Theory

For the wetting phase (w) and non-wetting phase (n) we have:

$$S_n + S_w = 1 \tag{1}$$

The pressure in the wetting fluid is less than in the nonwetting fluid. This difference in pressures is known as the capillary pressure, p_c, and it's function of saturation:

$$p_c(S_w) = p_n - p_w. \tag{2}$$

Relative permeability
The relative permeability for an isotropic medium is defined as:

$$k_{r\alpha} = K^e_\alpha / K.$$

The relative permeabilities will generally be functions of saturation, generally nonlinear. It is common to use analytic relationships to represent relative permeabilities. These are usually stated using normalized or effective saturations $\hat{S}_w$. The simplest model possible is called the Corey model:

$$\begin{aligned} k_{rw} &= (\hat{S}_w)^{n_w} k^0_w, \\ k_{ro} &= (1 - \hat{S}_w)^{n_n} k^0_o. \end{aligned} \tag{3}$$

where $n_w > 1$, $n_o > 1$ and k^0_α are fitting parameters.
Immiscible two-phase flow
For the case of immiscible fluids, the flow equations are:

$$\frac{\partial(\phi \rho_w S_w)}{\partial t} + \nabla \cdot (\rho_w \mathbf{v}_w) = \rho_w q_w, \tag{4}$$

$$\frac{\partial(\phi \rho_n S_n)}{\partial t} + \nabla \cdot (\rho_n \mathbf{v}_n) = \rho_n q_n. \tag{5}$$

$$\mathbf{v}_\alpha = -\frac{k_{r\alpha}}{\mu_\alpha} K(\nabla p_\alpha - \rho_\alpha g \nabla z).$$

To simplify notation, the phase mobilities ($\lambda_\alpha = K k_{r\alpha}/\mu_\alpha$) or relative phase mobilities ($\lambda_\alpha = \lambda_\alpha K$) are used.
Fractional flow formulation
A common choice is to use p_n and S_w which gives the following system

$$\frac{\partial(\phi \rho_w S_w)}{\partial t} + \nabla \cdot (\rho_w \mathbf{v}_w) = \rho_w q_w, \tag{6}$$

$$\frac{\partial(\phi \rho_n (1 - S_w))}{\partial t} + \nabla \cdot (\rho_n \mathbf{v}_n) = \rho_n q_n. \tag{7}$$

with

$$\mathbf{v}_w = \frac{K k_{rw}}{\mu_w}(\nabla p_n - \nabla P_c(S_w) - \rho_w g \nabla z), \tag{8}$$

$$\mathbf{v}_n = \frac{K k_{rn}}{\mu_n}(\nabla p_n - \rho_n g \nabla z). \tag{9}$$

Incompressible flow
For incompressible flow, only the Saturation S is a function of time and the fluid densities ρ_α are constant. Therefore, the mass-balance equations are:

$$\phi \frac{\partial S_\alpha}{\partial t} + \nabla \cdot \mathbf{v}_\alpha = q_\alpha. \tag{10}$$

If we reareange terms, using $\mathbf{v} = \mathbf{v}_w + \mathbf{v}_n$, the total Darcy velocity, $\lambda = \lambda_n + \lambda_w = \lambda K$, the total mobility, $q = q_n + q_w$, $\Delta\rho = \rho_w - \rho_n$ and the fractional flow function f_w:

$$f_w = \frac{\lambda_w}{\lambda_n + \lambda_w} = \frac{\lambda_w}{\lambda},$$

for the wetting phase we have:

$$\phi \frac{\partial (S_w)}{\partial t} + \nabla \cdot [f_w(\mathbf{v} + \lambda_n \Delta\rho g \nabla z)] = q_w - \nabla \cdot (f_w \lambda_n p_c \nabla S_w), \tag{11}$$

and

$$\mathbf{v} = -\lambda(\nabla p_n - f_w \nabla p_w - (f_n \rho_n + f_w \rho_w) g \nabla z).$$

Sequential solution procedures
The two-phase, incompressible model will be solved using the fractional-flow formulation. This fractional flow model consists of an elliptic pressure equation

$$\nabla \cdot \mathbf{v} = q, \qquad \mathbf{v} = -\lambda(\nabla p_n - f_w \nabla p_c - (f_n \rho_n + f_w \rho_w) g \nabla z) \tag{12}$$

and a parabolic transport equation (11)

$$\phi \frac{\partial (S_w)}{\partial t} + \nabla \cdot [f_w(\mathbf{v} + \lambda_n(\Delta\rho g \nabla z) + \nabla P_c(S_w))] = q_w. \tag{13}$$

Where the capillary pressure $p_c = p_w - p_n$ is assumed to be a known function P_c of the wetting saturation S_w.
Saturation solvers
The saturation equation depends on the time, using backward Euler discretization for the time derivative in Equation 13, we have:

$$\phi \frac{(S_w^{n+1} - S_w^n)}{\Delta t} + \nabla \cdot [f_w(S_w)(\mathbf{v} + \lambda_n(S_w)(\Delta\rho g \nabla z + \nabla P_c(S_w)))] = q_w, \tag{14}$$

or

$$S_w^{n+1} = S_w^n - \frac{\Delta t}{\phi} \nabla \cdot [f_w(S_w)(\mathbf{v} + \lambda_n(S_w)(\Delta\rho g \nabla z + \nabla P_c(S_w)))] + q_w,$$

which can be computed explicitly:

$$S_w^{n+1} = S_w^n - \mathcal{F}(S_w^n, S_w^n),$$

or implicitly:

$$S_w^{n+1} = S_w^n - \mathcal{F}(S_w^{n+1}, S_w^n).$$

The nonlinear system can be solved with NR method, where, for the $(k+1)$-th iteration we have:

$$\mathbf{J}(\mathbf{S}^k)\delta\mathbf{S}^{k+1} = -\mathbf{F}(\mathbf{S}^k;\mathbf{S}^n), \qquad \mathbf{S}^{k+1} = \mathbf{S}^k + \delta\mathbf{S}^{k+1},$$

where $\mathbf{J}(\mathbf{S}^k) = \frac{\partial\mathbf{F}(\mathbf{S}^k;\mathbf{S}^n)}{\partial\mathbf{S}^k}$ is the Jacobian matrix, and $\delta\mathbf{S}^{k+1}$ is the NR update at iteration step $k+1$.
Therefore, the linear system to solve is:

$$\mathbf{J}(\mathbf{S}^k)\delta\mathbf{S}^{k+1} = \mathbf{b}(\mathbf{S}^k). \tag{15}$$

with $\mathbf{b}(\mathbf{S}^k)$ being the function evaluated at iteration step k, $\mathbf{b}(\mathbf{S}^k) = -\mathbf{F}(\mathbf{S}^k;\mathbf{S}^n)$.

Heterogeneous permeability layers, BC

We simulate flow through a porous media with the following characteristics: size 64 x 64

Boundary conditions: Injection of water y direction, $rate_{y_0} = 0.1 meter^3/day$, $p_{y_{max}} = 0 bars$
$injRate = -0.1 * meter^3/day;$
$bc = fluxside([], G,' ymin', -injRate,' sat', [1, 0]);$
$bc = pside(bc, G,' ymax', 0 * barsa,' sat', [0, 1]);$
$T = 450 * day();$
$dT = T/60;$

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG	% of total ICCG
10^1	4458	DICCG$_{10}$	700	306	1006	23
10^1	4458	DICCG$_{POD_{10}}$	700	280	980	22
10^1	4458	DICCG$_{POD_5}$	700	380	1080	24
10^2	6157	DICCG$_{10}$	975	291	1266	21
10^2	6157	DICCG$_{POD_{10}}$	975	287	1262	20
10^2	6157	DICCG$_{POD_5}$	975	359	1334	22
10^3	6635	DICCG$_{10}$	1077	292	1369	21
10^3	6635	DICCG$_{POD_{10}}$	1077	311	1388	21
10^3	6635	DICCG$_{POD_5}$	1077	403	1480	22

Table 1: Comparison between the ICCC and DICCG methods of the average number of linear iterations for the second NR iteration for various contrast between permeability layers, no cp, water injection through the y boundary.

Capillary pressure

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG	% of total ICCG
10^1	4469	DICCG$_{10}$	704	367	1071	24
10^1	4469	DICCG$_{POD_{10}}$	704	367	1071	24
10^1	4469	DICCG$_{POD_5}$	704	433	1137	25
10^2	6149	DICCG$_{10}$	976	295	1271	21
10^2	6149	DICCG$_{POD_{10}}$	976	274	1250	20
10^2	6149	DICCG$_{POD_5}$	976	299	1275	21
10^3	6637	DICCG$_{10}$	1078	269	1347	20
10^3	6637	DICCG$_{POD_{10}}$	1078	311	1389	21
10^3	6637	DICCG$_{POD_5}$	1078	254	1332	20

Table 2: Comparison between the ICCC and DICCG methods of the average number of linear iterations for the second NR iteration for various contrast between permeability layers, cp, water injection through the y boundary.

Boundary conditions: Injection of water y direction, $rate_{x_0} = 0.1 meter^3/day$, $p_{x_{max}} = 0 bars$
$injRate = -0.1 * meter^3/day$;
$bc = fluxside([], G,' xmin', -injRate,' sat', [1, 0])$;
$bc = pside(bc, G,' xmax', 0 * barsa,' sat', [0, 1])$;
$T = 450 * day()$;
$dT = T/60$;

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG	% of total ICCG
10^1	4573	DICCG$_{10}$	701	499	1200	26
10^1	4573	DICCG$_{POD_{10}}$	701	476	1177	26
10^1	4573	DICCG$_{POD_5}$	701	590	1291	28
10^2	4596	DICCG$_{10}$	727	442	1169	25
10^2	4596	DICCG$_{POD_{10}}$	727	442	1169	25
10^2	4596	DICCG$_{POD_5}$	727	483	1210	26
10^3	3154	DICCG$_{10}$	431	381	812	26
10^3	3154	DICCG$_{POD_{10}}$	431	381	812	26
10^3	3154	DICCG$_{POD_5}$	431	413	844	27

Table 3: Comparison between the ICCC and DICCG methods of the average number of linear iterations for the second NR iteration for various contrast between permeability layers, no cp , water injection through the x boundary.

Capillary pressure

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG	% of total ICCG
10^1	4687	DICCG$_{10}$	726	593	1319	28
10^1	4687	DICCG$_{POD_{10}}$	726	593	1319	28
10^1	4687	DICCG$_{POD_5}$	726	527	1253	27
10^2	4551	DICCG$_{10}$	724	408	1132	25
10^2	4551	DICCG$_{POD_{10}}$	724	408	1132	25
10^2	4551	DICCG$_{POD_5}$	724	399	1123	25
10^3	3025	DICCG$_{10}$	430	410	840	28
10^3	3025	DICCG$_{POD_{10}}$	430	410	840	28
10^3	3025	DICCG$_{POD_5}$	430	381	811	27

Table 4: Comparison between the ICCC and DICCG methods of the average number of linear iterations for the second NR iteration for various contrast between permeability layers, cp, water injection through the x boundary.

3D Gravity 10 z-cells

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG	% of total ICCG
10^1	5123	DICCG_{10}	850	182	1032	20
10^1	5123	$\text{DICCG}_{POD_{10}}$	850	58	908	18
10^1	5123	DICCG_{POD_5}	850	52	902	18
10^2	5641	DICCG_{10}	940	214	1154	20
10^2	5641	$\text{DICCG}_{POD_{10}}$	940	79	1019	18
10^2	5641	DICCG_{POD_5}	940	99	1039	18
10^3	3360	DICCG_{10}	560	305	865	26
10^3	3360	$\text{DICCG}_{POD_{10}}$	560	148	708	21
10^3	3360	DICCG_{POD_5}	560	167	727	22

Table 5: Comparison between the ICCC and DICCG methods of the average number of linear iterations for the second NR iteration for various contrast between permeability layers, no cp , water injection through the x boundary.

Capillary pressure

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG	% of total ICCG
10^1	5100	DICCG_{10}	850	608	1458	29
10^1	5100	$\text{DICCG}_{POD_{10}}$	850	78	928	18
10^1	5100	DICCG_{POD_5}	850	92	942	18
10^2	5640	DICCG_{10}	940	179	1119	20
10^2	5640	$\text{DICCG}_{POD_{10}}$	940	133	1073	19
10^2	5640	DICCG_{POD_5}	940	148	1088	19
10^3	3360	DICCG_{10}	560	377	937	28
10^3	3360	$\text{DICCG}_{POD_{10}}$	560	172	732	22
10^3	3360	DICCG_{POD_5}	560	238	798	24

Table 6: Comparison between the ICCC and DICCG methods of the average number of linear iterations for the second NR iteration for various contrast between permeability layers, cp, water injection through the x boundary.

Heterogeneous permeability layers, Wells

We simulate flow through a porous media with the following characteristics: size 64 x 64

Boundary conditions: Injection of water y direction, $W_1 = 100bars$, $W_2 = 100bars$
$T = 450 * day()$;
$dT = T/60$;

Capillary pressure

Layers with different permeability

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG	% of total ICCG
10^1	4799	DICCG$_{10}$	799	113	912	19
10^1	4799	DICCG$_{POD_{10}}$	799	113	912	19
10^1	4799	DICCG$_{POD_5}$	799	162	961	20
10^2	6060	DICCG$_{10}$	1080	87	1167	19
10^2	6060	DICCG$_{POD_{10}}$	1080	87	1167	19
10^2	6060	DICCG$_{POD_5}$	1080	101	1181	19
10^3	5803	DICCG$_{10}$	1080	121	1201	21
10^3	5803	DICCG$_{POD_{10}}$	1080	98	1178	20
10^3	5803	DICCG$_{POD_5}$	1080	108	1188	20

Table 7: Comparison between the ICCC and DICCG methods of the average number of linear iterations for the second NR iteration for various contrast between permeability layers, no cp, water injection through the y boundary.

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG	% of total ICCG
10^1	4802	DICCG$_{10}$	801	140	941	20
10^1	4802	DICCG$_{POD_{10}}$	801	140	941	20
10^1	4802	DICCG$_{POD_5}$	801	154	955	20
10^2	6203	DICCG$_{10}$	1099	87	1186	19
10^2	6203	DICCG$_{POD_{10}}$	1099	87	1186	19
10^2	6203	DICCG$_{POD_5}$	1099	98	1197	19
10^3	5867	DICCG$_{10}$	1080	196	1276	22
10^3	5867	DICCG$_{POD_{10}}$	1080	95	1175	20
10^3	5867	DICCG$_{POD_5}$	1080	125	1205	21

Table 8: Comparison between the ICCC and DICCG methods of the average number of linear iterations for the second NR iteration for various contrast between permeability layers, cp, water injection through the y boundary.

Property	Water	Units
Porosity	0.2	
Permeability	10	*millidarcy*
ρ	1000	$kilogram/meter^3$
k_r	$(1 - S_w)^2$	

Property	Water	Oil	Units
μ	1	10	$centi * poise$
ρ	1000	700	$kilogram/meter^3$
k_r	$(S_w)^2$	$(1 - S_w)^2$	