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# GALERKIN PROPER ORTHOGONAL DECOMPOSITION METHODS FOR A GENERAL EQUATION IN FLUID DYNAMICS*

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**Abstract.** Error estimates for Galerkin proper orthogonal decomposition (POD) methods for nonlinear parabolic systems arising in fluid dynamics are proved. For the time integration the backward Euler scheme is considered. The asymptotic estimates involve the singular values of the POD snapshot set and the grid-structure of the time discretization as well as the snapshot locations.

**Key words.** proper orthogonal decomposition, evolution problems, Navier–Stokes equations, error estimates

**AMS subject classifications.** 35K20, 65N

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## 1. Introduction.

Proper orthogonal decomposition (POD) provides a method for deriving low order models of dynamical systems. It can be thought of as a Galerkin approximation in the spatial variable, built from functions corresponding to the solution of the physical system at prespecified time instances. These are called the snapshots. Due to possible linear dependence or almost linear dependence, the snapshots themselves are not appropriate as a basis. Instead, a singular value decomposition is carried out and the leading generalized eigenfunctions are chosen as a basis, referred to as the POD basis.

POD was successfully used in a variety of fields including signal analysis and pattern recognition (see, e.g., [10]), fluid dynamics and coherent structures (see, e.g., [7, 22, 23, 24]), and more recently in control theory (see, e.g., [1, 3, 6, 15, 17, 18, 21]) and inverse problems (see [5]). Good approximation properties are reported for POD based schemes in several articles; see [8, 9, 14, 19], for example. Symmetry preserving properties of POD approximations are analyzed in [4].

As soon as one uses POD, questions concerning the quality of the approximation properties, convergence, and rate of convergence become relevant. It appears that, except for the work in [16], these issues have not been addressed. This may be due, in part, to the fact that for POD based approximation of partial differential equations one cannot rely on results clarifying the approximation properties of the POD-subspaces to elements in function spaces as, e.g., $L^p$ or $C$. Such results are an essential building block for, e.g., finite element approximations to partial differential equations. In our work we propose a strategy to describe and analyze convergence and rate of convergence approximations based on POD approximation in space and backwards Euler discretization in time for a class of nonlinear evolutionary partial differential equations including the Navier–Stokes equation in two dimensions. Due to the lack of typical function space approximation results described above, these results are not a priori in the format that one is familiar with from finite difference, finite element, or spectral approximations, for example. While we believe that the estimates that we propose reflect well the properties of POD based approximations,

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they are certainly up for discussion and future improvement. Roughly, it will be shown that the approximation error can be decomposed in a contribution that arises due to the POD approximation in space, which is measured in terms of spectral properties specifying the POD basis, and in the usual approximation error due to the backwards Euler scheme with respect to time integration.

Concerning the availability of snapshots, two situations can be considered: in the first one the snapshots are obtained by an independent numerical method and then used within a POD approach for the sake of system reduction. In the second situation the snapshots could be obtained and digitalized from actual physical phenomena.

In this article we analyze the case in which snapshots are assumed to be available for the same system as that for which the approximation properties are analyzed. The value of this analysis is to understand and justify analytically the observed high approximation quality of POD schemes. Certainly the problem of quantifying the approximation properties of POD based schemes where the snapshots are taken from processes which are possibly "nearby," but different from the system under consideration, is of considerable interest. This situation can occur, for example, in the case of control and optimal control of systems: snapshots are taken from a system at a nominal control value, and a POD model reduction and subsequent optimal control step are performed. The dynamics of the optimally controlled system then differ from the original system. The analysis of these problems can be the focus of future research.

As already mentioned, the present work is a continuation of our efforts to approximation properties of POD based schemes. We extend our earlier results of [16] in three directions. First, the results of the present paper are asymptotic results in the sense that the constants appearing in the estimates do not depend on the snapshot set. Second, we now utilize two time discretizations, one for the set of snapshots and a second one for the numerical integration. The effect of the two grids on the convergence rate is kept separate in the estimates. Third, we focus in this paper on a different class of nonlinearities, including the Navier–Stokes equations in dimension two, which were not included in [16].

The paper is organized as follows. In section 2 the nonlinear evolution problem is introduced and necessary prerequisites are given. The POD method is reviewed in section 3. Convergence of the backward Euler scheme is studied in section 4. Technical proofs are deferred to appendices.

**2. General equations in fluid dynamics.** In this section we specify the abstract nonlinear evolution problem that will be considered in this paper and present an existence and uniqueness result.

Let $V$ and $H$ be real separable Hilbert spaces and suppose that $V$ is dense in $H$ with compact embedding. By $\langle \cdot\,,\cdot\rangle_H$ we denote the inner product in $H$. The inner product in $V$ is given by a symmetric bounded, coercive, bilinear form $a : V \times V \to \mathbb{R}$:

$$\langle \varphi, \psi\rangle_V = a(\varphi, \psi) \quad \text{for all } \varphi, \psi \in V \tag{2.1}$$

with an associated norm given by $\|\cdot\|_V = \sqrt{a(\cdot\,,\cdot)}$. Since $V$ is continuously injected into $H$, there exists a constant $c_V > 0$ such that

$$\|\varphi\|_H \le c_V\ \|\varphi\|_V \quad \text{for all } \varphi \in V. \tag{2.2}$$

We associate with $a$ the linear operator $A$:

$$\langle A\varphi, \psi\rangle_{V',V} = a(\varphi, \psi) \quad \text{for all } \varphi, \psi \in V,$$

where $\langle \cdot\,,\cdot\rangle_{V',V}$ denotes the duality pairing between $V$ and its dual. Then $A$ is an isomorphism from $V$ onto $V'$. Alternatively, $A$ can be considered as a linear unbounded self-adjoint operator in $H$ with domain

$$D(A) = \{\varphi \in V : A\varphi \in H\}.$$

By identifying $H$ and its dual $H'$ it follows that

$$D(A) \hookrightarrow V \hookrightarrow H = H' \hookrightarrow V',$$

each embedding being continuous and dense, when $D(A)$ is endowed with the graph norm of $A$.

We introduce the continuous operator $R : V \to V'$, which maps $D(A)$ into $H$ and satisfies

$$\begin{aligned} \|R\varphi\|_H &\le c_R\, \|\varphi\|_V^{1-\delta_1} \|A\varphi\|_H^{\delta_1} \quad \text{for all } \varphi \in D(A), \\ |\langle R\varphi,\varphi\rangle_{V',V}| &\le c_R\, \|\varphi\|_V^{1+\delta_2} \|\varphi\|_H^{1-\delta_2} \quad \text{for all } \varphi \in V \end{aligned} \tag{2.3}$$

for a constant $c_R > 0$ and for $\delta_1, \delta_2 \in [0, 1)$. We also assume that $A + R$ is coercive on $V$; i.e., there exists a constant $\eta > 0$ such that

$$a(\varphi,\varphi) + \langle R\varphi,\varphi\rangle_{V',V} \ge \eta\, \|\varphi\|_V^2 \quad \text{for all } \varphi \in V. \tag{2.4}$$

Moreover, let $B : V \times V \to V'$ be a bilinear continuous operator mapping $D(A) \times D(A)$ into $H$ such that there exist constants $c_B > 0$ and $\delta_3, \delta_4, \delta_5 \in [0, 1)$ satisfying

$$\begin{aligned} \langle B(\varphi,\psi),\psi\rangle_{V',V} &= 0, \\ \left|\langle B(\varphi,\psi),\phi\rangle_{V',V}\right| &\le c_B\, \|\varphi\|_H^{\delta_3}\|\varphi\|_V^{1-\delta_3}\|\psi\|_V\|\phi\|_V^{\delta_3}\|\phi\|_H^{1-\delta_3}, \\ \|B(\varphi,\chi)\|_H + \|B(\chi,\varphi)\|_H &\le c_B\, \|\varphi\|_V\|\chi\|_V^{1-\delta_4}\|A\chi\|_H^{\delta_4}, \\ \|B(\varphi,\chi)\|_H &\le c_B\, \|\varphi\|_H^{\delta_5}\|\varphi\|_V^{1-\delta_5}\|\chi\|_V^{1-\delta_5}\|A\chi\|_H^{\delta_5} \end{aligned} \tag{2.5}$$

for all $\varphi, \psi, \phi \in V$, for all $\chi \in D(A)$. To simplify the notation we set $B(\varphi) = B(\varphi,\varphi)$ for $\varphi \in V$.

For given $f \in L^2(0, T; H)$ and $y_0 \in V$ we consider the nonlinear evolution problem

$$\frac{d}{dt}\, \langle y(t),\varphi\rangle_H + a(y(t),\varphi) + \left\langle B(y(t)) + Ry(t),\varphi\right\rangle_{V',V} = \langle f(t),\varphi\rangle_H \tag{2.6a}$$

for all $\varphi \in V$ and $t \in (0, T]$ a.e. and

$$y(0) = y_0 \quad \text{in } H. \tag{2.6b}$$

The following theorem guarantees the existence of a unique solution to (2.6).

THEOREM 2.1. *Assume that* (2.3) *and* (2.5) *hold. Then for every* $f \in L^2(0, T; H)$ *and* $y_0 \in V$ *there exists a unique solution of* (2.6) *satisfying*

$$y \in C([0, T]; V) \cap L^2(0, T; D(A)) \cap H^1(0, T; H). \tag{2.7}$$

*Proof.* The proof is analogous to that of Theorem 2.1 in [25, p. 111], where the case with time-independent $f$ was treated. □

Condition (2.4) will not be needed before section 4. Let us present an example for the nonlinear evolution system (2.6).

EXAMPLE 2.2. *Let $\Omega$ denote a bounded domain in $\mathbb{R}^2$ with boundary $\Gamma$ and let $T > 0$. The two-dimensional Navier–Stokes equations are given by*

$$\varrho\,\big(u_t + (u\cdot\nabla)u\big) - \nu\Delta u + \nabla p = f \quad \text{in } Q = (0,T)\times\Omega, \tag{2.8a}$$

$$\operatorname{div}\, u = 0 \quad \text{in } Q, \tag{2.8b}$$

*where $\varrho > 0$ is the density of the fluid, $\nu > 0$ is the kinematic viscosity, $f$ represents volume forces, and*

$$(u\cdot\nabla)u = \Big(u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2}, u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2}\Big)^{\mathsf{T}}.$$

*The unknowns are the velocity field $u = (u_1, u_2)$ and the pressure $p$. Together with* (2.8) *we consider nonslip boundary conditions*

$$u = u_d \quad \text{on } \Sigma = (0,T)\times\Gamma \tag{2.8c}$$

*and the initial condition*

$$u(0,\cdot) = u_0 \quad \text{in } \Omega. \tag{2.8d}$$

*In* [25, pp. 104–107, 116–117] *it was proved that* (2.8) *can be written in the form* (2.6).

Next we recall Young's inequality, which will frequently be used in our work. For a proof we refer to [2, p. 28], for instance.

LEMMA 2.3 (Young's inequality). *For all $a, b, \varepsilon > 0$ and for all $p \in (1,\infty)$ we have*

$$ab \le \frac{\varepsilon a^p}{p} + \frac{b^q}{q\varepsilon^{q/p}},$$

*where $q = p/(p-1)$.*

## 3. The POD method.

This section is devoted to a discussion of the POD method for the nonlinear evolution problem (2.6). Throughout we denote by $y$ the unique solution to (2.6) satisfying (2.7). Moreover, we suppose that $f \in C([0,T];H)$.

### 3.1. Computation of the POD basis.

For given $n \in \mathbb{N}$ let

$$0 = t_0 < t_2 < \cdots < t_n \le T$$

denote a grid in the interval $[0,T]$ and set $\delta t_j = t_j - t_{j-1}$, $j = 1,\dots,n$. Define

$$\Delta t = \max\ (\delta t_1,\dots,\delta t_n) \quad \text{and} \quad \delta t = \min\ (\delta t_1,\dots,\delta t_n).$$

Suppose that the snapshots $y(t_j)$ of (2.6) at the given time instances $t_j$, $j = 0,\dots,n$, are known. We set

$$\mathcal{V} = \text{span}\ \{y(t_0),\dots,y(t_n)\}$$

and refer to $\mathcal{V}$ as the ensemble consisting of the snapshots $\{y(t_j)\}_{j=0}^n$, at least one of which is assumed to be nonzero. Notice that $\mathcal{V} \subset V$ by construction. Throughout the remainder of this section we let $X$ denote either the space $V$ or $H$.

Let $\{\psi_i\}_{i=1}^d$ denote an orthonormal basis for $\mathcal{V}$ with $d = \dim \mathcal{V}$. Then each member of the ensemble can be expressed as

$$y(t_j) = \sum_{i=1}^{d} \langle y(t_j), \psi_i \rangle_X \psi_i \quad \text{for } j = 0, \dots, n. \tag{3.1}$$

The method of POD consists of choosing an orthonormal basis such that for every $\ell \in \{1, \dots, d\}$ the mean square error between the elements $y(t_j)$, $0 \le j \le n$, and the corresponding $\ell$th partial sum of (3.1) is minimized on average:

$$\begin{gathered}\min_{\{\psi_i\}_{i=1}^{\ell}} \sum_{j=0}^{n} \alpha_j \left\| y(t_j) - \sum_{i=1}^{\ell} \langle y(t_j), \psi_i \rangle_X \psi_i \right\|_X^2 \\ \text{subject to } \langle \psi_i, \psi_j \rangle_X = \delta_{ij} \quad \text{for } 1 \le i \le \ell, 1 \le j \le i.\end{gathered} \tag{3.2}$$

Here $\{\alpha_j\}_{j=0}^n$ are positive weights, which for our purposes are chosen to be

$$\alpha_0 = \frac{\delta t_1}{2}, \quad \alpha_j = \frac{\delta t_j + \delta t_{j+1}}{2} \text{ for } j = 1, \dots, n-1, \quad \text{and} \quad \alpha_n = \frac{\delta t_n}{2}.$$

A solution $\{\psi_i\}_{i=1}^{\ell}$ to (3.2) is called a POD basis of rank $\ell$. The subspace spanned by the first $\ell$ POD basis functions is denoted by $V^\ell$.

*Remark* 3.1. Note that

$$\mathcal{I}_n(y) = \sum_{j=0}^{n} \alpha_j \left\| y(t_j) - \sum_{i=1}^{\ell} \langle y(t_j), \psi_i \rangle_X \psi_i \right\|_X^2$$

is the trapezoidal approximation for the integral

$$\mathcal{I}(y) = \int_0^T \left\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \psi_i \rangle_X \psi_i \right\|_X^2 dt.$$

For all $y \in C([0,T]; X)$ it follows that $\lim_{n\to\infty} \mathcal{I}_n(y) = \mathcal{I}(y)$.

The solution of (3.2) is characterized by the necessary optimality condition. For that purpose we endow $\mathbb{R}^{n+1}$ with the weighted inner product

$$\langle v, w \rangle_{\mathbb{R}^{n+1}} = \sum_{j=0}^{n} \alpha_j v_j w_j \quad \text{for } v = (v_0, \dots, v_n)^\mathsf{T},\ w = (w_0, \dots, w_n)^\mathsf{T} \in \mathbb{R}^{n+1}.$$

Let us introduce the bounded linear operator $\mathcal{Y}_n : \mathbb{R}^{n+1} \to X$ by

$$\mathcal{Y}_n v = \sum_{j=0}^{n} \alpha_j v_j y(t_j) \quad \text{for } v \in \mathbb{R}^{n+1}.$$

Then the adjoint $\mathcal{Y}_n^* : X \to \mathbb{R}^{n+1}$ is given by

$$\mathcal{Y}_n^* z = \left(\langle z, y(t_0) \rangle_X, \dots, \langle z, y(t_n) \rangle_X\right)^\mathsf{T} \quad \text{for } z \in X.$$

It follows that $\mathcal{R}_n = \mathcal{Y}_n \mathcal{Y}_n^* \in \mathcal{L}(X)$ and $\mathcal{K}_n = \mathcal{Y}_n^* \mathcal{Y}_n \in \mathbb{R}^{(n+1)\times(n+1)}$ are given by

$$\mathcal{R}_n z = \sum_{j=0}^{n} \alpha_j \langle z, y(t_j) \rangle_X y(t_j) \text{ for } z \in X \quad \text{and} \quad \left(\mathcal{K}_n\right)_{ij} = \langle y(t_j), y(t_i) \rangle_X,$$

respectively. Here $\mathcal{L}(X)$ denotes the Banach space of all bounded linear operators on $X$.

Using a Lagrangian framework we derive the following optimality conditions for the optimization problem (3.2):

$$\mathcal{R}_n\psi = \lambda\psi; \tag{3.3}$$

compare, e.g., [7, 26]. Note that $\mathcal{R}_n$ is a bounded, self-adjoint and nonnegative operator. Moreover, since the image of $\mathcal{R}_n$ has finite dimensions, $\mathcal{R}_n$ is also compact. By Hilbert–Schmidt theory (see, e.g., [20, p. 203]) there exist an orthonormal basis $\{\psi_i\}_{i\in\mathbb{N}}$ for $X$ and a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of nonnegative real numbers so that

$$\mathcal{R}_n\psi_i = \lambda_i\psi_i, \quad \lambda_1 \geq \cdots \geq \lambda_d > 0, \quad \text{and } \lambda_i = 0 \text{ for } i > d. \tag{3.4}$$

Moreover, $\mathcal{V} = \operatorname{span}\,\{\psi_i\}_{i=1}^d$.

Note that $\{\lambda_i\}_{i\in\mathbb{N}}$ as well as $\{\psi_i\}_{i\in\mathbb{N}}$ depend on $n$. Contents permitting the notation of this dependence are dropped.

*Remark* 3.2. Setting

$$v_i = \frac{1}{\sqrt{\lambda_i}}\,\mathcal{Y}_n^*\psi_i \quad \text{for } i = 1, \ldots, d$$

we find $\mathcal{K}_n v_i = \lambda_i v_i$ and $\langle v_i, v_j\rangle_{\mathbb{R}^{n+1}} = \delta_{ij}$ for $1 \leq i, j \leq d$. Thus, $\{v_i\}_{i=1}^d$ is an orthonormal basis of eigenvectors of $\mathcal{K}_n$ for the image of $\mathcal{K}_n$. Conversely, if $\{v_i\}_{i=1}^d$ is a given orthonormal basis for the image of $\mathcal{K}_n$, then it follows that the first $d$ eigenfunctions of $\mathcal{R}_n$ can be determined by

$$\psi_i = \frac{1}{\sqrt{\lambda_i}}\,\mathcal{Y}_n v_i \quad \text{for } i = 1, \ldots, d.$$

The sequence $\{\psi_i\}_{i=1}^\ell$ solves the optimization problem (3.2). This fact as well as the error formula below were proved in [7, 26], for example.

Proposition 3.3. *Let $\lambda_1 \geq \cdots \geq \lambda_d > 0$ denote the positive eigenvalues of $\mathcal{R}^n$ with the associated eigenvectors $\psi_1, \ldots, \psi_d \in X$. Then, $\{\psi_i^n\}_{i=1}^\ell$ is a POD basis of rank $\ell \leq d$, and we have the error formula*

$$\sum_{j=0}^n \alpha_j \left\| y(t_j) - \sum_{i=1}^\ell \langle y(t_j), \psi_i\rangle_X \psi_i \right\|_X^2 = \sum_{i=\ell+1}^d \lambda_i. \tag{3.5}$$

**3.2. Perturbation analysis for $\sum_{i=\ell+1}^d \lambda_i$.** The eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ depend on the time instances $\{t_j\}_{j=0}^n$. Next we investigate $\sum_{i=\ell+1}^d \lambda_i$ as $\Delta t$ tends to zero, i.e., $n \to \infty$. Let us define the bounded linear operator $\mathcal{Y} : L^2(0, T; \mathbb{R}) \to X$ by

$$\mathcal{Y}\varphi = \int_0^T \varphi(t)y(t)\,dt \quad \text{for } \varphi \in L^2(0, T; \mathbb{R}).$$

The adjoint $\mathcal{Y}^* : X \to L^2(0, T; \mathbb{R})$ is given by

$$\big(\mathcal{Y}^* z\big)(t) = \big\langle z, y(t)\big\rangle_X \quad \text{for } z \in X.$$

For $\mathcal{R} = \mathcal{Y}\mathcal{Y}^* \in \mathcal{L}(X)$ we find

$$\mathcal{R}z = \int_0^T \big\langle z\,, y(t)\big\rangle_X y(t)\,dt \quad \text{for } z \in X. \tag{3.6}$$

Notice that $\mathcal{R}_n\varphi$ is the trapezoidal approximation for the integral $\mathcal{R}\varphi$. If $y_t \in L^2(0,T;X)$, then we obtain

$$\lim_{\Delta t\to\infty} \|\mathcal{R}_n - \mathcal{R}\|_{\mathcal{L}(X)} = 0. \tag{3.7}$$

Let us mention that as far as the following analysis is concerned any other choice of positive weights $\alpha_j$ is possible provided that (3.7) holds.

We proceed to investigate the relationship between $\mathcal{R}_n$ and $\mathcal{R}$. Notice that $\mathcal{R}$ is self-adjoint and nonnegative. Since $y \in C([0,T];V)$, the Kolmogorov compactness criterion in $L^2(0,T;\mathbb{R})$ implies that $\mathcal{Y}^* : X \to L^2(0,T;X)$ is compact. Boundedness of $\mathcal{Y}$ implies that $\mathcal{R}$ is a compact operator as well. From the Hilbert–Schmidt theorem it follows that there exists a complete orthonormal basis $\{\psi_i^\infty\}_{i\in\mathbb{N}}$ for $X$ and a sequence $\{\lambda_i^\infty\}_{i\in\mathbb{N}}$ of nonnegative real numbers so that

$$\mathcal{R}\psi_i^\infty = \lambda_i^\infty\psi_i^\infty, \quad \lambda_1^\infty \geq \lambda_2^\infty \geq \cdots, \quad \text{and } \lambda_i^\infty \to 0 \text{ as } i\to\infty. \tag{3.8}$$

*Remark* 3.4. Analogous to Remark 3.2 we set

$$v_i^\infty = \frac{1}{\sqrt{\lambda_i^\infty}}\,\mathcal{Y}^*\psi_i^\infty = \frac{1}{\sqrt{\lambda_i^\infty}}\,\langle\psi_i^\infty, y(t)\rangle_X\,dt \quad \text{for } i \in \{j\in\mathbb{N} : \lambda_j^\infty > 0\}.$$

Let $\mathcal{K} = \mathcal{Y}^*\mathcal{Y} \in \mathcal{L}(L^2(0,T;\mathbb{R}))$ be given by

$$\mathcal{K}\varphi = \int_0^T \langle y(s), y(t)\rangle_X\varphi(s)\,ds \quad \text{for } \varphi \in L^2(0,T;\mathbb{R}).$$

It follows that

$$\begin{aligned}
\big(\mathcal{K}v_i^\infty\big)(t) &= \int_0^T \langle y(s), y(t)\rangle_X v_i^\infty(s)\,ds \\
&= \frac{1}{\sqrt{\lambda_i^\infty}}\left\langle \int_0^T \langle\psi_i^\infty, y(s)\rangle_X y(s)\,ds, y(t)\right\rangle_X = \frac{1}{\sqrt{\lambda_i^\infty}}\,\langle\mathcal{R}\psi_i^\infty, y(t)\rangle_X \\
&= \frac{1}{\sqrt{\lambda_i^\infty}}\,\langle\mathcal{R}\psi_i^\infty, y(t)\rangle_X = \lambda_i^\infty v_i^\infty(t)
\end{aligned}$$

and, consequently, the $v_i^\infty$'s are the eigenfunctions of $\mathcal{K}$ for $i\in\mathbb{N}$ with $\lambda_i^\infty > 0$.

The spectra of $\mathcal{R}$ and $\mathcal{R}_n$ are pure point spectra except for possibly 0. Each nonzero eigenvalue of $\mathcal{R}$ has finite multiplicity and 0 is the only possible accumulation point of the spectrum of $\mathcal{R}$; see [13, p. 185]. These facts together with (3.7) will allow us to draw important conclusions on the term $\sum_{i=\ell+1}^d \lambda_i^n$ in our estimates below. Henceforth we denote by $\{\lambda_i^n\}_{i=1}^{d(n)}$ the positive eigenvalues of $\mathcal{R}_n$ with associated eigenfunctions $\{\psi_i^n\}_{i=1}^{d(n)}$. Similarly $\{\lambda_i^\infty\}_{i\in\mathbb{N}}$ denotes the positive eigenvalues of $\mathcal{R}$ with associated eigenfunctions $\{\psi_i^\infty\}_{i\in\mathbb{N}}$. In each case the eigenvalues are considered according to their multiplicity. Let us note that

$$\int_0^T \|y(t)\|_X^2\,dt = \sum_{i=1}^\infty \lambda_i^\infty. \tag{3.9}$$

In fact,

$$\mathcal{R}\psi_i^\infty = \int_0^T \langle\psi_i^\infty, y(t)\rangle_X y(t)\,dt \quad \text{for every } i\in\mathbb{N}.$$

Taking the inner product with $\psi_i^\infty$ and summing over $i$ we arrive at

$$\sum_{i=1}^{\infty} \int_0^T \left| \langle \psi_i^\infty, y(t) \rangle_X \right|^2 dt = \sum_{i=1}^{\infty} \langle \mathcal{R}\psi_i^\infty, \psi_i^\infty \rangle_X = \sum_{i=1}^{\infty} \lambda_i^\infty.$$

Expanding $y(t) \in X$ in terms of $\{\psi_i^\infty\}_{i\in\mathbb{N}}$ we have

$$y(t) = \sum_{i=1}^{\infty} \langle \psi_i^\infty, y(t) \rangle_X \psi_i^\infty$$

and hence

$$\int_0^T \|y(t)\|_X^2 \, dt = \sum_{i=1}^{\infty} \int_0^T \left| \langle \psi_i^\infty, y(t) \rangle_X \right|^2 dt = \sum_{i=1}^{\infty} \lambda_i^\infty,$$

which is (3.9). From Proposition 3.3 and (3.4) we obtain

$$\sum_{j=0}^{n} \alpha_j \, \|y(t_j)\|_X^2 = \sum_{i=1}^{\infty} \lambda_i^n \quad \text{for every } n \in \mathbb{N}. \tag{3.10}$$

For convenience we do not indicate the dependence of $\alpha_j$ on $n$. Note that for $y \in C([0,T], X)$

$$\sum_{j=0}^{n} \alpha_j \, \|y(t_j)\|_X^2 \to \int_0^T \|y(t)\|_X^2 \, dt \quad \text{as } \Delta t \to 0.$$

Combining this fact with (3.9) and (3.10) we find

$$\sum_{i=1}^{\infty} \lambda_i^n \to \sum_{i=1}^{\infty} \lambda_i^\infty \quad \text{as } \Delta t \to 0. \tag{3.11}$$

Now choose and fix

$$\ell \quad \text{such that} \quad \lambda_\ell^\infty \neq \lambda_{\ell+1}^\infty. \tag{3.12}$$

Then by spectral analysis of compact operators [13, pp. 212–214] and (3.7) it follows that

$$\lambda_i^n \to \lambda_i^\infty \quad \text{for } 1 \leq i \leq \ell \text{ as } \Delta t \to 0. \tag{3.13}$$

Combining (3.11) and (3.13) there exists $\overline{\Delta t} > 0$ such that

$$\sum_{i=\ell+1}^{\infty} \lambda_i^n \leq 2 \sum_{i=\ell+1}^{\infty} \lambda_i^\infty \quad \text{for all } \Delta t \leq \overline{\Delta t} \tag{3.14}$$

if $\sum_{i=\ell+1}^{\infty} \lambda_i^\infty \neq 0$. Moreover, for $\ell$ as above, $\overline{\Delta t}$ can also be chosen such that

$$\sum_{i=\ell+1}^{d(n)} \left| \langle \psi_i^n, y_0 \rangle_X \right|^2 \leq 2 \sum_{i=\ell+1}^{\infty} \left| \langle \psi_i^\infty, y_0 \rangle_X \right|^2 \quad \text{for all } \Delta \leq \overline{\Delta t}, \tag{3.15}$$

provided that $\sum_{i=\ell+1}^{\infty} |\langle y_0, \psi_i^\infty \rangle_X|^2 \neq 0$. To verify (3.15) let us first note that $y_0 = y(0) \in \overline{\text{range}\,\mathcal{R}} = (\ker\,\mathcal{R})^\perp$. In fact, if $v \in \ker\,\mathcal{R}$, then $t \mapsto \langle v, y(t)\rangle_X$ is the zero function in $L^2(0,T;X)$. Since by assumption $y \in C([0,T];X)$ it follows that $\langle v, y(0)\rangle_X = 0$. But $v \in \ker\,\mathcal{R}$ was chosen arbitrarily and hence $y_0 \in (\ker\,\mathcal{R})^\perp$. As a consequence we have

$$\|y_0\|_X^2 = \sum_{i=1}^{\infty} \left|\langle y_0, \psi_i^\infty\rangle_X\right|^2. \tag{3.16}$$

Since $t_0 = 0$ holds, we have $y_0 \in \mathcal{V}^{(n)}$ for every $n$ and

$$\|y_0\|_X^2 = \sum_{i=1}^{d(n)} \left|\langle y_0, \psi_i^n\rangle_X\right|^2. \tag{3.17}$$

Therefore, for $\ell < d(n)$ by (3.16) and (3.17)

$$\begin{aligned}
\sum_{i=\ell+1}^{d(n)} \left|\langle y_0, \psi_i^n\rangle_X\right|^2 &= \sum_{i=1}^{d(n)} \left|\langle y_0, \psi_i^n\rangle_X\right|^2 - \sum_{i=1}^{\ell} \left|\langle y_0, \psi_i^n\rangle_X\right|^2 + \sum_{i=1}^{\ell} \left|\langle y_0, \psi_i^\infty\rangle_X\right|^2 \\
&\quad + \sum_{i=\ell+1}^{\infty} \left|\langle y_0, \psi_i^\infty\rangle_X\right|^2 - \sum_{i=1}^{\infty} \left|\langle y_0, \psi_i^\infty\rangle_X\right|^2 \\
&= \sum_{i=1}^{\ell} \left( \left|\langle y_0, \psi_i^\infty\rangle_X\right|^2 - \left|\langle y_0, \psi_i^n\rangle_X\right|^2 \right) + \sum_{i=\ell+1}^{\infty} \left|\langle y_0, \psi_i^\infty\rangle_X\right|^2.
\end{aligned}$$

As a consequence of (3.7) and (3.12) we have $\lim_{\Delta t \to 0} \psi_i^n = \psi_i^\infty$ for $i = 1, \ldots, \ell$ and hence (3.15) follows.

## 4. Backward Euler Galerkin method.

This section is devoted to error estimates for the Galerkin POD method applied to (2.6) combined with the backward Euler method for the time integration. Throughout, (2.3)–(2.5) are assumed to hold.

### 4.1. Case $X = V$.

Let us choose $X = V$ in the context of section 3. To study the backward Euler Galerkin POD method for (2.6), we introduce the Ritz projection $P^\ell : V \to V^\ell$, $1 \le \ell \le d$, by

$$a(P^\ell \varphi, \psi) = a(\varphi, \psi) \quad \text{for all } \psi \in V^\ell, \tag{4.1}$$

where $\varphi \in V$. Since the Hilbert space $V$ is endowed with the inner product (2.1), $P^\ell$ is the orthogonal projection of $V$ on $V^\ell$. In particular, this implies that $P^\ell$ has norm one.

LEMMA 4.1. *For every $\ell \in \{1, \ldots, d\}$ the projection operators $P^\ell$ satisfy*

$$\sum_{j=0}^{n} \alpha_j \, \|y(t_j) - P^\ell y(t_j)\|_V^2 \le \sum_{i=\ell+1}^{d} \lambda_i, \tag{4.2}$$

*where $\lambda_i$ denote the eigenvalues introduced in* (3.4).

*Proof.* For arbitrary $\varphi \in V$ we deduce from (2.1) and (4.1) that

$$\|\varphi - P^\ell \varphi\|_V^2 = a(\varphi - P^\ell\varphi, \varphi - P^\ell\varphi) = a(\varphi - P^\ell\varphi, \varphi - \psi) \le \|\varphi - P^\ell\varphi\|_V \|\varphi - \psi\|_V$$

for all $\psi \in V^\ell$ so that

$$\|\varphi - P^\ell\varphi\|_V \le \|\varphi - \psi\|_V \quad \text{for all } \psi \in V^\ell. \tag{4.3}$$

Using (4.3) and (3.5) we obtain

$$\sum_{j=0}^{n} \alpha_j \, \|y(t_j) - P^\ell y(t_j)\|_V^2 \le \sum_{j=0}^{n} \alpha_j \, \left\| y(t_j) - \sum_{i=1}^{\ell} a(y(t_j),\psi_i)\psi_i \right\|_V^2 = \sum_{i=\ell+1}^{d} \lambda_i,$$

which is estimate (4.2). ∎

The Galerkin POD method for (2.6) is described next. For $m \in \mathbb{N}$ we introduce the time grid

$$0 = \tau_0 < \tau_1 < \cdots < \tau_m = T, \qquad \delta\tau_j = \tau_j - \tau_{j-1} \text{ for } j = 1,\dots,m$$

and set

$$\delta\tau = \min\{\delta\tau_j : 1 \le j \le m\} \quad \text{and} \quad \Delta\tau = \max\{\delta\tau_j : 1 \le j \le m\}.$$

Throughout we assume that $\Delta\tau/\delta\tau$ is bounded uniformly with respect to $m$. To relate the two time discretizations $\{t_j\}_{j=0}^n$ and $\{\tau_j\}_{j=0}^m$ we set for every $\tau_k$, $0 \le k \le m$, an associated index $\bar{k} = \text{argmin}\,\{|\tau_k - t_j| : 0 \le j \le n\}$ and define $\sigma_n \in \{1,\dots,n\}$ as the maximum of the occurrence of the same value $t_{\bar{k}}$ as $k$ ranges over $0 \le k \le m$.

The problem consists of finding a sequence $\{Y_k\}_{k=0}^m$ in $V^\ell$ satisfying

$$\langle Y_0, \psi\rangle_H = \langle y_0, \psi\rangle_H \quad \text{for all } \psi \in V^\ell \tag{4.4a}$$

and

$$\langle \overline{\partial}_\tau Y_k, \psi\rangle_H + a(Y_k,\psi) + \langle B(Y_k) + RY_k, \psi\rangle_{V',V} = \langle f(\tau_k), \psi\rangle_H \tag{4.4b}$$

for all $\psi \in V^\ell$ and $k = 1,\dots,m$, where we have set

$$\overline{\partial}_\tau Y_k = \frac{Y_k - Y_{k-1}}{\delta\tau_k}.$$

In the following theorem, existence and a priori estimates for the solution $\{Y_k\}_{k=0}^m$ are established. For the proof we refer to Appendix A.

Theorem 4.2. *For every $k = 1,\dots,m$ there exists at least one solution $Y_k$ of* (4.4b). *If $\Delta\tau$ is sufficiently small, the sequence $\{Y_k\}_{k=1}^m$ is uniquely determined. Moreover, the following estimates are satisfied:*

$$\|Y_k\|_H^2 \le (1+\gamma\delta\tau)e^{-\gamma k\delta\tau} \, \|y_0\|_H^2 + \frac{1 - e^{-\gamma k \Delta\tau}}{\gamma} \, \|f\|_{C([0,T];H)}^2 \tag{4.5a}$$

*for $k = 0,\dots,m$, where $\gamma = \eta/c_V^2$, $c_V, \eta$ were introduced in* (2.2) *and* (2.4), *respectively, and*

$$\sum_{k=1}^{m} \|Y_k - Y_{k-1}\|_H^2 + \eta \sum_{k=1}^{m} \delta\tau_k \, \|Y_k\|_V^2 \le \|y_0\|_H^2 + \frac{T}{\gamma} \, \|f\|_{C([0,T];H)}^2. \tag{4.5b}$$

Our next goal is to derive an error estimate for the expression

$$\sum_{k=0}^{m} \beta_k \, \|Y_k - y(\tau_k)\|_H^2,$$

where $y(\tau_k)$ is the solution of (2.6) at the time instances $t = \tau_k$, $k = 1, \dots, m$, and the positive weights $\beta_j$ are given by

$$\beta_0 = \frac{\delta\tau_1}{2}, \quad \beta_j = \frac{\delta\tau_j + \delta\tau_{j+1}}{2} \text{ for } j = 1, \dots, m-1, \quad \text{and} \quad \beta_m = \frac{\delta\tau_m}{2}. \tag{4.6}$$

We make use of the following assumptions:

(A1) $y_t \in L^2(0,T;V)$ and $y_{tt} \in L^2(0,T;H)$.

(A2) There exists a normed linear space $W$ continuously embedded in $V$ and a constant $c_a > 0$ such that $y \in C([0,T];W)$ and

$$a(\varphi, \psi) \le c_a \, \|\varphi\|_H \|\psi\|_W \quad \text{for all } \varphi \in V \text{ and } \psi \in W. \tag{4.7}$$

(A3) $y \in W^{2,2}(0,T;V)$.

EXAMPLE 4.3. *For $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, with $\Omega$ a bounded domain in $\mathbb{R}^l$ and*

$$a(\varphi, \psi) = \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx \quad \textit{for all } \varphi, \psi \in H_0^1(\Omega),$$

*choosing $W = H^2(\Omega) \cap H_0^1(\Omega)$ implies $a(\varphi,\psi) \le \|\varphi\|_W \|\psi\|_H$ for all $\varphi \in W$, $\psi \in V$, and (4.7) holds with $c_a = 1$.*

*Remark* 4.4. Note that (A2) implies the existence of a constant $c_P > 0$ depending on $\ell$ and $\lambda_\ell$ such that

$$\|P^\ell\|_{\mathcal{L}(H)} \le c_P \quad \text{for all } 1 \le \ell \le d. \tag{4.8}$$

In fact, using (2.2) and (4.7) we find

$$\|P^\ell \varphi\|_H \le \sum_{i=1}^{\ell} |a(\psi_i, \varphi)| \, \|\psi_i\|_H \le c_a c_V \, \|\varphi\|_H \sum_{i=1}^{\ell} \|\psi_i\|_W.$$

Now we estimate the term $\|\psi_i\|_W$ for $i = 1, \dots, \ell$. Using $\sum_{j=0}^n \alpha_j = T$ and (3.4) we have

$$\begin{aligned}\|\psi_i\|_W &= \frac{1}{\lambda_i} \, \|\mathcal{R}_n \psi_i\|_W \le \frac{1}{\lambda_\ell} \sum_{j=0}^{n} \alpha_j \big|a(\psi_i, y(t_j))\big| \|y(t_j)\|_W \\ &\le \frac{1}{\lambda_\ell} \, \|y\|_{C([0,T];W)} \sum_{j=0}^{n} \alpha_j \|y(t_j)\|_V \le \frac{T}{\lambda_\ell} \, \|y\|_{C([0,T];W)} \|y\|_{C([0,T];V)}.\end{aligned}$$

This bound implies

$$\|P^\ell \varphi\|_H \le \frac{c\ell}{\lambda_\ell} \, \|\varphi\|_H \tag{4.9}$$

with $c = c_a c_V T \, \|y\|_{C([0,T];W)} \|y\|_{C([0,T];V)}$.

Throughout we shall use the decomposition

$$Y_k - y(\tau_k) = Y_k - P^\ell y(\tau_k) + P^\ell y(\tau_k) - y(\tau_k) = \vartheta_k + \varrho_k, \tag{4.10}$$

where $\vartheta_k = Y_k - P^\ell y(\tau_k)$ and $\varrho_k = P^\ell y(\tau_k) - y(\tau_k)$. The following lemma establishes an error estimate for $\vartheta_k$. For the proof we refer to Appendix B.

LEMMA 4.5. *Assume that $\Delta\tau$ is sufficiently small and that* (A1), (A2) *hold. Then there exist constants $C_1, C_2 > 0$ independent of the grids $\{t_j\}_{j=0}^n$ and $\{\tau_j\}_{j=0}^m$ such that*

$$\begin{aligned} \|\vartheta_k\|_H^2 \quad \le \quad C_1 e^{C_2 k\delta\tau} \Bigg( & \|y_0 - P^\ell y_0\|_H^2 + \frac{\sigma_n}{\delta t}\Big(\frac{1}{\delta\tau} + \Delta\tau\Big) \sum_{i=\ell+1}^d \lambda_i \\ & + \sigma_n \Delta\tau \big(1 + c_P^2\big)(\Delta\tau + \Delta t) \|y_{tt}\|^2_{L^2(0,t_{\bar{k}+1};H)} \\ & + \sigma_n \Delta\tau\Delta t \; \|y_t\|^2_{L^2(0,t_{\bar{k}+1};V)} \Bigg) \end{aligned} \tag{4.11}$$

*for each $1 \le k \le m$.*

*Remark* 4.6. Since $y_0 \in \mathcal{V}$ we infer from (2.2) that

$$\|y_0 - P^\ell y_0\|_H^2 \le c_V^2 \sum_{i=\ell+1}^d \big|\langle \psi_i, y_0\rangle_V\big|^2.$$

We turn to the term $\|\varrho_k\|_H^2$. Observe that

$$\begin{aligned} &\|\varrho_k\|_H^2 = \|P^\ell y(\tau_k) - y(\tau_k)\|_H^2 \\ &\le 3\left(\|P^\ell y(\tau_k) - P^\ell y(t_{\bar{k}})\|_H^2 + \|P^\ell y(t_{\bar{k}}) - y(t_{\bar{k}})\|_H^2 + \|y(t_{\bar{k}}) - y(\tau_k)\|_H^2\right) \\ &\le 3\big(1 + c_P^2\big)\; \|y(t_{\bar{k}}) - y(\tau_k)\|_H^2 + 3\; \|P^\ell y(t_{\bar{k}}) - y(t_{\bar{k}})\|_H^2 \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \|y(t_{\bar{k}}) - y(\tau_k)\|_H^2 \quad & \le \quad \left(\int_{t_{\bar{k}-1}}^{t_{\bar{k}+1}} \|y_t(s)\|_H \, ds\right)^2 \\ & \le \quad (\delta t_{\bar{k}} + \delta t_{\bar{k}+1})\; \|y_t\|^2_{L^2(t_{\bar{k}-1},t_{\bar{k}+1};H)}, \end{aligned} \tag{4.13}$$

where we set $t_{m+1} = T$ whenever $\bar{k} = m$. Using (4.13) and $\beta_k \le \Delta\tau$ we obtain

$$\sum_{k=0}^m \beta_k \; \|y(t_{\bar{k}}) - y(\tau_k)\|_H^2 \le 2\sigma_n \Delta\tau\Delta t \; \|y_t\|^2_{L^2(0,T;H)}.$$

From (2.2), $\beta_k \le \Delta\tau$, $\alpha_j \ge \delta t/2$, and Lemma 4.1 we infer that

$$\begin{aligned} \sum_{k=0}^m \beta_k \; \|P^\ell y(t_{\bar{k}}) - y(t_{\bar{k}})\|_H^2 \quad & \le \quad \frac{2c_V^2 \sigma_n \Delta\tau}{\delta t} \sum_{j=0}^n \alpha_j \; \|P^\ell y(t_j) - y(t_j)\|_V^2 \\ & \le \quad \frac{2c_V^2 \sigma_n \Delta\tau}{\delta t} \sum_{i=\ell+1}^d \lambda_i. \end{aligned} \tag{4.14}$$

Combining the last two bounds and (4.12) it follows that

$$\sum_{k=0}^{m} \beta_k \, \|\varrho_k\|_H^2 \le 6\sigma_n (1 + c_P^2)\Delta\tau\Delta t \, \|y_t\|^2_{L^2(0,T;H)} + \frac{6c_V^2 \sigma_n \Delta\tau}{\delta t} \sum_{i=\ell+1}^{d} \lambda_i. \tag{4.15}$$

Note that $\sum_{k=0}^m \beta_k = T$ holds. By Lemma 4.5 we have

$$\begin{aligned} \sum_{k=0}^{m} \beta_k \, \|\vartheta_k\|_H^2 \quad &\le \quad C_3 \left( \|\vartheta_0\|_H^2 + \frac{\sigma_n}{\delta t}\Big(\frac{1}{\delta\tau} + \Delta\tau\Big) \sum_{i=\ell+1}^{d} \lambda_i \right) \\ &\quad + C_3 \Delta\tau (1 + c_P^2)(\Delta\tau + \sigma_n \Delta t) \|y_{tt}\|^2_{L^2(0,T;H)} \\ &\quad + C_3 \sigma_n \Delta\tau\Delta t \, \|y_t\|^2_{L^2(0,T;V)}, \end{aligned} \tag{4.16}$$

where $C_3 = C_1 T e^{C_2 T}$. From (4.10), (4.15), (4.16), and Remark 4.4 we obtain the first part of the following theorem.

THEOREM 4.7.

(a) *Assume that* (A1), (A2) *hold and that* $\Delta\tau$ *is sufficiently small. Then there exists a constant* $C$ *depending on* $T$, *but independent of the grids* $\{t_j\}_{j=0}^n$ *and* $\{\tau_j\}_{j=0}^m$, *such that*

$$\begin{aligned} &\sum_{k=0}^{m} \beta_k \, \|Y_k - y(\tau_k)\|_H^2 \\ &\le C \sum_{i=\ell+1}^{d} \left( \big|\langle \psi_i, y_0\rangle_V\big|^2 + \frac{\sigma_n}{\delta t}\Big(\frac{1}{\delta\tau} + \Delta\tau\Big)\lambda_i \right) + C\sigma_n \Delta\tau\Delta t \, \|y_t\|^2_{L^2(0,T;V)} \\ &\quad + C\sigma_n (1 + c_P^2)\Delta\tau \left( \Delta t \|y_t\|^2_{L^2(0,T;H)} + (\Delta\tau + \Delta t)\|y_{tt}\|^2_{L^2(0,T;H)} \right). \end{aligned} \tag{4.17}$$

(b) *If* (A3) *is satisfied and* $\Delta\tau$ *sufficiently small, then there exists a constant* $C$ *depending on* $T$, *but independent of the grids* $\{t_j\}_{j=0}^n$ *and* $\{\tau_j\}_{j=0}^m$, *such that*

$$\begin{aligned} &\sum_{k=0}^{m} \beta_k \, \|Y_k - y(\tau_k)\|_H^2 \le C\sigma_n \Delta\tau(\Delta\tau + \Delta t)\|y_{tt}\|^2_{L^2(0,T;V)} \\ &+ C \left( \sum_{i=\ell+1}^{d} \left( \big|\langle \psi_i, y_0\rangle_V\big|^2 + \frac{\sigma_n}{\delta t}\Big(\frac{1}{\delta\tau} + \Delta\tau\Big)\lambda_i \right) + \sigma_n \Delta\tau\Delta t \|y_t\|^2_{L^2(0,T;V)} \right). \end{aligned} \tag{4.18}$$

*Proof.* The proof of part (b) is obtained from that for (a) by utilizing (2.2) and $\|P^\ell\|_{\mathcal{L}(V)} = 1$ and by simple modifications of the estimates for the two terms $\sum_{k=0}^m \beta_k \|\vartheta_k\|_H^2$ and $\sum_{j=1}^k \delta\tau_j \|z_j\|_H^2$ in (B.16) of Appendix B. □

Compared to standard finite difference, finite element, or spectral element approximation results in the basic Galerkin POD backward Euler convergence, the result of Theorem 4.7 has an unusual format. This is due, in part, to the fact that one cannot rely on function space rate of convergence results, which are typically the basis for approximation theory of partial differential equations. The terms in the second line of (4.17) depend (through $\psi_i$, $\lambda_i$, $d$) on the way in which the snapshots are taken, on the number $\ell$ of basis elements, and on the relative locations of the snapshots and the time discretization (through $\sigma_n$). In the remainder of this section we shall analyze these terms and show how they can be simplified if further assumptions are admitted.

*Remark* 4.8. In (4.17) and (4.18) the eigenvalues and eigenfunctions depend on $n$, i.e., $\lambda_i = \lambda_i^n$ and $\psi_i = \psi_i^n$. As proved in section 3, if $\ell$ satisfies (3.12) and $\sum_{i=\ell+1}^{\infty} \lambda_i^{\infty} \neq 0$ or $\sum_{i=\ell+1}^{\infty} |\langle \psi_i, y_0 \rangle_V|^2 \neq 0$, then by (3.14), (3.15) we have

$$\begin{aligned}\sum_{i=\ell+1}^{d} &\Big( \big|\langle \psi_i, y_0 \rangle_V\big|^2 + \frac{\sigma_n}{\delta t}\Big(\frac{1}{\delta\tau} + \Delta\tau\Big)\lambda_i\Big) \\ &\leq 2 \sum_{i=\ell+1}^{\infty} \Big( \big|\langle \psi_i^{\infty}, y_0 \rangle_V\big|^2 + \frac{\sigma_n}{\delta t}\Big(\frac{1}{\delta\tau} + \Delta\tau\Big)\lambda_i^{\infty}\Big) \quad \text{for all } \Delta t \leq \overline{\Delta t},\end{aligned}$$

and the dependence of the estimates of eigenvalues and eigenfunctions on $n$ in (4.17) and (4.18) is thus eliminated.

Let us next derive some corollaries to the proof of Theorem 4.7. At first we consider the case in which the two grids coincide so that $n = m$ and $\tau_j = t_j$ for $j = 0, \dots, m$.

COROLLARY 4.9. *Suppose that the assumptions of Theorem* 4.7(a) *hold. If the two time discretizations coincide, then there exists a constant $C > 0$ depending on $T$, but independent of the grid $\{\tau_j\}_{j=0}^m$, such that*

$$\begin{aligned}\sum_{k=0}^{m} \beta_k \ \|Y_k - y(\tau_k)\|_H^2 &\leq C(1 + c_P^2)\Delta\tau^2 \ \|y_{tt}\|_{L^2(0,T;H)}^2 \\ &+ C\left( \sum_{i=\ell+1}^{d} \Big( \big|\langle \psi_i, y_0 \rangle_V\big|^2 + \Big(\frac{1}{\delta\tau^2} + 1\Big)\lambda_i\Big) + \Delta\tau^2 \ \|y_t\|_{L^2(0,T;V)}\right). \end{aligned} \tag{4.19}$$

*Proof.* We proceed as in the proof of Theorem 4.2. Since the two time discretizations coincide, we obtain $n = m$, $\sigma_n = 1$, $\delta t = \delta\tau$, and $\alpha_j = \beta_j$ for $j = 0, \dots, n$. In place of the estimate (4.15) we now have

$$\sum_{k=0}^{m} \beta_k \ \|\varrho_k\|_H^2 \leq c_V^2 \sum_{i=\ell+1}^{d} \lambda_i,$$

which gives the claim. ◻

*Remark* 4.10. Again, as in Theorem 4.7(b) compared to (a), the factor $1 + c_P^2$ can be avoided in (4.19) if in place of (A1), (A2) we assume (A3) and replace the term $\|y_{tt}\|_{L^2(0,T;H)}$ with $\|y_{tt}\|_{L^2(0,T;V)}$.

Let us briefly reflect on the behavior of the right-hand side of (4.17) and (4.18). First we note that if the number of POD elements for the Galerkin scheme coincides with the dimension of $\mathcal{V}$, then the first additive term on the right-hand side disappears. Second, if the number of snapshots is refined so that $\Delta t \to 0$, then the factor multiplying $\sum_{i=\ell+1}^{d} \lambda_i$ blows up. As noted above, the term $\sum_{i=\ell+1}^{d} \lambda_i$ itself changes as the snapshots are refined. While computations for many concrete situations show that $\sum_{i=\ell+1}^{d} \lambda_i$ is small compared to $\Delta\tau$, the question nevertheless arises of whether the term $1/(\delta\tau\delta t)$ can be avoided in the estimates. For this purpose we choose

$$\mathcal{V} = \text{span } \{y(t_0), \dots, y(t_n), \overline{\partial}_t y(t_1), \dots, \overline{\partial}_t y(t_n)\}, \tag{4.20}$$

where

$$\overline{\partial}_t y(t_j) = \frac{y(t_j) - y(t_{j-1})}{\delta t_j} \quad \text{for } j = 1, \dots, n.$$

Equation (3.5) must be replaced by

$$\sum_{j=0}^{n} \alpha_j \left\| y(t_j) - \sum_{i=1}^{\ell} \langle y(t_j), \hat{\psi}_i \rangle_V \hat{\psi}_i \right\|_V^2 + \sum_{j=1}^{n} \alpha_j \left\| \overline{\partial}_t y(t_j) - \sum_{i=1}^{\ell} \langle \overline{\partial}_t y(t_j), \hat{\psi}_i \rangle_V \hat{\psi}_i \right\|_V^2 = \sum_{i=\ell+1}^{d} \hat{\lambda}_i,$$

where $\{\hat{\lambda}_i\}_{i\in\mathbb{N}}$, $\{\hat{\psi}_i\}_{i\in\mathbb{N}}$ are the eigenvalues and eigenfunctions of $\hat{\mathcal{R}}_n \in \mathcal{L}(V)$ given by

$$\hat{\mathcal{R}}_n z = \sum_{j=0}^{n} \alpha_j \left( \langle z, y(t_j) \rangle_V y(t_j) + \langle z, \overline{\partial}_t y(t_j) \rangle_V \overline{\partial}_t y(t_j) \right)$$

and satisfying

$$\hat{\mathcal{R}}_n \hat{\psi}_i = \hat{\lambda}_i \hat{\psi}_i, \quad \hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_{d(n)} > 0, \quad \text{and } \lambda_i = 0 \text{ for } i > d(n).$$

As a consequence, estimate (B.16) in Appendix B can be replaced by

$$\sum_{j=1}^{k} \delta\tau_j \, \|z_j\|_H^2 \leq 14\sigma_n (1 + c_P^2)(\Delta\tau^2 + \Delta\tau\Delta t) \|y_{tt}\|^2_{L^2(0,t_{\bar{k}+1};H)} + \frac{14\sigma_n c_V^2 \Delta\tau}{\delta t} \sum_{i=\ell+1}^{d} \hat{\lambda}_i$$

in the case of (A1), (A2) holding, and by

$$\sum_{j=1}^{k} \delta\tau_j \, \|z_j\|_H^2 \leq 28\sigma_n (\Delta\tau^2 + \Delta\tau\Delta t) \|y_{tt}\|^2_{L^2(0,t_{\bar{k}+1};V)} + \frac{28\sigma_n c_V^2 \Delta\tau}{\delta t} \sum_{i=\ell+1}^{d} \hat{\lambda}_i$$

in the case of (A3). We obtain the following corollary.

COROLLARY 4.11. *If in addition to the assumptions of Theorem* 4.7(a) *the snapshots set is taken as in* (4.20)*, then*

$$\begin{aligned} &\sum_{k=0}^{m} \beta_k \, \|Y_k - y(\tau_k)\|_H^2 \\ &\leq C \sum_{i=\ell+1}^{d} \left( \left| \langle \hat{\psi}_i, y_0 \rangle_V \right|^2 + \frac{\sigma_n \Delta\tau}{\delta t} \hat{\lambda}_i \right) + C\sigma_n \Delta\tau\Delta t \, \|y_t\|^2_{L^2(0,T;V)} \\ &\quad + C(1 + c_P^2)\Delta\tau \big( (\Delta\tau + \sigma_n \Delta t) \|y_{tt}\|^2_{L^2(0,T;H)} + \sigma_n \Delta t \|y_t\|^2_{L^2(0,T;H)} \big), \end{aligned} \tag{4.21}$$

*where $C$ has the same properties as in Theorem* 4.7.

*Remark* 4.12. In [12] a laser surface hardening problem was considered. The numerical experiments show that the inclusion of the difference quotients into the snapshot set leads to better results.

In estimate (4.21) the term $1 + c_P^2$ can be avoided if (A3) in place of (A1), (A2) holds and $\|y_{tt}\|_{L^2(0,T;H)}$ is replaced by $\|y_{tt}\|_{L^2(0,T;V)}$. Note that the terms $\{\hat{\lambda}_i\}_{i\in\mathbb{N}}$,

$\{\hat{\psi}_i\}_{i\in\mathbb{N}}$, and $\sigma_n$ depend on the time discretization of $[0,T]$ for the snapshots as well as the numerical integration. We address this dependence next.

If we suppose that

$$\Delta t = O(\delta\tau) \quad \text{and} \quad \Delta\tau = O(\delta t), \tag{4.22}$$

then there exists a constant $c_1 > 0$ independent of $\{t_j\}_{j=0}^n$ and $\{\tau_j\}_{j=0}^m$ such that

$$\max\Big(\sigma_n, \frac{\sigma_n \Delta\tau}{\delta t}\Big) \le c_1. \tag{4.23}$$

To obtain an estimate that is independent of the spectral values of a specific snapshot set $\{y(t_j)\}_{j=0}^n$ we follow the analysis of section 3.2. We assume that $y \in W^{2,2}(0,T;V)$, so that in particular (A3) holds, and introduce the operator $\hat{\mathcal{R}} \in \mathcal{L}(V)$ corresponding to $\mathcal{R}$ by

$$\hat{\mathcal{R}}z = \int_0^T \langle z, y(t)\rangle_V y(t) + \langle z, y_t(t)\rangle_V y_t(t)\, dt \quad \text{for } z \in V.$$

Note that $\hat{\mathcal{R}} = \hat{\mathcal{Y}}\hat{\mathcal{Y}}^*$, where $\hat{\mathcal{Y}}^* : V \to W^{1,2}(0,T;\mathbb{R})$ is given by

$$(\hat{\mathcal{Y}}^* z)(t) = \langle z, y(t)\rangle_V.$$

Since $y \in W^{2,2}(0,T;V)$ it is simple to argue that $\hat{\mathcal{Y}}^*$ is compact and hence $\hat{\mathcal{R}}$ is compact. Let us denote the positive eigenvalues and corresponding eigenfunctions of $\hat{\mathcal{R}}$ by $\{\hat{\lambda}_i^\infty\}_{i\in\mathbb{N}}$ and $\{\hat{\psi}_i^\infty\}_{i\in\mathbb{N}}$. Since $t_0 = 0$, we proceed as in section 3.2, that $y_0 \in \text{range}\, \hat{\mathcal{R}}_n$ for all $n$ and $y_0 \in \overline{\text{range}\, \hat{\mathcal{R}}}$. The assumption $y \in W^{2,2}(0,T;V)$ allows us to argue that the analogue of (3.7), i.e.,

$$\lim_{\Delta t\to 0} \|\hat{\mathcal{R}}_n - \hat{\mathcal{R}}\|_{\mathcal{L}(V)} = 0,$$

holds. Let us choose and fix $\ell$ such that

$$\hat{\lambda}_\ell^\infty \ne \hat{\lambda}_{\ell+1}^\infty. \tag{4.24}$$

We can now proceed precisely as in section 3.2 to assert that there exists $\overline{\Delta t} > 0$ such that

$$\sum_{i=\ell+1}^{d(n)} \hat{\lambda}_i^n \le 2\sum_{i=\ell+1}^{\infty} \hat{\lambda}_i^\infty \quad \text{and} \quad \sum_{i=\ell+1}^{d(n)} \big|\langle y_0, \hat{\psi}_i^n\rangle_V\big|^2 \le 2\sum_{i=\ell+1}^{\infty} \big|\langle y_0, \hat{\psi}_i^\infty\rangle_V\big|^2 \tag{4.25}$$

for all $\Delta t \le \overline{\Delta t}$, provided, of course, that the terms on the right-hand side of (4.25) are different from zero. We summarize the above discussion in the following corollary.

COROLLARY 4.13. *Assume that $y \in W^{2,2}(0,T;V)$ and let the snapshots be chosen as in* (4.20). *If* (4.22) *holds and $\ell$ satisfies* (4.24)*, then there exists a constant $C > 0$, independent of $\ell$ and the grids $\{t_j\}_{j=0}^n$ and $\{\tau_j\}_{j=0}^m$, and a $\overline{\Delta t} > 0$, depending on $\ell$, such that*

$$\begin{aligned}\sum_{k=0}^{m} \beta_k \, \|Y_k - y(\tau_k)\|_H^2 \le{}& C\sum_{i=\ell+1}^{\infty}\Big(\big|\langle y_0, \hat{\psi}_i^\infty\rangle_V\big|^2 + \hat{\lambda}_i^\infty\Big) \\ &+ C\Big(\Delta\tau\Delta t\, \|y_t\|_{L^2(0,T;V)}^2 + \Delta\tau(\Delta\tau + \Delta t)\, \|y_{tt}\|_{L^2(0,T;V)}^2\Big)\end{aligned} \tag{4.26}$$

*for all* $\Delta t \le \overline{\Delta t}$.

*Remark* 4.14. In (4.26) the first term on the right-hand side of the inequality reflects the spatial approximation error of the Galerkin POD scheme and the second reflects the approximation error due to the temporal backward Euler scheme. If the latter is replaced by the Crank–Nicolson method, then, assuming $\Delta\tau = \Delta t$ and appropriate regularity on $y$, it can be shown with the techniques of this section that an estimate analogous to (4.26) holds with the first additive term on the right-hand side unchanged and the second one of fourth order in $\Delta\tau$.

**4.2. Case $X = H$.** Here we consider the case in which the POD basis is constructed with respect to the $H$-norm. Differently from the situation where the POD basis was constructed in $V$, the right-hand side of the estimate will involve the stiffness matrix

$$S = ((S_{ij})) \in \mathbb{R}^{d\times d} \quad \text{with} \quad S_{ij} = a(\psi_j, \psi_i).$$

We shall require the following lemma.

LEMMA 4.15. *For every* $\ell \in \{1, \dots, d\}$ *the projection operator* $P^\ell : V \to V^\ell$ *satisfies*

$$\sum_{j=0}^{n} \alpha_j \, \left\| y(t_j) - P^\ell y(t_j) \right\|_V^2 \le \|S\|_2 \sum_{i=\ell+1}^{d} \lambda_i, \tag{4.27}$$

*where* $\lambda_i$ *denote the eigenvalues introduced in* (3.4) *and* $\|\cdot\|_2$ *stands for the spectral norm for symmetric matrices.*

*Proof.* Using the fact that $\|\varphi\|_V^2 \le \|S\|_2 \, \|\varphi\|_H^2$ for all $\varphi \in \mathcal{V}$ (see [16, Lemma 2]), we can proceed as in the proof of Lemma 4.1 and, utilizing (3.5) with $X = H$, we obtain the desired result. $\square$

THEOREM 4.16. *Suppose that* (A3) *holds and that* $\Delta\tau$ *is sufficiently small. Then there exists a constant* $C > 0$ *depending on* $T$*, but independent of the grids* $\{t_j\}_{j=0}^n$ *and* $\{\tau_j\}_{j=0}^m$*, such that*

$$\begin{aligned} \sum_{k=0}^{m} \beta_k \, \|Y_k - y(\tau_k)\|_H^2 \le C \sum_{i=\ell+1}^{d} \|S\|_2 \Big( \big|\langle \psi_i, y_0\rangle_H\big|^2 + \frac{\sigma_n}{\delta t}\Big(\frac{1}{\delta\tau} + \Delta\tau\Big)\lambda_i \Big) \\ + C\sigma_n \Delta\tau \left( (\Delta\tau + \Delta t)\|y_{tt}\|_{L^2(0,T;V)}^2 + \Delta t \, \|y_t\|_{L^2(0,T;V)}^2 \right). \end{aligned} \tag{4.28}$$

*Proof.* We proceed as in the proofs of Lemma 4.5 and Theorem 4.7 and indicate only the necessary changes. Estimate (B.15) requires no change. For (B.16) we utilize $\|P^\ell\varphi\| \le c_V\|\varphi\|_V$ for $\varphi \in V$ and obtain, by applying Lemma 4.15,

$$\begin{aligned} \sum_{j=1}^{k} \delta\tau_j \, \|z_j\|_H^2 \quad &\le \quad 14\sigma_n(1 + c_V^2)(\Delta\tau^2 + \Delta\tau\Delta t)\|y_{tt}\|_{L^2(0,t_{\bar{k}+1};V)}^2 \\ &\quad + \frac{56\sigma_n c_V^2 \|S\|_2}{\delta t \delta\tau} \sum_{i=\ell+1}^{d} \lambda_i. \end{aligned} \tag{4.29}$$

The analogue of (B.17) is again obtained by Lemma 4.15. Summarizing the $\vartheta_k$-terms we have

$$\begin{aligned} \sum_{j=1}^{k} \delta\tau_j \, \|\vartheta_k\|_H^2 \le C \left( \|\vartheta_0\|_H^2 + \frac{\sigma_n \|S\|_2}{\delta t}\Big(\frac{1}{\delta\tau} + \Delta\tau\Big) \sum_{i=\ell+1}^{d} \lambda_i \right) \\ + C\sigma_n(1 + c_V^2)\Delta\tau\big((\Delta\tau + \Delta t)\|y_{tt}\|_{L^2(0,T;V)}^2 + \Delta t \, \|y_t\|_{L^2(0,T;V)}^2\big). \end{aligned} \tag{4.30}$$

Turning to the $\varrho_k$-terms we find, following the estimates after (4.12),

$$\begin{aligned}\sum_{k=0}^{m} \beta_k \, \|\varrho_k\|_H^2 &\\ &\le 6\sigma_n(1+c_V^2)\Delta\tau\Delta t \, \|y_t\|^2_{L^2(0,T;V)} + 3\sum_{k=0}^{m}\beta_k\|P^\ell y(t_{\bar k}) - y(t_{\bar k})\|_H^2 \\ &\le 6\sigma_n(1+c_V^2)\Delta\tau\Delta t \, \|y_t\|^2_{L^2(0,T;V)} + \frac{6c_V^2\sigma_n\Delta\tau}{\delta t}\sum_{j=0}^{n}\alpha_j \, \|P^\ell y(t_j) - y(t_j)\|_V^2 .\end{aligned}$$

Thus by Lemma 4.15

$$\sum_{k=0}^{m}\beta_k \, \|\varrho_k\|_H^2 \le 6\sigma_n(1+c_V^2)\Delta\tau\Delta t \, \|y_t\|^2_{L^2(0,T;V)} + \frac{6c_V^2\sigma_n\Delta\tau\|S\|_2}{\delta t}\sum_{i=\ell+1}^{d}\lambda_i . \tag{4.31}$$

Finally $\vartheta_0 = y_0 - P^\ell y_0$ can be estimated as follows:

$$\begin{aligned}\|y_0 - P^\ell y_0\|_H &\le c_V \, \|y_0 - P^\ell y_0\|_V \le c_V \, \left\| y_0 - \sum_{i=1}^{\ell}\langle y_0,\psi_i\rangle_H\psi_i\right\|_V \\ &= c_V\left(\|S\|_2\sum_{i=\ell+1}^{d}\left|\langle y_0,\psi\rangle_H\right|^2\right)^{1/2}.\end{aligned}$$

Combining the last estimate with (4.30)–(4.31) we obtain (4.28). ☐

*Remark* 4.17. Let us briefly discuss the asymptotic properties of the expression on the right-hand side of (4.28), which are restricted due to the appearance of $\delta t\delta\tau$ in the denominator and the terms $\sigma_n$ and $\|S\|_2$. As in section 4.1 the factor $1/\delta\tau$ can be eliminated by adding the set $\{\overline{\partial} y(t_j)\}_{j=1}^n$ to the set of snapshots. Assuming that $\Delta t = O(\delta\tau)$ and $\Delta\tau = O(\delta t)$ implies (4.23), and consequently, $\sigma_n$ and $\sigma_n\Delta\tau/\delta t$ are uniformly bounded with respect to refinement of the $t$- and $\tau$-grids. The factor $\|S\|_2$, which tends to infinity as $m\to\infty$, appears to be unavoidable in case the POD basis is computed in $H$.

## Appendix A. Proof of Theorem 4.2.

**A.1. Existence.** Existence of a solution $\{Y_k\}_{k=1}^m$ can be proved by using the Schauder fixed point theorem; see [11, p. 222], for instance. For that purpose we define $z = \mathcal{T}_k w$ via the mappings $\mathcal{T}_k : V^\ell \to V^\ell$, $k = 1,\dots,m$, as follows: $z \in V^\ell$ is the solution to

$$\langle z,\psi\rangle_H + \delta\tau_k\big(a(z,\psi) + \langle B(w,z) + Rz,\psi\rangle_{V',V}\big) = \langle \delta\tau_k f(\tau_k) + Y_{k-1},\psi\rangle_H \tag{A.1}$$

for all $\psi \in V^\ell$. The bilinear form

$$\langle\cdot\,,\cdot\rangle_H + \delta\tau_k\big(a(\cdot\,,\cdot) + \langle B(w,\cdot) + R(\cdot),\cdot\rangle_{V',V}\big)$$

is continuous and coercive in $V^\ell\times V^\ell$ by (2.3)–(2.5). The existence and uniqueness of a solution to (A.1) can thus be shown by the Lax–Milgram theorem. The fixed points of $\mathcal{T}_k$ are the solutions of (4.4b). Taking $\psi = z$ in (A.1) above and using (2.2) and (2.4) we derive

$$\|z\|_V \le \frac{c_V}{\eta}\,\Big(\|f(\tau_k)\|_H + \frac{1}{\delta\tau_k}\,\|Y_{k-1}\|_H\Big). \tag{A.2}$$

Let us introduce the set

$$M_k = \Big\{ w \in V^\ell : \|w\|_V \le \frac{c_V}{\eta} \, \Big( \|f(\tau_k)\|_H + \frac{1}{\delta\tau_k} \, \|Y_{k-1}\|_H \Big) \Big\} \subset V^\ell.$$

From (A.2) we infer that $\mathcal{T}_k$ maps $M_k$ into itself. Since $M_k$ is a closed ball in $V^\ell$, the set $M_k$ is bounded, closed, and convex. Since the image of $\mathcal{T}_k$ is finite dimensional, $\mathcal{T}_k$ is compact. Thus, the existence of a fixed point $Y_k$ follows from the Schauder fixed point theorem.

**A.2. Uniqueness.** To prove the uniqueness we assume that the two sequences $\{Y_k^1\}_{k=0}^m$, $\{Y_k^2\}_{k=0}^m$ in $V^\ell$ are solutions of (4.4b). Then $\delta Y_k = Y_k^1 - Y_k^2 \in V^\ell$ solves

$$\langle \delta Y_k, \psi \rangle_H + \delta\tau_k \big( a(\delta Y_k, \psi) + \langle R\delta Y_k, \psi \rangle_{V',V} \big) = \delta\tau_k \, \langle B(Y_k^2) - B(Y_k^1), \psi \rangle_{V',V}$$

for all $\psi \in V^\ell$. Setting $\psi = \delta Y_k$ and using (2.4), (2.5), and Young's inequality we obtain

$$\begin{aligned}
\|\delta Y_k\|_H^2 + \eta\delta\tau_k \, \|\delta Y_k\|_V^2 &\le \delta\tau_k \, \langle B(Y_k^2) - B(Y_k^1), \delta Y_k \rangle_{V',V} \\
&= -\delta\tau_k \, \langle B(\delta Y_k, Y_k^2) + B(Y_k^1, \delta Y_k), \delta Y_k \rangle_{V',V} \\
&= -\delta\tau_k \, \langle B(\delta Y_k, Y_k^2), \delta Y_k \rangle_{V',V} \\
&\le c_B \delta\tau_k \, \|Y_k^2\|_V \|\delta Y_k\|_H \|\delta Y_k\|_V \\
&\le \|\delta Y_k\|_H^2 + \frac{c_B^2 \delta\tau_k^2}{4} \, \|Y_k^2\|_V^2 \|\delta Y_k\|_V^2.
\end{aligned}$$

It follows that

$$\Big( 1 - \frac{c_B^2 \delta\tau_k}{4\eta} \, \|Y_k^2\|_V^2 \Big) \, \|\delta Y_k\|_V^2 \le 0.$$

Let $c = \max\{\|Y_k^2\|_V : k = 1, \dots, m\}$. Then $\delta Y_k = 0$ and hence $Y_k^1 = Y_k^2$, provided that $\Delta\tau \le 4\eta/(c^2 c_B^2)$.

**A.3. A priori estimates.** To prove the estimates (4.5) we take $\psi = Y_k$ in (4.4b). Due to (2.3)–(2.5) and the identity

$$2 \, \langle \varphi - \psi, \varphi \rangle_H = \|\varphi\|_H^2 - \|\psi\|_H^2 + \|\varphi - \psi\|_H^2 \quad \text{for all } \varphi, \psi \in H \tag{A.3}$$

we obtain

$$\|Y_k\|_H^2 - \|Y_{k-1}\|_H^2 + \|Y_k - Y_{k-1}\|_H^2 + 2\eta\delta\tau_k \, \|Y_k\|_V^2 \le 2\delta\tau_k \, \|f(\tau_k)\|_H \|Y_k\|_H.$$

Using (2.2) and Young's inequality it follows that

$$\|Y_k\|_H^2 + \|Y_k - Y_{k-1}\|_H^2 + \eta\delta\tau_k \, \|Y_k\|_V^2 \le \|Y_{k-1}\|_H^2 + \frac{c_V^2 \delta\tau_k}{\eta} \, \|f(\tau_k)\|_H^2. \tag{A.4}$$

From (A.4) and (2.2) we infer that

$$(1 + \gamma\delta\tau_k) \, \|Y_k\|_H^2 \le \|Y_{k-1}\|_H^2 + \frac{\delta\tau_k}{\gamma} \, \|f(\tau_k)\|_H^2,$$

where $\gamma = \eta/c_V^2$, which yields

$$\|Y_k\|_H^2 \le \frac{1}{1 + \gamma\delta\tau} \, \|Y_{k-1}\|_H^2 + \frac{\delta\tau_k}{\gamma(1 + \gamma\delta\tau_k)} \, \|f(\tau_k)\|_H^2. \tag{A.5}$$

From

$$\frac{\delta\tau_k}{1+\gamma\delta\tau_k} = \frac{1}{\gamma}\Big(1-\frac{1}{1+\gamma\delta\tau_k}\Big) \le \frac{1}{\gamma}\Big(1-\frac{1}{1+\gamma\Delta\tau}\Big) = \frac{\Delta\tau}{1+\gamma\Delta\tau}$$

and (A.5) we infer upon summation that

$$\|Y_k\|_H^2 \le \Big(\frac{1}{1+\gamma\delta\tau}\Big)^k \|Y_0\|_H^2 + \frac{\Delta\tau}{\gamma}\ \|f\|^2_{C([0,T];H)} \sum_{j=1}^k \Big(\frac{1}{1+\gamma\Delta\tau}\Big)^j. \tag{A.6}$$

Recall that

$$\Big(\frac{1}{1+\gamma\delta\tau}\Big)^k \le (1+\gamma\delta\tau)e^{-\gamma k\delta\tau} \quad \text{and} \quad \Big(\frac{1}{1+\gamma\Delta\tau}\Big)^k \ge e^{-\gamma k\Delta\tau}. \tag{A.7}$$

Moreover, setting $\zeta = 1/(1+\gamma\Delta\tau)$ we find

$$\Delta\tau\sum_{j=1}^k \Big(\frac{1}{1+\gamma\Delta\tau}\Big)^j = \Delta\tau\ \frac{1-\zeta^k}{\zeta^{-1}-1} = \frac{1-\zeta^k}{\gamma} \le \frac{1-e^{-\gamma k\Delta\tau}}{\gamma}.$$

Inserting this estimate and (A.7) in (A.6) and utilizing the fact that $\|Y_0\|_H \le \|y_0\|_H$ yield (4.5a). Summing (A.4) over $k$ we find

$$\|Y_m\|_H^2 + \sum_{k=1}^m \|Y_k - Y_{k-1}\|_H^2 + \eta\sum_{k=1}^m \delta\tau_k\|Y_k\|_V^2 \le \|Y_0\|_H^2 + \frac{c_V T}{\gamma}\|f\|^2_{C([0,T];H)},$$

which is estimate (4.5b).

**Appendix B. Proof of Lemma 4.5.** Using the notation $\overline{\partial}_\tau\vartheta_k = (\vartheta_k - \vartheta_{k-1})/\delta\tau_k$, $k=1,\dots,m$, we obtain

$$\begin{aligned}&\langle\overline{\partial}_\tau\vartheta_k,\psi\rangle_H + a(\vartheta_k,\psi) + \langle R\vartheta_k,\psi\rangle_{V',V}\\ &\qquad = \langle v_k,\psi\rangle_H + \langle B(y(\tau_k)) - B(Y_k) + R(y(\tau_k) - P^\ell y(\tau_k)),\psi\rangle_{V',V},\end{aligned} \tag{B.1}$$

where

$$v_k = y_t(\tau_k) - \overline{\partial}_\tau P^\ell y(\tau_k) = y_t(\tau_k) - \overline{\partial}_\tau y(\tau_k) + \overline{\partial}_\tau y(\tau_k) - \overline{\partial}_\tau P^\ell y(\tau_k).$$

We put $w_k = y_t(\tau_k) - \overline{\partial}_\tau y(\tau_k)$ and $z_k = \overline{\partial}_\tau y(\tau_k) - \overline{\partial}_\tau P^\ell y(\tau_k)$. Choosing $\psi = \vartheta_k \in V^\ell$ in (B.1), using (2.4) and (A.3) we infer that

$$\begin{aligned}&\|\vartheta_k\|_H^2 - \|\vartheta_{k-1}\|_H^2 + \|\vartheta_k - \vartheta_{k-1}\|_H^2 + 2\eta\delta\tau_k\ \|\vartheta_k\|_V^2\\ &\le 2\delta\tau_k\big(\|v_k\|_H\|\vartheta_k\|_H + \big|\langle B(y(\tau_k)) - B(Y_k),\vartheta_k\rangle_{V',V}\big| + \|R\varrho_k\|_{V'}\|\vartheta_k\|_V\big).\end{aligned} \tag{B.2}$$

Applying Young's inequality it follows that

$$\|R\varrho_k\|_{V'}\|\vartheta_k\|_V \le \|R\|_{\mathcal{L}(V,V')}\|\varrho_k\|_V\|\vartheta_k\|_V \le \frac{\eta}{4}\ \|\vartheta_k\|_V^2 + c_0\ \|\varrho_k\|_V^2 \tag{B.3}$$

for a constant $c_0 > 0$ depending on $\|R\|_{\mathcal{L}(V,V')}$ and $\eta$. We proceed by estimating the nonlinear terms on the right-hand side of (B.2). Note that

$$\begin{aligned}&B(y(\tau_k)) - B(Y_k)\\ &\qquad = -B(y(\tau_k), Y_k - y(\tau_k)) - B(Y_k - y(\tau_k)) - B(Y_k - y(\tau_k), y(\tau_k)).\end{aligned} \tag{B.4}$$

Applying (2.5), (2.2), and Young's inequality we obtain the existence of two constants $c_1, c_2 > 0$ satisfying

$$\begin{aligned}
&\left|\langle B(y(\tau_k), Y_k - y(\tau_k)), \vartheta_k\rangle_{V',V}\right| \\
&= \left|\langle B(y(\tau_k), \varrho_k), \vartheta_k\rangle_{V',V}\right| \leq c_B c_V^{\delta_3}\ \|y\|_{C([0,T];V)}\|\varrho_k\|_V\|\vartheta_k\|_H^{1-\delta_3}\|\vartheta_k\|_V^{\delta_3} \\
&\leq \frac{\eta}{4}\ \|\vartheta_k\|_V^2 + c_1\ \|\vartheta_k\|_H^2 + c_2\ \|\varrho_k\|_V^2.
\end{aligned} \tag{B.5}$$

Again utilizing (2.5), Young's inequality, and (2.2) we find that there exist constants $c_3, c_4 > 0$ such that

$$\begin{aligned}
&\left|\langle B(Y_k - y(\tau_k), y(\tau_k)), \vartheta_k\rangle_{V',V}\right| \\
&\quad = \left|\langle B(\vartheta_k, y(\tau_k)) + B(\varrho_k, y(\tau_k)), \vartheta_k\rangle_{V',V}\right| \\
&\quad \leq c_B\ \|y\|_{C([0,T];V)}\left(\|\vartheta_k\|_H\|\vartheta_k\|_V + c_V^{\delta_3}\|\varrho_k\|_V\|\vartheta_k\|_H^{1-\delta_3}\|\vartheta_k\|_V^{\delta_3}\right) \\
&\quad \leq \frac{\eta}{4}\ \|\vartheta_k\|_V^2 + c_3\ \|\vartheta_k\|_H^2 + c_4\ \|\varrho_k\|_V^2.
\end{aligned} \tag{B.6}$$

From $y \in C([0,T];V)$ it follows that there exists a constant $c_5 > 0$ such that

$$\max_{1\leq k\leq m}\left(\|\varrho_k\|_H^{\delta_3}\|\varrho_k\|_V^{1-\delta_3}, \|\varrho_k\|_V\right) \leq c_5. \tag{B.7}$$

Using (2.5) and (4.10) we conclude that

$$\langle B(Y_k - y(\tau_k)), \vartheta_k\rangle_{V',V} = \langle B(\vartheta_k, \varrho_k) + B(\varrho_k, \varrho_k), \vartheta_k\rangle_{V',V}. \tag{B.8}$$

Applying (2.5), (B.7), (B.8), and Young's inequality we find that

$$\begin{aligned}
&\left|\langle B(Y_k - y(\tau_k)), \vartheta_k\rangle_{V',V}\right| \\
&\qquad \leq c_B c_5\left(\|\vartheta_k\|_H\|\vartheta_k\|_V + \|\varrho_k\|_V\|\vartheta_k\|_H^{1-\delta_3}\|\vartheta_k\|_V^{\delta_3}\right) \\
&\qquad \leq \frac{\eta}{4}\ \|\vartheta_k\|_V^2 + c_6\ \|\vartheta_k\|_H^2 + c_7\ \|\varrho_k\|_V^2
\end{aligned} \tag{B.9}$$

for two constants $c_6, c_7 > 0$. From (B.2)–(B.9), Young's inequality, and $v_k = w_k + z_k$ we obtain

$$\|\vartheta_k\|_H^2 \leq \|\vartheta_{k-1}\|_H^2 + \delta\tau_k\big(\|w_k\|_H^2 + \|z_k\|_H^2 + c_8\ \|\vartheta_k\|_H^2 + c_9\ \|\varrho_k\|_V^2\big), \tag{B.10}$$

where $c_8 = 2 + c_1 + c_3 + c_6$ and $c_9 = c_0 + c_2 + c_4 + c_7$. Suppose that

$$\Delta\tau \leq \frac{1}{2c_8}. \tag{B.11}$$

With (B.11) holding we have $0 < 1 - c_8\delta\tau_k \leq 1/2$ and

$$\frac{1}{1 - c_8\delta\tau_k} \leq \frac{1}{1 - c_8\Delta\tau} \leq 1 + 2c_8\Delta\tau. \tag{B.12}$$

From (B.10) and (B.12) we find that

$$\|\vartheta_k\|_H^2 \leq (1 + 2c_8\Delta\tau)\big(\|\vartheta_{k-1}\|_H^2 + \delta\tau_k\big(\|w_k\|_H^2 + \|z_k\|_H^2 + c_9\ \|\varrho_k\|_V^2\big)\big) \tag{B.13}$$

holds. By summation on $k$ we obtain

$$
\begin{aligned}
\|\vartheta_k\|_H^2 &\le \Big(1+\frac{2c_8\Delta\tau}{\delta\tau}\frac{k\delta\tau}{k}\Big)^k \\
&\quad\cdot\left(\|\vartheta_0\|_H^2+\sum_{j=1}^k \delta\tau_j\left(\|w_j\|_H^2+\|z_j\|_H^2+c_9\ \|\varrho_j\|_V^2\right)\right) \\
&\le e^{c_{10}k\delta\tau}\left(\|\vartheta_0\|_H^2+\sum_{j=1}^k \delta\tau_j\left(\|w_j\|_H^2+\|z_j\|_H^2+c_9\ \|\varrho_k\|_V^2\right)\right),
\end{aligned}
\tag{B.14}
$$

where $c_{10}=2c_8\Delta\tau/\delta\tau$. Recall that by assumption, $\Delta\tau/\delta\tau$ is bounded uniformly with respect to $m$. We next estimate the terms involving $w_j$ and $z_j$:

$$
\begin{aligned}
\sum_{j=1}^k \delta\tau_j\|w_j\|_H^2 &= \sum_{j=1}^k \delta\tau_j\ \|y_t(\tau_j)-\overline{\partial}_\tau y(\tau_j)\|_H^2 \\
&= \sum_{j=1}^k \frac{1}{\delta\tau_j}\ \|\delta\tau_j y_t(\tau_j)-(y(\tau_j)-y(\tau_{j-1}))\|_H^2 \\
&= \sum_{j=1}^k \frac{1}{\delta\tau_j}\left\|\int_{\tau_{j-1}}^{\tau_j}(s-\tau_{j-1})y_{tt}(s)\,ds\right\|_H^2 \\
&\le \sum_{j=1}^k \frac{1}{\delta\tau_j}\int_{\tau_{j-1}}^{\tau_j}(s-\tau_{j-1})^2\,ds\int_{\tau_{j-1}}^{\tau_j}\|y_{tt}(s)\|_H^2\,ds \\
&= \sum_{j=1}^k \frac{\delta\tau_j^2}{3}\ \|y_{tt}\|^2_{L^2(\tau_{j-1},\tau_j;H)} \le \frac{\Delta\tau^2}{3}\ \|y_{tt}\|^2_{L^2(0,\tau_k;H)}.
\end{aligned}
\tag{B.15}
$$

The term $\|z_j\|_H^2$ can be estimated as follows:

$$
\begin{aligned}
\|z_j\|_H^2 &= \|\overline{\partial}_\tau y(\tau_j)-\overline{\partial}_\tau P^\ell y(\tau_j)\|_H^2 \\
&= \big\|\overline{\partial}_\tau y(\tau_j)-y_t(\tau_j)+y_t(\tau_j)-y_t(t_{\bar{j}})+y_t(t_{\bar{j}})-\overline{\partial}_\tau y(t_{\bar{j}}) \\
&\qquad +\overline{\partial}_\tau y(t_{\bar{j}})-\overline{\partial}_\tau P^\ell y(t_{\bar{j}})+\overline{\partial}_\tau P^\ell y(t_{\bar{j}})-P^\ell y_t(t_{\bar{j}}) \\
&\qquad +P^\ell y_t(t_{\bar{j}})-P^\ell y_t(\tau_j)+P^\ell y_t(\tau_j)-\overline{\partial}_\tau P^\ell y(\tau_j)\big\|_H^2 \\
&\le 7\big(1+\|P^\ell\|^2_{\mathcal{L}(H)}\big)\ \|\overline{\partial}_\tau y(\tau_j)-y_t(\tau_j)\|_H^2 \\
&\quad +7\ \big(1+\|P^\ell\|^2_{\mathcal{L}(H)}\big)\ \|y_t(t_{\bar{j}})-\overline{\partial}_\tau y(t_{\bar{j}})\|_H^2 \\
&\quad +7\big(1+\|P^\ell\|^2_{\mathcal{L}(H)}\big)\ \|y_t(\tau_j)-y_t(t_{\bar{j}})\|_H^2 \\
&\quad +7\ \|\overline{\partial}_\tau y(t_{\bar{j}})-\overline{\partial}_\tau P^\ell y(t_{\bar{j}})\|_H^2.
\end{aligned}
$$

Note that

$$
\|\overline{\partial}_\tau y(\tau_j)-y_t(\tau_j)\|_H^2=\frac{1}{\delta\tau_j^2}\ \left\|\int_{\tau_{j-1}}^{\tau_j}(t-\tau_{j-1})y_{tt}(t)\,dt\right\|_H^2\le\frac{\delta\tau_j}{3}\ \|y_{tt}\|^2_{L^2(\tau_{j-1},\tau_j;H)}.
$$

Analogously, we find

$$
\|y_t(t_{\bar{j}})-\overline{\partial}_\tau y(t_{\bar{j}})\|_H^2\le\frac{\delta\tau_j}{3}\ \|y_{tt}\|^2_{L^2(t_{\bar{j}-1},t_{\bar{j}};H)}.
$$

From

$$\begin{aligned}\|y_t(\tau_j)-y_t(t_{\bar j})\|_H^2 &\le \left(\int_{t_{\bar j-1}}^{t_{\bar j+1}}\|y_{tt}(s)\|_H\,ds\right)^2 \le \big(\delta t_{\bar j}+\delta t_{\bar j+1}\big)\|y_{tt}\|^2_{L^2(t_{\bar j-1},t_{\bar j+1};H)}\\ &\le 2\Delta t\ \|y_{tt}\|^2_{L^2(t_{\bar j-1},t_{\bar j+1};H)},\end{aligned}$$

where we set $t_{m+1}=T$ whenever $j=m$, we find

$$\begin{aligned}\|z_j\|_H^2 \le{}& \frac{7}{3}\big(1+c_P^2\big)\delta\tau_j\big(\|y_{tt}\|^2_{L^2(\tau_{j-1},\tau_j;H)}+\|y_{tt}\|^2_{L^2(t_{\bar j-1},t_{\bar j};H)}\big)\\ &+14\big(1+c_P^2\big)\Delta t\ \|y_{tt}\|^2_{L^2(t_{\bar j-1},t_{\bar j+1};H)}\\ &+\frac{14}{\delta\tau_j^2}\ \big(\|y(t_{\bar j})-P^\ell y(t_{\bar j})\|_H^2+\|y(t_{\bar j-1})-P^\ell y(t_{\bar j-1})\|_H^2\big),\end{aligned}$$

where we set $t_{n+1}=T$. Note that $\alpha_j\ge\delta t/2$. Using (2.2) and Lemma 4.1 we estimate

$$\sum_{j=1}^k\frac{1}{\delta\tau_j}\ \|y(t_{\bar j})-P^\ell y(t_{\bar j})\|_H^2\le\frac{2\sigma_n}{\delta\tau\delta t}\sum_{j=0}^n\alpha_j\|y(t_j)-P^\ell y(t_j)\|_H^2\le\frac{2\sigma_n c_V^2}{\delta\tau\delta t}\sum_{i=\ell+1}^d\lambda_i$$

and, analogously,

$$\sum_{j=1}^k\frac{1}{\delta\tau_j}\ \|y(t_{\bar j-1})-P^\ell y(t_{\bar j-1})\|_H^2\le\frac{2\sigma_n c_V^2}{\delta\tau\delta t}\sum_{i=\ell+1}^d\lambda_i.$$

Hence,

$$\begin{aligned}\sum_{j=1}^k\delta\tau_j\ \|z_j\|_H^2\le{}&14\sigma_n(1+c_P^2)(\Delta\tau^2+\Delta\tau\Delta t)\|y_{tt}\|^2_{L^2(0,t_{\bar k+1};H)}\\ &+\frac{56\sigma_n c_V^2}{\delta t\delta\tau}\sum_{i=\ell+1}^d\lambda_i.\end{aligned}\tag{B.16}$$

Using $\alpha_j\ge 2/\delta t$ and $\|P^\ell\|_{\mathcal{L}(V)}=1$ we obtain for the terms $\|\varrho_k\|_V^2$

$$\begin{aligned}\|\varrho_j\|_V^2&=\|P^\ell y(\tau_j)-y(\tau_j)\|_V^2\\ &=\|P^\ell y(\tau_j)-P^\ell y(t_{\bar j})+P^\ell y(t_{\bar j})-y(t_{\bar j})+y(t_{\bar j})-y(\tau_j)\|_V^2\\ &\le 4\ \|y(t_{\bar j})-y(\tau_j)\|_V^2+2\ \|P^\ell y(t_{\bar j})-y(t_{\bar j})\|_V^2\\ &\le 8\Delta t\ \|y_t\|^2_{L^2(t_{\bar j-1},t_{\bar j+1};V)}+\frac{4\alpha_{\bar j}}{\delta t}\ \|P^\ell y(t_{\bar j})-y(t_{\bar j})\|_V^2.\end{aligned}$$

Thus, we get

$$\sum_{j=1}^k\delta\tau_j\ \|\varrho_j\|_V^2\le 4\sigma_n\Delta\tau\Delta t\ \|y_t\|^2_{L^2(0,t_{\bar k+1};V)}+\frac{4\sigma_n\Delta\tau}{\delta t}\sum_{i=\ell+1}^d\lambda_i.\tag{B.17}$$

Combining (B.14)–(B.17) the claim follows.

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