Physics-based preconditioners for large-scale subsurface flow simulation.

G. B. Diaz Cortes, C. Vuik, J. D. Jansen

TU Delft

Theory

Linear System

$$Ax = b.$$

Iterative Methods.

Initial guess solution x^0, residual $r^k = b - Ax^k$.

Krylov subspace of dimension k.

$$\kappa_k(M^{-1}A, M^{-1}r^0) = span\{M^{-1}r^0, \ldots, (M^{-1}A)^{k-1}(M^{-1}r^0)\}.$$

Conjugate Gradient

$$min_{x^k \in \kappa_j(A,r^0)} ||x - x^k||_A, \qquad ||x||_A := \sqrt{(x, Ax)}.$$

Iteration

$$x^{k+1} = x^k + \alpha_k p^k, \qquad \alpha^k = \frac{(r^k, r^k)}{(Ap^k, p^k)}, \qquad (Ap^i, p^j) = 0, \;\; i \neq j.$$

Convergence:

$$||x - x^{k+1}||_A \leq ||x - x^0||_A \left(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^{k+1}.$$

Preconditioning

The same solution of the original system, but a better spectrum.

$$M^{-1}Ax = M^{-1}b$$

Deflation [1]

Deflated System:

$$PA\hat{x} = Pb, \qquad \hat{x} \text{ is the deflated solution.}$$

$$x = Qb + P^T\hat{x}, \qquad x \text{ is the solution of the original system.}$$

$$P = I - AQ, \qquad Q = ZE^{-1}Z^T, \qquad P \in \mathbf{R}^{n \times n}, \qquad Q \in \mathbf{R}^{n \times n},$$

$$E = Z^TAZ, \qquad E \in \mathbf{R}^{k \times k}, \qquad Z \in \mathbf{R}^{n \times k}.$$

Convergence (Deflation + Preconditioning):

$$||x - x^{i+1}||_A \leq 2||x - x^0||_A \left(\frac{\sqrt{\kappa(M^{-1}PA)} - 1}{\sqrt{\kappa(M^{-1}PA)} + 1} \right)^{i+1}, \qquad \kappa(M^{-1}PA) < \kappa(A).$$

Proper Orthogonal Decomposition (POD) [2, 3]

The POD method is a ROM which basis functions are obtained from 'Snapshots' by simulation or experiment ,

$$X := [\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_m], \qquad \mathbf{x}_i \in \mathbf{R}^n.$$

The basis functions are a set of l, $l \leq m << n$, orthogonal vectors, $\{\phi_j\}_{j=1}^{l}$, that correspond to the l eigenvectors of the largest eigenvalues

$$\frac{\sum_{j=1}^{l} \lambda_j}{\sum_{j=1}^{m} \lambda_j} \leq \alpha, \qquad 0 < \alpha \leq 1,$$

of the data snapshot correlation matrix,

$$\mathbf{R} := \frac{1}{m}\mathbf{X}\mathbf{X}^T \equiv \frac{1}{m}\sum_{i=1}^{m} \mathbf{x}_i\mathbf{x}_i^T.$$

Model

Single-Phase flow

The governing partial differential equations for single-phase flow result in a system of ordinary differential equations [4],

$$\mathbf{V}\dot{\mathbf{p}} + \mathbf{T}\mathbf{p} = \mathbf{q},$$

neglecting gravity and restricting the analysis to slightly compressible flow, the system is linear and $\mathbf{p}$ is a vector of grid block pressures, $\mathbf{q}$ is a vector of grid block source terms (wells), the dot represents differentiation with respect to time, while $\mathbf{T}$ and $\mathbf{V}$ are the transmissibility and accumulation matrices. We use the Peaceman well model which gives:

$$\mathbf{V}\dot{\mathbf{p}} + \mathbf{T}\mathbf{p} = \mathbf{J}(\mathbf{p} - \mathbf{p}_{well}),$$

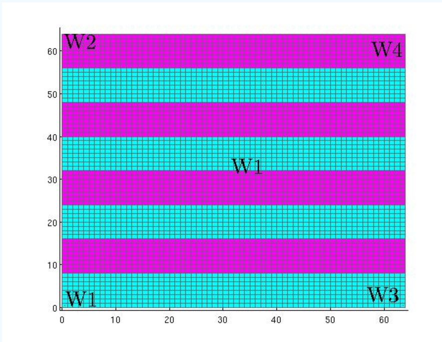
where $\mathbf{J}$ is a matrix with well indices in the appropriate positions and $\mathbf{p}_{well}$ is a vector of well bore pressures [4].

For incompressible flow we have:

$$\mathbf{T}\mathbf{p} = \mathbf{J}(\mathbf{p} - \mathbf{p}_{well}).$$

Snapshots as deflation vectors

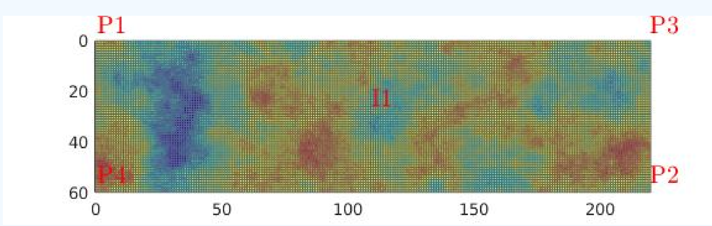
Heterogeneous permeability layers problem (4 snapshots)



κ_2 (mD)	10^{-1}	10^{-3}
ICCG	75	110
DICCG	1	1

Number of iterations for the ICCG and DICCG methods ($tol = 10^{-11}$), varying permeability contrast between layers .

SPE10 (2nd layer, 4 snapshots)

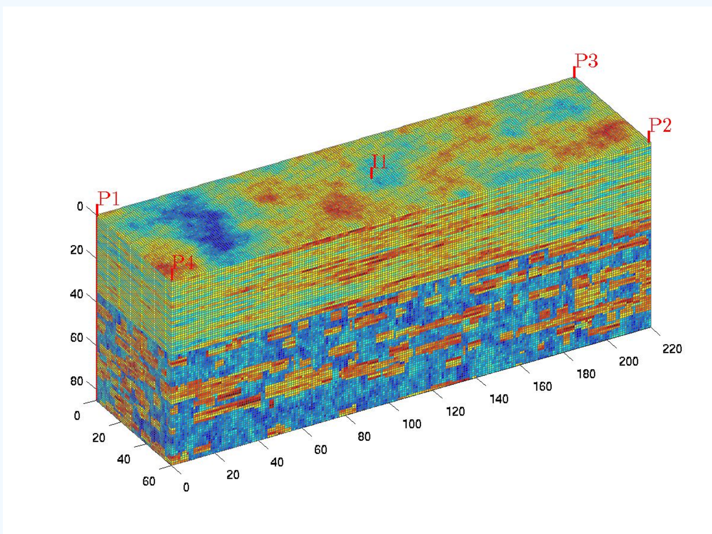


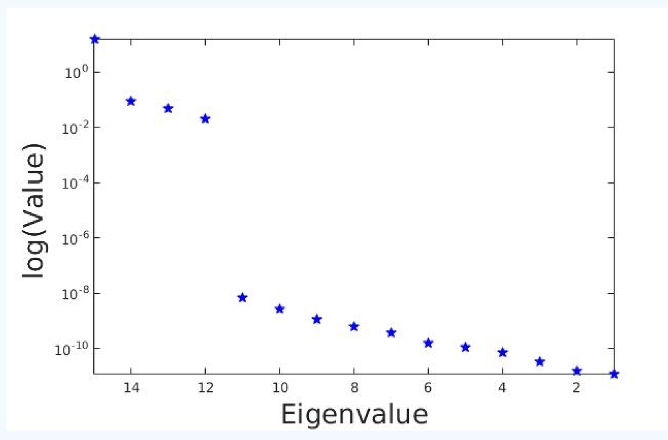
Method	16x56	30x110	46x166	60x220
ICCG	34	73	126	159
DICCG	1	1	1	1

Number of iterations for ICCG and DICCG methods ($tol = 10^{-11}$), various grid sizes.

POD-based deflation vectors

SPE10 (15 snapshots)





Eigenvalues of 15 snapshots obtained with ICCG ($tol = 10^{-11}$).

Method	Iterations
ICCG	1011
DICCG_{15}	2000
DICCG_{POD}	2

Number of iterations for the ICCG, DICCG_{15} and DICCG_{POD} methods, tolerance of solvers and snapshots 10^{-11}.

References

[1] J. Tang. *Two-Level Preconditioned Conjugate Gradient Methods with Applications to Bubbly Flow Problems.* PhD thesis, Delft University of Technology, 2008.

[2] J. D. Jansen R. Markovinović. Accelerating iterative solution methods using reduced-order models as solution predictors. *International journal for numerical methods in engineering*, 68(5):525–541, 2006.

[3] P. Astrid; G. Papaioannou; J. C Vink; J.D. Jansen. Pressure Preconditioning Using Proper Orthogonal Decomposition. In *2011 SPE Reservoir Simulation Symposium, The Woodlands, Texas, USA*, January 2011.

[4] J.D. Jansen. *A systems description of flow through porous media.* Springer, 2013.