COM361 — Introdução a Otimização — 2022, Prof. Amit

Event triggered social media chatter model - Julia implementation

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1. Introduction and problem description

This project's goal is to describe and explore a social marketing-based model for information spread, testing different control techniques so to determine which one provides better results. An important constraint for most marketing campaigns nowadays is the advertising cost, therefore a key goal for dispersing a message is to do that as efficiently and possible respecting budget and time constraints. From that, many optimization problems can be formulated. The following question, described in [1] gives the reason for such models to exist:

> At a high level, how does an organization sell to someone? Typically, individuals or groups (like marketing agencies) are enlisted as message spreaders who broadcast that information in a variety of formats including billboards, social media posts, and television advertisements.

So in order to sell, it's necessary to spread information and the further question would be **how to maximize sells (and information spread) while minimizing costs**.

In [1], the approach to design an advertising model starts from the well established Vidale-Wolfe marketing model with some tweaks to better handle the social media dynamics. Quoting [2]:

> The Vidale-Wolfe marketing model is a first-order, linear, nonhomogeneous ordinary differential equation (ODE) where the forcing term is proportional to advertising expenditure. With an initial response in sales as the initial

> condition, the solution of the initial value problem is straightforward for a first undergraduate ODE course.

Mathematically, this model is described as follows:

$\frac{dS(t)}{dt} = \beta u(t)[M(t) - S(t)] - \delta S(t)$

where

- S(t) → Sales at time t;
- M(t) → Market size at time t;
- β → Advertising constant;
- δ → Rate of brand sale decay;
- $u(t)$ → Control action at time t.

Most of the terms above are self-explanatory but β, which is described in [1] as "[...] the rate of decay of brand sale given no active advertising."

From that, the Event-triggered Social Media Chatter Model is derived by doing the following:

1. Normalizing Vidale-Wolfe's model by setting $M(t) = 1$;
2. Breaking β into two other constants:
   A. β_1 → the social marketing campaign constant;
   B. β_2 → the social interaction constant;
3. Generalizing the sales term $S(t)$ to an information spread value X_t.

In [1] the meaning of these constants is explained in the following paragraph

> The effectiveness of social marketing is affected by dynamic resource spending and promotion over the network to convince people to purchase a product, uphold a social or political movement, or join in an activity. The social marketing constant can be associated with a traditional advertising campaign or an event that triggers similar social media interest. Once people become exposed to an advertisement and decide to share the message, the social interaction constant may be viewed as the natural tendency of the social media network to advertise internally through posts, tweets, and likes without external influences and advertising.

The final mathematical model is

$$dX_t = \beta_1 u(t)[1 - X_t] + \beta_2[1 - X_t]X_t - \delta X_t$$

or visually

Model

The final definition to be presented is the concept of socio-equilibrium threshold. It basically describes the equilibrium level of social media chatter after the control (promotion) goes to 0 or mathematically

$$X_{eqb} = 1 - \frac{\delta}{\beta_2}$$

In [1], β_2 is multiplied by a factor k, but in this project the value of k will be embedded in β_2. Finally, the "[...] goal of social media marketing is to increase peoples' attention and interest beyond the natural equilibrium point through the control action of spending resources on ads..." and at the same time minimizing the associated costs.

2. Mathematical model

During a marketing campaign, **the goal is to achieve social craze status as fast as possible while keeping costs as low as possible**. From that the optimization problem will be derived. The cost function for the Event-triggered Social Media Chatter Model is defined in [1] as

$$J = \int_0^{t_f} [u^2(t) + (x - x_d)^2 + \lambda] dt$$

where

- λ → weight placed on time. Meaning the importance of how long it takes to get to equilibrium;
- x_d → desired amount of activity.

when there's no need to maintain x_d, $u(t) = 0$. Now the mentioned equations will be discretized, so they can be used in future simulations:

- Event-triggered Social Media Chatter Model:

$$x(t+1) - x(t) = \beta_1 u(t)[1 - x(t)] + \beta_2[1 - x(t)]x(t) - \delta x(t)$$

- Cost function:

$$J = \sum_{t=0}^{T_F} (u^2(t) + (x(t) - x_d(t))^2 + \lambda)$$

- Equilibrium point:

$$X_{eqb} = 1 - \frac{\delta}{\beta_2}$$

Gathering these equations, the optimization problem can be formulated as

$$
\begin{array}{lll}
\underset{u}{\text{minimize}} & J(u,x) & \\
\text{subjected to:} & u_i \geq 0 & i = 1, \ldots, m \\
& x_{i+1} = x_i + \beta_1 u_i[1-x_i] + \beta_2[1-x_i]x_i - \delta x_i & \\
& x_i \geq 0 &
\end{array}
$$

More constraints may be added according to the control technique used. The constants used are defined in the cell below, following the same as used in the snipet on page 151 of [1].

The first formulation proposed tries to get to the equibilibrium minimizing the associated expenses. However, another goal could be to get to the X_{eqb} as fast as possible. For this second goal proposed in this project, the objective function can be reduced to

$$
J = \sum_{t=0}^{T_F} (x(t) - X_{eqb})^2
$$

In [8]:

```
#dependencies

import Pkg;
Pkg.add("JuMP");
Pkg.add("Clp");
Pkg.add("PyPlot");
Pkg.add("Ipopt");

@time using Clp;
@time using JuMP;
@time using PyPlot;
@time using Ipopt;
```

```
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  No Changes to `~/.julia/environments/v1.8/Project.toml`
  No Changes to `~/.julia/environments/v1.8/Manifest.toml`
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  0.000285 seconds (180 allocations: 13.742 KiB)
  0.000284 seconds (180 allocations: 13.742 KiB)
  0.000296 seconds (180 allocations: 13.742 KiB)
```

In [57]:

```
δ = 0.1; #forgetting factor
β1 = 0.1; #spreading constant
β2 = 0.15; #social spreading
λ = 1; #time cost weight
xEquilibrium = 1-(δ/β2);
```

```
ρβ1 = 0.02;
ρβ2 = 0.03;
ρδ = 0.025;

function getTimeXeqReached(x)
    for t in 1:numberOfIterations
        if x[t] >= 0.33
            return t;
        end
    end
    return 0;
end;
```

3.1 Optimal control

In [115…

```
function optimalControl()
    m = Model(Ipopt.Optimizer);
    set_silent(m);
    numberOfIterations = 100;
    @variable(m, xOptimalControl[1:numberOfIterations] >= 0);
    @variable(m, uOptimalControl[1:numberOfIterations] >= 0);

    for t in 1:numberOfIterations-1
        @constraint(m, xOptimalControl[t+1] == xOptimalControl[t] + β1*uOpti
    end

    @constraint(m, sum(xOptimalControl) >= 0);
    @constraint(m, sum(uOptimalControl) >= 0);
    @constraint(m, xOptimalControl[1] == 0) #initial point


    @objective(m, Min, sum(
        uOptimalControl .^2 .+ (xOptimalControl .- xEquilibrium).^2 .+ λ
    ));
    optimize!(m);
    return JuMP.objective_value.(m), JuMP.value.(xOptimalControl), JuMP.valu
end;
```

In [116…

```
####### exporting variables for further analysis ########
costOptimalControlResult, xOptimalControlResult, uOptimalControlResult, cont
println(costOptimalControlResult);
println(xOptimalControlResult);
println(uOptimalControlResult);
println(controlAmountOptimalControlResult);

##### simple plot #######
clf(); #required for vscode on mac
fig = plt.figure();

ax1 = fig.add_subplot(2, 1, 1);
ax1.title.set_text("x vs step");
plt.plot(range(1, numberOfIterations), transpose(JuMP.value.(xOptimalControl
plt.plot(range(1, numberOfIterations), xEquilibrium .* ones(numberOfIteratic
```

```
ax2 = fig.add_subplot(2, 1, 2);
ax2.title.set_text("u vs step");
plt.plot(range(1, numberOfIterations), transpose(JuMP.value.(uOptimalControl

fig.tight_layout();
show()
gcf()
```

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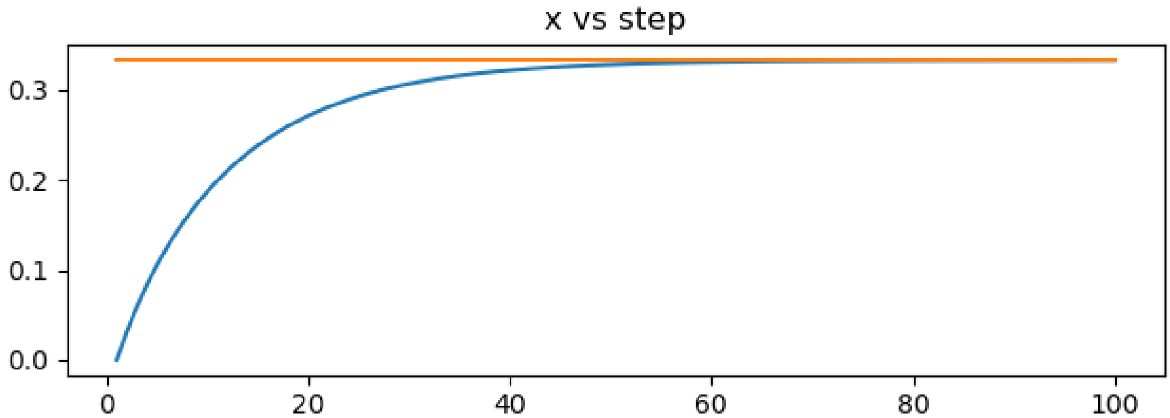

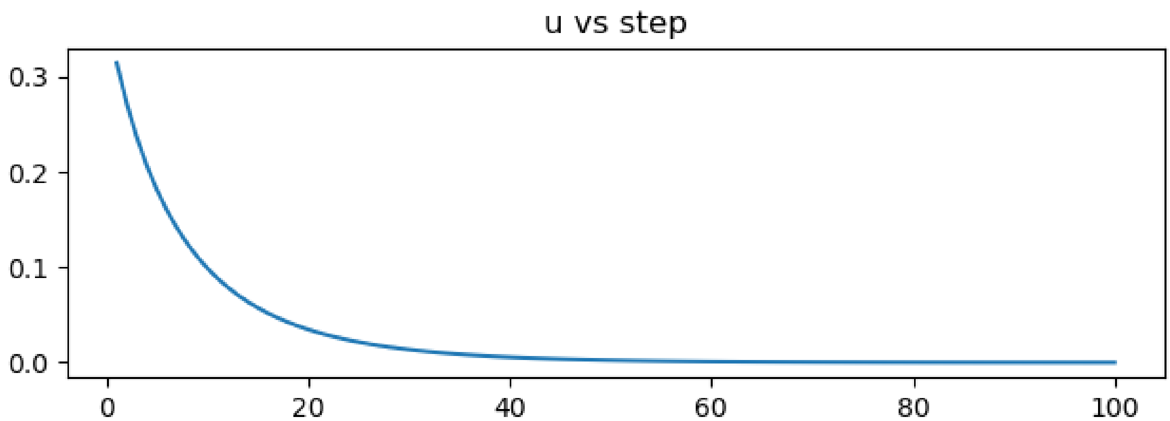

3.2 Optimal control with max instantaneous and total control input

In [79]:
```
function optimalControlWithCostConstraints()
    m = Model(Ipopt.Optimizer);
    set_silent(m);
    numberOfIterations = 100;
    @variable(m, xOptimalControlLimitedCosts[1:numberOfIterations] >= 0);
    @variable(m, uOptimalControlLimitedCosts[1:numberOfIterations] >= 0);

    for t in 1:numberOfIterations-1
        @constraint(m, xOptimalControlLimitedCosts[t+1] == xOptimalControlLi
    end

    @constraint(m, uOptimalControlLimitedCosts[numberOfIterations] == 0);
    @constraint(m, sum(xOptimalControlLimitedCosts) >= 0);
    @constraint(m, sum(uOptimalControlLimitedCosts) >= 0);

    @constraint(m, xOptimalControlLimitedCosts[1] == 0) #prevent explosion
    @constraint(m, uOptimalControlLimitedCosts[1] <= 1) #prevent explosion

    ##### introduce max possible control input ###########################
    @constraint(m, sum(uOptimalControlLimitedCosts) <= 1.8)
```

```julia
    @constraint(m, uOptimalControlLimitedCosts .<= 0.1)
    ##### introduce max possible control input ###########################

    advertisingCost = @expression(m, sum(uOptimalControlLimitedCosts .^2 .+
    @objective(m, Min, advertisingCost);
    optimize!(m);

    return JuMP.objective_value.(m), JuMP.value.(xOptimalControlLimitedCosts
end;
```

In [80]:

```julia
####### exporting variables for further analysis ########
costOptimalControlLimitedCostsResult, xOptimalControlLimitedCostsResult, uOp
println(costOptimalControlLimitedCostsResult);
println(xOptimalControlLimitedCostsResult);
println(uOptimalControlLimitedCostsResult);
println(controlAmountOptimalControlLimitedCostsResult);

##### simple plotting #######
clf(); #required for vscode on mac
fig = plt.figure();

ax1 = fig.add_subplot(2, 1, 1);
ax1.title.set_text("x vs step");
plt.plot(range(1, numberOfIterations), transpose(JuMP.value.(xOptimalControl
plt.plot(range(1, numberOfIterations), xEquilibrium .* ones(numberOfIteratio

ax2 = fig.add_subplot(2, 1, 2);
ax2.title.set_text("u vs step");
plt.plot(range(1, numberOfIterations), transpose(JuMP.value.(uOptimalControl

fig.tight_layout();
show()
gcf()
```

```
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```
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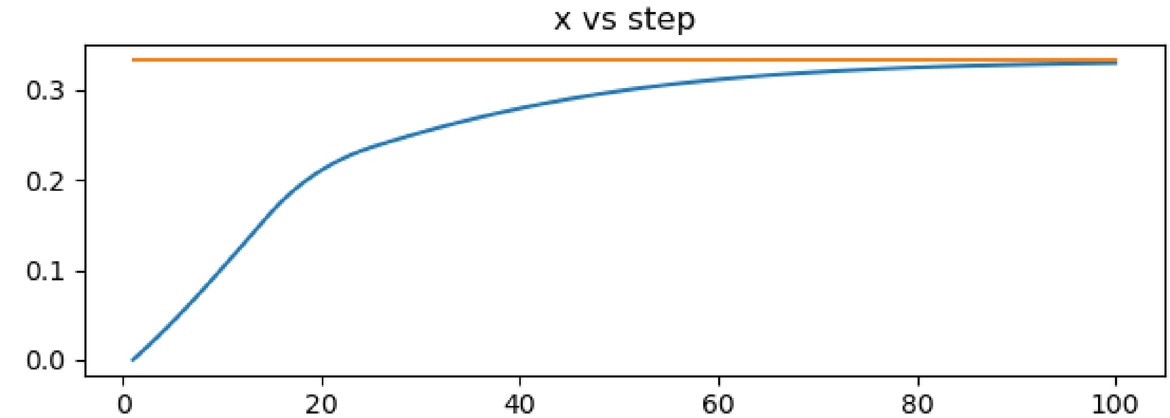
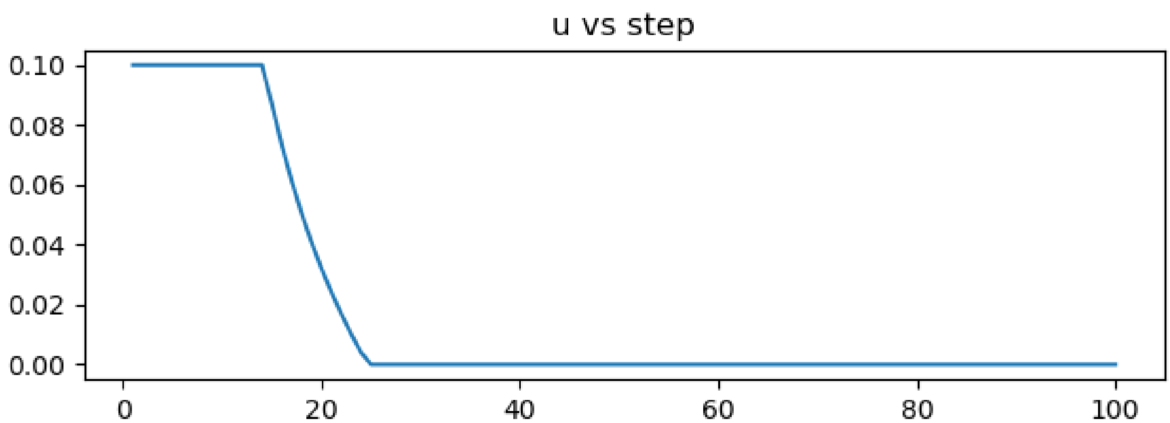
8976e-9, 2.176183674897004e-9, 2.1524333355733357e-9, 2.128981834710744e-9,
2.105763984416521e-9, -4.948958256977865e-43]
1.8000000059172825
```





3.3 Optimal control without cost concerns

In the problems formulated above, a maximum advertising time has been established and the goal is to get to the X_{eqb} using the minum amount of resources (control input). However, another problem could be formulated with the goal to reach X_{eqb} and fast as possible regarless of the cost.

In [82]:

```
function fastestEquilibriumControl()
    m = Model(Ipopt.Optimizer);
    set_silent(m);
    numberOfIterations = 100;
    @variable(m, xFastestEquilibrium[1:numberOfIterations] >= 0);
    @variable(m, uFastestEquilibrium[1:numberOfIterations] >= 0);

    for t in 1:numberOfIterations-1
        @constraint(m, xFastestEquilibrium[t+1] == xFastestEquilibrium[t] +
    end

    @constraint(m, uFastestEquilibrium[numberOfIterations] == 0);
    @constraint(m, xFastestEquilibrium[numberOfIterations] == xEquilibrium);
    @constraint(m, sum(xFastestEquilibrium) >= 0);
    @constraint(m, sum(uFastestEquilibrium) >= 0);
```

```
    @constraint(m, xFastestEquilibrium[1] == 0) #prevent explosion
    @constraint(m, uFastestEquilibrium .<= 0.1) #prevent explosion

    @objective(m, Min, sum(
        sum((xFastestEquilibrium .- xEquilibrium).^ 2)
    ));
    optimize!(m);

    return JuMP.objective_value.(m), JuMP.value.(xFastestEquilibrium), JuMP.
end;
```

In [83]:
```
####### exporting variables for further analysis ########
costFastestEquilibriumResult, xFastestEquilibriumResult, uFastestEquilibrium
println(costFastestEquilibriumResult);
println(xFastestEquilibriumResult);
println(uFastestEquilibriumResult);
println(controlAmountFastestEquilibriumResult);

##### plotting #######
clf(); #required for vscode on mac
fig = plt.figure();

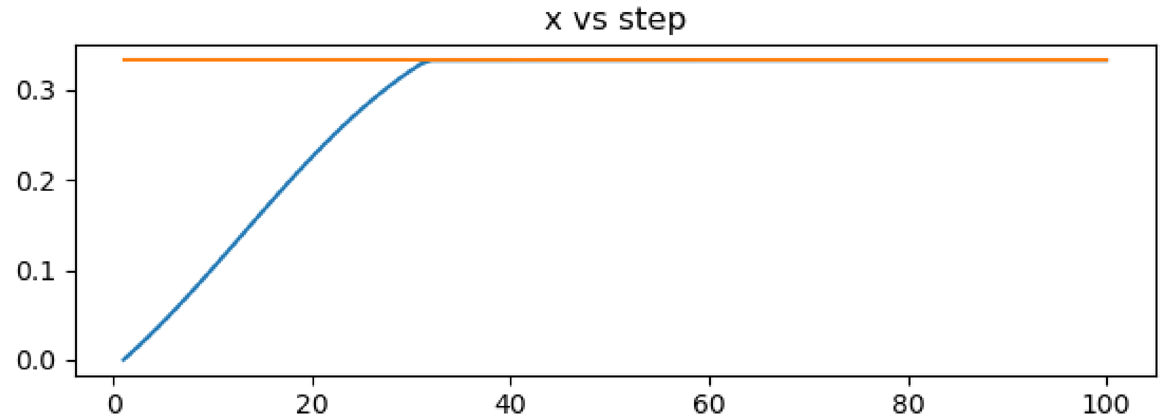
ax1 = fig.add_subplot(2, 1, 1);
ax1.title.set_text("x vs step");
plt.plot(range(1, numberOfIterations), transpose(JuMP.value.(xFastestEquilib
plt.plot(range(1, numberOfIterations), xEquilibrium .* ones(numberOfIteratio

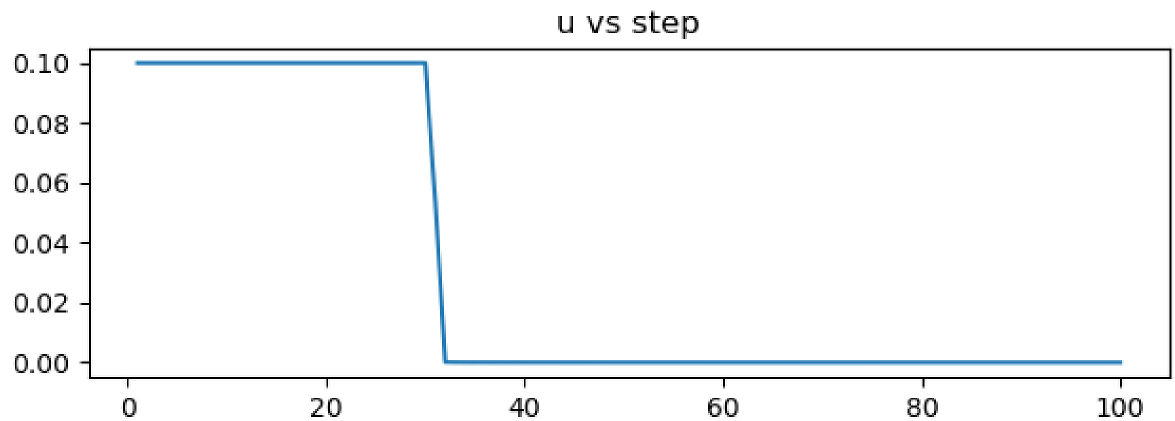
ax2 = fig.add_subplot(2, 1, 2);
ax2.title.set_text("u vs step");
plt.plot(range(1, numberOfIterations), transpose(JuMP.value.(uFastestEquilib

fig.tight_layout();
show()
gcf()
```

```
1.1287786900059669
[3.7489278434030316e-42, 0.010000000914057257, 0.020385001842951563, 0.0311
38070546259118, 0.04223815730502341, 0.0536600751359588, 0.0653745683945469
7, 0.07734847677589778, 0.08954499856168169, 0.10192405319891759, 0.1144427
3910638592, 0.12705587822168138, 0.1397166345067561, 0.15237718972580927,
0.1649894565952891, 0.17750580714480457, 0.18987979301832525, 0.20206683459
710909, 0.214024857246354, 0.2257148555779517, 0.23710137018720223, 0.24815
286559186028, 0.258842002760866, 0.2691458043372428, 0.2790457151226628, 0.
28852756433172444, 0.29758143933047604, 0.30620148286998594, 0.314385626886
29595, 0.3221352736915778, 0.32945489791439153, 0.33324242964152984, 0.3332
5435310645074, 0.3332622199922162, 0.33326846995020915, 0.3332737764543353,
0.33327843027129483, 0.333282587961871, 0.33328634607088775, 0.333289769894
4788, 0.333292906642185, 0.3332957921990959, 0.33329845491343996, 0.3333009
17865674, 0.3333032003034099, 0.3333053185902248, 0.33330728685703576, 0.33
33091174638717, 0.3333108213364755, 0.33331240821772934, 0.3333138868595585
4, 0.33331526517225374, 0.3333165503426856, 0.333317748929362, 0.3333188669
39951, 0.3333199098953184, 0.3333208828830478, 0.33332179060264894, 0.33332
263740411816, 0.33332342732112363, 0.33332416409979754, 0.3333248512239042,
0.33332549193699085, 0.33332608926200585, 0.33332664601877326, 0.3333271648
3963964, 0.33332764818355254, 0.33332809834878213, 0.33332851748446446, 0.3
333289076011122, 0.33332927058021816, 0.333329608183056, 0.3333299220587674
4, 0.33333021375181204, 0.33333048470884585, 0.3333307362850846, 0.33333096
97502015, 0.33333118629380265, 0.3333313870305173, 0.33333157300473604, 0.3
333317451950272, 0.3333319045182568, 0.3333320518334358, 0.333332187945314
6, 0.3333323136077449, 0.333332429526824, 0.33333253636383714, 0.3333326347
380116, 0.3333327252290936, 0.3333328083797587, 0.3333328846978658, 0.33333
295465856405, 0.3333330187062664, 0.3333330772565168, 0.33333313069781406,
0.3333331793935617, 0.33333322368458934, 0.3333332638934456, 0.333333300333
75397, 0.33333333333333326]
[0.10000000914057258, 0.10000000904115343, 0.10000000892809999, 0.100000008
79914342, 0.100000086515563, 0.10000000848204033, 0.10000000828658082, 0.1
0000000806025608, 0.10000000779698497, 0.10000000748918848, 0.1000000071273
3005, 0.10000000669928201, 0.10000000618943886, 0.10000000557745438, 0.1000
0000483640893, 0.10000000393009253, 0.10000000280888151, 0.1000000014033115
7, 0.0999999996137522, 0.09999999729322995, 0.0999999421767532, 0.09999999
003187027, 0.09999998414549331, 0.09999997551880245, 0.0999999621807184
7,
0.09999994001681634, 0.09999989922518243, 0.0999981144778697, 0.0999995637
303755, 0.0999982807567566, 0.05362601647758054, 0.00011067618723298284, 5.
877252214945719e-5, 4.041878234287372e-5, 3.095406201144896e-5, 2.514318237
7802584e-5, 2.1191212809619083e-5, 1.8315358483364304e-5, 1.611930311474912
4e-5, 1.4380718873742571e-5, 1.2965142830526683e-5, 1.178643190967346e-5,
1.0786826732166343e-5, 9.926141249922707e-6, 9.175547287207137e-6, 8.513824
627172805e-6, 7.925005490387722e-6, 7.396843285967514e-6, 6.919787377086979
e-6, 6.486278795757438e-6, 6.090255415695814e-6, 5.726797301629509e-6, 5.39
1867987347675e-6, 5.082122725440028e-6, 4.7947643394093445e-6, 4.5274334639
94498e-6, 4.2781239960185364e-6, 4.045117276804987e-6, 3.82693036363288e-6,
3.622275017722289e-6, 3.4300249277150242e-6, 3.2491893220327045e-6, 3.07889
15807609352e-6, 2.9183517912156876e-6, 2.766872437279839e-6, 2.623826595828
2146e-6, 2.4886481513880724e-6, 2.3608236448004022e-6, 2.2398854517340966e-
6, 2.1254060487102574e-6, 2.0169931723597837e-6, 1.9142857152979735e-6, 1.8
169502317181392e-6, 1.7246779494184122e-6, 1.6371822038536202e-6, 1.5541962
249809596e-6, 1.4754712199235918e-6, 1.400774704401899e-6, 1.32988904392956
25e-6, 1.2626101722868664e-6, 1.198746460034658e-6, 1.1381177100269382e-6,
1.0805542601827564e-6, 1.025896176320564e-6, 9.739925197494937e-7, 9.247006
756537763e-7, 8.778857292356062e-7, 8.334198774156694e-7, 7.911818655675755
e-7, 7.510564440749896e-7, 7.129338563870178e-7, 6.767094203843465e-7, 6.42
```

```
283405897246e-7, 6.095617942100757e-7, 5.784595250166943e-7, 5.489105402177
94e-7, 5.20896251502479e-7, 4.945246858616271e-7, 4.7024399229581973e-7, -
1.2641595362669033e-44]
3.0541570417764654
```





3.4 Model predective control

In [84]:

```
function MPC()
    t = collect(2:100) #eixo x
    loopCount = 100
    uMPC = Array{Float64}(undef, loopCount)
    xMPC = Array{Float64}(undef, loopCount)
    uMPC[1] = 0
    xMPC[1] = 0

    for i in t
        ################ optimization routine #####################
        m = Model(Ipopt.Optimizer);
        set_silent(m);
        movingTimeHorizon = 10; #prediction window
        @variable(m, xMPCLocal[1:movingTimeHorizon] >= xMPC[i-1]);
        @variable(m, uMPCLocal[1:movingTimeHorizon] >= 0);
        @constraint(m, xMPCLocal[1] == xMPC[i]) #initial point
        @constraint(m, xMPCLocal[10] == xEquilibrium) #initial point
        @constraint(m, uMPCLocal[1] == uMPC[i]) #initial point
        @constraint(m, uMPCLocal .<= 0.1)
        for t in 1:movingTimeHorizon-1
            @constraint(m, xMPCLocal[t+1] == xMPCLocal[t] + β1*uMPCLocal[t]*
        end
```

```
        @objective(m, Min, sum(
            uMPCLocal .* uMPCLocal .+ (xMPCLocal .- xEquilibrium).^2 .+ λ
        ));
        optimize!(m);
        ############################################################
        uMPC[i] = JuMP.value.(uMPCLocal)[2]
        xMPC[i] = xMPC[i - 1] + β1*uMPC[i - 1]*(1 - xMPC[i - 1]) + β2*(1 - x
    end

    return JuMP.objective_value.(m), xMPC, uMPC, sum(uMPCResult);
end;
```

In [85]:

```
costMPCResult,xMPCResult,uMPCResult,controlAmountMPCResult = MPC();
println(costMPCResult);
println(xMPCResult);
println(uMPCResult);
println(controlAmountMPCResult);

clf(); #required for vscode on mac
fig = plt.figure();

ax1 = fig.add_subplot(2, 1, 1);
ax1.title.set_text("x vs step");
plt.plot(range(1, loopCount), transpose(xMPC));
plt.plot(range(1, loopCount), xEquilibrium .* ones(loopCount));

ax2 = fig.add_subplot(2, 1, 2);
ax2.title.set_text("u vs step");
plt.plot(range(1, loopCount), transpose(uMPC));

fig.tight_layout();
show()
gcf()
```

```
101.43442942075757
[0.0, 0.0, 0.01000000099868024, 0.020385002024324844, 0.031138070837779364,
0.04223815771891406, 0.05366007568828731, 0.06537456910241679, 0.0773484776
5687619, 0.08954499963484423, 0.10192405448446289, 0.11444274062747162, 0.1
2705588000295542, 0.13971663657856248, 0.15237719211235365, 0.1649894593420
5317, 0.17750581029561022, 0.18987979662555782, 0.20206683872462228, 0.2140
2486197359472, 0.2257148610059425, 0.2371013764474497, 0.24815287286005241,
0.25884201127876816, 0.26914581444939667, 0.2790457273416028, 0.28704237655
88757, 0.293668625925108, 0.2993453644162042, 0.3042463857975898, 0.3084889
8064792496, 0.31216827491577903, 0.3153616227065969, 0.318142995534 6933, 0.
3205669327291344, 0.32268164825446793, 0.3249151693672486, 0.32650265986862
52, 0.328228524658 1623, 0.3297219187201048, 0.330934360328611, 0.3318678947
805426, 0.33253275427947676, 0.33292330237156453, 0.33312827521826155, 0.33
323950288999116, 0.33329399519732783, 0.3333187023551317, 0.333328928769428
6, 0.3333326939935829, 0.3333337960869559, 0.33333393174665504, 0.333333991
7595427, 0.3333340752516587, 0.3333341682634692, 0.3333342729466131, 0.3333
343907412268, 0.3333345232865955, 0.33333467242682674, 0.3333348402397605,
0.3333350290690523, 0.33333524166873735, 0.33333548077146835, 0.33333574985
210196, 0.333336052648191, 0.33333639338786947, 0.33333677682487584, 0.3333
37208308656, 0.33333769385178597, 0.3333382402506079, 0.33333885512004147,
0.3333395470412498, 0.3333403256701751, 0.3333412021130066, 0.3333421884999
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0.33335608135425426, 0.33335893258734833, 0.33336214068847886, 0.3333657510
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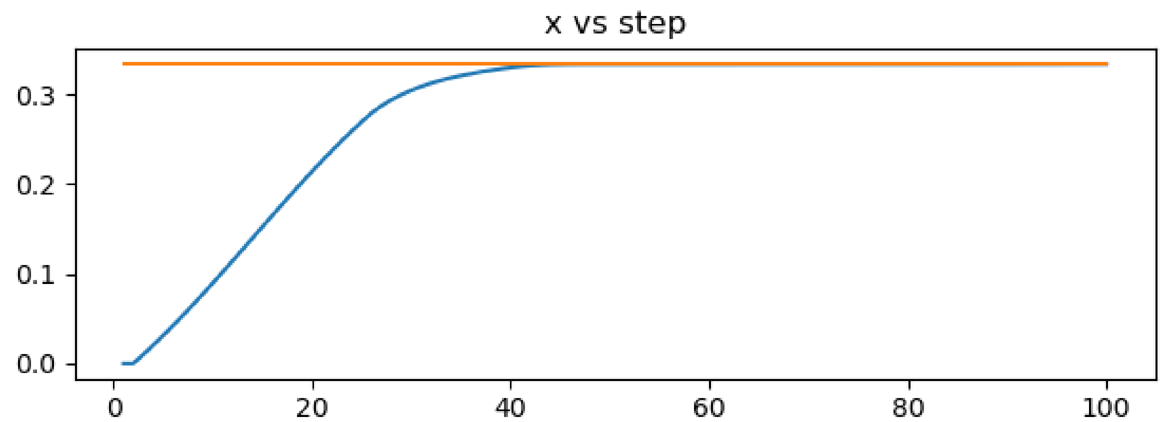
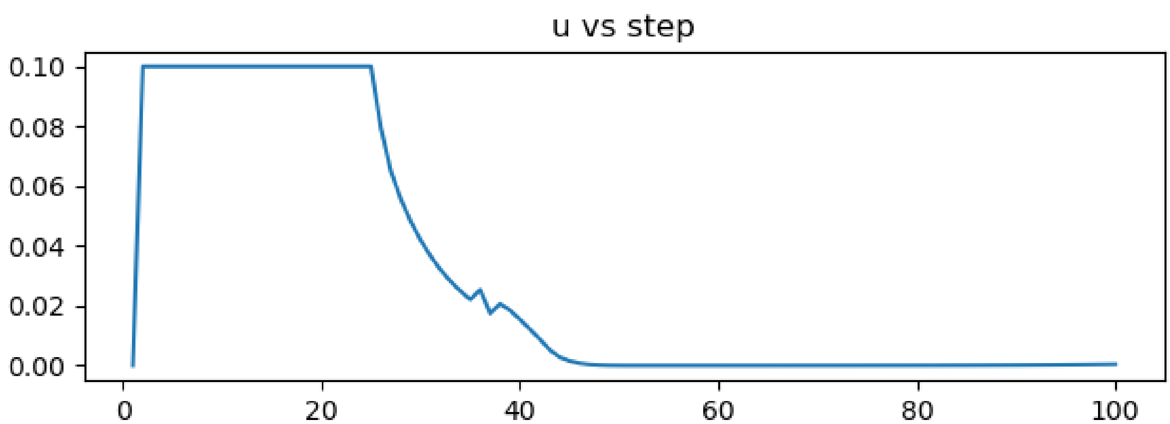
5322297521, 0.33339183902503355, 0.3333991729537022, 0.33340742664145606,
0.33341672862401717, 0.3334271944791847, 0.33343897290777563, 0.33345222809
54399, 0.3334671455011394, 0.3334839338525353, 0.3335025487426157]
[0.0, 0.10000000998680239, 0.10000000998680239, 0.10000000998975404, 0.1000
000099699274, 0.10000000997888855, 0.10000000998614632, 0.1000000099843392
8, 0.10000000998284196, 0.10000000997485345, 0.10000000997702911, 0.1000000
0996811832, 0.10000000998904311, 0.10000000989409542, 0.100000000998 80766,
0.10000000998756683, 0.10000000998704113, 0.10000000998650108, 0.1000000099
8594846, 0.10000000998538536, 0.10000000998481388, 0.10000000998423642, 0.1
0000000998365537, 0.10000000998307321, 0.10000000998249246, 0.0793994953553
1331, 0.06498463971829868, 0.05563247813957788, 0.0481678143381417, 0.04189
9244094217435, 0.03658167175712501, 0.0320178430900 8523, 0.0282080981469670
7, 0.024917720984724296, 0.022089632004667414, 0.025364078998641848, 0.0174
37988121477357, 0.020658282496373328, 0.018489360363111298, 0.0154238554720
00385, 0.01217293213289298, 0.008859171609827306, 0.005252921108355742, 0.0
027657466985860535, 0.0015142503355 30465, 0.0007469264602241208, 0.00034108
712030690437, 0.00014242033682523563, 5.3174631719658446e-5, 1.605188130830
852e-5, 2.381962839490817e-6, 1.349005322588036e-6, 1.7462040974569055e-6,
1.9516193122472187e-6, 2.196449080459465e-6, 2.471634634799612e-6, 2.781243
3776735537e-6, 3.12957718624131e-6, 3.5215252344226224e-6, 3.96263326362994
95e-6, 4.4608148817760285e-6, 5.017815075583173e-6, 5.646816670682363e-6,
6.35436658328814e-6, 7.150627125542681e-6, 8.04665400150099e-6, 9.054948810
1008e-6, 1.0189471452567304e-5, 1.146648894952125e-5, 1.2903378608410682e-
5, 1.452034702669946e-5, 1.6339953986809885e-5, 1.83911980092999e-5, 2.0697
77307182111e-5, 2.3287886283868658e-5, 2.6156164429525153e-5, 2.94699212771
9697e-5, 3.3148177309828016e-5, 3.7293886806430064e-5, 4.195664655320906e-
5, 4.7206154354406665e-5, 5.3129745080596103e-5, 5.9832718028758994e-5, 6.7
32501714848811e-5, 7.576565431119757e-5, 8.526329752029411e-5, 9.5952450025
7486e-5, 0.00010798238307859159, 0.00012152151325235132, 0.0001367593822148
2288, 0.00015390940723044758, 0.00017321189131776256, 0.0001951337587582776
2, 0.0002195774113509388, 0.0002471269039820237, 0.0002781266769409964, 0.0
```

```
0031301978761256815, 0.00035229539672581363, 0.00039231339567123567, 0.0004
409920078887757]
2.9671206150474885
```





3.5 Uncertanties

In [138…

```
function optimalControlWithUncertanties()
    m = Model(Ipopt.Optimizer);
    set_silent(m);
    numberOfIterations = 200;
    @variable(m, xOptimalControl[1:numberOfIterations] >= 0);
    @variable(m, uOptimalControl[1:numberOfIterations] >= 0);
    @variable(m, t1[1:numberOfIterations] >= 0);
    @variable(m, t2[1:numberOfIterations] >= 0);
    @variable(m, t3[1:numberOfIterations] >= 0);

    for t in 1:numberOfIterations-1
        @constraint(
            m,
            xOptimalControl[t+1] == xOptimalControl[t]
            + β1*uOptimalControl[t]*(1 - xOptimalControl[t]) - ρβ1*(β1*uOpti
            + β2*(1 - xOptimalControl[t])*xOptimalControl[t] - ρβ2*(1 - xOpt
            - (δ*xOptimalControl[t] - ρδ*(δ*xOptimalControl[t]))
        );
    end

    @constraint(m, sum(xOptimalControl) >= 0);
    @constraint(m, sum(uOptimalControl) >= 0);
    @constraint(m, xOptimalControl[1] == 0) #initial point
```

```
    @constraint(m, xOptimalControl[200] == xEquilibrium) #final point

    @objective(m, Min, sum(
        uOptimalControl .^2 .+ (xOptimalControl .- xEquilibrium).^2 .+ λ
    ));
    optimize!(m);
    return JuMP.objective_value.(m), JuMP.value.(xOptimalControl), JuMP.valu
end;
```

In [139…

```
####### exporting variables for further analysis ########
costOptimalControlWithUncertantiesResult, xOptimalControlWithUncertantiesRes
println(controlAmountOptimalControlWithUncertantiesResult)
println(xOptimalControlWithUncertantiesResult)

##### simple plot #######
clf(); #required for vscode on mac
fig = plt.figure();

ax1 = fig.add_subplot(2, 1, 1);
ax1.title.set_text("x vs step");
plt.plot(range(1, 200), transpose(JuMP.value.(xOptimalControlWithUncertantie
plt.plot(range(1, 200), xEquilibrium .* ones(200));

ax2 = fig.add_subplot(2, 1, 2);
ax2.title.set_text("u vs step");
plt.plot(range(1, 200), transpose(JuMP.value.(uOptimalControlWithUncertantie

fig.tight_layout();
show()
gcf()
```

```
13.736652085184138
[8.299344638756079e-41, 0.030558169989827313, 0.057434585427295896, 0.08119
50749639536, 0.10228992215834218, 0.12108346899053188, 0.13787491412453176,
0.15291321308924685, 0.16640794266587414, 0.17853735148884037, 0.1894544158
4846853, 0.19929145990209493, 0.20816372847159653, 0.21617218587022546, 0.2
2340573593217292, 0.2299430042227449, 0.23585378537846755, 0.24120023152281
406, 0.24603783832218515, 0.2504162712012395, 0.2543800639672196, 0.257969
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0.2692474185575, 0.27143684325786077, 0.2734207827952834, 0.275218647879922
5, 0.27684798877834343, 0.27832467954566825, 0.2796630826880036, 0.28087619
66304094, 0.2819757880137002, 0.28297251055592165, 0.28387601197753737, 0.2
846950302925904, 0.28543748060323587, 0.28611053339580594, 0.28672068521809
085, 0.2872738225159972, 0.2877752793201876, 0.28822988939734256, 0.2886420
334144341, 0.2890156816063315, 0.28935443238595776, 0.289661547291061, 0.28
99399826216455, 0.2901924180865239, 0.29042128274574264, 0.2906287785073069
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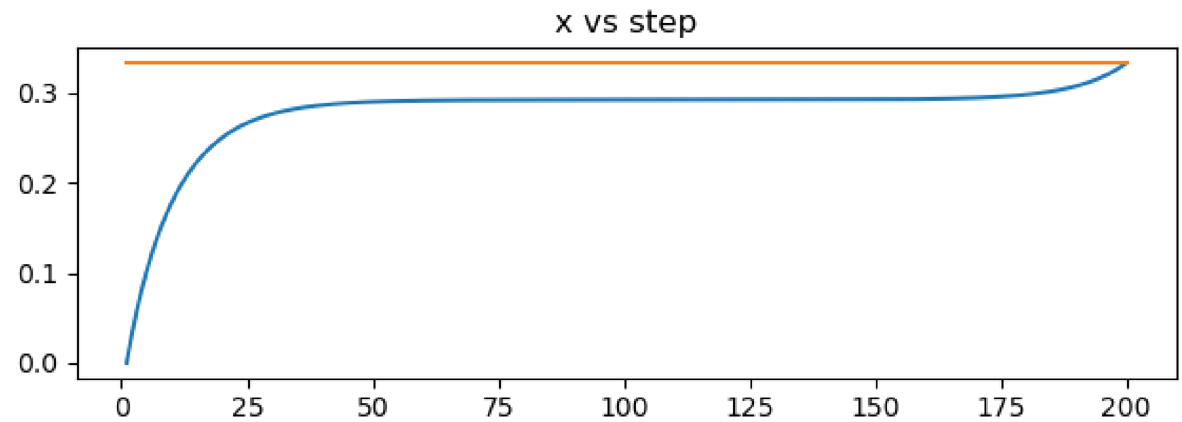

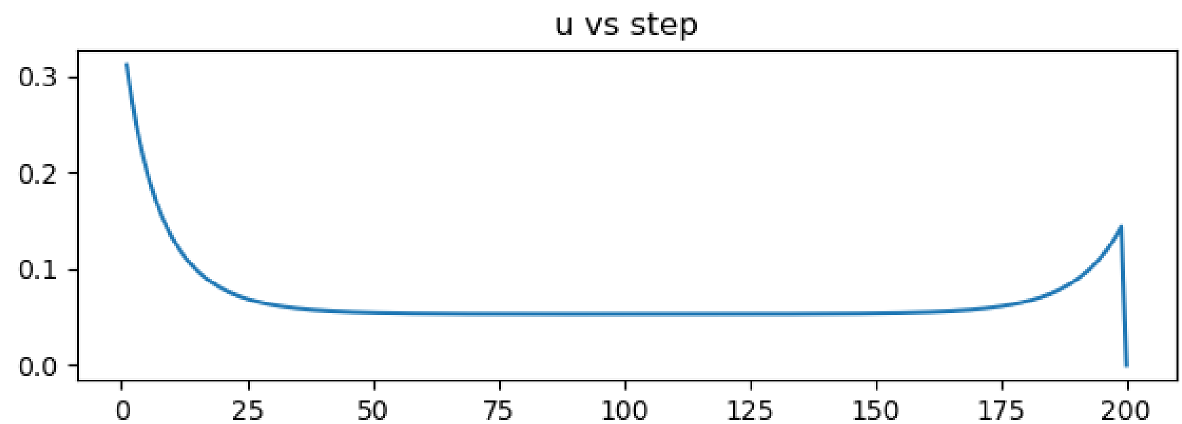

In [134…

```
function MPCWithUncertanties()
    t = collect(2:200) #eixo x
    loopCount = 200
    uMPC = Array{Float64}(undef, loopCount)
    xMPC = Array{Float64}(undef, loopCount)
    uMPC[1] = 0
    xMPC[1] = 0

    for i in t
        ################ optimization routine ####################
        m = Model(Ipopt.Optimizer);
        set_silent(m);
        movingTimeHorizon = 10; #prediction window
        @variable(m, xMPCLocal[1:movingTimeHorizon] >= xMPC[i-1]);
        @variable(m, uMPCLocal[1:movingTimeHorizon] >= 0);
        @constraint(m, xMPCLocal[1] == xMPC[i]) #initial point
        @constraint(m, xMPCLocal[10] == xEquilibrium) #initial point
        @constraint(m, uMPCLocal[1] == uMPC[i]) #initial point
        @constraint(m, uMPCLocal .<= 0.1)
        ####### have to force the setpoint constraint #########
        if (i >= 100)
            @constraint(m, xMPCLocal[2:10] .== xEquilibrium) #initial point
        end
        #######################################################
        for t in 1:movingTimeHorizon-1
            @constraint(
                m,
                xMPCLocal[t+1] == xMPCLocal[t]
                + β1*uMPCLocal[t]*(1 - xMPCLocal[t]) - ρβ1*(β1*uMPCLocal[t]*
```

```
                + β2*(1 - xMPCLocal[t])*xMPCLocal[t] - ρβ2*(1 - xMPCLocal[t]
                - (δ*xMPCLocal[t] - ρδ*(δ*xMPCLocal[t]))
            );
        end
        @objective(m, Min, sum(
            uMPCLocal .* uMPCLocal .+ (xMPCLocal .- xEquilibrium).^2 .+ λ
        ));
        optimize!(m);
        ##############################################################
        uMPC[i] = JuMP.value.(uMPCLocal)[2]
        xMPC[i] = xMPC[i - 1] + β1*uMPC[i - 1]*(1 - xMPC[i - 1]) + β2*(1 - x
    end

    return JuMP.objective_value.(m), xMPC, uMPC, sum(uMPCResult);
end;
MPCWithUncertanties()
```

```
(101.43442942075757, [0.0, 0.0, 0.010000000996644748, 0.0203850020201989,
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8742326, 0.07734847764089581, 0.08954499961880802  …  0.3339940982615921,
0.3339609945236356, 0.333929552370335, 0.3338996880969139, 0.33387132224507
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9060514044, 0.3337494589574241], [0.0, 0.10000000996644748, 0.1000000099664
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  …  0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0], 2.9671206150474885)
```

In [135…

```
loopCount = 200
costMPCWithUncertantiesResult,xMPCWithUncertantiesResult,uMPCWithUncertantie
println(costMPCWithUncertantiesResult);
println(xMPCWithUncertantiesResult);
println(uMPCWithUncertantiesResult);
println(controlAmountMPCWithUncertantiesResult);

clf(); #required for vscode on mac
fig = plt.figure();

ax1 = fig.add_subplot(2, 1, 1);
ax1.title.set_text("x vs step");
plt.plot(range(1, loopCount), transpose(xMPCWithUncertantiesResult));
plt.plot(range(1, loopCount), xEquilibrium .* ones(loopCount));

ax2 = fig.add_subplot(2, 1, 2);
ax2.title.set_text("u vs step");
plt.plot(range(1, loopCount), transpose(uMPCWithUncertantiesResult));

fig.tight_layout();
show()
gcf()
```

101.43442942075757
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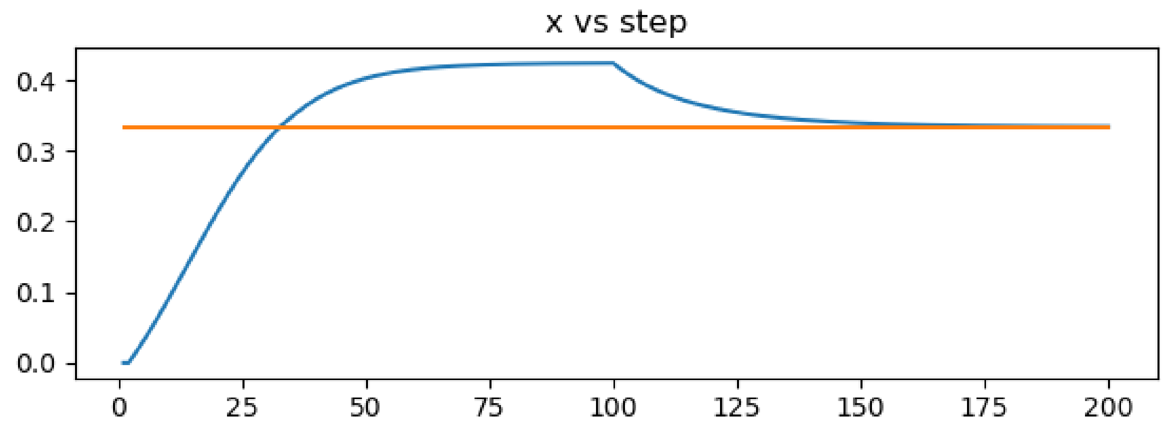
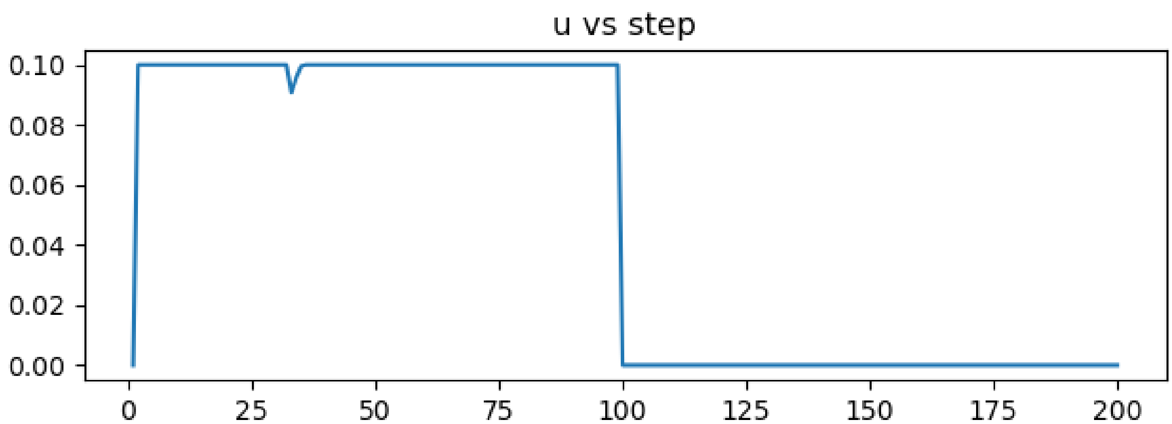




4. Results and discussion

Results summary

Control	Control amount	Control amount normalized*	Time to reach equilibrium
Optimal control	2.776115114183729	90.89627927478766	55
Optimal control with cost limit	1.8000000059172825	58.93606587008714	98
No cost concerns	3.0541570417764654	100.0	32
Predictive control	2.9671206150474885	97.15023079892606	41

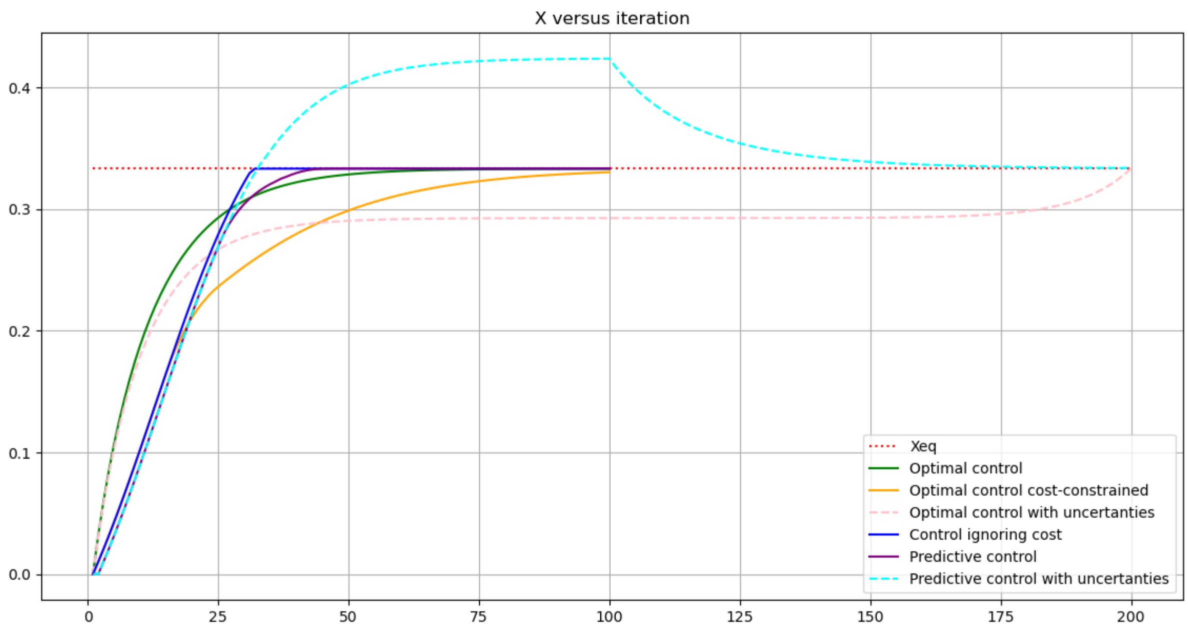
normalization = original control amount (100 / highest control amount)

In [150…

```
#X graph
#costOptimalControlResult, xOptimalControlResult, uOptimalControlResult, cor
#costOptimalControlLimitedCostsResult, xOptimalControlLimitedCostsResult, uC
#costFastestEquilibriumResult, xFastestEquilibriumResult, uFastestEquilibriu
#costMPCResult,xMPCResult,uMPCResult,controlAmountMPCResult = MPC();
#costOptimalControlWithUncertantiesResult, xOptimalControlWithUncertantiesRe

clf()
t1 = 1:100
t2 = 1:200
figure(figsize=(14,7))
```

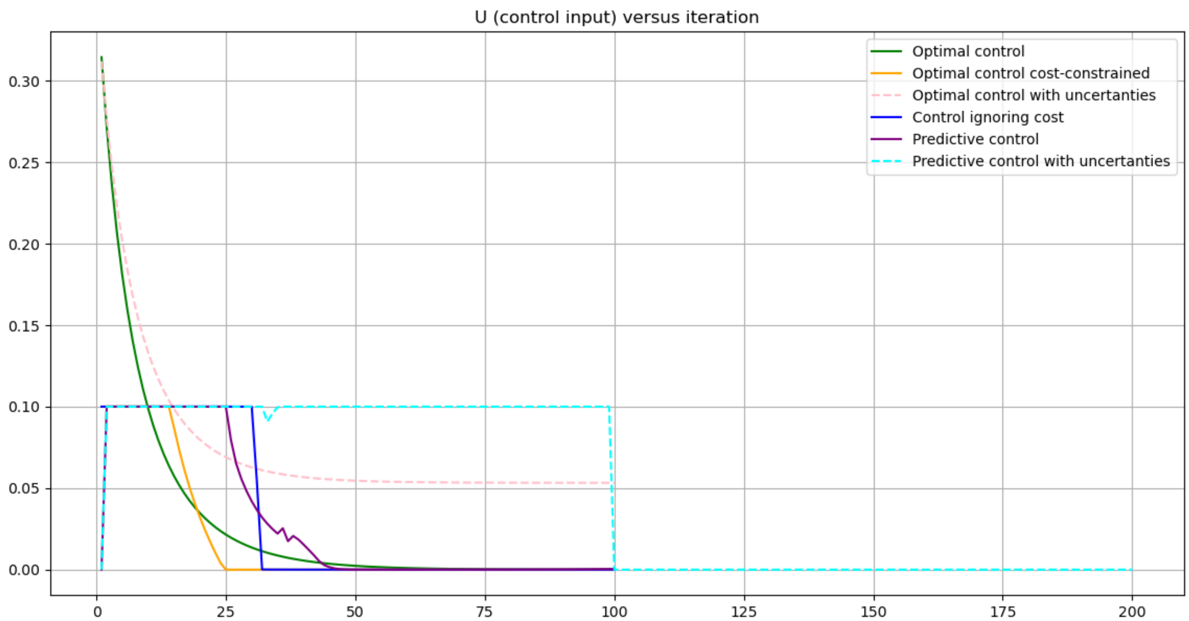
```
plot(t2, xEquilibrium .* ones(200), color="red", linestyle="dotted", label="
plot(t1, xOptimalControlResult, color="green", label="Optimal control")
plot(t1, xOptimalControlLimitedCostsResult, color="orange", label="Optimal c
plot(t2, xOptimalControlWithUncertantiesResult, color="pink", label="Optimal
plot(t1, xFastestEquilibriumResult, color="blue", label="Control ignoring co
plot(t1, xMPCResult, color="purple", label="Predictive control")
plot(t2, xMPCWithUncertantiesResult, color="cyan", label="Predictive control
legend()
title("X versus iteration");
grid("on")
gcf()
```



In [151…

```
#cost graph
#costOptimalControlResult, xOptimalControlResult, uOptimalControlResult, con
#costOptimalControlLimitedCostsResult, xOptimalControlLimitedCostsResult, uO
#costFastestEquilibriumResult, xFastestEquilibriumResult, uFastestEquilibriu
#costMPCResult,xMPCResult,uMPCResult,controlAmountMPCResult = MPC();
#costOptimalControlWithUncertantiesResult, xOptimalControlWithUncertantiesRe

clf()
figure(figsize=(14,7))
t = 1:100
plot(t, uOptimalControlResult, color="green", label="Optimal control")
plot(t, uOptimalControlLimitedCostsResult, color="orange", label="Optimal co
plot(t, uOptimalControlWithUncertantiesResult[1:100], color="pink", label="O
plot(t, uFastestEquilibriumResult, color="blue", label="Control ignoring cos
plot(t, uMPCResult, color="purple", label="Predictive control")
plot(t2, uMPCWithUncertantiesResult, color="cyan", label="Predictive control
legend()
title("U (control input) versus iteration");
grid("on")
gcf()
```



5. Conclusion

Based on the results presented above some conclusions can be draw.

All control models were able to reach the equilibrium within 100 iterations when there were no uncertainties to the predefined constants.

However, some models accomplish this task faster and more efficient than others. From the table presented in the previous section, two control techniques appear to be most suitable for a marketing campaign:

- **Regular optimal control[1]**
  - **Control amount = 2.776115114183729**
  - **Time to reach equilibrium = 55 iterations**
- **Optimal control with no cost concerns [2]**
  - **Control amount = 3.0541570417764654**
  - **Time to reach equilibrium = 32 iterations**

Comparing these two, it is clear that even though [1] uses 10% less control input, it takes 70% more time to reach X_{eq} making [2] the overall best solution.

Another important fact to be highlighted is that all control techniques struggle to deal with uncertainties in the estimation of $\beta 1$, $\beta 2$ and δ. The analysis was made based on the worst case scenario which is when these constants are underestimated and slow down the information spread dynamics. What is happens in this case is that the model stabilizes around a different set-point other than X_{eq}. To solve this issue, it's necessary for "force" $x(t)$ to be equals X_{eq} after a certain number of iterations by using the following constraint:

```
@constraint(m, xOptimalControl[200] == xEquilibrium)
```

Without these uncertainties, all models naturally converge to X_{eq} in order to minimize the cost function while the control input goes to 0 as theoretically expected.

6. Bibliography

1. Information Spread in a Social Media Age Modeling and Control, Michael Muhlmeyer and Shaurya Agarwal
2. Barg, Michael C. (2016) "Find, Process, and Share: An Optimal Control in the Vidale-Wolfe Marketing Model," CODEE Journal: Vol. 11, Article 1