Coupled Markov chains with applications to Approximate Bayesian Computation for model based clustering

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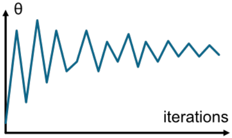
A complex problem



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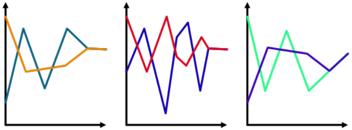
Unbiased Markov chain Monte Carlo methods with couplings



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Approximate Bayesian Computation

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Time-averaged estimator

1. draw X_0 and Y_0 from an initial distribution π_0 and draw $X_1 \sim P(X_0, \cdot)$;
2. set $t = 1$: while $t < \max\{m, \tau\}$ and:
   a. draw $(X_{t+1}, Y_t) \sim \bar{P}\{(X_t, Y_{t-1}), \cdot\}$;
   b. set $t \leftarrow t + 1$;
3. compute the time-averaged estimator:

$$H_{k:m}(X, Y) = \frac{1}{m-k+1} \sum_{l=k}^{m} h(X_l) + \sum_{l=k+1}^{\tau-1} \min(1, \frac{l-k}{m-k+1})\{h(X_l) - h(Y_{l-1})\}.$$

Metropolis–Hasting algorithm for a coupled kernel

Metropolis–Hasting algorithm allow us to calculate the coupled kernel $\bar{P}\{(X_t, Y_{t-1}), \cdot\}$:

1. sample $(X^\star, Y^\star)|(X_t, Y_{t-1})$ from a maximal coupling of $q(X_t, \cdot)$ and $q(Y_{t-1}, \cdot)$;
2. sample $U \sim \mathcal{U}([0, 1])$;
3. if
$$U \leq \min\left\{1, \frac{\pi(X^\star)q(X^\star, X_t)}{\pi(X_t)q(X_t, X^\star)}\right\}$$
then $X_{t+1} = X^\star$; otherwise $X_t = X_{t-1}$;
4. if
$$U \leq \min\left\{1, \frac{\pi(Y^\star)q(Y^\star, Y_t)}{\pi(Y_t)q(Y_t, Y^\star)}\right\}$$
then $Y_{t+1} = Y^\star$; otherwise $Y_t = Y_{t-1}$.

Maximal coupling

The algorithm:

Set $p = \mathcal{N}(X_{t-1}, 1)$ and $q = \mathcal{N}(Y_{t-1}, 1)$,

1. sample $X_t \sim p$;
2. sample $W | X_t \sim \mathcal{U}\{[0, p(X_t)]\}$;
3. if $W \leq q(X_t)$ then output (X_t, X_t), otherwise:
    1. sample $Y_t \sim q$;
    2. sample $W^\star | Y_t \sim \mathcal{U}\{[0, q(Y_t)]\}$ until $W^\star > p(Y_t)$ and output (X_t, Y_t).

Study case

Model

$$Y_i|\mu \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2_{obs})$$

$$\mu \sim \mathcal{N}(\mu_0, \sigma^2_0)$$
$$\mu_0 = 38, \quad \sigma^2_0 = 4$$

Dataset

1000 samples generated from a Gaussian distribution:

$$Y_{obs} \sim \mathcal{N}(\mu_{obs}, \sigma^2_{obs})$$
$$\mu_{obs} = 43, \quad \sigma^2_{obs} = 5$$

Implementation and parallelization

Results

$\mathcal{N}(\mu_n, \sigma_n^2)$
$\mu_n = y = tfd.Normal(1/(1/sigma0 * 2 + n/sigma_obs2)(mu0/sigma0 * 2 + (np.sum(y)/sigma_obs2)), 1/(1/sigma02 + n/sigma_obs * 2))$

- posterior
- risultato del time averaged
- grafico catene che si incontrano
- istogramma dei samplings (sotto alla posterior)

Approximate Bayesian Computation

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POLITECNICO MILANO 1863

ABC rejection sampling algorithm

NO, UTILIZZARE ALGORITMO DI PAGINA 103 *Inputs:*

- a target posterior density $\pi(\theta|y_{obs}) \propto p(y_{obs}|\theta)\pi(\theta)$, consisting of a prior distribution $\pi(\theta)$ and a procedure of generating data under the model $p(y_{obs}|\theta)$;
- a proposal density $g(\theta)$, with $g(\theta) > 0$ if $\pi(\theta|y_{obs}) > 0$;
- an integer $N > 0$;
- a kernel function $K_h(u)$ and a scale parameter $h > 0$;
- a low dimensional vector of summary statistics $s = S(y)$.

ABC rejection sampling algorithm

NO, UTILIZZARE ALGORITMO DI PAGINA 103 *Sampling* for $i = 1, ..., N$:

1. generate $\theta^{(i)} \sim g(\theta)$ from sampling density g;
2. generate $y \sim p(y|\theta^{(i)})$ from the likelihood;
3. compute summary statistic $s = S(y)$;
4. accept $\theta^{(i)}$ with probability $\frac{K_h(\|s - s_{obs}\|)\pi(\theta^{(i)})}{Kg(\theta^{(i)})}$, where $K \geq K_h(0) \max_\theta \frac{\pi(\theta)}{g(\theta)}$; else go to 1.

Output:

- a set of parameter vectors $\theta^{(1)}, ..., \theta^{(N)} \sim \pi_{ABC}(\theta|S_{obs})$.

summary statistics: media distanza: norma (modulo della differenza) (mahalanobis nel caso multivariato) kernel: 1/(np.sqrt(2*math.pi))np.exp(-1/2*u*2)

Study case

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Conclusions

Next steps

The next step will be the conclusion of the **separate multivariate implementation** of both solution to be tested on simulated data and the **parallelized multivariate implementation**.

Further steps will be testing on more complex data.

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