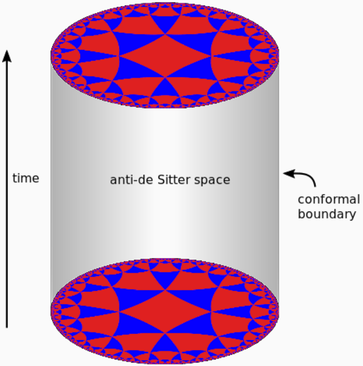
- $-X_0^2 + \sum_{i=1}^{d-1} X_i^2 - X_{d+1}^2 = -R^2$ and has $\mathrm{SO}(2, d-1)$ isometry
- AdS global coordinate metric can be written as $ds^2 = \frac{R^2}{\cos^2\rho}(d\tau^2 + d\rho^2 + \sin^2\rho d\Omega_{d-2}^2)$
- as $\rho \to 0$ we see the spacetime is topologically $S^1 \times \mathbb{R}^{d-1}$ with boundary $\mathbb{R} \times S_{d-2}$
- Light can reach boundary in finite time but massive particle cannot



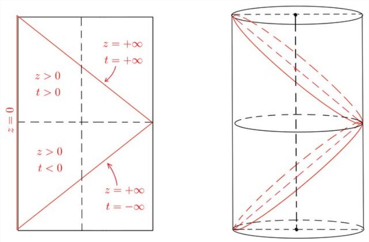
# Anti-de Sitter space II

- AdS has a boundary (not true for dS)
- In Poincare patch,

$$ds^2 = \frac{R^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu)$$

$$\lim_{z\to 0} \simeq \eta_{\mu\nu}dx^\mu dx^\nu$$

which is conformally flat



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# Conformal Invariance I

- Conformal transformations $x \to x' = f(x)$ change metric as $G_{\mu\nu}(x) \to \Omega(x)^{-2} g_{\mu\nu}(x)$
- Leaves angles unchanged
- Fields change as $\phi(x) \to \phi'(x') = \lambda^{-\Delta}\phi(x)$ for some scaling dimension $\Delta$
- Strongly fixes correlators

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C}{(x_1 - x_2)^{2\Delta}} \tag{1}$$

| Name | $\epsilon^\mu(x)$ | $\sigma(x)$ | Operator |
|---|---|---|---|
| Translation | $a^\mu$ | 0 | $P_\mu$ |
| Lorentz transformations | $\omega^\mu{}_\nu x^\nu,\ \omega_{\mu\nu} = -\omega_{\nu\mu}$ | 0 | $J_{\mu\nu}$ |
| Dilatation | $\lambda x^\mu$ | $\lambda$ | $D$ |
| Special conformal transformation | $b^\mu x^2 - 2(b\cdot x)x^\mu$ | $-2(b\cdot x)$ | $K_\mu$ |

**Figure 1:** Conformal transformations

# Conformal Invariance II

- In $d > 2$, conformal group is $\mathrm{SO}(d, 2)$
- So 5d AdS has same conformal group as 4d CFT, $\mathrm{SO}(4, 2)$

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# Motivation

- QCD Lagrangian is $\mathcal{L} = g_{YM}^{-2}\left(\frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - fermions\right)$ with $A_{a\mu}$ as $3 \times 3$ matrix
- At low energies, coupling is strong. Solution?

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- QCD Lagrangian is $\mathcal{L} = g_{YM}^{-2}\left(\frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - \mathit{fermions}\right)$ with $A_{a\mu}$ as $3 \times 3$ matrix
- At low energies, coupling is strong. Solution?
- Promote $A_\mu$ to $N \times N$ matrix, take $N \to \infty$, expand in $1/N$

# Toy Model

- $\mathcal{L} = g^{-2}\mathrm{Tr}\left[\frac{1}{2}(\partial\Phi^a_b)^2 + \frac{1}{4}\Phi^4\right]$ where $\Phi$ is $N \times N$ matrix and has $\mathrm{U}(N)$ symmetry
- First diagram contributes $N^3g^2$, 2nd contributes $Ng^2$
- Cannot draw 2nd diagram without crossing
- What is a general rule to count these diagrams?

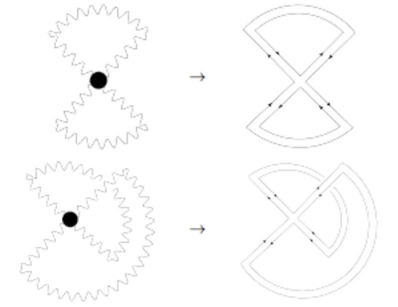


Figure 2: Vacuum Diagrams

# Counting Diagrams

- In general vacuum contribution is $A \sim (g^2)^{E-V} N^F$

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- $A \sim (g^2 N)^{E-V} N^{F+V-E} = \lambda^{L-1} N^{F+V-E}$

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- Euler character $\chi = F + V - E = 2 - 2g \implies A \sim \lambda^{L-1} N^{2-2g}$
- Sum of all diagrams $W = \log Z = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda)$
- 

Figure 3: genus 0-2 surfaces

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Figure 3: genus 0-2 surfaces

- Generally, correlators look like

$$\langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle = \sum_{n=0}^{\infty} N^{2-n-2g} F_n^{(g)}(x_1, \dots, x_n; \lambda) \qquad (2)$$

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# Strings I

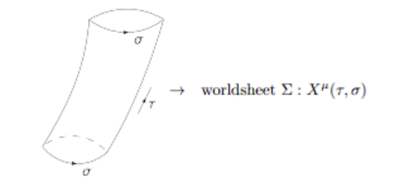
- Fundamental object is a string

$$Z = \int DX^{\mu}(\tau) e^{\frac{i}{2\pi\alpha'} \int_{\Sigma} dA + \lambda R} \tag{3}$$

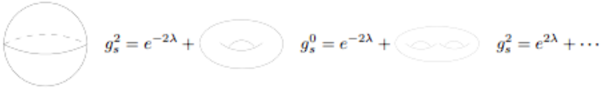
- To evaluate, go to euclidean signature

$$Z_E = \sum_{g=0} e^{-\lambda\chi} \sum_{\text{surface w/ given topology}} e^{-S_{string}} \tag{4}$$

- String theory has interactions built in (only depends on topology of world sheet)



- Define $g_s = e^{\lambda}$, then




- Summing over all topologies sums over interactions

# Strings II

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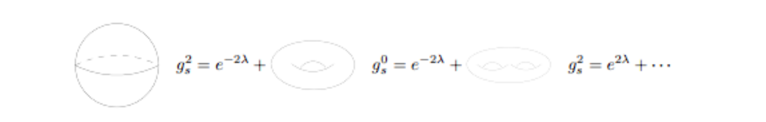



- Summing over all topologies sums over interactions
- Generally for $n$-scattering,

$$A_n = \sum_{g=0} g_s^{n-2+2g} F_n^{(g)} \tag{5}$$

# Strings III

So we can identify a large N gauge theory with a string theory

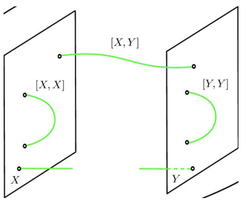
$$
\begin{aligned}
g_s &\leftrightarrow \frac{1}{N} \\
\text{external strings} &\leftrightarrow \mathcal{O}_i \left|0\right\rangle \\
\text{topology of worldsheet} &\leftrightarrow \text{topology of Feynman diagram}
\end{aligned}
$$

# D-Branes I

- Open strings end on D-branes
- D-Branes are dynamical and are described by

$$S_{\text{eff}} = -T_p \int d^{p+1}x \left(1 + \frac{1}{4}F^2 + \frac{1}{2}(\partial_\alpha \Phi^a)^2 + \dots\right)$$

- Can stack N D-branes on top of each other and get $N \times N$ representation $(A_\alpha)^I_J$



# D-Branes II

- On the worldsheet, $\mathrm{U}(N)$ global symmetry, but from spacetime POV, gauge symmetry
- $$S \to -\frac{1}{g_{YM}^2} \int d^{p+1}x \mathrm{Tr} \left( \frac{1}{4} F^2 + \frac{1}{2} (D_\alpha \Phi^a)^2 + [\Phi^a, \Phi^b]^2 + \dots \right)$$

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- Identify $g_{YM}^2 \sim g_s$ and $\alpha' \to 0$, this is just $\mathcal{N} = 4$ SYM!
- Considering open string perspective ($g_s N << 1$)
- Note that I have left off the sypersymmetric part, but they are there!

# D-Branes III

- Consider closed string perspective $g_s N \to \infty$
- For N D3-branes (type IIB superstring) in $\mathbb{R}^{9,1}$
- D-branes gravitate, in SUGRA approximation, they source a metric

$$\begin{aligned} ds^2 =& H(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{1/2}\delta_{ij}dx^i dx^j \\ H(r) =& 1 + \left(\frac{R}{r}\right)^4 \end{aligned}$$

- For $r >> R$,

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) + R^5 d\Omega_5^2 \tag{6}$$

which is $AdS_5 \times S^5$

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# AdS/CFT Correspondence I

- Closed string perspective: IIB SUGRA on $AdS_5 \times S^5 \leftrightarrow \mathbb{R}^{9,1}$
- Open string perspective: $\mathcal{N} = 4$ SYM in 4D $\leftrightarrow$ IIB SUGRA on $\mathbb{R}^{9,1}$

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- Open string perspective: $\mathcal{N} = 4$ SYM in 4D $\leftrightarrow$ IIB SUGRA on $\mathbb{R}^{9,1}$
- So conjecture $\mathrm{SU}(N)$ $\mathcal{N} = 4$ SYM in 4D $\leftrightarrow$ IIB supertring theory on $AdS_5 \times S^5$ with

$$g_{YM}^2 = 4\pi g_s \text{ and } g_{YM}^2 N = R^4/\alpha'^2$$

# AdS/CFT Correspondence II

**Table 5.1** Different forms of the AdS/CFT correspondence

| | $\mathcal{N} = 4$ Super Yang–Mills theory | IIB theory on $AdS_5 \times S^5$ |
|---|---|---|
| Strongest form | any $N$ and $\lambda$ | Quantum string theory, $g_s \neq 0, \alpha'/L^2 \neq 0$ |
| Strong form | $N \to \infty, \lambda$ fixed but arbitrary | Classical string theory, $g_s \to 0, \alpha'/L^2 \neq 0$ |
| Weak form | $N \to \infty, \lambda$ large | Classical supergravity, $g_s \to 0, \alpha'/L^2 \to 0$ |

Figure 4: Statements of AdS/CFT from Ammon and Erdmenger

# Dictionary I

| $\mathcal{N}=4$ SYM | | IIB string theory in $AdS_5 \times S^5$ |
|---|---|---|
| Conformal SO(4,2) | $\leftrightarrow$ | isometry of AdS: SO(4,2) |
| Global SO(6) | $\leftrightarrow$ | isometry of $S^5$: SO(6) |
| Global SUSY | $\leftrightarrow$ | Local SUSY |
| Conformal SO(4,2) | $\leftrightarrow$ | isometry of AdS: SO(4,2) |
| $g_{YM}^2$ | $\leftrightarrow$ | $4\pi g_s$ |
| $\pi^4/(2N^2)$ | $\leftrightarrow$ | $R^4/\alpha'^2$ |
| $\lambda = g_{YM}^2 N$ | $\leftrightarrow$ | $G_N/R^8$ |

# Dictionary II

2 Limits

- 1) Semiclassical: $\hbar = 1, G_N, \lambda \to 0$
- Then $N^{-2}$ corrections are quantum gravity corrections and $\lambda^{-1/2}$ are $\alpha'$ corrections
- 2) Classical string: $\alpha'$ arbitrary and $N \to \infty$ or $g_s \to 0$
- Corresponds to t'Hooft limit

# Dictionary III

| boundary theory | | bulk theory |
|---|---|---|
| repr. of conformal SO(4,2) | $\leftrightarrow$ | repr. of isometry SO(4,2) |
| conformal local operators | $\leftrightarrow$ | bulk fields |
| scalar operator $\mathcal{O}$ | $\leftrightarrow$ | scalar field $\phi$ |
| vector operator $J_\mu$ | $\leftrightarrow$ | vector field $A_\mu$ |
| tensor operator $T_{\mu\nu}$ | $\leftrightarrow$ | tensor field $h_{\mu\nu}$ |

# Example: Scalar field

- Action for scalar in AdS:

$$S \sim \int d^{d+1}x\sqrt{-g}(g^{MN}\partial_M\Phi\partial_N\Phi + m^2\Phi^2) + O(N^{-1}) \quad (7)$$

- $\Phi(x,z) = A(x)z^{d-\Delta} + B(x)z^{\Delta}$ as $z \to 0$
- $A(x)$ is boundary "value" $\implies$ add $\int d^dx\phi_0(x)\mathcal{O}(x)$ to action
- $\int d^dx\phi_0(x)\mathcal{O}(x) \Leftrightarrow \phi_0(x) = \lim_{z\to 0} z^{\Delta-d}\Phi(x,z)$
- Can reconstruct bulk fields from the boundary

# Dictionary V

- In short, we can summarize as

$$Z_{gravity}[\Phi|_{\partial AdS=\phi(x)}] = Z_{CFT}[\phi(x)] \tag{8}$$

- Can calculate Euclidean correlators as

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \frac{\delta^n \log Z_{CFT}}{\delta\phi_1 \ldots \delta\phi_n} = \frac{\delta^n \log Z_{gravity}[\Phi]}{\delta\phi_1 \ldots \delta\phi_n} \tag{9}$$

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- ex: $\langle \mathcal{O}(x) \rangle = 2(\Delta - d/2)B(x)$
- or 2-pt function $\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = \frac{1}{|x_1 - x_2|^{\Delta}}$

# Spontaneous Extra Dimension?

- Where does the extra dimension come from?

## Spontaneous Extra Dimension?

- Where does the extra dimension come from?
- Can view it as energy scale of boundary theory!

$$E_{YM} = \frac{R}{z} E_{local} \quad d_{YM} = \frac{z}{R} d_{local}$$

- or

$$\begin{aligned} z \to 0 \implies E_{YM} \to \infty, d_{YM} \to 0 \quad & \text{UV process of SYM} \\ z \to \infty \implies E_{YM} \to 0, d_{YM} \to \infty \quad & \text{IR process of SYM} \end{aligned}$$

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# Other Examples of Gauge/Gravity Duality

- M-Theory $AdS_4 \times S^7/CFT_3$
- Thermofield double and Eternal Black holes
- $AdS_3/CFT_2$
- JT gravity/SYK model
- Topological gravity/Narain CFTs
- + more

- Quantum Phase transitions
- Holographic Superconductors/Superfluids
- Transport Properties

$$\mathcal{L} = R + 6/L^2 - \frac{1}{4}F^2 - |\nabla\phi - iqA\phi|^2 - m^2|\phi|^2$$

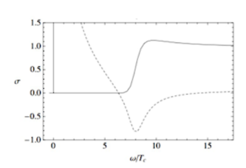


**Figure 5:** Low temperature limit of $\sigma(\omega) = -\lim_{k\to 0}\frac{1}{\omega}G^R(k,\omega)$ [1]

[1]Gary T Horowitz and Matthew M Roberts. Holographic superconductors with various condensates. Physical Review D, 78(12):126008, 2008.

# QIS Applications

- Entanglement Entropy and Complexity

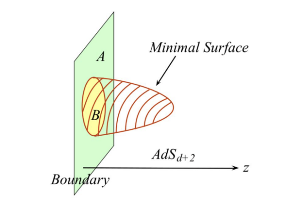


**Figure 6:** Entanglement Entropy given by $S = \frac{Area(\gamma_A)}{4G_N}$

- Quantum Error Correcting codes

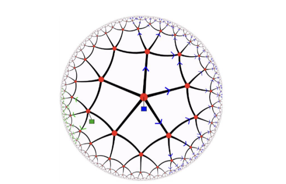

**Figure 7:** AdS-Rindler in HaPPY tensor network code.

# QCD Applications

- Solutions where $A(r)$ diverges at finite $r$ describes a confining theory
- AdS/QCD gives pretty good predictions

**Table 13.4** Meson masses and decay constants in the hard wall AdS/QCD model

| Observable | Measured (MeV) | AdS/QCD (MeV) |
|---|---|---|
| $m_\pi$ | $139.6 \pm 0.0004$ | 141 |
| $m_\rho$ | $775.8 \pm 0.5$ | 832 |
| $m_{a_1}$ | $1230 \pm 40$ | 1220 |
| $f_\pi$ | $92.4 \pm 0.35$ | 84.0 |
| $F_\rho^{1/2}$ | $345 \pm 8$ | 353 |
| $F_{a_1}^{1/2}$ | $433 \pm 13$ | 440 |

Figure 8: From Ammon

# Hydrodynamics Applications

- Relativistic hydrodynamics: $\exists U(1)$ current anomolies that directly modifies equations proposed by Landau
- Holographic calculations of the diffusion constant, shear viscosity, etc.