Physics-based preconditioners for large-scale subsurface flow simulation.

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Theory

Linear System

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

Iterative Methods.

Initial guess solution $\mathbf{x}^0$, residual $\mathbf{r}^k = \mathbf{b} - \mathbf{A}\mathbf{x}^k$.

Krylov subspace of dimension k.

$$\mathcal{K}_k(\mathbf{M}^{-1}\mathbf{A}, \mathbf{M}^{-1}\mathbf{r}^0) = span\{\mathbf{M}^{-1}\mathbf{r}^0, \ldots, (\mathbf{M}^{-1}\mathbf{A})^{k-1}(\mathbf{M}^{-1}\mathbf{r}^0)\}.$$

Conjugate Gradient

$$min_{\mathbf{x}^k \in \mathcal{K}_j(\mathbf{A},\mathbf{r}^0)} ||x - \mathbf{x}^k||_{\mathbf{A}}, \qquad ||x||_{\mathbf{A}} := \sqrt{(x, Ax)}.$$

Iteration

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha_k p^k, \qquad \alpha^k = \frac{(\mathbf{r}^k, \mathbf{r}^k)}{(Ap^k, p^k)}, \qquad (Ap^i, p^j) = 0, \ i \neq j.$$

Convergence:

$$||\mathbf{x} - \mathbf{x}^{k+1}||_{\mathbf{A}} \leq ||\mathbf{x} - \mathbf{x}^0||_{\mathbf{A}} \left(\frac{\sqrt{\mathbf{C}(\mathbf{A})} - 1}{\sqrt{\mathbf{C}(\mathbf{A})} + 1} \right)^{k+1},$$

where $\mathbf{C}(\mathbf{A})$ is the condition number of $\mathbf{A}$.

Preconditioning

The same solution of the original system, but a better spectrum.

$$\mathbf{M}^{-1}Ax = \mathbf{M}^{-1}\mathbf{b}$$

Deflation [1]

Deflated System:

$$PA\hat{\mathbf{x}} = Pb, \qquad \hat{\mathbf{x}} \text{ is the deflated solution.}$$

$$\mathbf{x} = Qb + P^T\hat{\mathbf{x}}, \qquad \mathbf{x} \text{ is the solution of the original system.}$$

$$P = I - AQ, \qquad Q = ZE^{-1}Z^T, \qquad P \in \mathbf{R}^{n\times n}, \qquad Q \in \mathbf{R}^{n\times n},$$

$$E = Z^TAZ, \qquad E \in \mathbf{R}^{k\times k}, \qquad Z \in \mathbf{R}^{n\times k}.$$

Convergence (Deflation + Preconditioning):

$$||\mathbf{x} - \mathbf{x}^{i+1}||_{\mathbf{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathbf{A}} \left(\frac{\sqrt{\mathbf{C}(\mathbf{M}^{-1}PA)} - 1}{\sqrt{\mathbf{C}(\mathbf{M}^{-1}PA)} + 1} \right)^{i+1}, \qquad \mathbf{C}(\mathbf{M}^{-1}PA) < \mathbf{C}(\mathbf{A}).$$

Proper Orthogonal Decomposition (POD) [2, 3]

The POD method is a ROM which basis functions are obtained from 'Snapshots' by simulation or experiment ,

$$X := [\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_m], \qquad \mathbf{x}_i \in \mathbf{R}^n.$$

The basis functions are a set of l, $l \leq m << n$, orthogonal vectors, $\{\phi_j\}_{j=1}^l$, that correspond to the l eigenvectors of the largest eigenvalues

$$\frac{\sum_{j=1}^{l} \lambda_j}{\sum_{j=1}^{m} \lambda_j} \leq \alpha, \qquad 0 < \alpha \leq 1,$$

of the data snapshot correlation matrix,

$$\mathbf{R} := \frac{1}{m}\mathbf{X}\mathbf{X}^T \equiv \frac{1}{m}\sum_{i=1}^{m} \mathbf{x}_i\mathbf{x}_i^T.$$

Model

Single-Phase flow

The governing partial differential equations for single-phase flow result in a system of ordinary differential equations [4],

$$\mathbf{V}\dot{\mathbf{p}} + \mathbf{T}\mathbf{p} = \mathbf{q},$$

neglecting gravity and restricting the analysis to slightly compressible flow, the system is linear and $\mathbf{p}$ is a vector of grid block pressures, $\mathbf{q}$ is a vector of grid block source terms (wells), the dot represents differentiation with respect to time, while $\mathbf{T}$ and $\mathbf{V}$ are the transmissibility and accumulation matrices. We use the Peaceman well model which gives:

$$\mathbf{V}\dot{\mathbf{p}} + \mathbf{T}\mathbf{p} = \mathbf{J}(\mathbf{p} - \mathbf{p}_{well}),$$

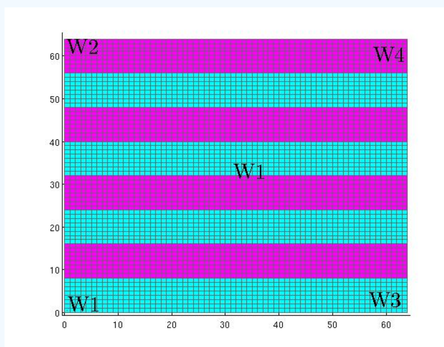
where $\mathbf{J}$ is a matrix with well indices in the appropriate positions and $\mathbf{p}_{well}$ is a vector of well bore pressures [4].

For incompressible flow we have:

$$\mathbf{T}\mathbf{p} = \mathbf{J}(\mathbf{p} - \mathbf{p}_{well}).$$

Snapshots as deflation vectors

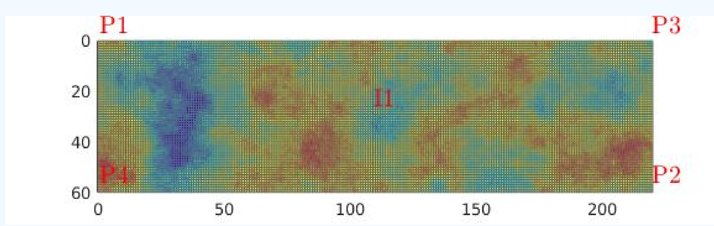
Heterogeneous permeability layers problem (4 snapshots)



κ_2 (mD)	10^{-1}	10^{-3}
ICCG	75	110
DICCG	1	1

Number of iterations for the ICCG and DICCG methods ($tol = 10^{-11}$), varying permeability contrast between layers .

SPE10 (2nd layer, 4 snapshots)

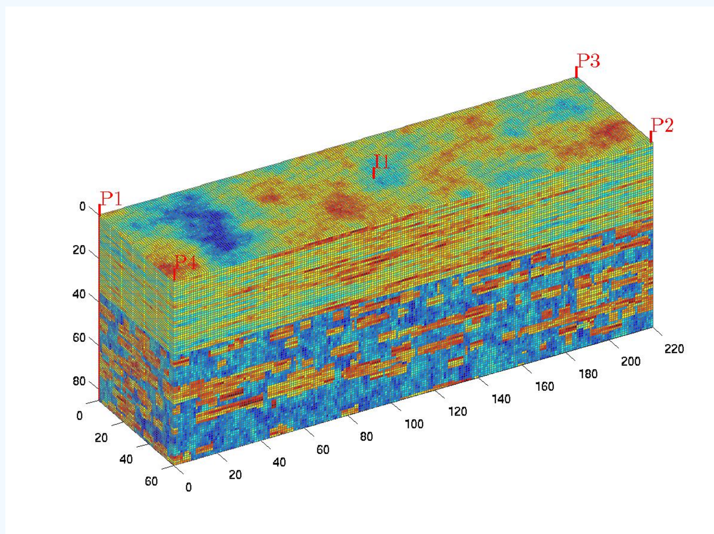


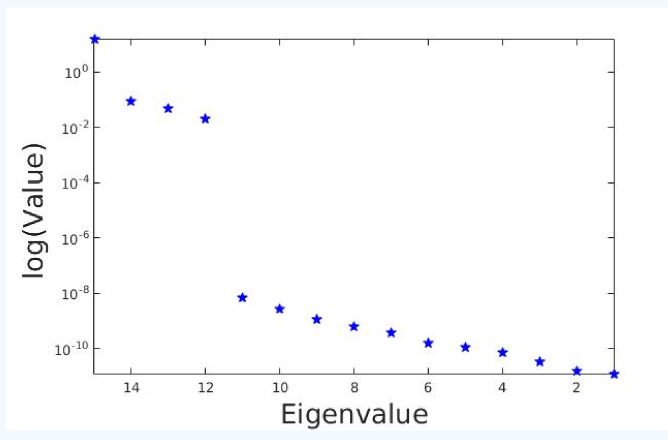
Method	16x56	30x110	46x166	60x220
ICCG	34	73	126	159
DICCG	1	1	1	1

Number of iterations for ICCG and DICCG methods ($tol = 10^{-11}$), various grid sizes.

POD-based deflation vectors

SPE10 (15 snapshots)





Eigenvalues of 15 snapshots obtained with ICCG ($tol = 10^{-11}$).

Method	Iterations
ICCG	1011
DICCG_{15}	2000
DICCG_{POD}	2

Number of iterations for the ICCG, DICCG_{15} and DICCG_{POD} methods, tolerance of solvers and snapshots 10^{-11}.

References

[1] J. Tang. *Two-Level Preconditioned Conjugate Gradient Methods with Applications to Bubbly Flow Problems.* PhD thesis, Delft University of Technology, 2008.

[2] J. D. Jansen R. Markovinović. Accelerating iterative solution methods using reduced-order models as solution predictors. *International journal for numerical methods in engineering*, 68(5):525–541, 2006.

[3] P. Astrid; G. Papaioannou; J. C Vink; J.D. Jansen. Pressure Preconditioning Using Proper Orthogonal Decomposition. In *2011 SPE Reservoir Simulation Symposium, The Woodlands, Texas, USA*, January 2011.

[4] J.D. Jansen. *A systems description of flow through porous media.* Springer, 2013.