1.

Expand as Taylor polynomials :

$$f(x_0+\delta) = f(x_0) + f'(x_0)\delta + \tfrac{1}{2}f''(x_0)\delta^2 + \tfrac{1}{6}f'''(x_0)\delta^3 + \tfrac{1}{24}f^{(4)}(x_0)\delta^4$$
$$+ \tfrac{1}{120}f^{(5)}(x_0)\delta^5 + \ldots + g_1\varepsilon f$$

$$f(x_0-\delta) = f(x_0) - f'(x_0)\delta + \tfrac{1}{2}f''(x_0)\delta^2 - \tfrac{1}{6}f'''(x_0)\delta^3 + \tfrac{1}{24}f^{(4)}(x_0)\delta^4$$
$$- \tfrac{1}{120}f^{(5)}(x_0)\delta^5 + \ldots + g_2\varepsilon f$$

$$f(x_0+\delta) - f(x_0-\delta) = 2f'(x_0)\delta + \tfrac{1}{3}f'''(x_0)\delta^3 + \tfrac{1}{60}f^{(5)}(x_0)\delta^5$$

$$(*)\quad f'(x_0) = \frac{1}{2\delta}\left[f(x_0+\delta) - f(x_0-\delta)\right] - \tfrac{1}{6}f'''(x_0)\delta^2 - \tfrac{1}{120}f^{(5)}(x_0)\delta^4$$
$$+ \bar{g}\varepsilon f/2\delta \qquad \bar{g} = g_1 - g_2$$

For $x = x \pm 2\delta$ just replace δ with 2δ

$$(**)\quad f'(x_0) = \frac{1}{4\delta}\left[f(x_0+2\delta) - f(x_0-2\delta)\right] - \tfrac{4}{6}f'''(x_0)\delta^2 - \tfrac{16}{120}f^{(5)}(x_0)\delta^4 + \tilde{g}\frac{\varepsilon f}{4\delta}$$

Back to $(*)$

$$4f'(x_0) = \frac{2}{\delta}\left[f(x_0+\delta) - f(x_0-\delta)\right] - \tfrac{4}{6}f''(x_0)\delta^2 - \tfrac{4}{120}f^{(5)}(x_0)\delta^4 + \frac{2\bar{g}\varepsilon f}{\delta}$$

Subtract $(**)$

$$3f'(x_0) = \frac{2}{\delta}\left[f(x_0+\delta) - f(x_0-\delta)\right] - \frac{1}{4\delta}\left[f(x_0+2\delta) - f(x_0-2\delta)\right] + \tfrac{1}{10}f^{(5)}(x_0)\delta^4$$
$$+ \frac{\varepsilon f}{\delta}\left(2\bar{g} - \tfrac{1}{4}\tilde{g}\right)$$

$$f'(x_0) = \frac{2}{3\delta}\left[f(x_0+\delta) - f(x_0-\delta)\right] - \frac{1}{12\delta}\left[f(x_0+2\delta) - f(x_0-2\delta)\right]$$
$$+ \frac{1}{30} f^{(5)}(x_0)\delta^4 + \frac{\varepsilon f}{\delta} g' \qquad g' = \frac{2\bar{g} - \frac{1}{4}\tilde{g}}{3}$$

$$\text{Error} = \frac{1}{30} f^{(5)}(x_0)\delta^4 + g'\frac{\varepsilon f}{\delta}$$

$$\frac{\partial \text{Error}}{\partial \delta} = \frac{2}{15} f^{(5)}(x_0)\delta^3 - \frac{g'\varepsilon f}{\delta^2}$$

$$\frac{\partial \text{Error}}{\partial \delta} = 0 \Rightarrow \delta = \left(\frac{15}{2}\frac{g'\varepsilon f}{f^{(5)}}\right)^{1/5}$$ to minimize the error

For $f(x) = e^x$, $f^{(5)}(x) = e^x$

For $f(x) = e^{0.01x}$, $f^{(5)}(x) = (0.01)^5 e^{0.01x}$

Double precision $\Rightarrow \varepsilon \sim 10^{-16}$

$\Rightarrow \delta \sim 10^{-16/5} \sim 10^{-3.2}$