# Physics-based preconditioners for large-scale subsurface flow simulation
# Mo P058

G. B. Diaz Cortes, C. Vuik, J. D. Jansen

TU Delft

EAGE  ECMOR XV

## Theory

Linear System

$$Ax = b.$$

Iterative Methods.

Initial guess solution $x^0$, residual $r^k = b - Ax^k$.
Krylov subspace of dimension $k$.

$$\mathcal{K}_k(M^{-1}A, M^{-1}r^0) = span\{M^{-1}r^0, \ldots, (M^{-1}A)^{k-1}(M^{-1}r^0)\}.$$

Conjugate Gradient

$$min_{x^k \in \mathcal{K}_j(A, r^0)} ||x - x^k||_A, \qquad ||x||_A := \sqrt{(x, Ax)}.$$

Iteration

$$x^{k+1} = x^k + \alpha_k p^k, \qquad \alpha^k = \frac{(r^k, r^k)}{(Ap^k, p^k)}, \qquad (Ap^i, p^j) = 0, \ \ i \neq j.$$

Convergence:

$$||x - x^{k+1}||_A \leq ||x - x^0||_A \left(\frac{\sqrt{C(A)} - 1}{\sqrt{C(A)} + 1}\right)^{k+1},$$

where $C(A)$ is the condition number of $A$.

Preconditioning

The same solution of the original system, but a better spectrum.

$$M^{-1}Ax = M^{-1}b$$

Deflation [1]

Deflated System:

$$PA\hat{x} = Pb, \qquad \hat{x} \text{ is the deflated solution.}$$

$$x = Qb + P^T\hat{x}, \qquad x \text{ is the solution of the original system.}$$

$$P = I - AQ, \qquad Q = ZE^{-1}Z^T, \qquad P \in \mathbb{R}^{n \times n}, \qquad Q \in \mathbb{R}^{n \times n},$$

$$E = Z^TAZ, \qquad E \in \mathbb{R}^{k \times k}, \qquad Z \in \mathbb{R}^{n \times k}.$$

Convergence (Deflation + Preconditioning):

$$||x - x^{i+1}||_A \leq 2||x - x^0||_A \left(\frac{\sqrt{C(M^{-1}PA)} - 1}{\sqrt{C(M^{-1}PA)} + 1}\right)^{i+1}, \qquad C(M^{-1}PA) < C(A).$$

## Proper Orthogonal Decomposition (POD) [2, 3]

The POD method is a ROM which basis functions are obtained from 'Snapshots' by simulation or experiment ,

$$X := [x_1, x_2, \ldots x_m], \qquad x_i \in \mathbb{R}^n.$$

The basis functions are a set of $l$, $l \leq m << n$, orthogonal vectors, $\{\phi_j\}_{j=1}^{l}$, that correspond to the $l$ eigenvectors of the largest eigenvalues

$$\frac{\sum_{j=1}^{l}\lambda_j}{\sum_{j=1}^{m}\lambda_j} \leq \alpha, \qquad 0 < \alpha \leq 1,$$

of the data snapshot correlation matrix,

$$R := \frac{1}{m}XX^T \equiv \frac{1}{m}\sum_{i=1}^{m} x_i x_i^T.$$

## Model

Single-Phase flow

The governing partial differential equations for single-phase flow result in a system of ordinary differential equations [4],

$$V\dot{p} + Tp = q,$$

neglecting gravity and restricting the analysis to slightly compressible flow, the system is linear and **p** is a vector of grid block pressures, **q** is a vector of grid block source terms (wells), the dot represents differentiation with respect to time, while **T** and **V** are the transmissibility and accumulation matrices. We use the Peaceman well model which gives:

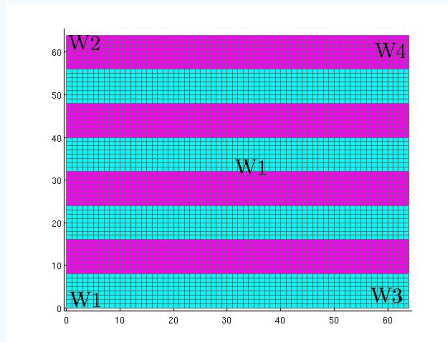
$$V\dot{p} + Tp = J(p - p_{well}),$$

where **J** is a matrix with well indices in the appropriate positions and $p_{well}$ is a vector of well bore pressures [4].
For incompressible flow we have:

$$Tp = J(p - p_{well}).$$

## Snapshots as deflation vectors

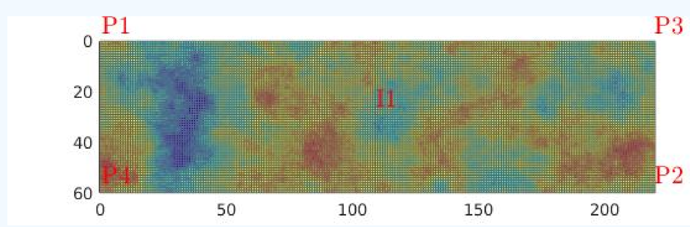
Heterogeneous permeability layers problem (4 snapshots)




| $\kappa_2$ (mD) | $10^{-1}$ | $10^{-3}$ |
|---|---|---|
| ICCG | 75 | 110 |
| DICCG | 1 | 1 |

Number of iterations for the ICCG and DICCG methods ($tol = 10^{-11}$), varying permeability contrast between layers .

SPE10 (2nd layer, 4 snapshots)

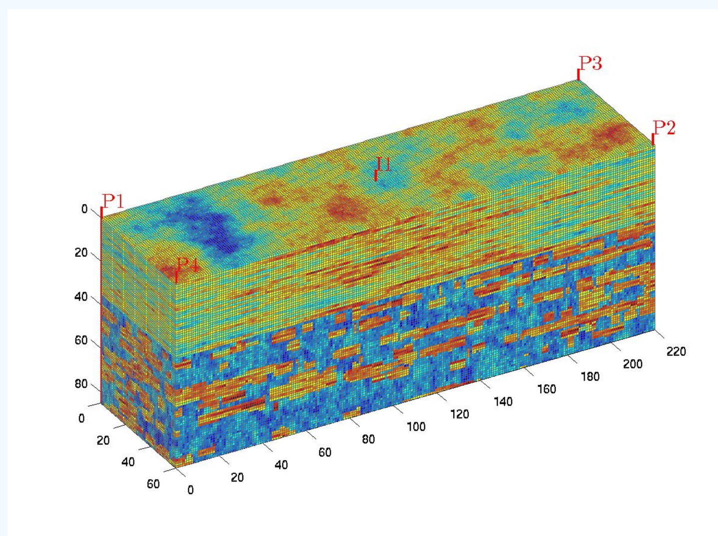


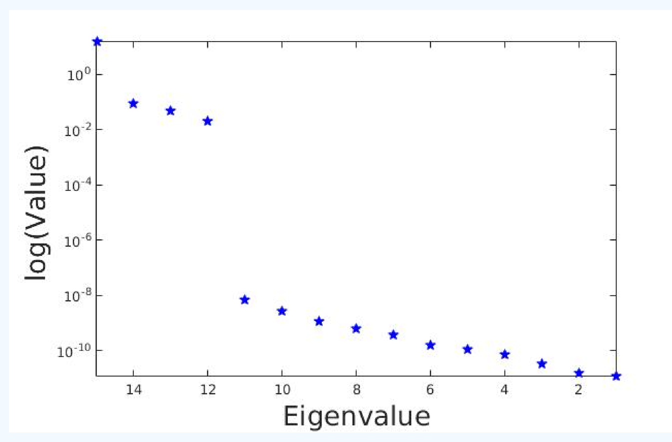

| Method | 16x56 | 30x110 | 46x166 | 60x220 |
|---|---|---|---|---|
| ICCG | 34 | 73 | 126 | 159 |
| DICCG | 1 | 1 | 1 | 1 |

Number of iterations for ICCG and DICCG methods ($tol = 10^{-11}$), various grid sizes.

## POD-based deflation vectors

SPE10 (15 snapshots)




Eigenvalues of 15 snapshots obtained with ICCG ($tol = 10^{-11}$).

| Method | Iterations |
|---|---|
| ICCG | 1011 |
| $DICCG_{15}$ | 2000 |
| $DICCG_{POD}$ | 2 |

Number of iterations for the ICCG, $DICCG_{15}$ and $DICCG_{POD}$ methods, tolerance of solvers and snapshots $10^{-11}$.

## References

[1] J. Tang. *Two-Level Preconditioned Conjugate Gradient Methods with Applications to Bubbly Flow Problems.* PhD thesis, Delft University of Technology, 2008.

[2] J. D. Jansen R. Marković. Accelerating iterative solution methods using reduced-order models as solution predictors. *International journal for numerical methods in engineering*, 68(5):525–541, 2006.

[3] P. Astrid; G. Papaioannou; J. C Vink; J.D. Jansen. Pressure Preconditioning Using Proper Orthogonal Decomposition. In *2011 SPE Reservoir Simulation Symposium, The Woodlands, Texas, USA*, January 2011.

[4] J.D. Jansen. *A systems description of flow through porous media.* Springer, 2013.