Notes on Baire Category Theorem

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Definition (Nowhere dense). *Let X be a topological space. A subset A is called nowhere dense in X if the interior of the closure of A is empty, i.e. Int $(\overline{A}) = \emptyset$.*

Proposition. *Let X be a topological space, then*

1. *Any subset of a nowhere dense set is nowhere dense.*
2. *The union of finitely many nowhere dense sets is nowhere dense.*
3. *A subset is nowhere dense iff its closure is nowhere dense.*
4. *A subset is nowhere dense iff its complement contains a dense set.*

Definition (Meagre set/First category). *In a topological space X, a subset S is a meagre subset (or of the first category) if it is the union of countably many nowhere dense subsets in X.*

Definition (Second category). *In a topological space X, a subset S is of the second category if it is not of first category.*

Proposition. *Let X be a topological space, then*

1. *Any subset of a meager set is meager.*
2. *The union of finitely many meagre sets is nowhere meagre.*

Theorem (Baire Category Theorem). *Let X be a complete metric space, then*

1. *Let $G_1, G_2 \ldots$ be a sequence of dense open subsets X. Then $G = \bigcap_{n=1}^{\infty} G_n$ is dense in X.*
2. *If X is the union of countably many closed sets, then at least one of the closed sets has non-empty interior. This means that a complete metric space is of the second category.*
3. *The complement of a meagre subset is dense and of the second category.*

Theorem (Principle of uniform boundness (Osgood)). *Let U be a set of the second category in a metric space X and let $\mathcal{F}$ be a family of continuous functions $f : X \to \mathbb{R}$ such that $f(u) : f \in \mathcal{F}$ is bounded for every $u \in U$. Then the elements of $\mathcal{F}$ are uniformly bounded for in some ball B in X, i.e. $|f(x)|$ is bounded by some n for all $f \in \mathcal{F}$ and all $x \in B$.*