

# Conditional Expectation and Martingales\*

T. Bucci<sup>†</sup>, G. Corbo<sup>‡</sup>

June 11, 2022

## 1 Conditional Expectation

$(\Omega, \mathcal{F}, \mathbb{P})$  probability space.

$\mathcal{G} \subseteq \mathcal{F}$   $\mathcal{G}$  sub- $\sigma$ -algebra of  $\mathcal{F}$ .

$X : \Omega \rightarrow \mathbb{R}$  random variable with  $\mathbb{E}[|X|] < +\infty$ .

**Definition 1.1.** We call conditional expectation of  $X$  w.r.t.  $\mathcal{G}$  any  $\mathcal{G}$ -measurable random variable  $V$  s.t.:

$$\int_G X d\mathbb{P} = \int_G V d\mathbb{P} \quad \forall G \in \mathcal{G} \quad (1)$$

**Notation.**  $V = \mathbb{E}[X|\mathcal{G}]$ .

**Example 1.1.**  $X_1, \dots, X_n$   $B(1, p)$  independent  $S_n = X_1 + \dots + X_n$   
 $\mathcal{G} = \sigma(S_n)$   $\sigma$ -algebra generated by the events  $\{S_n = k\} \ k = 0, \dots, n$

$$\forall i \in 1, \dots, n \quad \mathbb{E}[X_i | \sigma(S_n)] = \frac{S_n}{n} \left( \text{another notation } \mathbb{E}[X_i | S_n = k] = \frac{k}{n} \right)$$

*Proof.* Any set in  $\mathcal{G}$  is union of  $\{S_n = k\}$  for some  $k$ 's. Therefore it suffices to show that:

$$\int_{S_n=k} X_i d\mathbb{P} = \int_{S_n=k} \frac{S_n}{n} d\mathbb{P} \quad \forall k = 0, \dots, n$$

$X_i$  has values in  $\{0, 1\}$  so

$$\int_{\{S_n=k\}} \mathbb{1}_{\{X_i=1\}} d\mathbb{P} = \frac{k}{n} \cdot \mathbb{P}\{S_n = k\} \quad \forall k = 0, \dots, n$$

We call  $\hat{S}_n = X_1 + \dots + X_{i-1} + X_{i+1} + \dots + X_n$ , then:

$$\begin{aligned} \int_{\{S_n=k\}} \mathbb{1}_{X_i} d\mathbb{P} &= \mathbb{P}\{S_n = k, X_i = 1\} \\ &= \mathbb{P}\{\hat{S}_n = k - 1, X_i = 1\} \\ &= \mathbb{P}\{\hat{S}_n = k - 1\} \mathbb{P}\{X_i = 1\} \\ &= \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} p \quad \forall k = 0, \dots, n \end{aligned}$$

---

\*Taken during the course of Stochastic Dynamical Models by Professor Franco Fagnola, Politecnico di Milano, A.Y. 2021-22. Errors and updates can be proposed through the repository at: <https://github.com/gabrielecorbo/Martingales>

<sup>†</sup>teo.bucci@mail.polimi.it

<sup>‡</sup>gabriele.corbo@mail.polimi.it

We have to check that

$$\binom{n-1}{k-1} p^k (1-p)^{n-k} = \frac{k}{n} \binom{n}{k} p^k (1-p)^{n-k} \left( = \frac{k}{n} \mathbb{P}\{S_n = k\} \right)$$

indeed

$$\frac{k}{n} \binom{n}{k} = \frac{k}{n} \frac{n!}{k!(n-k)!} = \binom{n-1}{k-1}.$$

Then

$$\int_{S_n=k} X_i d\mathbb{P} = \int_{S_n=k} \frac{S_n}{n} d\mathbb{P} \quad \forall k = 0, \dots, n$$

So

$$\mathbb{E}[X_i | \sigma(S_n)] = \frac{S_n}{n}$$

□

**Example 1.2.**  $X, Y$  real random variable with joint distribution  $f_{X,Y}$  and conditional distribution:

$$f_{Y|X}(y|x) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_X(x)} & \text{if } f_X(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} \mathbb{E}[Y | \sigma(X)] &= \int y \cdot f_{Y|X}(y|X) dy \\ \text{r.v. } \omega &\rightarrow \int y \cdot f_{Y|X}(y|X(\omega)) dy \end{aligned}$$

**Theorem 1.1.** Given  $X$  with  $\mathbb{E}[|X|] < +\infty$  and  $\mathcal{G}$  sub- $\sigma$ -algebra of  $\mathcal{F}$ , one can always find  $\mathbb{E}[X|\mathcal{G}]$ .

*Proof.* Suppose  $X \geq 0$  (if not we write  $X = X^+ - X^-$  and work on the two parts separately).

Consider the following measures on  $\mathcal{G}$ :

- $\mathbb{P}|_{\mathcal{G}}$  ( $\mathbb{P}$  is defined on  $\mathcal{F}$ ).
- $\mathbb{Q}(G) = \int_G X d\mathbb{P}$ .

Note that  $\mathbb{Q}$  is absolutely continuous w.r.t  $\mathbb{P}|_{\mathcal{G}}$  ( $\mathbb{Q} \ll \mathbb{P}|_{\mathcal{G}}$ ).

Then, by Radon-Nykodim theorem, there exists the density  $V$  of  $\mathbb{Q}$  w.r.t.  $\mathbb{P}|_{\mathcal{G}}$

$$\left( V = \frac{d\mathbb{Q}}{d\mathbb{P}|_{\mathcal{G}}} \right).$$

$V$  is  $\mathcal{G}$ -measurable because  $\mathbb{Q}$  and  $\mathbb{P}|_{\mathcal{G}}$  are measures on  $\mathcal{G}$ .

Then:

$$\mathbb{Q}(G) = \int_G X d\mathbb{P} = \int_G V d\mathbb{P}$$

□

**Example 1.3.**  $(X_n)_{n \geq 0}$  Markov chain with state space  $I$  and transition matrix  $P$ .

$$f : I \rightarrow \mathbb{R} \quad \text{s.t.} \quad \mathbb{E}[|f(X_n)|] < +\infty \quad \forall n \in \mathbb{N}.$$

Then:

$$\mathbb{E}[f(X_{n+1}) | \sigma(X_1, \dots, X_n)] = (Pf)(X_n)$$

Where  $(Pf)(j) = \sum_{j' \in I} p_{jj'} f(j')$ .

*Proof.* We want to show that  $\forall G \in \sigma(X_1, \dots, X_n)$

$$\int_G f(X_{n+1}) d\mathbb{P} = \int_G (Pf)(X_n) d\mathbb{P}$$

Each  $G$  is the countable union of a set  $\{X_1 = i_1, \dots, X_n = i_n\}$ , therefore we must show that:

$$\int_{\{X_1=i_1, \dots, X_n=i_n\}} f(X_{n+1}) d\mathbb{P} = \int_{\{X_1=i_1, \dots, X_n=i_n\}} (Pf)(X_n) d\mathbb{P}$$

The LHS can be written as follows:

$$\begin{aligned} \int_{\{X_1=i_1, \dots, X_n=i_n\}} f(X_{n+1}) d\mathbb{P} &= \sum_{j \in I} \int_{\{X_1=i_1, \dots, X_n=i_n\}} f(j) \mathbf{1}_{\{X_{n+1}=j\}} d\mathbb{P} \\ &= \sum_{j \in I} f(j) \mathbb{P}\{X_{n+1} = j, X_n = i_n, \dots, X_1 = i_1\} \\ &= \sum_{j \in I} f(j) \mathbb{P}\{X_{n+1} = j | X_n = i_n, \dots, X_1 = i_1\} \mathbb{P}\{X_n = i_n, \dots, X_1 = i_1\} \\ &= \sum_{j \in I} f(j) p_{i_n j} \int_{\{X_1=i_1, \dots, X_n=i_n\}} d\mathbb{P} \\ &= \int_{\{X_1=i_1, \dots, X_n=i_n\}} \sum_{j \in I} f(j) p_{X_n j} d\mathbb{P} \\ &= \int_{\{X_1=i_1, \dots, X_n=i_n\}} (Pf)(X_n) d\mathbb{P} \end{aligned}$$

□

**Remark** (for CTMC).

$$s < t \quad \mathbb{E}[f(X_t) | \sigma(X_r | r \leq s)] = (P_{t-s}f)(X_s)$$

## 1.1 Properties of $\mathbb{E}[\cdot | \mathcal{G}]$

1. **(linear)**  $\mathbb{E}[\alpha X + \beta Y | \mathcal{G}] = \alpha \cdot \mathbb{E}[X | \mathcal{G}] + \beta \cdot \mathbb{E}[Y | \mathcal{G}]$
2. **(positive)**  $X \geq 0 \implies \mathbb{E}[X | \mathcal{G}] \geq 0$
3. **(normalized)**  $\mathbb{E}[1 | \mathcal{G}] = 1$
4. **Projective Property**

**Theorem 1.2.** If  $\mathcal{H}$  is a sub- $\sigma$ -algebra of  $\mathcal{G}$  then:

$$\mathbb{E}[\mathbb{E}[X | \mathcal{G}] | \mathcal{H}] = \mathbb{E}[X | \mathcal{H}]$$

*Proof.* We must check that  $\forall H \in \mathcal{H}$

$$\int_H \mathbb{E}[\mathbb{E}[X|\mathcal{G}] | \mathcal{H}] d\mathbb{P} = \int_H \mathbb{E}[X|\mathcal{H}] d\mathbb{P}$$

Since  $H \in \mathcal{H}$

$$\int_H \mathbb{E}[X|\mathcal{H}] d\mathbb{P} = \int_H X d\mathbb{P}$$

and since  $\mathcal{H} \subseteq \mathcal{G}$ ,  $H \in \mathcal{G}$ , so:

$$\int_H \mathbb{E}[\mathbb{E}[X|\mathcal{G}] | \mathcal{H}] d\mathbb{P} = \int_H \mathbb{E}[X|\mathcal{G}] d\mathbb{P} = \int_H X d\mathbb{P}$$

□

**Definition 1.2.**  $X$  real r.v. with  $\mathbb{E}[|X|] < +\infty$ ,  $\mathcal{G}$  sub- $\sigma$ -algebra of  $\mathcal{F}$

$$X \perp\!\!\!\perp \mathcal{G} \iff \mathbb{P}(\{X \in B\} \cap G) = \mathbb{P}\{X \in B\} \cdot \mathbb{P}\{G\} \quad \forall B \in \mathcal{B}(\mathbb{R}) \quad \forall G \in \mathcal{G}$$

**Remark.** If  $\mathcal{G} = \sigma(Y)$  the definition matches with  $X \perp\!\!\!\perp Y$  because every  $G \in \mathcal{G}$  is in the form  $\{Y \in B'\}$  with  $B' \in \mathcal{B}(\mathbb{R})$ .

**Theorem 1.3.**  $X \perp\!\!\!\perp \mathcal{G} \implies \mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$

*Proof.* The constant r.v.  $\mathbb{E}[X]$  is  $\mathcal{G}$ -measurable  $\forall G \in \mathcal{G}$   
Note that  $\forall B \in \mathcal{B}(\mathbb{R})$

$$\begin{aligned} \int_G \mathbb{1}_B(X) d\mathbb{P} &= \mathbb{P}(\{X \in B\} \cap G) = \mathbb{P}\{X \in B\} \cdot \mathbb{P}\{G\} = \\ &= \int_G \mathbb{P}\{X \in B\} d\mathbb{P} = \int_G \mathbb{E}[\mathbb{1}_B(X)] d\mathbb{P} \end{aligned}$$

Then  $\forall f : \mathbb{R} \rightarrow \mathbb{R}$  measurable and bounded

$$\int_G f(X) d\mathbb{P} = \int_G \mathbb{E}[f(X)] d\mathbb{P}$$

Hence we can construct a sequence of bounded functions to approximate the identity  $x \rightarrow x$ , defined as:

$$f_n(x) = \max\{-n, \min\{x, n\}\}$$

s.t.:

$$\int_G f_n(X) d\mathbb{P} = \int_G \mathbb{E}[f_n(X)] d\mathbb{P}$$

Finally, applying the monotone convergence theorem as  $n \rightarrow \infty$

$$\int_G X d\mathbb{P} = \int_G \mathbb{E}[X] d\mathbb{P}$$

□

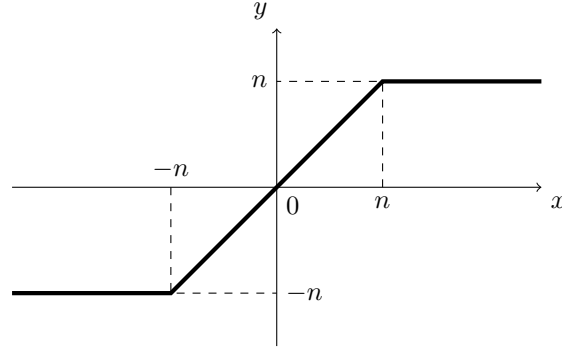


Figure 1: Approximation of the identity map.

5. **Contractivity in  $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ ,  $p \geq 1$**

$$X \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}) \implies \mathbb{E}[X|\mathcal{G}] \in \mathcal{L}^p(\Omega, \mathcal{G}, \mathbb{P})$$

Indeed, is a consequence of the Jensen's inequality: If  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  is convex and  $\mathbb{E}[|\Phi(X)|] < +\infty$ , then:

$$\Phi(\mathbb{E}[X]) \leq \mathbb{E}[\Phi(X)]$$

(that also holds for conditional expectations  $\Phi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\Phi(X)|\mathcal{G}]$ ). In our case  $\Phi(X) = |X|^p$ .

**Theorem 1.4.**  $X$  r.v. with  $\mathbb{E}[|X|^p] < +\infty$

$V$   $\mathcal{G}$ -measurable random variable with  $\mathbb{E}[|V|^q] < +\infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$  conjugate index. Then

$$\mathbb{E}[V \cdot X|\mathcal{G}] = V \cdot \mathbb{E}[X|\mathcal{G}]$$

*Proof.*  $V$  indicator function  $V = \mathbb{1}_G$   $G \in \mathcal{G}$  ( $V$   $\mathcal{G}$ -measurable), then, by definition of conditional expectation,  $\forall G' \in \mathcal{G}$ :

$$\begin{aligned} \int_{G'} \mathbb{E}[\mathbb{1}_G \cdot X|\mathcal{G}] d\mathbb{P} &= \int_{G'} \mathbb{1}_G \cdot X d\mathbb{P} = \int_{G' \cap G} X d\mathbb{P} = \\ &= \int_{G' \cap G} \mathbb{E}[X|\mathcal{G}] d\mathbb{P} = \int_{G'} \mathbb{1}_G \cdot \mathbb{E}[X|\mathcal{G}] d\mathbb{P} \end{aligned}$$

by the arbitrariness of  $G'$ :

$$\mathbb{E}[\mathbb{1}_G \cdot X|\mathcal{G}] = \mathbb{1}_G \cdot \mathbb{E}[X|\mathcal{G}]$$

by linearity:

$$\mathbb{E}[V \cdot X|\mathcal{G}] = V \cdot \mathbb{E}[X|\mathcal{G}] \quad \forall V \text{ simple and } \mathcal{G}\text{-measurable}$$

If  $V > 0$ ,  $V = \sup_{n \geq 1} V_n$  with  $V_n$  simple  $\mathcal{G}$ -measurable, so:

$$\mathbb{E}[V \cdot X|\mathcal{G}] = \sup_{n \geq 1} \mathbb{E}[V_n \cdot X|\mathcal{G}] = \sup_{n \geq 1} V_n \cdot \mathbb{E}[X|\mathcal{G}] = V \cdot \mathbb{E}[X|\mathcal{G}]$$

If  $V$  is real we can write  $V = V^+ - V^-$  and the thesis follows.  $\square$

**Theorem 1.5.** If  $X$  r.r.v. with  $\mathbb{E}[|X|^2] < +\infty$ , then:  
 $Z$   $\mathcal{G}$ -measurable random variable with  $\mathbb{E}[|Z|^2] < +\infty$

$$\mathbb{E}[|X - Z|^2] = \mathbb{E}[|X - \mathbb{E}[X|\mathcal{G}]|^2] + \mathbb{E}[|\mathbb{E}[X|\mathcal{G}] - Z|^2]$$

In particular:

$$\mathbb{E}[|X - \mathbb{E}[X|\mathcal{G}]|^2] = \min_{Z \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})} \mathbb{E}[|X - Z|^2]$$

*Proof.*

$$\begin{aligned} \mathbb{E}[|X - Z|^2] &= \mathbb{E}[|(X - \mathbb{E}[X|\mathcal{G}]) + (\mathbb{E}[X|\mathcal{G}] - Z)|^2] \\ &= \mathbb{E}[|X - \mathbb{E}[X|\mathcal{G}]|^2] + \mathbb{E}[|\mathbb{E}[X|\mathcal{G}] - Z|^2] \\ &\quad + 2 \cdot \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])(\mathbb{E}[X|\mathcal{G}] - Z)] \end{aligned}$$

We only need to prove that the last term is zero. By Projective property:

$$\mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])(\mathbb{E}[X|\mathcal{G}] - Z)] = \mathbb{E}[\mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])(\mathbb{E}[X|\mathcal{G}] - Z)] | \mathcal{G}]$$

$\mathbb{E}[X|\mathcal{G}] - Z$  is  $\mathcal{G}$ -measurable, so:

$$\mathbb{E}[\mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])(\mathbb{E}[X|\mathcal{G}] - Z)] | \mathcal{G}] = \mathbb{E}[(\mathbb{E}[X|\mathcal{G}] - Z) \cdot \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])] | \mathcal{G}]$$

Finally:

$$\begin{aligned} \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])] | \mathcal{G}] &= \mathbb{E}[X|\mathcal{G}] - \mathbb{E}[(\mathbb{E}[X|\mathcal{G}])] | \mathcal{G}] \\ &= \mathbb{E}[X|\mathcal{G}] - \mathbb{E}[X|\mathcal{G}] = 0 \end{aligned}$$

□

## 2 Martingales

### 2.1 Introduction

$(\Omega, \mathcal{F}, \mathbb{P})$   $T$  set of times either  $\mathbb{N}$  or  $[0, \infty)$ .

**Definition 2.1.** A **filtration** on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is a sequence  $\mathcal{F}_t : t = 0, 1, 2, \dots$  of sub-sigma algebras of  $\mathcal{F}$ , such that for all  $t$ ,  $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ .

The filtration represent our knowledge at successive times. This increases with time (i.e. we don't forget things).

**Definition 2.2.**  $(M_t)_{t \in T}$  is a Martingale w.r.t. a filtration  $(\mathcal{F}_t)_{t \in T}$  of sub- $\sigma$ -algebras of  $\mathcal{F}$  if:

1. **(adaptedness)**  $M_t$  is  $\mathcal{F}_t$ -measurable  $\forall t \in T$
2. **(integrability)**  $\mathbb{E}[|M_t|] < +\infty \quad \forall t$
3. **(martingale property)**  $\forall s < t \quad \mathbb{E}[M_t | \mathcal{F}_s] = M_s$

**Example 2.1.**  $(X_n)_{n \geq 0}$  symmetric RW on  $\mathbb{Z}$  starting from 0.

$X_n = \sum_{k=1}^n Y_k \quad Y_k$  iid with  $\mathbb{P}\{Y_k = 1\} = \frac{1}{2} = \mathbb{P}\{Y_k = -1\}$

$(X_n)_{n \geq 0}$  is a martingale w.r.t.  $\{\mathcal{F}_n\}_{n \geq 0}$  with  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ .

*Proof.* We check the definition.

1.  $X_n$  is  $\mathcal{F}_n$ -measurable.
2.  $X_n$  is integrable ( $|X_n| \leq n$ ).
3. If  $m < n$ ,

$$\mathbb{E}[X_n|\mathcal{F}_m] = \mathbb{E}\left[\sum_{k=m+1}^n Y_k + \sum_{k=1}^m Y_k|\mathcal{F}_m\right]$$

The first sum is independent of  $\mathcal{F}_m$ , the second one is  $\mathcal{F}_n$ -measurable, so:

$$\mathbb{E}[X_n|\mathcal{F}_m] = \mathbb{E}\left[\sum_{k=m+1}^n Y_k\right] + \sum_{k=1}^m Y_k = \sum_{k=m+1}^n \mathbb{E}[Y_k] + X_m = X_m$$

Since  $\mathbb{E}[Y_k] = 0$ .

□

**Example 2.2** (Another martingale).  $M_n = X_n^2 - n \quad (M_n)_{n \geq 0}$

1.  $M_n$  is  $\mathcal{F}_n$ -measurable.
2.  $|M_n| \leq |X_n^2 - n| \leq n^2 + n < +\infty \implies \mathbb{E}[|M_n|] < +\infty$ .
- 3.

$$\begin{aligned} \mathbb{E}[X_n|\mathcal{F}_m] &= \mathbb{E}[X_n^2 - n|\mathcal{F}_m] \\ &= \mathbb{E}[X_n^2|\mathcal{F}_m] - \mathbb{E}[n|\mathcal{F}_m] = \\ &= \mathbb{E}\left[\left(\sum_{k=1}^m Y_k + \sum_{k=m+1}^n Y_k\right)^2|\mathcal{F}_m\right] - n = \\ &= \mathbb{E}\left[\left(\sum_{k=1}^m Y_k\right)^2|\mathcal{F}_m\right] + 2 \cdot \mathbb{E}\left[\left(\sum_{k=1}^m Y_k\right) \cdot \left(\sum_{k=m+1}^n Y_k\right)|\mathcal{F}_m\right] \\ &\quad + \mathbb{E}\left[\left(\sum_{k=m+1}^n Y_k\right)^2|\mathcal{F}_m\right] - n = \\ &= X_m^2 + 2 \cdot X_m \cdot \mathbb{E}\left[\sum_{k=m+1}^n Y_k|\mathcal{F}_m\right] + \mathbb{E}\left[\left(\sum_{k=m+1}^n Y_k\right)^2\right] - n = \\ &= X_m^2 + 2 \cdot X_m \cdot 0 + \sum_{k=m+1}^n \mathbb{E}[Y_k^2] - n = \\ &= X_m^2 + (n - m) - n = X_m^2 - m = M_m \end{aligned}$$

Since  $\mathbb{E}[Y_k] = 0$  and  $\mathbb{E}[Y_k^2] = 1$ .

## 2.2 Stopping time

$T$  stopping time of the filtration  $(\mathcal{F}_t)_{t \in T}$  (i.e.  $\{T \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0$ )

We call **stopped martingale at time  $T$** :  $M^{IT} = (M_{\min(T, t)})_{t \geq 0}$ .

**Theorem 2.1** (Stopping Theorem).  $(M_{\min(T, t)})_{t \geq 0}$  is a martingale w.r.t. the same filtration  $(\mathcal{F}_t)_{t \in T}$  and in particular:

$$\mathbb{E}[M_{\min(T, t)}] = \mathbb{E}[M_0] \quad (2)$$

**Remark.** A martingale  $(M_t)_{t \in T}$  necessarily has constant mean by:

$$\mathbb{E}[M_T] = \mathbb{E}[\mathbb{E}[M_T | \mathcal{F}_s]] = \mathbb{E}[M_0] \quad \forall s < t$$

**Example 2.3.**  $(X_n)_{n \geq 0}$  symmetric RW on  $\mathbb{Z}$  starting from 0 and  $a, b \in \mathbb{N}$  or  $a, b > 0$ .

$$X_n = \sum_{k=1}^n Y_k \quad Y_k \text{ iid with } \mathbb{P}\{Y_k = 1\} = \frac{1}{2} = \mathbb{P}\{Y_k = -1\}$$

We want to compute the exit probability from  $(-a, b)$  either from  $-a$  or  $b$  and the mean exit time  $T = \min(n \geq 1 | X_n = -a \text{ or } X_n = b)$ .

Consider the martingales  $(X_n)_{n \geq 0}$  and  $(X_n^2 - n)_{n \geq 0}$  and apply the stopping theorem:

$$\begin{aligned} \mathbb{E}[X_{T \wedge n}^2 - T \wedge n] &= \mathbb{E}[0] = 0 \\ \mathbb{E}[X_{T \wedge n}^2] &= \mathbb{E}[T \wedge n] \end{aligned} \quad (3)$$

Thus  $\mathbb{E}[T \wedge n] \leq \max(a^2, b^2)$  because  $|X_{T \wedge n}^2| \leq \max(a^2, b^2)$ , thus we have:  $\mathbb{E}[T] = \sup_{n \geq 1} \mathbb{E}[T \wedge n]$

Applying the stopping theorem to  $(X_n)_{n \geq 0}$ :

$$\mathbb{E}[X_{T \wedge n}] = \mathbb{E}[X_0] = 0$$

$$\mathbb{E}[X_{T \wedge n}] \leq \max(a, b)$$

Using dominated convergence:

$$\mathbb{E}[X_T] = \lim_{n \rightarrow \infty} \mathbb{E}[X_{T \wedge n}] = 0$$

$$0 = -a \cdot \mathbb{P}\{X_T = -a\} + b \cdot \mathbb{P}\{X_T = b\}$$

Recalling that:

$$\mathbb{P}\{X_T = -a\} + \mathbb{P}\{X_T = b\} = 1$$

Solving the system:

$$\mathbb{P}\{X_T = -a\} = \frac{b}{a+b} \quad \mathbb{P}\{X_T = b\} = \frac{a}{a+b}$$

Then using monotone convergence on the RHS on (3) and dominated convergence on the LHS as  $n \rightarrow \infty$  we obtain:

$$\begin{aligned} \mathbb{E}[T] &= \mathbb{E}[X_T^2] = a^2 \cdot \mathbb{P}\{X_T = -a\} + b^2 \cdot \mathbb{P}\{X_T = b\} = \\ &= \frac{a^2 b}{a+b} + \frac{a b^2}{a+b} = ab \end{aligned}$$

**Remark.** Note the resemblance of this result with the mean ending time starting from  $i$  in the gambler's ruin problem:

$$\mathbb{E}_i[T] = i(N - i)$$



**Remark (Warning).** It is not always possible to take the limit  $n \rightarrow \infty$  to find  $\mathbb{E}[M_{\min}(T,t)] = \mathbb{E}[M_t]$

**Example 2.4.**  $(X_n)_{n \geq 0}$  symmetric RW on  $\mathbb{Z}$  starting from 0.  
 $S = \min \{n \geq 1 : X_n = 1\}$   
 Stopping theorem  $\mathbb{E}[X_{\min}(S,n)] = 0$  but  $\mathbb{E}[X_S] = 1$ , therefore:

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_{\min}(S,n)] \neq \mathbb{E}[X_S]$$

### How to find martingales of a discrete time MC?

**Theorem 2.2.**  $(X_n)_{n \geq 0}$  MC with values in a discrete set  $I$   
 $f : I \rightarrow \mathbb{R}$  s.t.  $\mathbb{E}[|f(X_n)|] < +\infty$ , then:

$$M_n = f(X_n) - \sum_{k=0}^{n-1} (Pf(X_k) - f(X_k)) \quad (4)$$

$(M_n)_{n \geq 0}$  is a martingale w.r.t. the natural filtration of  $(X_n)_{n \geq 0}$ . Where  $(Pf)(j) = \sum_{j' \in I} p_{jj'} f(j')$ .

*Proof.* We check the definition:

1.  $M_n$  is  $\mathcal{F}_n$ -measurable.

2.  $M_n$  is integrable.

$\mathbb{E}[|f(X_n)|] < +\infty$  by assumption  $\forall n$ .

$\mathbb{E}[Pf(X_k)] = \mathbb{E}[\mathbb{E}[f(X_{k+1})|\mathcal{F}_k]] < +\infty$  The equality holds since:

$$\mathbb{E}[f(X_{k+1})|\mathcal{F}_k] = Pf(X_k)$$

indeed for  $m < n$ :

$$\mathbb{E}[f(X_n)|\mathcal{F}_m] = P^{n-m}f(X_m)$$

because:

$$\begin{aligned} \mathbb{E}[f(X_n)|\mathcal{F}_m] &= \mathbb{E}[\mathbb{E}[f(X_n)|\mathcal{F}_{n-1}|\mathcal{F}_m]] \\ &= \mathbb{E}[Pf(X_{n-1})|\mathcal{F}_m] \\ &= \mathbb{E}[P^2f(X_{n-2})|\mathcal{F}_m] = \dots \end{aligned}$$

3. For  $n > m$ :

$$\begin{aligned} \mathbb{E}[M_n|\mathcal{F}_m] &= \mathbb{E}\left[f(X_n) - \sum_{k=0}^{n-1} (Pf(X_k) - f(X_k)) \middle| \mathcal{F}_m\right] \\ &= \mathbb{E}\left[f(X_n) - f(X_m) - \sum_{k=m}^{n-1} (Pf(X_k) - f(X_k)) \middle| \mathcal{F}_m\right] \\ &\quad + \underbrace{f(X_m) - \sum_{k=0}^{m-1} (Pf(X_k) - f(X_k))}_{M_m} \\ &= (*) + M_m \end{aligned}$$

The last term is  $M_m$  so it suffices to show that the first term is zero. It is equal to:

$$\begin{aligned}
(*) &= \mathbb{E} \left[ \mathbb{E} \left[ f(X_n) - f(X_m) - \sum_{k=m}^{n-1} (Pf(X_k) - f(X_k)) \middle| \mathcal{F}_{n-1} \right] \middle| \mathcal{F}_m \right] = \\
&= \mathbb{E} \left[ \cancel{Pf(X_{n-1})} - f(X_m) - \cancel{Pf(X_{n-1})} + f(X_{n-1}) - \sum_{k=m}^{n-2} (\dots) \middle| \mathcal{F}_m \right] = \\
&= \mathbb{E} \left[ f(X_{n-1}) - f(X_m) - \sum_{k=m}^{n-2} (Pf(X_k) - f(X_k)) \middle| \mathcal{F}_m \right] = \\
&= \text{iterating...} = \mathbb{E}[f(X_m) - f(X_m)] = 0
\end{aligned}$$

□

**Example 2.5.**  $(X_n)_{n \geq 0}$  symmetric RW on  $\mathbb{Z}$  starting from 0.

$f(x) = x^2$ .

$$(M_n = f(X_n) - \sum_{k=1}^{n-1} (Pf(X_k) - f(X_k)))$$

$$Pf(i) = p_{i,i+1}f(i+1) + p_{i,i-1}f(i-1) = \frac{1}{2}(i+1)^2 + \frac{1}{2}(i-1)^2 = i^2 + 1$$

$$Pf(i) - f(i) = 1$$

$$\sum_{k=0}^{n-1} Pf(i) - f(i) = n \implies M_n = X_n^2 - n$$

**Remark.** If you add a constant to a martingale it remains a martingale.

**Theorem 2.3** (Martingale stopping theorem). Let  $\mathbb{T}$  be a set of times,  $(M_t)_{t \in \mathbb{T}}$  a Martingale w.r.t. a filtration  $(\mathcal{F}_t)_{t \in \mathbb{T}}$  and  $T$  a stopping time w.r.t.  $(\mathcal{F}_t)_{t \in \mathbb{T}}$ . Then  $(M_{T \wedge t})_{t \in \mathbb{T}}$  is still a Martingale w.r.t. the same filtration  $(\mathcal{F}_t)_{t \in \mathbb{T}}$ . In particular:

$$\mathbb{E}[M_{T \wedge t}] = \mathbb{E}[M_0]$$

*Proof.* (case with  $T$  discrete stopping time<sup>1</sup>) We have to check the three properties.

1. (adaptedness) i.e.  $\forall t$ ,  $M_{T \wedge t}$  is  $\mathcal{F}_t$ -measurable.

$$T \text{ discrete} \implies T = \sum_{r \in D} r \mathbb{1}_{\{T=r\}}$$

$\{T = r\} \in \mathcal{F}_r$  because  $T$  is a stopping time.<sup>2</sup> We can decompose the RV, one up to time  $t$  and one after  $t$ :

$$\begin{aligned}
M_{T \wedge t} &= \sum_{r \in D \cap [0, t]} \underbrace{M_r \mathbb{1}_{\{T=r\}}}_{\substack{\mathcal{F}_r\text{-measurable} \\ r \leq t \\ \implies \mathcal{F}_t\text{-measurable}}} + \underbrace{M_t}_{\mathcal{F}_t\text{-measurable}} \underbrace{\mathbb{1}_{\{T > t\}}}_{\{T \leq t\}^C \in \mathcal{F}_t}
\end{aligned}$$

<sup>1</sup>Otherwise approximation stuff...

<sup>2</sup> $\{T = r\} = \{T \leq r\} \setminus \bigcup_{n \geq 1} \{t \leq r - \frac{1}{n}\}$ . We have to use inequalities.

as a result,  $M_{T \wedge t}$  is  $\mathcal{F}_t$ -measurable.

2. (integrability) i.e.  $\mathbb{E}[|M_{T \wedge t}|] < +\infty$ . We can use again the decomposition. Moreover, since  $D$  is discrete, this implies that  $D \cap [0, t]$  is *finite* for any  $t$ . Thus:

$$|M_{T \wedge t}| \leq \sum_{r \in D \cap [0, t]} |M_r| \mathbb{1}_{\{T=r\}} + |M_t| \mathbb{1}_{\{T < t\}}$$

is a finite sum of integrable RVs.

3.  $\mathbb{E}[M_{T \wedge t} | \mathcal{F}_s] = M_{T \wedge s}$  for any  $s < t$ , i.e. we have to check that

$$\forall A \in \mathcal{F}_s : \int_A M_{T \wedge s} d\mathbb{P} = \int_A M_{T \wedge t} d\mathbb{P}$$

Notice that both  $A \cap \{T \leq s\}$  and  $A \cap \{T > s\}$  belong to  $\mathcal{F}_s$ .

Computing:

$$\begin{aligned} \int_A M_{T \wedge s} d\mathbb{P} &= \int_{A \cap \{T \leq s\}} M_T d\mathbb{P} + \int_{A \cap \{T > s\}} M_s d\mathbb{P} \\ &= \int_{A \cap \{T \leq s\}} M_T d\mathbb{P} + \int_{A \cap \{T > s\}} M_t d\mathbb{P} \\ &= \int_{A \cap \{T \leq s\}} M_T d\mathbb{P} + \underbrace{\int_{A \cap \{s < T \leq t\}} M_t d\mathbb{P}}_{(*)} + \int_{A \cap \{T > t\}} M_t d\mathbb{P} \end{aligned}$$

where in the second step we used the Martingale property of  $(M_t)_{t \in \mathbb{T}}$  and in the third we split the set. Notice that

$$(*) = \int_{A \cap \{s < T \leq t\}} M_t d\mathbb{P} = \sum_{r \in D \cap (s, t]} \int_{A \cap \{T=r\}} M_t d\mathbb{P}$$

recall that  $A \in \mathcal{F}_s$ ,  $\{T = r\} \in \mathcal{F}_r$ ,  $s < r$  then  $A \cap \{T = r\} \in \mathcal{F}_r$  (the biggest). So using again the Martingale property for the given Martingale the integral in the sum can be written as

$$\int_{A \cap \{T=r\}} M_t d\mathbb{P} = \int_{A \cap \{T=r\}} M_r d\mathbb{P}$$

Summing up

$$\begin{aligned}
\int_A M_{T \wedge s} d\mathbb{P} &= \int_{A \cap \{T \leq s\}} M_T d\mathbb{P} + \sum_{r \in D \cap (s, t]} \int_{A \cap \{T=r\}} M_r d\mathbb{P} + \int_{A \cap \{T > t\}} M_t d\mathbb{P} \\
&= \int_{A \cap \{T \leq s\}} M_T d\mathbb{P} + \sum_{r \in D \cap (s, t]} \int_{A \cap \{T=r\}} \textcolor{red}{M}_T d\mathbb{P} + \int_{A \cap \{T > t\}} \textcolor{red}{M}_{T \wedge t} d\mathbb{P} \\
&= \int_{A \cap \{T \leq s\}} M_{T \wedge t} d\mathbb{P} + \int_{A \cap \{s < T \leq t\}} M_{T \wedge t} d\mathbb{P} + \int_{A \cap \{T > t\}} M_{T \wedge t} d\mathbb{P} \\
&= \int_A M_{T \wedge t} d\mathbb{P}
\end{aligned}$$

□

### Martingales of a continuous time Markov Chain

**Theorem 2.4.**  $(X_t)_{t \geq 0}$  continuous time MC (regular..., FKE/BKE have a unique solution...). Consider  $f : I \rightarrow \mathbb{R}$  such that  $\mathbb{E}[|f(X_t)|] < +\infty$ ,  $Q$  transition rate matrix, and  $\mathbb{E}[|Qf(X_s)|] < +\infty \forall s$ , then defining

$$M_t := f(X_t) - \int_0^t (Qf)(X_s) ds \quad (5)$$

$(M_t)_{t \geq 0}$  is a Martingale w.r.t. the natural filtration of the given MC  $(X_t)_{t \geq 0}$ , namely the collection  $(\mathcal{F}_t)_{t \geq 0}$  of  $\mathcal{F}_t := \sigma(X_s | s \leq t)$ .

**Remark.** Compare (5) with the discrete-time analog:

$$f(X_n) - \sum_{k=0}^{n-1} (Pf(X_k) - f(X_k))$$

*Proof.* Recall that  $\mathbb{E}[f(X_t) | \mathcal{F}_s] = (P_{t-s}f)(X_s)$  where  $P_{t-s} = (p_{ij}(t-s))_{i,j \in I}$ . We check the Martingale property:

$$\mathbb{E}[M_t | \mathcal{F}_s] \stackrel{?}{=} M_s$$

We begin the computation

$$\begin{aligned}
\mathbb{E}[M_t | \mathcal{F}_s] &= \mathbb{E} \left[ f(X_t) - \int_0^t Qf(X_r) dr | \mathcal{F}_s \right] \stackrel{?}{=} f(X_s) - \int_0^s Qf(X_r) dr = M_s \\
\int_0^t &= \int_0^s + \int_s^t
\end{aligned}$$

$$\begin{aligned}
\mathbb{E} \left[ f(X_t) - f(X_s) - \int_s^t Qf(X_r) dr | \mathcal{F}_s \right] &\stackrel{?}{=} 0 \\
\mathbb{E}[f(X_t) - f(X_s) | \mathcal{F}_s] &\stackrel{?}{=} \mathbb{E} \left[ \int_s^t Qf(X_r) dr | \mathcal{F}_s \right] = (*)
\end{aligned}$$

then since the conditional expectation is essentially an integral, by Fubini's theorem and remembering that  $(P_{r-s}Qf)(j) = \sum_{jk} p_{ij}(r-s)q_{jk}f(k)$ , we have

the FKE,  $\dots = \frac{d}{dr} \sum_k p_{ik}(r-s)f(k)$

$$\begin{aligned}
(*) &= \int_s^t \mathbb{E}[Qf(X_r)|\mathcal{F}_s]dr \\
&= \int_s^t (P_{r-s}Qf)(X_s)dr \\
&= \int_s^t \frac{d}{dr}(P_{r-s}f)(X_s)dr \\
&= (P_{r-s}f)(X_s)|_s^t \\
&= P_{t-s}f(X_s) - f(X_s) \\
&= \mathbb{E}[f(X_t)|\mathcal{F}_s] - f(X_s) \\
&= \mathbb{E}[f(X_t) - f(X_s)|\mathcal{F}_s]
\end{aligned}$$

□

#### Application/example.

Computation of the mean depletion time of an  $M/M/1$  queue (i.e. mean extinction time of a population).

$(X_t)_{t \geq 0}$  where  $X_t$  is the number of individuals in the system, with 1 counter. Suppose the system can host at most  $N$  individuals<sup>3</sup> (it's not exactly a  $M/M/1$  queue). The set of states is then  $\{0, 1, \dots, N\}$ .

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots & 0 \\ \mu & -(\lambda + \mu) & \lambda & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mu & -(\lambda + \mu) & \lambda \\ 0 & \dots & 0 & 0 & \mu & -\mu \end{bmatrix}, \quad \lambda < \mu$$

We consider as initial state  $n$ , where  $n \ll N$ , and define

$$T = \min\{t \geq 0 | X_t = 0\},$$

we want to compute  $\mathbb{E}_n[T]$  using the Martingales theorem.

Consider  $f(n) = n$ :

$$\begin{aligned}
\text{if } n > 0, \quad Qf(n) &= \mu f(n-1) - (\lambda + \mu)f(n) + \lambda f(n+1) \\
&= \mu(n-1) + (\lambda + \mu)n + \lambda(n+1) \\
&= \lambda - \mu \\
\text{if } n = 0, \quad Qf(0) &= \lambda(f(1) - f(0)) = \lambda
\end{aligned}$$

The martingale is

$$X_t - \int_0^t (\lambda \mathbb{1}_{\{X_s=0\}} + (\lambda - \mu) \mathbb{1}_{\{X_s>0\}}) ds$$

Martingale stopping theorem:

$$\mathbb{E}_n \left[ X_{T \wedge t} - \int_0^{T \wedge t} (\lambda \mathbb{1}_{\{X_s=0\}} + (\lambda - \mu) \mathbb{1}_{\{X_s>0\}}) ds \right] = n$$

---

<sup>3</sup>to avoid technical problems of integrability.

when  $s < T$  you have not yet reached 0, so the first  $\mathbf{1}$  is zero and the second is one.

$$\mathbb{E}_n[\underbrace{X_{T \wedge t}}_{\geq 0} + (\mu - \lambda)(T \wedge t)] = n \quad (6)$$

thus

$$(\mu - \lambda)\mathbb{E}_n[T \wedge t] \leq n \Rightarrow \text{when } t \rightarrow \infty : \mathbb{E}_n[T] \leq \frac{n}{\mu - \lambda}$$

so  $T$  is integrable. Back to (6)

$$X_{T \wedge t} \xrightarrow[\text{a.s.}]{t \rightarrow \infty} X_T = 0 \Rightarrow (\mu - \lambda)\mathbb{E}_n[T] = n \Rightarrow \mathbb{E}_n[T] = \frac{n}{\mu - \lambda}$$

**Theorem 2.5** (Martingale Convergence Theorem). Given  $(M_t)_{t \geq 0}$  (with either  $t \in \mathbb{N}$  or  $t \in \mathbb{R}^+$ ) Martingale w.r.t. some filtration  $(\mathcal{F}_t)_{t \geq 0}$ . If

$$\sup_{t \geq 0} \mathbb{E}[|M_t|] < +\infty$$

then

$$\lim_{t \rightarrow +\infty} M_t$$

exists almost surely (may not be in  $L^1$ ).

**Example 2.6.** Consider  $(X_n)_{n \geq 0}$  symmetric RW on  $\mathbb{Z}$ .

$$X_0 = 0 \quad X_n = \sum_{k=1}^n Y_k \quad Y_k \text{ i.i.d. } \mathbb{P}(Y_k = \pm 1) = \frac{1}{2}$$

Let

$$M_n = e^{\vartheta X_n - cn}, \quad c, \vartheta \in \mathbb{R}$$

1. Find  $c$  such that  $(M_n)_{n \geq 0}$  is a Martingale.
2. Show that for such  $c$  and all  $\vartheta$ ,  $M_n$  converges a.s., but not in  $L^1$ .

For  $(M_n)_{n \geq 0}$  to be a Martingale, beside checking the two basic properties, we have to check that if  $\mathbb{E}[M_{n+1}|\mathcal{F}_n] \stackrel{?}{=} M_n$ . The first quantity is equal to:

$$\begin{aligned} \mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[e^{\vartheta X_{n+1} - c(n+1)}|\mathcal{F}_n\right] \\ &= \mathbb{E}\left[e^{\vartheta Y_{n+1} - c} e^{\vartheta X_n - cn}|\mathcal{F}_n\right] \\ &= \mathbb{E}\left[e^{\vartheta Y_{n+1}}|\mathcal{F}_n\right] e^{\vartheta X_n - c(n+1)} \end{aligned}$$

thus

$$\begin{aligned} \mathbb{E}\left[e^{\vartheta Y_{n+1}}|\mathcal{F}_n\right] e^{\vartheta X_n - c(n+1)} &\stackrel{?}{=} e^{\vartheta X_n - cn} \\ \mathbb{E}\left[e^{\vartheta Y_{n+1}}\right] e^{-c} &\stackrel{?}{=} 1 \\ e^c &\stackrel{?}{=} \mathbb{E}\left[e^{\vartheta Y_{n+1}}\right] = \frac{1}{2}(e^\vartheta + e^{-\vartheta}) = \cosh \vartheta \end{aligned}$$

so

$$c = \log \cosh \vartheta \quad (= c(\vartheta))$$

With such  $c$ ,  $(M_n)_{n \geq 0}$  is a Martingale

$$|M_n| = M_n \Rightarrow \mathbb{E}[|M_n|] = \mathbb{E}[M_n] = \mathbb{E}[M_0] = 1$$

By Martingale convergence theorem,  $M_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} k$  and by the Law of Large Numbers we can compute  $k$ .

$$X_n = \sum_{k=1}^n Y_k \Rightarrow \frac{1}{n} \sum_{k=1}^n Y_k \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mathbb{E}[Y_1] = 0$$

then

$$e^{\vartheta X_n - c(\vartheta)n} = \left( e^{\overbrace{\vartheta \frac{1}{n} \sum_{k=1}^n Y_k}^{\rightarrow 0} - c(\vartheta)} \right)^n \sim \left( e^{-c(\vartheta)} \right)^n \rightarrow 0 = k$$

since  $c(\vartheta) > 0$ ,  $e^{-c(\vartheta)} < 1$ .

Summing up:

$$M_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0 \quad \text{but} \quad \mathbb{E}[M_n] = 1 \Rightarrow M_n \not\rightarrow 0 \text{ in } L^1$$

## 2.3 Supermartingales and Submartingales

Consider  $(X_t)_{t \geq 0}$  with either  $t \in \mathbb{N}$  or  $t \in \mathbb{R}^+$  and a filtration  $(\mathcal{F}_t)_{t \geq 0}$ . We say that  $(X_t)_{t \geq 0}$  is a supermartingale (resp. submartingale) if

- (adaptedness)  $X_t$  is  $\mathcal{F}_t$ -measurable for all  $t$ .
- (integrability)  $\mathbb{E}[|X_t|] < +\infty$
- $\forall s < t$ ,

$$\mathbb{E}[X_t | \mathcal{F}_s] \leq X_s, \text{ for supermartingales}$$

$$\mathbb{E}[X_t | \mathcal{F}_s] \geq X_s, \text{ for submartingales}$$

**Remark.** A process is a Martingale if and only if it's both a sub- and super-Martingale.

**Theorem 2.6.** If  $(M_t)_{t \geq 0}$  is a Martingale and  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a **convex** (resp. **concave**) function such that  $\mathbb{E}[|\varphi(M_t)|] < +\infty$  for any  $t$ , then

$$(\varphi(M_t))_{t \geq 0}$$

is a **submartingale** (resp. **supermartingale**) w.r.t. the same filtration of  $(M_t)_{t \geq 0}$ .

*Proof.* Consider  $\varphi$  convex, using the Jensen inequality

$$s < t \quad \varphi(M_s) = \varphi(\mathbb{E}[M_t | \mathcal{F}_s]) \leq \mathbb{E}[\varphi(M_t) | \mathcal{F}_s]$$

□

**Examples.**

$$\varphi(x) = |x|, \varphi(x) = |x|^2, \varphi(x) = \max\{0, x\}.$$

## 2.4 Doob–Meyer decomposition.

Consider a submartingale  $(X_n)_{n \geq 0}$  with  $n \in \mathbb{N}$  w.r.t. a filtration  $(\mathcal{F}_n)_{n \geq 0}$  then it can be written as

$$X_n = M_n + A_n$$

where  $(M_n)_{n \geq 0}$  is a martingale and  $A_n$  is  $\mathcal{F}_{n-1}$ -measurable and increasing  $A_{n+1} \geq A_n$  a.s.<sup>4</sup>

---

<sup>4</sup> $A_n$  is a *predictable* process.