

# Physics-based preconditioners for large-scale subsurface flow simulation.

G. B. Diaz Cortes, C. Vuik, J. D. Jansen



## Theory

### Linear System

$$Ax = b.$$

### Iterative Methods.

Initial guess solution  $x^0$ , residual  $r^k = b - Ax^k$ .  
Krylov subspace of dimension  $k$ .

$$\kappa_k(M^{-1}A, M^{-1}r^0) = span\{M^{-1}r^0, \dots, (M^{-1}A)^{k-1}(M^{-1}r^0)\}.$$

### Conjugate Gradient

$$\min_{x^k \in K_j(A, r^0)} \|x - x^k\|_A, \quad \|x\|_A := \sqrt{(x, Ax)}.$$

Iteration

$$x^{k+1} = x^k + \alpha_k p^k, \quad \alpha^k = \frac{(r^k, r^k)}{(Ap^k, p^k)}, \quad (Ap^i, p^j) = 0, \quad i \neq j.$$

### Convergence:

$$\|x - x^{k+1}\|_A \leq \|x - x^0\|_A \left( \frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^{k+1}.$$

### Preconditioning

The same solution of the original system, but a better spectrum.

$$M^{-1}Ax = M^{-1}b$$

### Deflation [1]

Deflated System:

$$PA\hat{x} = Pb, \quad \hat{x} \text{ is the deflated solution.}$$
$$x = Qb + P^T\hat{x}, \quad x \text{ is the solution of the original system.}$$
$$P = I - AQ, \quad Q = ZE^{-1}Z^T, \quad P \in \mathbb{R}^{n \times n}, \quad Q \in \mathbb{R}^{n \times n},$$
$$E = Z^T AZ, \quad E \in \mathbb{R}^{k \times k}, \quad Z \in \mathbb{R}^{n \times k}.$$

### Convergence (Deflation + Preconditioning):

$$\|x - x^{i+1}\|_A \leq 2\|x - x^0\|_A \left( \frac{\sqrt{\kappa(M^{-1}PA)} - 1}{\sqrt{\kappa(M^{-1}PA)} + 1} \right)^{i+1}, \quad \kappa(M^{-1}PA) < \kappa(A).$$

## Proper Orthogonal Decomposition (POD) [2, 3]

The POD method is a ROM which basis functions are obtained from ‘Snapshots’ by simulation or experiment ,

$$X := [x_1, x_2, \dots, x_m], \quad x_i \in \mathbb{R}^n.$$

The basis functions are a set of  $l$ ,  $l \leq m \ll n$ , orthogonal vectors,  $\{\phi_j\}_{j=1}^l$ , that correspond to the  $l$  eigenvectors of the largest eigenvalues

$$\frac{\sum_{j=1}^l \lambda_j}{\sum_{j=1}^m \lambda_j} \leq \alpha, \quad 0 < \alpha \leq 1,$$

of the data snapshot correlation matrix,

$$R := \frac{1}{m}XX^T \equiv \frac{1}{m} \sum_{i=1}^m x_i x_i^T.$$

## Model

### Single-Phase flow

The governing partial differential equations for single-phase flow result in a system of ordinary differential equations [4],

$$\mathbf{V}\dot{\mathbf{p}} + \mathbf{T}\mathbf{p} = \mathbf{q},$$

neglecting gravity and restricting the analysis to slightly compressible flow, the system is linear and  $\mathbf{p}$  is a vector of grid block pressures,  $\mathbf{q}$  is a vector of grid block source terms (wells), the dot represents differentiation with respect to time, while  $\mathbf{T}$  and  $\mathbf{V}$  are the transmissibility and accumulation matrices. We use the Peaceman well model which gives:

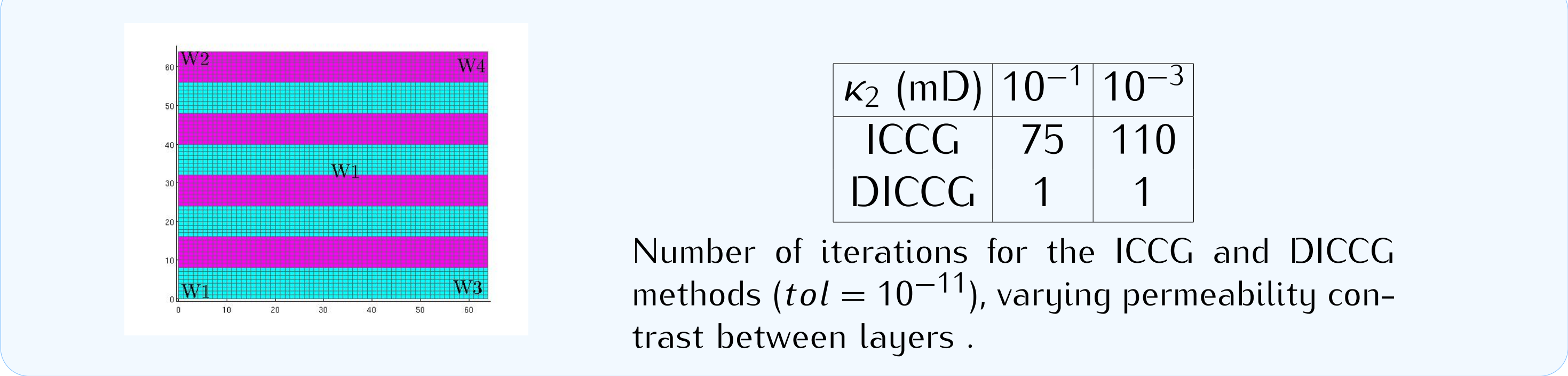
$$\mathbf{V}\dot{\mathbf{p}} + \mathbf{T}\mathbf{p} = \mathbf{J}(\mathbf{p} - \mathbf{p}_{well}),$$

where  $\mathbf{J}$  is a matrix with well indices in the appropriate positions and  $\mathbf{p}_{well}$  is a vector of well bore pressures [4].  
For incompressible flow we have:

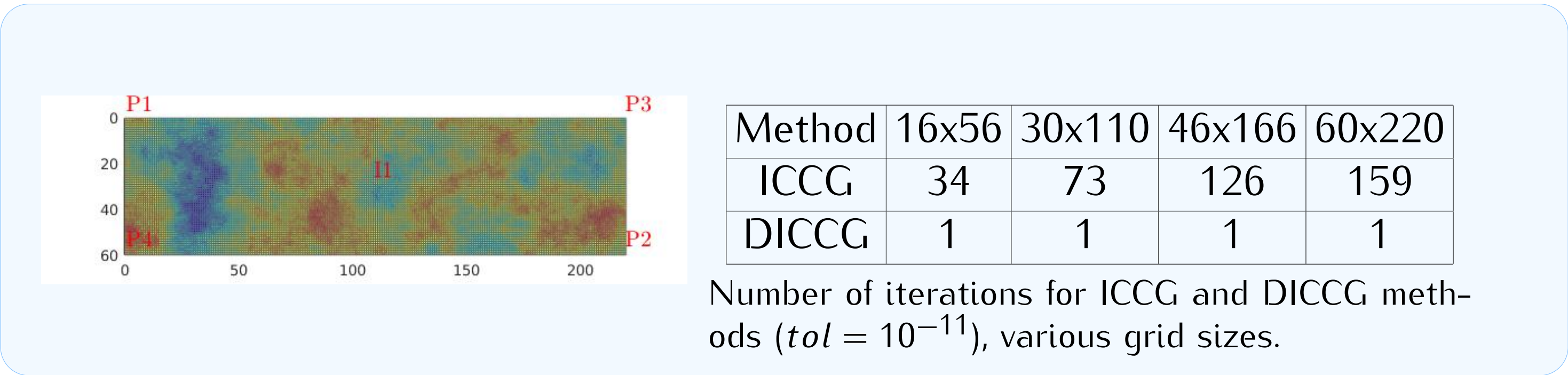
$$\mathbf{T}\mathbf{p} = \mathbf{J}(\mathbf{p} - \mathbf{p}_{well}).$$

## Snapshots as deflation vectors

### Heterogeneous permeability layers problem (4 snapshots)

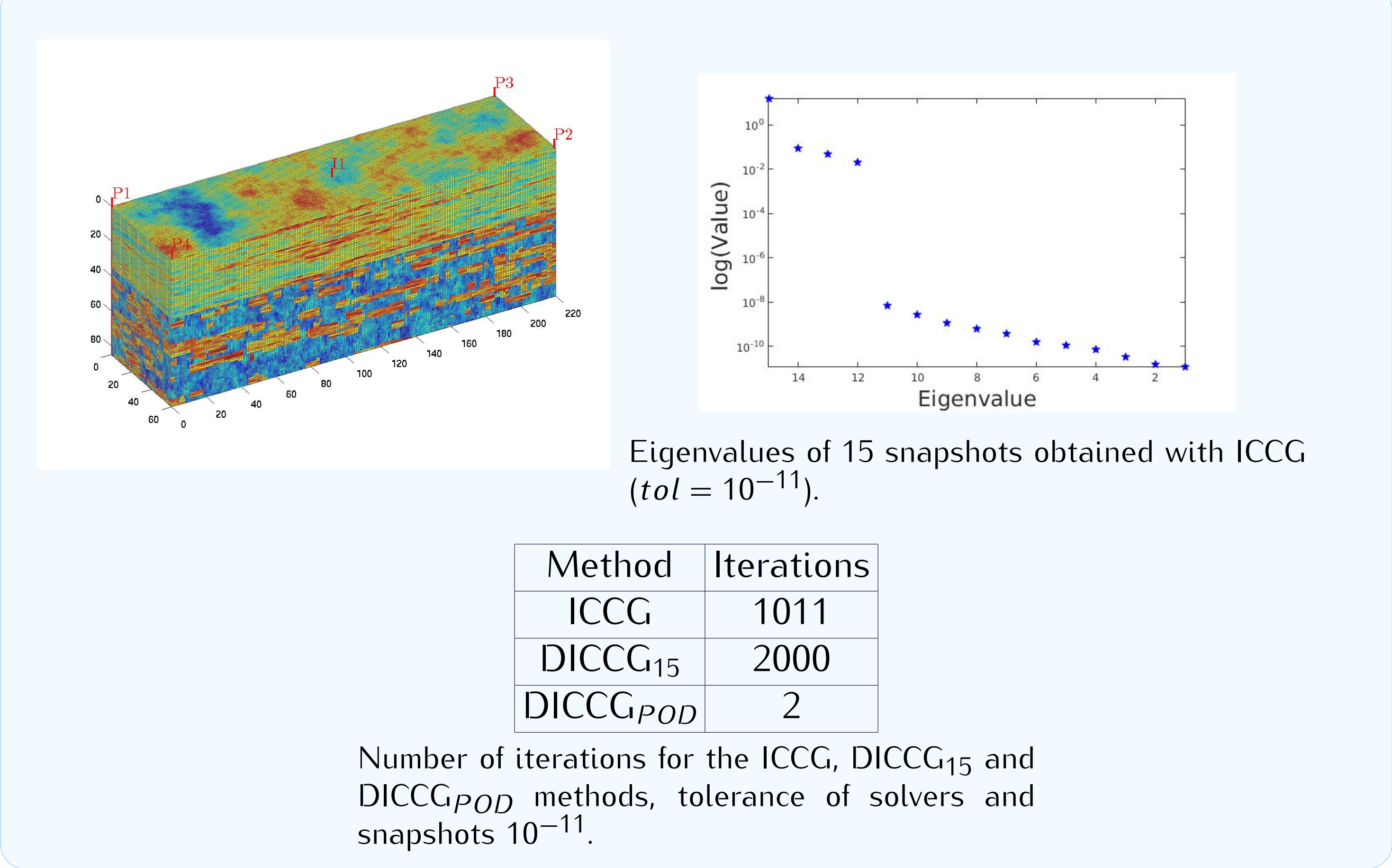


### SPE10 (2nd layer, 4 snapshots)



## POD-based deflation vectors

### SPE10 (15 snapshots)



## References

[1] J. Tang. *Two-Level Preconditioned Conjugate Gradient Methods with Applications to Bubbly Flow Problems*. PhD thesis, Delft University of Technology, 2008.

[2] J. D. Jansen R. Markovinović. Accelerating iterative solution methods using reduced-order models as solution predictors. *International journal for numerical methods in engineering*, 68(5):525–541, 2006.

[3] P. Astrid; G. Papaioannou; J. C Vink; J.D. Jansen. Pressure Preconditioning Using Proper Orthogonal Decomposition. In *2011 SPE Reservoir Simulation Symposium, The Woodlands, Texas, USA*, January 2011.

[4] J.D. Jansen. *A systems description of flow through porous media*. Springer, 2013.