Analysis of Gas-Cap or Dissolved-Gas Drive Reservoirs

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ABSTRACT

A numerical method of solving the partial differential equations which describe the one-dimensional displacement of oil by gas has been presented. Possible extension of the method to treat multidimensional flow is discussed, and the limitations of this extension are indicated. Using this method, it is possible to allow for the existence of a gas cap, the presence of any number of gas-injection or oil-production wells and the evolution of dissolved gas from the oil. It is also possible to allow for variation in the cross-sectional area, elevation, porosity and permeability of the reservoir. The influence of relative permeability and the force of gravity in the direction of flow upon the displacement is considered.

The influence of capillary pressure upon the flow and the effect of gravity in the direction perpendicular to flow are neglected. The physical properties of the fluids are considered to be functions of pressure only, and equilibrium between contiguous phases is assumed.

The numerical calculations can be readily carried out by the use of a digital computer. Several example analyses have been performed using the IBM 704 computer, and about one-third of an hour of computing time was required per case. Reservoir behavior predicted by use of this numerical method was compared to data obtained by other methods for three cases – complete pressure maintenance, dissolved-gas drive and gas-cap drive. The independent solutions to these problems were obtained by analytical solution, laboratory experiment and field data, respectively. Agreement of the numerical solution with data from these sources was good; this agreement establishes the convergence and accuracy of the numerical method.

INTRODUCTION

Most petroleum reservoirs can be produced by any one of several alternative programs. When a reservoir is produced by primary methods, production economics can be influenced by controlling the number and location of wells and the flow rate of each well. An even greater influence may be achieved by augmenting the recovery of oil obtainable by primary methods. This can be accomplished by injection of fluids such as water, natural or enriched gas or a bank of light liquid hydrocarbons. Selection of the most desirable operation requires a means of predicting the reservoir behavior which will result from each of the several alternative programs. The purpose of this paper is to present a mathematical method for predicting the behavior of reservoirs produced by gas-cap drive, dissolved-gas drive or pressure maintenance by gas injection.

The method described herein takes cognizance of phase changes caused by a decline or an increase (due to gas injection) in reservoir pressure, of the presence of a gas cap and of the effect of gravity on the flow of gas and oil. Relative permeability relationships are used to define the flow properties of the rock. Allowance is made for variation in cross-sectional area, elevation, permeability and porosity of the reservoir. Both the influence of capillary pressure upon the flow and pressure gradients in the gas cap are neglected. Whenever a liquid phase and a gas phase are in contact, they are assumed to be in equilibrium. The physical properties of the fluids are considered to be functions of pressure only. Therefore, if the method is to be used to predict the effects of a gas-injection program, mixtures of the injected gas and formation crude should have the same physical properties as mixtures of formation gas and crude.

The equations to be presented in this paper apply only to a one-dimensional case; therefore, they neglect the influence of gravity in the direction transverse to the flow. As is well known, this gravitational influence may lead to overriding of oil by gas. Consequently, this procedure as presented is most applicable to long, thin reservoirs for which gravity overriding is not important. On the other hand, the equations presented can be generalized to treat multidimensional flow and, hence, to consider gravity overriding, if desired. A word of caution on two points is advisable here, however. First, the authors have not demonstrated the accuracy of the numerical technique for multidimensional flow. Second, and more important, capillary pressure will often be of importance in multidimensional problems. Obviously, in such cases a generalization of the

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present method would be inadequate.

These limitations of the present method are less restrictive than limitations on previously published methods of predicting gas-drive reservoir behavior.[1-9] For example, the method of Ref. 1 is restricted by the assumption that pressures and saturations are uniform throughout the reservoir, and the method of Ref. 2 is limited to dissolved-gas-drive reservoirs involving a single well. Ref. 3 through 8 are suitable only for complete pressure maintenance by gas injection. Ref. 9 presents a method capable of handling pressure decline, but the mathematical relations used are not rigorous, the method of computation is tedious and it does not provide a way of calculating pressure distribution throughout the reservoir.

The method presented in the paper differs from that of Ref. 9 by being based on solution of a more rigorous mathematical statement of the problem. This statement consists of two simultaneous partial differential equations and suitable boundary conditions. These equations are solved by a systematic approximation technique which has been adapted to solution by electronic computation. This relieves the engineer of most of the labor involved and makes feasible the completion of a reservoir analysis in approximately 20 minutes on an IBM 704 computer. The calculations provide complete pressure and saturation profiles at frequent time intervals, as well as gas-oil ratios and cumulative production data for each well.

Derivation of the differential equations stating the problem is presented first. This is followed by transformation of the equations into finite-difference equations. Finally, the convergence and accuracy of the method is evaluated by comparing computed results with results obtained from analytical solution, laboratory experiment and field observation.

THE DIFFERENTIAL SYSTEM

The fundamental equations used to define the reservoir behavior are the equations of continuity for gas and stock-tank oil and Darcy's law of flow for each phase. The continuity equation for gas allows for the presence of a distributed sink for fluid production of strength V_o for oil and V_g for gas. This equation is

$$-\frac{\partial}{\partial x}(u_g A\delta + u_o AB_o R_s) - V_o R_s - V_g = \phi A \frac{\partial}{\partial t}(S_g\delta + S_o B_o R_s) \quad \ldots\ldots\ldots\ldots \quad (1)$$

where δ = the standard cubic feet of gas per cubic feet of gas at reservoir conditions,
V_g = the rate of free-gas production per unit length of the reservoir, and
V_o = the rate of oil production per unit length of the reservoir.

Eq. 2 is the continuity equation for oil. In this expression, the volatility of the stock-tank oil has been neglected.

$$-\frac{\partial}{\partial x}(u_o AB_o) - V_o = \phi A \frac{\partial}{\partial t}(S_o B_o) \quad \ldots \quad (2)$$

Darcy's law of flow for each phase,

$$u_o = -\frac{kk_{ro}}{\mu_o}\left(\frac{\partial P}{\partial x} + \rho_o g \frac{\partial z}{\partial x}\right), \quad \ldots\ldots \quad (3)$$

and

$$u_g = -\frac{kk_{rg}}{\mu_g}\left(\frac{\partial P}{\partial x} + \rho_g g \frac{\partial z}{\partial x}\right), \quad \ldots\ldots \quad (4)$$

where z = the elevation of the reservoir above an arbitrary datum plane, and the equation,

$$S_o = 1 - S_{wc} - S_g, \quad \ldots\ldots\ldots\ldots \quad (5)$$

complete the statement of the differential problem except for boundary and initial conditions.

Eqs. 3, 4 and 5 may be used to eliminate u_o, u_g and S_o from Eqs. 1 and 2. This would result in two differential equations containing two dependent variables, S_g and P. These two equations may be solved simultaneously to obtain the desired solution. Instead of taking this approach, it is desirable first to combine Eqs. 1 and 2 to eliminate time derivatives of saturation and, then, to use the new equation and Eqs. 2, 3, 4 and 5 to define the problem. The advantage of doing this is that the new equation can be approximated by a difference equation having only pressures at an advanced time level as unknowns. This gives rise to a set of equations which can be solved to obtain a complete pressure distribution at the new time. These new pressures can be used in a difference approximation of Eq. 2 to yield saturations at the new time. Thus, the solution of the two equations is divided into two steps, solution of a pressure equation and solution of a saturation equation. The necessity of solving for pressures and saturations simultaneously is avoided by this procedure.

The desired combination of Eqs. 1 and 2 is effected by multiplying Eq. 1 by B_o, multiplying Eq. 2 by $(\delta - B_o R_s)$, replacing S_o by $(1 - S_{wc} - S_g)$, expanding the product derivatives on the right-hand side of the equation and replacing terms of the form

$$\frac{\partial C(P)}{\partial t} \text{ by } \frac{dC(P)}{dP}\frac{\partial P}{\partial t} = C'\frac{\partial P}{\partial t}.$$

Here C or $C(P)$ indicates a function of pressure only, and C' denotes $\frac{dC}{dP}$. Adding the resulting equations gives

[1] References given at end of paper.

$$- B_o \frac{\partial}{\partial x}(u_g A\delta + u_o AB_o R_s) - (\delta - B_o R_s) \cdot \frac{\partial}{\partial x}(u_o AB_o) - B_o(V_o R_s + V_g) - (\delta - B_o R_s)V_o = \phi A\{S_g[B_o(\delta - B_o R_s)' - B_o'(\delta - B_o R_s)] + [1 - S_{wc}][B_o(B_o R_s)' + B_o'(\delta - B_o R_s)]\} \frac{\partial P}{\partial t} = \phi A[D(P, S_g)] \frac{\partial P}{\partial t} , \quad \ldots\ldots\ldots\ldots\ldots (6)$$

where the function, $D(P, S_g)$ is defined by the last equality.

Next, u_o, u_g and S_o are eliminated from Eqs. 6 and 2 by use of Eqs. 3, 4 and 5, which results in the following equations.

$$B_o \frac{\partial}{\partial x}\left[\left(\frac{kk_{rg}A\delta}{\mu_g}\right)\left(\frac{\partial P}{\partial x} + \rho_g g \frac{\partial z}{\partial x}\right)\left(\frac{kk_{ro}AB_o R_s}{\mu_o}\right)\left(\frac{\partial P}{\partial x} + \rho_o g \frac{\partial z}{\partial x}\right)\right] + (\delta - B_o R_s) \cdot \frac{\partial}{\partial x}\left[\left(\frac{kk_{ro}AB_o}{\mu_o}\right)\left(\frac{\partial P}{\partial x} + \rho_o g \frac{\partial z}{\partial x}\right)\right] - B_o(V_o R_s + V_g) - (\delta - B_o R_s)V_o = \phi AD \frac{\partial P}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

and

$$\frac{\partial}{\partial x}\left[\left(\frac{kk_{ro}AB_o}{\mu_o}\right)\left(\frac{\partial P}{\partial x} + \rho_o g \frac{\partial z}{\partial x}\right)\right] - V_o = \phi A\left[(1 - S_{wc} - S_g)B_o' \frac{\partial P}{\partial t} - B_o \frac{\partial S_g}{\partial t}\right] \quad \ldots (8)$$

All that is needed in addition to Eqs. 7 and 8 to define the reservoir behavior is a statement of suitable boundary and initial conditions.

The initial conditions in most gas-cap-drive situations are that the gas saturation throughout the reservoir is zero and that the pressure at the original gas-oil contact is specified. Since the reservoir fluids are initially at rest, the pressure gradient at any point in the reservoir is equal to the density of the oil times the elevation gradient at that point. This completely defines the initial pressure distribution in the reservoir. Stated symbolically, these initial conditions are

$$S_g = 0 \text{ for } 0 \le x \le X, \; t = 0 ;$$
$$P = P_o \text{ for } x = 0, \; t = 0 ;$$

and

$$\frac{\partial P(x)}{\partial x} = - \rho_o(x) g \frac{\partial z}{\partial x} \text{ for } 0 \le x \le X, \; t = 0$$

where X = total length of the reservoir, and
P_o = initial gas-cap pressure.

The boundary condition at the original gas-oil contact is that the rate of gas entry at that point, i_g, is determined by the expansion of the gas-cap gas which has a volume V. The amount of gas in the gas cap at any pressure is $V\,\delta\,(P)$, and the rate at which gas enters the oil zone is $-V\,\delta' \frac{\partial P}{\partial t} = i_g$.

At any production-well location, it is desirable to specify the rate of stock-tank oil production q_o provided this rate does not exceed the productivity of the well. This oil rate is related to the gas-production rate through the relative permeabilities and fluid properties existing at that time at the location in question. The quantitative relationship is derived from material balances on the differential element Δx of the reservoir which contains a production well. A material balance on the oil phase yields

$$B_o A(u_o)_1 = q_o + B_o A(u_o)_2 .$$

where subscript 1 indicates conditions at $x - \Delta x/2$ and subscript 2 conditions at $x + \Delta x/2$.

The gas-phase material balance yields

$$\delta A(u_g)_1 = q_g + \delta A(u_g)_2 .$$

Darcy's law of flow is used to eliminate $(u_g)_1$, $(u_g)_2$, $(u_o)_1$ and $(u_o)_2$ from these equations; then the resulting equations are combined to eliminate

$$\left(\frac{\partial P}{\partial x}\right)_1 - \left(\frac{\partial P}{\partial x}\right)_2 .$$

Assuming $\left(\frac{\partial s}{\partial x}\right)_1 = \left(\frac{\partial z}{\partial x}\right)_2$, the result is

$$q_g = q_o \frac{\delta}{B_o} \frac{k_{rg}}{k_{ro}} \frac{\mu_o}{\mu_g} \quad \ldots\ldots\ldots\ldots (9)$$

If the well is located at the end of a reservoir, then $(u_o)_2 = (u_g)_2 = 0$. In this case, Eq. 10 is the result of this analysis and should be used in place of Eq. 9.

$$\frac{q_g}{q_o} = \frac{\delta k_{rg}\mu_o}{B_o k_{ro}\mu_g} + (\rho_o - \rho_g) \frac{kk_{rg}A\delta g}{q_o \mu_g} \frac{\partial z}{\partial x} . \quad (10)$$

Note that in both Eqs. 9 and 10 q_g represents only gas produced as free gas. A quantity of dissolved gas equal to $q_o R_s$ is also produced with the oil.

If the productivity of the well is less than the desired production rate, the boundary condition becomes one of constant pressure at the well. Since the one-dimensional analysis presented in this paper neglects the pressure drop due to radial flow around the well, the constant pressure employed in the calculations should be greater than the actual bottom-hole pressure to be maintained in the well. The difference in these two pressures should equal the pressure drop caused by radial flow into

the well and can be estimated from radial flow formulas.

At the reservoir boundaries, there must be no flow of either gas or oil. This is accomplished by setting $u_o = u_g = 0$.

At injection wells (if there are any), the rate of gas injection may be arbitrarily specified. The gas saturation at gas-injection wells and at the gas-cap contact is set equal to that saturation existing when the rock contains residual oil.

Specifying the boundary conditions completes the statement of the problem in differential form. The next step is to write suitable difference approximations to Eqs. 7 and 8.

THE DIFFERENCE SYSTEM

There exist many difference equations which under casual investigation appear to be valid analogues of Eqs. 7 and 8. Some of these provide adequate approximation, while others do not. Since there is no rigorous analysis to demonstrate that the solution of these analogues converges to that of the differential system, the final test must be experimental computing. Analysis of similar, but simpler, systems serves as a useful guide in selecting equations to be tested.

The set of difference approximations to Eqs. 7 and 8 presented in this section have been subjected to extensive testing, and the results are discussed in the following section. This testing has shown the set to be suitable for the prediction of reservoir behavior for gas-cap drive, dissolved-gas drive and pressure maintenance by gas injection. However, no claim is made that this is the only set of equations which will satisfactorily approximate Eqs. 7 and 8.

In writing finite-difference approximations to Eqs. 7 and 8, the distance axis is divided into J increments Δx in length, and the time scale is divided into increments Δt in length. The following definitions are used.

$x_j = j\Delta x$,

$X = J\Delta x$ = total length of reservoir,

$t_n = n\Delta t$, and

$P_{jn} = P(x_j, t_n)$, etc.

Derivatives of the form,

$$\frac{\partial}{\partial x}\left[C(P)E(S_g)\frac{\partial P}{\partial x}\right]$$

to be evaluated at x_j and t_{n+1} are replaced by terms of the form,

$$\frac{C_{j+\frac{1}{2},n}E_{j,n}\dfrac{(P_{j+1,n+1}-P_{j,n+1})}{\Delta x} - C_{j-\frac{1}{2},n}E_{j-1,n}\dfrac{(P_{j,n+1}-P_{j-1,n+1})}{\Delta x}}{\Delta x}$$

in arriving at difference approximations of Eqs. 7 and 8.

Attention is called to the fact that the coefficient E, which is a function of saturation only, is evaluated at the upstream end of the interval. That is, if the pressure difference involves points j and $j+1$, E is evaluated at point j. This is the upstream end of the interval since in the problem being considered flow is from point j to $j+1$. If E were evaluated at the midpoint of the interval, like $C(P)$, then the numerical procedure would not give solutions which converge to a true solution of the differential system. If the equations presented herein are generalized to treat multidimensional flow, care must be taken to evaluate the function E at the upstream end of the interval whenever it appears in approximations of distance derivatives. Otherwise, the extension of

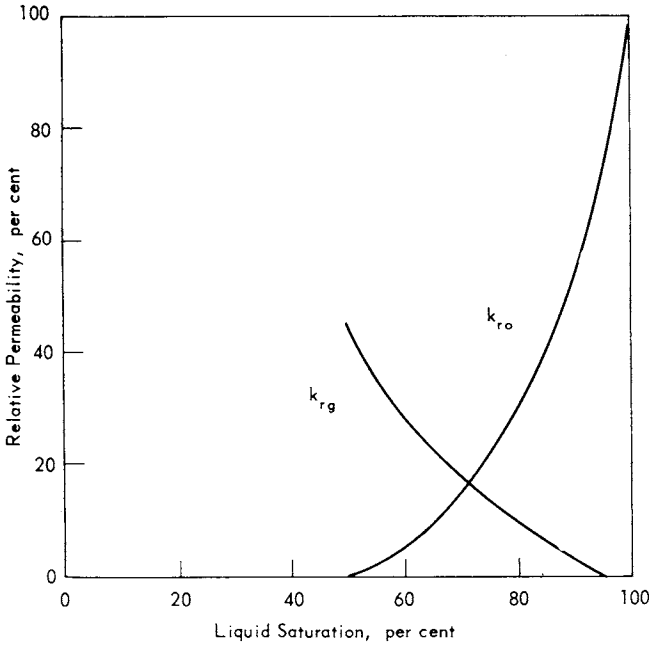

FIG. 1 – RELATIVE PERMEABILITY-SATURATION RELATION, PRESSURE MAINTENANCE.

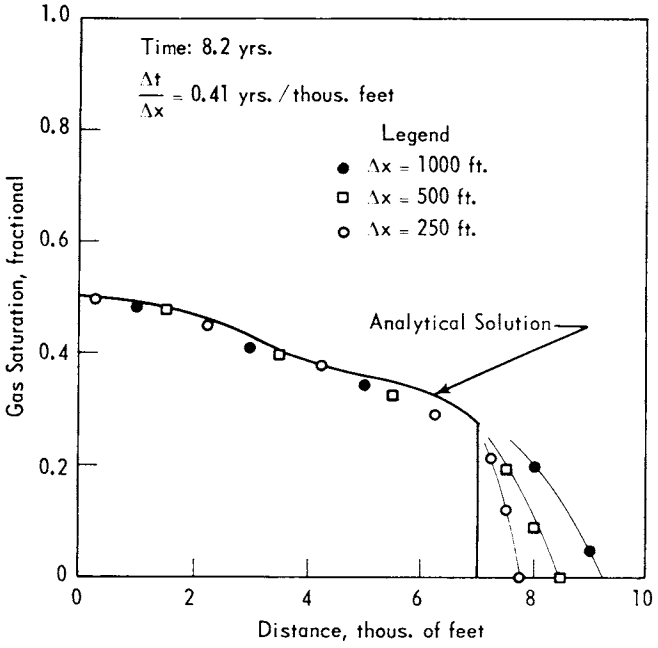

FIG. 2 – SATURATION PROFILE, PRESSURE MAINTENANCE.

the procedure to two or three dimensions is perfectly straight-forward.

Time derivatives such as $\left(\frac{\partial P}{\partial t}\right)_j$ are approximated as $\frac{P_{j,\,n+1} - P_{j,\,n}}{\Delta t}$. Production wells are approximated as distributed sinks of strength $\left(\frac{q_o}{\Delta x}\right)_{j,n}$ for oil and $\frac{q_o}{\Delta x}$ for gas. Injection wells are treated as sources of strength $\frac{i_g}{\Delta x}$. The inclusion of these source and sink terms satisfies the boundary conditions of specified rates at the wells.

In writing the difference equations, subscripts were dropped from B_o, R_s and S_g for simplicity. The approximation to Eq. 7 is implicit in pressures and is given by

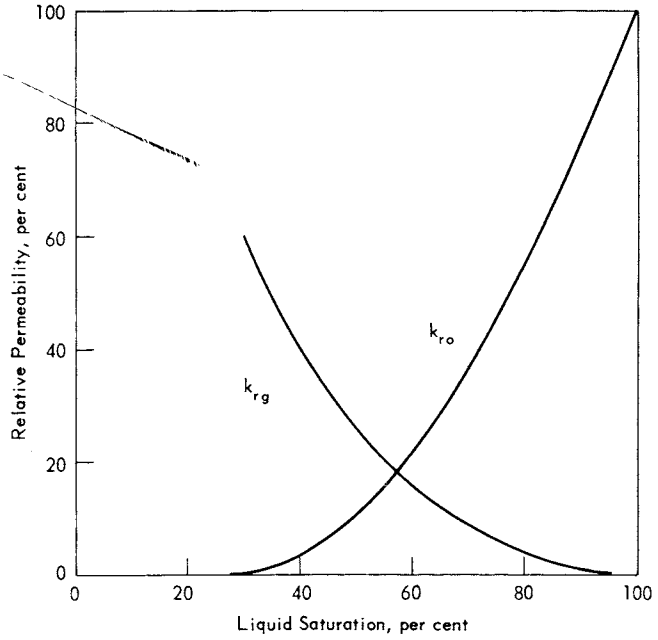

FIG. 3 – RELATIVE PERMEABILITY-SATURATION RELATION, DISSOLVED-GAS DRIVE.

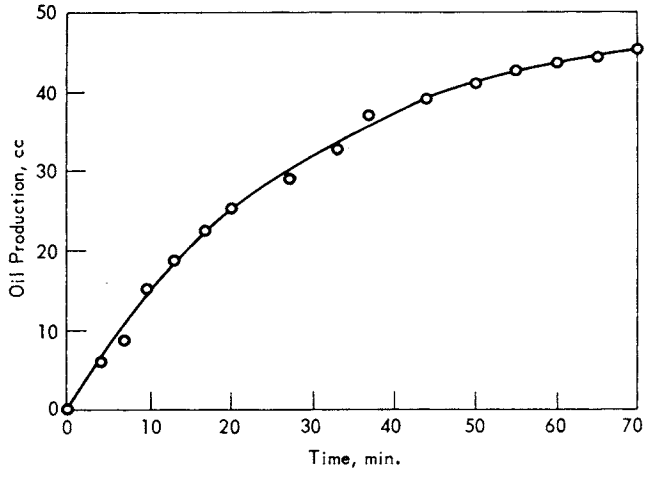

FIG. 4 – EXPERIMENTAL PRODUCTION HISTORY, USED AS BOUNDARY CONDITION TO COMPUTE BEHAVIOR OF DISSOLVED-GAS DRIVE.

$$
\begin{aligned}
&B_{j,n}\left\{(kk_{rg})_{j,n}\left(\frac{A\delta}{\mu_g}\right)_{j+\frac{1}{2},n}\right.\\
&\left[\frac{(P_{j+1,n+1}-P_{j,n+1})+(\rho_g g)_{j+\frac{1}{2},n}(z_{j+1}-z_j)}{(\Delta x)^2}\right]\\
&-(kk_{rg})_{j-1,n}\left(\frac{A\delta}{\mu_g}\right)_{j-\frac{1}{2},n}\\
&\left[\frac{(P_{j,n+1}-P_{j-1,n+1})+(\rho_g g)_{j-\frac{1}{2},n}(z_j-z_{j-1})}{(\Delta x)^2}\right]\\
&+(kk_{ro})_{j,n}\left(\frac{ABR}{\mu_o}\right)_{j+\frac{1}{2},n}\\
&\left[\frac{(P_{j+1,n+1}-P_{j,n+1})+(\rho_o g)_{j+\frac{1}{2},n}(z_{j+1}-z_j)}{(\Delta x)^2}\right]\\
&-(kk_{ro})_{j-1,n}\left(\frac{ABR}{\mu_o}\right)_{j-\frac{1}{2},n}\\
&\left.\left[\frac{(P_{j,n+1}-P_{j-1,n+1})+(\rho_o g)_{j-\frac{1}{2},n}(z_j-z_{j-1})}{(\Delta x)^2}\right]\right\}\\
&+(\delta-BR)_{j,n}\left\{(kk_{ro})_{j,n}\left(\frac{AB}{\mu_o}\right)_{j+\frac{1}{2},n}\right.\\
&\left[\frac{(P_{j+1,n+1}-P_{j,n+1})+(\rho_o g)_{j+\frac{1}{2},n}(z_{j+1}-z_j)}{(\Delta x)^2}\right]\\
&-(kk_{ro})_{j-1,n}\left(\frac{AB}{\mu_o}\right)_{j-\frac{1}{2},n}\\
&\left.\left[\frac{(P_{j,n+1}-P_{j-1,n+1})+(\rho_o g)_{j-\frac{1}{2},n}(z_j-z_{j-1})}{(\Delta x)^2}\right]\right\}\\
&+B_{j,n}\left(\frac{i_g}{\Delta x}\right)_{j,n}-B_{j,n}\left(\frac{q_o R+q_g}{\Delta x}\right)_{j,n}\\
&-(\delta-BR)_{j,n}\left(\frac{q_o}{\Delta x}\right)_{j,n}\\
&=(\phi A)_j D_{j,n}\,\frac{(P_{j,n+1}-P_{j,n})}{\Delta t}\,, \qquad \ldots\ldots\ldots \quad (11)
\end{aligned}
$$

where i_g, q_o, $q_g \geq 0$.

There are $(J + 1)$ grid points in the reservoir corresponding to $j = 0, 1, 2 \ldots J\text{-}1, J$. The reservoir may be divided into segments of length Δx, centered about grid points. Eq. 11 as applied to the point j represents a combination of the two material-balance equations for the segment centered on j. At the end points $j = 0$ and $j = J$, there exists only one-half of a segment; hence, the right-hand side of Eq. 11 should be divided by two when applying it to these points.

Maintaining the proper boundary conditions at the points $j = 0$ and $j = J$ also requires that $k - 1$ and k_J be set equal to zero in Eq. 11. The resulting form of the equation places a restriction on the pressure such that the rate of flow at these points is equal to that specified by the i_g, q_o, and q_g terms of Eq. 11. If no production or injection occurs at $j=J$, then $i_g=q_o=q_g=0$ and the no-flow boundary conditions of $u_o=u_g=0$ are satisfied. The boundary condition at the gas-cap contact is satisfied by setting i_g at $j = 0$ equal to

$$-V\delta' \frac{(P_{o,n+1} - P_{o,n})}{\Delta t}.$$

As stated earlier examination of Eq. 11 reveals that, if conditions of saturation and pressure are known at time level t_n, the only unknowns in Eq. 11 are pressures at time t_{n+1}. The equation may, therefore, be put in the form,

$$a_j P_{j-1,n+1} + b_j P_{j,n+1} + c_j P_{j+1,n+1} = d_j$$

where a_j, b_j, c_j and d_j are known constants for each point j. The equation for each point involves three unknowns, except for the first ($j=0$) and last ($j=J$) equations which contain only two. There are, altogether, J equations and J unknowns. This system of linear simultaneous equations can readily be solved by a method described in the literature[10] to yield a set of pressures at the new time level t_{n+1}. These new pressures are used in a difference approximation of Eq. 8 to find the new saturation at t_{n+1}. The approximation used for Eq. 8 is

$$(kk_{ro})_{j,n}\left(\frac{AB}{\mu_o}\right)_{j+\frac{1}{2},n}\left[\frac{(P_{j+1,n+1}-P_{j,n+1}) + (\rho_o g)_{j+\frac{1}{2},n}(z_{j+1}-z_j)}{(\Delta x)^2}\right] - (kk_{ro})_{j-1,n}\left(\frac{AB}{\mu_o}\right)_{j-\frac{1}{2},n}\left[\frac{(P_{j,n+1}-P_{j-1,n+1}) + (\rho_o g)_{j-\frac{1}{2},n}(z_j-z_{j-1})}{(\Delta x)^2}\right] - \left(\frac{q_o}{\Delta x}\right)_{j,n} = (\phi A)_j \left\{[(1 - S_{wc})B' - SB']_{j,n}\left[\frac{(P_{j,n+1}-P_{j,n})}{\Delta t}\right] - B_{j,n}\frac{(S_{j,n+1}-S_{j,n})}{\Delta t}\right\} \quad . . (12)$$

At $j = 0$, the original gas-oil contact, the gas saturation is held constant at a value corresponding to the residual oil saturation. At $j = J$, as for Eq. 11, it must be noted that $k_j = 0$ and that only one-half of a segment is present.

When the pressure at a well is specified rather than the flow rate, this pressure is first corrected to allow for the pressure drop due to radial flow into the well. Then this corrected pressure is used in lieu of Eq. 11 at the grid point corresponding to the well. Writing Eq. 11 for all points except those at which pressures are specified and combining this set of equations with the specified pressures results in a system of equations which can be solved to yield a complete pressure distribution at time level $n + 1$ (assuming pressures and saturations were known at time level n). Then Eq. 11 written for the points at which the pressure was arbitrarily specified will contain two unknowns, q_o and q_g, and can be combined with either Eq. 9 or Eq. 10, as appropriate, to yield q_o and q_g for the well.

Eq. 12 is solved explicitly for the saturation at time level $n + 1$. As with all explicit equations, there is an upper limit on the size of the time step that can be employed. This limit cannot be established precisely for the general case, but for the case of constant coefficients (i.e., the pressure-maintenance problem where fluid properties are considered invariant), this limit is given by

$$\Delta t \leq \left[\frac{\Delta x \phi}{(u_o + u_g) f_g'}\right]_{\text{minimum}}$$

The maximum value of $(u_o + u_g)$ is employed in this equation. The quantity f_g' is equal to $\frac{df_g}{dS_g}$ evaluated at the gas saturation existing at the gas front.

This limitation was derived from a heuristic stability analysis and verified by experimental computing for the case of pressure maintenance. While it was found that a slightly less stringent time-step limitation could be applied for gas-cap or dissolved-gas-drive calculations, this equation serves as a useful guide in selecting the time step.

MATERIAL BALANCE EQUATIONS

The difference equations presented in the preceding section are consistent with the conservation of matter. Thus, the amount of gas and oil should be conserved during the course of the calculations. It is advantageous to monitor the accuracy of the calculations by means of an over-all material balance on both gas and oil, since such balances represent independent checks on the accuracy with which the difference equations are being solved. In this way, machine or other errors may be detected.

To compute an over-all material balance it is necessary to integrate to obtain both the quantity of oil and the quantity of gas present in the reservoir as a function of time. It also is necessary to integrate to obtain the cumulative production of gas and oil. The integrals are of an elementary form and will not be repeated here. However, it should be noted that, to make the material balance exact except for machine round-off error, these integrals must be evaluated by the trapezoidal rule.

EVALUATION OF THE METHOD

The method described in the preceding sections is intended for application to a wide variety of gas-drive problems including dissolved-gas drive, gas-cap drive, complete pressure maintenance and pressure-buildup operations. Therefore, in evaluating the validity of the method, it was desirable to solve a number of problems by the numerical method of calculation and to compare the answers to ones obtained by analytical methods, laboratory experiment or field data. Three problems were selected – complete pressure maintenance, dissolved-gas drive and gas-cap drive. In all three cases, gravity was an important factor in the recovery mechanism. The first problem, complete pressure maintenance, was solved analytically by the Buckley-Leverett technique. Laboratory data were obtained for a dissolved-gas-drive reservoir, and field data were available for a gas-cap-drive reservoir.

The numerical calculations were programed for the IBM 704 in machine language using the SHARE assembly program. The total machine storage requirement for input-output routines, program and data storage is approximately 7,000 words. If the reservoir is divided into 20 grid intervals, which is adequate for the problems described, the computations require six seconds per time step. About 200 time steps are required in a typical case, and this results in a total computing time of approximately 20 minutes per case. Increasing the number of grid intervals increases the number of time steps required, so that total computing time is proportional to the square of the number of intervals used.

COMPLETE PRESSURE MAINTENANCE

The results of complete pressure maintenance were computed for a hypothetical reservoir 10,000-ft long, 25,000 sq ft in cross-sectional area and with an angle of inclination of 6° from the horizontal direction. The reservoir was initially 75 per cent saturated with oil; the other 25 per cent saturation was connate water. Production was taken only from the bottom extremity of the reservoir, and gas was injected into the upper extremity at a rate of 500,000 scf/day. This rate of injection just balanced the total rate of production so that the pressure at the lower end of the reservoir was constant at 1,000 psig. Reservoir permeability was 1 darcy, and the porosity was 25 per cent. The relative permeability-saturation relationship for the reservoir matrix is shown in Fig. 1.

This problem was solved under the assumption that the pressure drop in the reservoir was small enough so that the fluid properties could be treated as invariant. The fluid properties employed in the calculations were: oil viscosity, 0.5 cp; gas viscosity, 0.014 cp; oil density, 45.5 lb/cu ft; gas density, 0.5 lb/cu ft; oil formation volume factor, 0.8; gas solubility, 600 scf/STB; and gas expansion factor, 82.5 scf/reservoir cu ft.

The saturation profile after 8.2 years of injection, as calculated by the Buckley-Leverett[4,5] analytical technique, is shown as the solid curve in Fig. 2. Results of the numerical calculations are also shown in this figure. The open circles were computed with $\Delta x = 250$ ft, the squares with $\Delta x = 500$ ft and the solid circles with $\Delta x = 1{,}000$ ft. The ratio $\Delta t/\Delta x$ was equal to 0.41×10^{-3} years/ft in all three calculations. Behind the gas front, the data from all three numerical calculations give good agreement with the analytical solution, but at the front it is only fair for Δx equal to 500 or 1,000 ft. With Δx equal to 250 ft, the computed front is relatively accurate; further refinement of the size of Δx, while keeping $\Delta t/\Delta x$ constant, would further increase the accuracy.

Refinement of Δt while keeping Δx constant at 500 ft did not appreciably alter the solution shown in Fig. 2.

Application of the numerical technique to this problem constitutes a severe test of the method because of the saturation discontinuity which develops during the displacement. Nevertheless, the data given in Fig. 2 show that the approximate solution can be made to approach the analytical solution as closely as desired by refining the size of Δx.

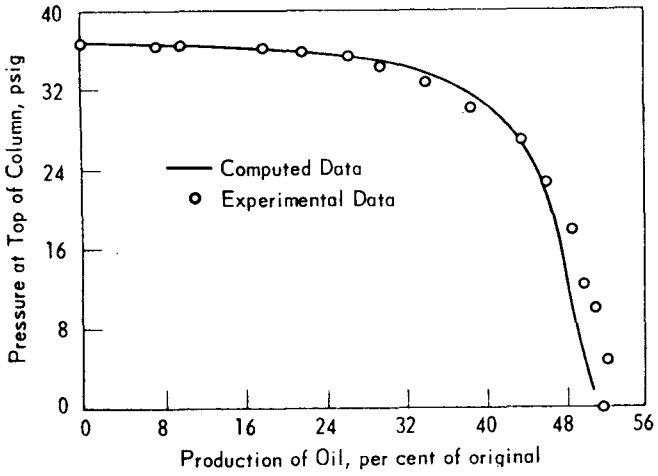

FIG. 5 – PRESSURE BEHAVIOR AT TOP OF COLUMN, DISSOLVED-GAS DRIVE.

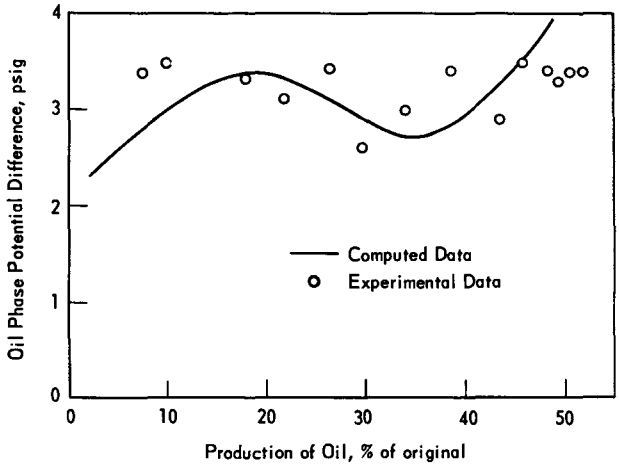

FIG. 6 – VARIATION OF THE DIFFERENCE IN OIL-PHASE POTENTIAL AT TOP AND BOTTOM OF COLUMN, DISSOLVED-GAS DRIVE.

TABLE 1 – PHYSICAL PROPERTIES OF FLUIDS FOR DISSOLVED-GAS-DRIVE SYSTEM, TEMPERATURE = 75°F

Pressure (psig)	Solubility*	Oil Formation Volume Factor	Expansibility of Gas**	Oil Viscosity (cp)	Gas Density (lb/cu ft)
0	0	1.000	1.00	1.55	0.115
5	6.26	0.991	–	–	–
10	11.73	0.980	1.70	1.31	0.198
15	17.48	0.967	–	–	–
20	23.58	0.950	2.43	1.11	0.280
25	30.32	0.930	–	–	–
30	38.00	0.908	3.17	0.93	0.364
35	47.62	0.882	–	–	–
40	62.06	0.854	3.94	0.76	0.450

*Cubic feet of gas at standard conditions per cubic foot of stock-tank oil.

**Cubic feet at standard conditions per cubic foot at reservoir conditions.

DISSOLVED-GAS DRIVE

Dissolved-gas-drive calculations were used to simulate the behavior of a laboratory model. This model consisted of a rectangular column packed with sand and mounted with the long dimension vertical. The dimensions of the model were 3/8 × 1/2 × 72 in. It was packed with 1/2 Wausau sand to a porosity of 44.1 per cent and a total permeability of 13.0 darcies. Relative permeability data for the sand are presented in Fig. 3. These data were determined experimentally by the displacement technique of Johnson, *et al.*[11]

The oil phase charged to the column was a light refined hydrocarbon (Bayol D) saturated with propane at a temperature of 75°F and a pressure of 36.64 psig. The experiment was conducted at a constant temperature of 75°F; some of the properties of the equilibrium phases of Bayol D-propane mixtures at this temperature are given in Table 1 as a function of pressure. These properties are the oil formation volume factor and viscosity, and the gas expansibility, solubility and density. The gas viscosity was constant at 0.008 cp. The oil density varied linearly from 47.5 lb/cu ft at a pressure of 0 psig to 45.0 lb/cu ft at a pressure of 40 psig. At the start of the experiment, the model was saturated with oil containing dissolved gas, there being no connate water present. Production of oil was initiated from the bottom of the column by reducing the back-pressure maintained at this point. The resulting cumulative production of stock-tank oil is shown in Fig. 4 as a function of elapsed time.

A material balance applied to the experimental data indicated that the oil and gas in the column after the initiation of production were not in complete equilibrium. This lack of equilibrium probably was caused by supersaturation of the liquid phase. The average degree of supersaturation during the early

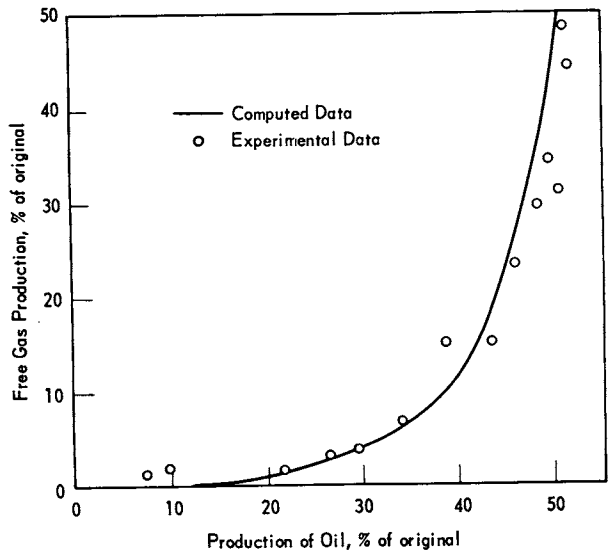

FIG. 7 – FREE-GAS PRODUCTION HISTORY, DISSOLVED-GAS DRIVE.

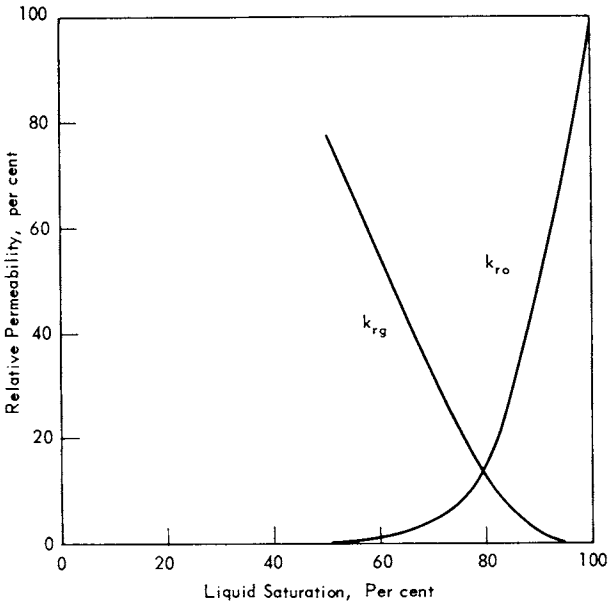

FIG. 8 – RELATIVE PERMEABILITY-SATURATION RELATION, GAS-CAP DRIVE.

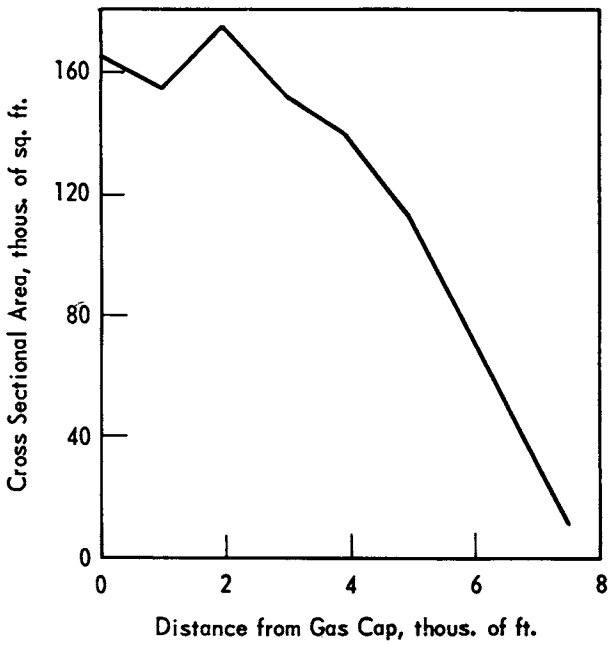

FIG. 9 – VARIATION OF CROSS-SECTIONAL AREA, GAS-CAP RESERVOIR.

TABLE 2 – PERTINENT RESERVOIR DATA – GAS-CAP DRIVE

Original and Saturation Pressure (psig)	2,700
Reservoir Temperature (°F)	181
Volume of Oil Zone (acre-ft)	20,986
Volume of Effective Gas Cap (acre-ft)	10,883
Formation Dip (ft/mile)	226
Average Permeability (md)	165
Average Porosity (per cent)	23
Average Connate-Water Content of Oil Zone (per cent)	28
Average Connate-Water Content of Gas Zone (per cent)	25
Original Oil in Place (STB)	19,840,000
Original Gas-Cap Gas in Place (Mscf)	13,969,000

part of the experiment was estimated to be slightly in excess of 1 psi. Inconsistencies in the material balance on gas of about 4 per cent prevented estimation of the degree of supersaturation late in the experiment.

In computing the behavior of this model, the rate of production of stock-tank oil at any time was derived from the data of Fig. 4. This rate schedule was entered into the calculations as the boundary condition at the production end of the model, and the resulting gas production and pressure behavior of the model was calculated by the previously described numerical technique.

Both the results of calculation and experiment are shown in Figs. 5, 6 and 7. In all three figures, the abscissa is the cumulative quantity of oil produced. The ordinate in Fig. 5 is the pressure at the top of the column; in Fig. 6 it is the difference in the oil-phase potential ($P + \rho_o g \Delta z$) measured at the top and bottom of the column, and in Fig. 7 it is the cumulative quantity of free (undissolved) gas produced. In these figures, the solid curves represent the calculated data; the open circles are data points which were experimentally measured.

The computed pressure behavior at the top of the column (Fig. 5) agrees quite well with that observed experimentally, especially early in the experiment. Agreement was not as good late in the experiment when the pressure was declining very rapidly, making it experimentally difficult to measure accurately the cumulative quantity of oil produced. Note, however, that the final cumulative oil recovery measured after the final blowdown to zero pressure agreed very well with the calculated ultimate recovery.

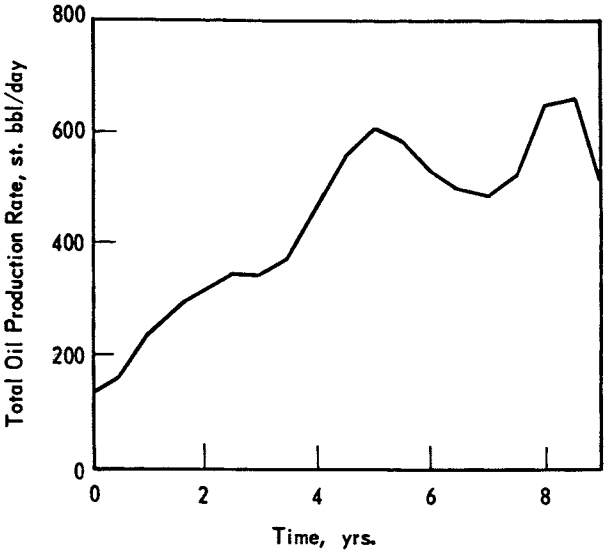

FIG. 10 – VARIATION OF TOTAL PRODUCTION RATE, GAS-CAP RESERVOIR.

The difference in oil-phase potential measured between the top and bottom of the column (Fig. 6) was relatively small, 2.5 to 3.5 psig, and the pressure difference was of course smaller, 0.6 to 1.6 psig. The experimental data exhibit appreciable scatter; but, even so, the average deviation of the experimental from the computed values is ± 12 per cent.

The data of Fig. 7 indicate that the first free-gas production occurred when about 15 per cent of the original oil had been produced. Thereafter, the agreement of the experimental and computed data is reasonably good. In fact, the comparisons shown on Figs. 5, 6 and 7 indicate that the computed and experimental reservoir behavior agree within the probable error of the experiment.

GAS-CAP DRIVE

Kirby, *et al*,[12] published a limited amount of production history for a gas-cap-drive reservoir in which there was no active water drive. Data for this reservoir are given in Table 2. Table 3 gives data on fluid properties as a function of pressure for the reservoir fluids. Relative permeability data for the reservoir are shown in Fig. 8. These

TABLE 3 – PHYSICAL PROPERTIES OF FLUIDS FOR GAS-CAP-DRIVE SYSTEM, TEMPERATURE = 181°F

Pressure (psig)	Solubility*	Oil Vol. Factor	Expansibility of Gas**	Oil Viscosity (cp)	Gas Viscosity (cp)	Oil Density (lb/cu ft)	Gas Density (lb/cu ft)
0	0	1.000	0.8	0.816	0.0117	50.3	0.03
300	17.0	0.930	17.0	0.664	0.0131	47.6	0.75
600	31.3	0.888	34.0	0.576	0.0145	46.5	1.48
900	42.8	0.865	51.5	0.515	0.0157	45.8	2.20
1200	51.0	0.845	69.3	0.470	0.0168	45.0	2.95
1500	63.0	0.823	87.2	0.435	0.0177	44.2	3.73
1800	75.5	0.802	106.5	0.407	0.0185	43.5	4.52
2100	88.5	0.782	126.0	0.383	0.0194	42.7	5.22
2400	102.8	0.760	144.0	0.360	0.0201	42.0	5.93
2700	119.0	0.738	159.0	0.338	0.0209	41.2	6.65

* Cubic feet of gas at standard conditions per cubic foot of stock-tank oil.

** Cubic feet at standard conditions per cubic foot at reservoir conditions.

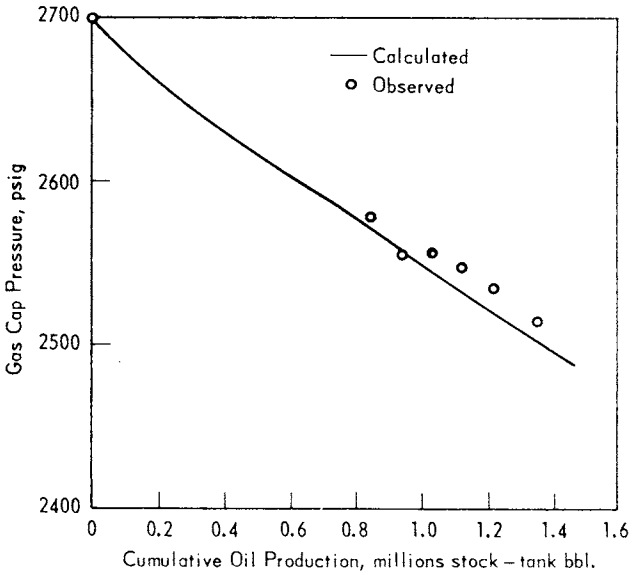

FIG. 11 – GAS-CAP PRESSURE, GAS-CAP DRIVE.

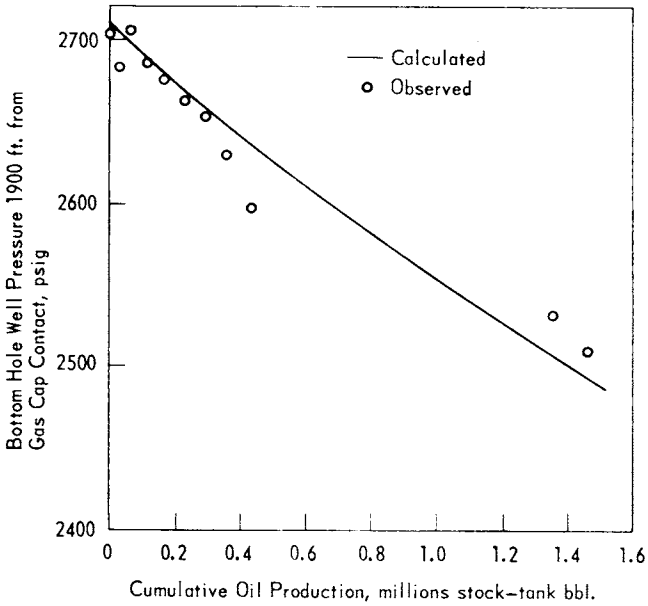

FIG. 12 – OIL-ZONE PRESSURE 1,900-FT FROM GAS-CAP CONTACT.

data were measured on cores taken from the field. Fig. 9 shows the variation of cross-sectional area with location in the reservoir, and Fig. 10 shows the oil-production schedule for the reservoir.

All of these data were employed in the numerical computation of the reservoir behavior. Producing wells were distributed throughout the reservoir as indicated by the field map given in the reference paper.

In Fig. 11 computed gas-cap pressures are compared to those observed in the field. The solid curve is computed, and the circles represent the field data. Agreement is quite good; the best curve through the experimental points would fall a little above the computed curve, with a maximum deviation of 15 psi.

For the production rates used in this reservoir, oil-zone pressures differ only slightly from the gas-cap pressure. This is illustrated in Fig. 12, which shows shut-in bottom-hole pressures for a well 1,900 ft from the gas cap as a function of total cumulative production. As in Fig. 11, the solid curve represents the computed data, and the circles are the observed data. The data presented in Fig. 12 were not presented *per se* by Kirby, *et al*, but were taken from their original worksheets. Kirby, *et al*, chose to present an average pressure of all the wells in the field, rather than the pressures for a single well. Also, they adjusted the observed pressures at all wells to a common elevation by subtracting the quantity $(\rho_o\ g\ \Delta\ z)$ from the observed pressures. Here $\Delta\ z$ is the difference in elevation between the well being observed and the reference elevation. The computed pressures in this figure are a little higher (15 psi or less) than the computed gas-cap pressure at corresponding amounts of production. As for the gas-cap pressures, the computed pressure data are in reasonably good agreement with the observed data.

CONCLUSIONS

A numerical method of solving the partial differential equations which describe the displacement of oil by gas has been presented. This method and the assumptions made in deriving it are described in the paper.

Results presented in the paper lead to the following conclusions.

1. Comparison of data computed by this method to the analytical solution for complete pressure maintenance demonstrates the convergence and accuracy of the numerical method.
2. Comparison of computed results to data obtained from a laboratory model produced by dissolved-gas drive shows the applicability of the method to this mode of production.
3. Finally, a comparison of predicted data with field data from a gas-cap-drive reservoir demonstrates its applicability to a field problem.

ACKNOWLEDGMENT

The assistance of J. R. Sheffield in providing the laboratory data on the dissolved-gas-drive model is gratefully acknowledged.

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DISCUSSION

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The authors have considered a very difficult problem and one in which many questions remain to be answered. The partial differential equations are very complex, and the finite-difference method of solution has not been established rigorously. It is known that finite-difference methods can fail to approximate the true solutions of partial differential equations. Thus, there is reason to doubt the validity of the method from a mathematical viewpoint.

The results of reservoir analysis may lead to large capital investments and may be a prime factor in determining the amount of oil recovery. Thus, there is an economic incentive for the results of reservoir analysis to reflect the true nature of the reservoir. Misleading results can be worse than no results at all if they lead to poor investments and the waste of important national resources. Thus, there is a considerable justification for being very critical of the validity of new methods of reservoir analysis.

The method of analysis presented by the authors involves the solution by finite-difference approximations of two simultaneous partial differential equations containing the fluid pressure and one of the fluid saturations as dependent variables, and distance and time as the independent variables. Since two dependent variables can vary with distance and time, it seems reasonable to expect that the authors would present examples where the pressure and the saturations have considerable variations with both time and distance. Also, one would expect that numerical results would be shown to approach the results obtained by analytical methods as the increments in time and distance are reduced.

In the example of complete pressure maintenance, the numerical results do not appear to converge to the analytical solution of the partial differential equations. The analytical solution is contained in the well known paper by Buckley and Leverett.[1] It is maintained by some that the Buckley-Leverett solution is not the true analytical solution and that the multivalues given by Buckley and Leverett occur because they made a mathematical transformation which is not valid in the vicinity of the saturation discontinuity. Thus, there is some disagreement concerning the Buckley-Leverett solution. For this reason, it is discussed in the following paragraphs.

The Buckley-Leverett solution can be obtained by well established mathematical methods of the type employed in Ref. 2. The partial differential equation for the gas saturation is

$$\frac{\partial}{\partial x}\left[V(t)\,F(S_g) - \frac{\alpha g(\rho_o - \rho_g)}{\mu_o}\,G(S_g)\right] = -\phi\frac{\partial S_g}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where α = the dip angle,

$V(t)$ = a known function of time,

$$F(S_g) \equiv \frac{1}{1 + \frac{\mu_g}{\mu_o}\frac{k_o}{k_g}}, \text{ and}$$

$$G(S_g) \equiv k_o/k\,F(S_g).$$

Eq. 1 is a first-order partial differential equation of a type for which there is a well known method of obtaining solutions (see, for example, Chapter 4 of Ref. 3). The analytical solution of Eq. 1 is

$$x = F'(S_g)\int_o^t \frac{V(t)}{\phi}\,dt - \frac{\alpha g(\rho_o - \rho_g)}{\phi\,\mu_o}\,G't + \psi(S_g) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where $\psi(S_g)$ is the initial gas saturation. For the case considered by the authors where the initial gas saturation is zero, $\psi(S_g)$ is zero. That Eq. 2 satisfies Eq. 1 can be verified by direct substitution. Eq. 2 yields the same multivalue saturations profiles as obtained by Buckley-Leverett. Thus, the Buckley-Leverett solution is the true analytical solution, and it contains multivalued saturations. Furthermore, the analytical solution does not contain a discontinuity.

In formulating the partial differential equation, it is assumed that the saturations are continuous

[1]References given at end of Discussion.

functions of distance. The presence of a discontinuity violates this assumption and across the discontinuity the partial differential equation breaks down. If it is assumed that a discontinuity exists at a given point and a given time, continuity considerations yield the relation for the velocity of the discontinuity. Buckley-Leverett, followed by a number of investigators, have used the procedure of first determining the analytical solution and then eliminating the presence of multivalue saturations by employing discontinuities which satisfy conditions.

The results presented in Fig. 2 do not approximate the true analytical solution but appear to approximate the combination of the analytical solution and the discontinuity which satisfies material balance. The authors should explain why the results do not approach the true analytical solution since the finite-difference relations are represented as being approximations to the partial differential equations. Furthermore, in the vicinity of the discontinuity the numerical results are a very poor approximation to the combination of the analytical solution and the discontinuity. Thus, it is not clear that the finite-difference method would yield results that approach the solid line in Fig. 2 as the increments in time and distance approach zero.

The Buckley-Leverett flow considered by the authors is a very simple flow compared to what the partial differences can represent. There may be many discontinuities present in the porous media. They can arise within the body, and they can begin at the boundaries even after the flow has been in progress for some time. Since the authors have failed to present a completely satisfactory demonstration that their method yields the correct results for a very simple case, how can one be sure the method will be applicable in a very involved case which may arise in the analysis of a natural reservoir?

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AUTHORS' REPLY TO J. C. MARTIN

Many of Martin's comments concern the example case of complete pressure maintenance. The analytical statement of this problem was formulated by Buckley and Leverett[1] as a first-order quasilinear hyperbolic differential equation. This class of differential equation often arises in simplified theories which ignore mechanisms of dissipation such as viscous stresses, heat conduction, ohmic loss or, as in the present case, capillary pressure. It is the rule, rather than the exception, that the physically meaningful solutions of these problems develop singularities (discontinuities) after a finite time.[2] Lax[3] notes that, while the classical solution of the differential equation cannot be continued after this time, it is possible to continue it in a generalized sense. This kind of generalization is dictated by the integral version of the conservation law. Lax refers to this generalized solution as a weak solution and gives a formal definition of it. It develops that these weak solutions are not determined uniquely by their initial values. Therefore, an additional principle is needed for developing a relevant subclass. In this aspect, the problem is similar to the solution of a quadratic equation. A unique answer cannot be obtained from the equation alone; additional information is necessary to reject the extraneous root.

With this background, we are prepared to consider Martin's comments. One of his objections is that, when the authors' numerical procedure is applied to the case of complete pressure maintenance, it does not approximate what he calls the "true analytical solution" of the differential equation. The solution to which he refers was given by Buckley and Leverett and contains multiple values of saturation at some locations in space. Obviously, this is not a physically significant solution of the problem and so must be rejected. The generalized solution which does occur in nature is formed by replacing the region of multiple-valued saturations by a saturation discontinuity so located that the resulting solution satisfies the law of conservation of mass. The authors selected the difference analogue of the differential equation with the objective of approximating the solution having physical significance.

The comparison of Fig. 2 gives a measure of the degree of achievement of this objective. Martin is critical of the accuracy of the approximation in the vicinity of the saturation discontinuity. He states, ". . . It is not clear that the finite-difference method would yield results that approach the solid line in Fig. 2 as the increments in time and distance approach zero". The authors take issue with this statement. For a perfect solution, the length of the region of the reservoir containing saturations between zero and the frontal saturation should be zero. The data of Fig. 2 show that when a distance increment of 1,000 ft was used in the numerical procedure, this region occupied 22 per cent of the reservoir (from 7,000 to 9,200 ft). For a distance increment of 500

ft, it occupied 14 per cent; for a distance increment of 250 ft, it occupied 8 per cent. Thus, halving the size of the distance increment while holding $\Delta t/\Delta x$ constant approximately doubles the slope of the saturation curve in the vicinity of the front. This experimental relationship of error to grid size is the same as predicted by an elemental truncation error analysis of the difference equations used. This analysis shows that the size of the error should be proportional to Δx. The authors feel that this evidence makes clear the convergence of the numerical method as the time and distance increments approach zero.

Martin's final criticism is that the testing of the procedure was not sufficient to verify its accuracy in "a very involved case which may arise in the analysis of a natural reservoir". While the testing of an experimental method is never complete, the validity of the present method has been demonstrated for a wide variety of conditions. To date, the authors have not encountered a case for which it fails. If evidence of such failure is obtained, the authors will welcome the information.

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