14.273 Industrial Organization: Pset4

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1. Model setup.

Following the notations in Rust (1987), HZ's flow utility is:

$$u(x_t, i_t, \theta_1) + \epsilon_t(i_t) = \begin{cases} -RC - c(0, \theta_1) + \epsilon_t(1) & i_t = 1 \\ -c(x_t, \theta_1) + \epsilon_t(0) & i_t = 0 \end{cases}$$

where RC is the replacement cost, x_t is the observed state variable for mileage, $c(\cdot)$ is cost function and i_t is the decision to replace engine and $\epsilon_t(\cdot)$ is action specific and type I extreme value distributed structural error (or unobserved state variable).

The state transition probability is given by:

$$\theta_{3j} = \mathbb{P}(x_{t+1} = x_t + j | x_t, i_t = 0)$$

$j \in \{0, 1, 2\}$ and if $i_t = 1$ then $x_{t+1} = 0$ with probability 1.

HZ chooses i_t in every period t to maximize an infinite sum of discounted flow utilities. The maximal value is defined as the value function (suppress the dependency on θ_1, θ_3):

$$V(x_1, \epsilon_1) := \max_{i_t, t \in \{1,2,..\}} \mathbb{E}[\sum_{t=1}^{\infty} \beta^{t-1}(u(x_t, i_t, \theta_1) + \epsilon_t(i_t))]$$

Rewrite the value function as in the Bellman optimality form:

$$V(x_t, \epsilon_t) = \max_{i_t} \ (u(x_t, i_t, \theta_1) + \epsilon_t(i_t)) + \beta \mathbb{E}[V(x_{t+1}, \epsilon_{t+1}) | x_t, i_t]$$

where the expectation is with respect to (conditional) state transition probability of both x and ϵ, see Rust (1987) equation (4.5). The Bellman equation breaks the dynamic optimization problem into an infinite series of static choices.

2. (1) The choice specific value function can be derived by plugging a specific action into the value function:

$$\tilde{V}(x_t, \epsilon_t, i_t) = \begin{cases} -RC - c(0, \theta_1) + \epsilon_t(1) + \beta\mathbb{E}[V(x_{t+1}, \epsilon_{t+1})|x_t, i_t = 1] \\ -c(x_t, \theta_1) + \epsilon_t(0) + \beta\mathbb{E}[V(x_{t+1}, \epsilon_{t+1})|x_t, i_t = 0] \end{cases}$$

$$V(x_t, \epsilon_t) = \max\{\tilde{V}(x_t, \epsilon_t, 1), \tilde{V}(x_t, \epsilon_t, 0)\}$$

HZ's decision is about trading off the total (future) cost of maintaining an old engine and the lump sum cost of replacing to a new one. The time to replace is the stopping time in this problem, so it can be thought as an optimal stopping time problem where the optimal policy is characterized by a cutoff in x, HZ would choose to replace the engine if x is above that threshold (the threshold depends on realized value of ϵ).

(2) It's clear from 2 (1) that the optimal stopping rule is:

$$\begin{aligned} &- RC - c(0, \theta_1) + \epsilon_t(1) + \beta\mathbb{E}[V(x_{t+1}, \epsilon_{t+1})|x_t, i_t = 1] > \\ &- c(x_t, \theta_1) + \epsilon_t(0) + \beta\mathbb{E}[V(x_{t+1}, \epsilon_{t+1})|x_t, i_t = 0] \end{aligned}$$

or,

$$\tilde{V}(x_t, \epsilon_t, 1) > \tilde{V}(x_t, \epsilon_t, 0)$$

therefore, because the errors are type I extreme value distributed:

$$\mathbb{P}(i_t = 1|x_t) = \frac{\exp(u(x_t, 1, \theta_1) + \beta\mathbb{E}[V_{t+1}|x_t, i_t = 1])}{\sum_{k=\{0,1\}} \exp(u(x_t, k, \theta_1) + \beta\mathbb{E}[V_{t+1}|x_t, i_t = k]} \tag{2.1}$$

where $u(x_t, i_t, \theta_1)$ is defined in 1 and for convenience:

$$V_{t+1} := V(x_{t+1}, \epsilon_{t+1})$$

(3) For discrete x, under the assumption that the errors are type I extreme value distributed, we have (Rust (1987) equation (4.14)):

$$EV(x, i) = \sum_y \log\{\sum_j \exp[u(y, j) + \beta EV(y, j)]\} \cdot p(y|x, i) \tag{2.2}$$

where

$$EV(x, i) := \mathbb{E}[V_{t+1}|x_t, i_t]$$

and x, i are the state and choice of current period and y, j are the state and choice of the next period. Also note that here the transition probability does not depend on x_t but only on j (or Δx). To compute expected value function, we first need to estimate transition probability from the data, this can be done simply by counting:

$$\hat{\theta}_{30} = \frac{\sum_b \sum_t 1_{\{x_{bt+1} - x_{bt} = 0, i_{bt} = 0\}}}{\sum_b \sum_t 1_{\{i_{bt} = 0\}}}$$

$$\hat{\theta}_{31} = \frac{\sum_b \sum_t 1_{\{x_{bt+1}-x_{bt}=1, i_{bt}=0\}}}{\sum_b \sum_t 1_{\{i_{bt}=0\}}}$$
$$\hat{\theta}_{32} = \frac{\sum_b \sum_t 1_{\{x_{bt+1}-x_{bt}=2, i_{bt}=0\}}}{\sum_b \sum_t 1_{\{i_{bt}=0\}}}$$

we compute the expected value function in the inner loop of the nested fixed point algorithm (holding the value of θ fixed), we first guess the initial values of $EV(x,i)$ for all possible values of x, i and use the equation (2.2) to iterate expected value function until it converges. The criterion is:

$$\max_{x,i} |EV^{T+1}(x,i) - EV^{T}(x,i)| < \eta$$

The plot for $x = 1 - 30$ at the true value of parameters are shown below:

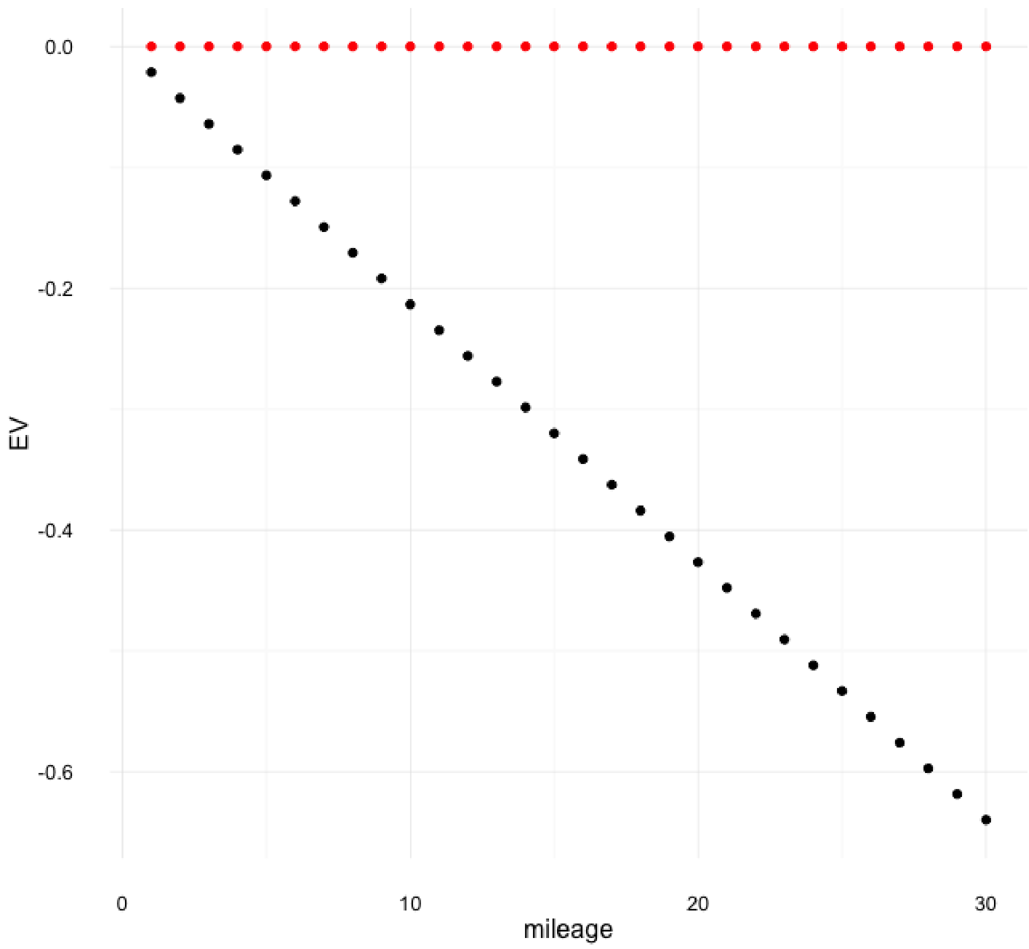

Figure 1: Expected Value Function (black for $i = 0$ and red for $i = 1$)

(4) Summary statistics.

3. (1) In the outer loop we search over a grid of values for (θ_1, β, RC), and compute the log likelihood function:

$$\log L = \sum_b \{\sum_t \log \mathbb{P}(i_{bt}|x_{bt}) + \sum_t \log \mathbb{P}(x_{bt}|x_{bt-1}, i_{t-1})\}$$

where b indexes for bus and t indexes for time period. We compute a log likelihood for each combination of values for (θ_1, β, RC) and choose the one that has the maximal value as our maximum likelihood estimation.

(2) In Hotz-Miller's approach, we will estimate the choice specific value function (as opposed to the expected value function as in Rust). We start by noting that conditional choice probability is observed directly from the data:

$$\hat{\mathbb{P}}(i = 1|x) = \frac{\sum_b \sum_t 1_{\{i_{bt}=1, x_{bt}=x\}}}{\sum_b \sum_t 1_{\{x_{bt}=x\}}}$$

The choice-specific value function (minus the structural error, and suppressing the dependency on θ_1, θ_3) can be presented recursively in the following form:

$$\tilde{V}(x_t, i_t) = u(x_t, i_t) + \beta \mathbb{E}_{x_{t+1}}[\mathbb{E}_{i_{t+1}}[\mathbb{E}_{\epsilon_{t+1}}[u(x_{t+1}, i_{t+1}) + \epsilon_{t+1} + \beta(\cdots)|i_{t+1}, x_{t+1}]|x_{t+1}]|x_t, i_t]$$

where $(\cdots)$ represents higher (two and above) period forward expectations. In principle it's an infinite loop but in practice we need to stop at some T, for example, when $T = 2$, $(\cdots)$ simplifies to:

$$(\cdots) = \mathbb{E}_{x_{t+2}}[\mathbb{E}_{i_{t+2}}[\mathbb{E}_{\epsilon_{t+2}}[u(x_{t+2}, i_{t+2}) + \epsilon_{t+2}|i_{t+2}, x_{t+2}]|x_{t+2}]|x_{t+1}, i_{t+1}]$$

For simplicity, in the code we use one-period forward simulation where:

$$\tilde{V}(x_t, i_t) = u(x_t, i_t) + \beta \mathbb{E}_{x_{t+1}}[\mathbb{E}_{i_{t+1}}[\mathbb{E}_{\epsilon_{t+1}}[u(x_{t+1}, i_{t+1}) + \epsilon_{t+1}|i_{t+1}, x_{t+1}]|x_{t+1}]|x_t, i_t]$$

it is estimated as:

$$\hat{\tilde{V}}(x_t, i_t) = \frac{1}{S} \sum_s [u(x_t, i_t) + \beta[u(x^s_{t+1}, i^s_{t+1}) + \gamma - \log(\hat{\mathbb{P}}(i^s_{t+1}|x^s_{t+1}))]]$$

where x^s_{t+1} is drawn from the transition probability $\hat{\theta}_{30}, \hat{\theta}_{31}, \hat{\theta}_{32}$, and i^s_{t+1} is drawn from $\hat{\mathbb{P}}(i|x)$, γ is the Euler's constant.