Preconditioning for Efficiently Applying Algebraic Multigrid in Fully Implicit Reservoir Simulations

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Summary

Fully implicit black-oil simulations result in huge, often very-ill-conditioned, linear systems of equations for different unknowns (e.g., pressure and saturations). It is well-known that the underlying Jacobian matrices contain both hyperbolic and nearly elliptic subsystems (corresponding to saturations and pressure, respectively). Because a reservoir simulation is typically driven by the behavior of the pressure, constrained-pressure-residual (CPR)-type two-stage preconditioning methods to solve the coupled linear systems are a natural choice and still belong to the most popular approaches. After a suitable extraction and decoupling, the computationally most costly step in such two-stage methods consists in solving the elliptic subsystems accurately enough. Algebraic multigrid (AMG) provides a technique to solve elliptic linear equations very efficiently. Hence, in recent years, corresponding CPR-AMG approaches have been extensively used in practice.

Unfortunately, if applied in a straightforward manner, CPR-AMG does not always work as expected. In this paper, we discuss the reasons for the lack of robustness observed in practice, and present remedies. More precisely, we will propose a preconditioning strategy (based on a suitable combination of left and right preconditioning of the Jacobian matrix) that aims at a compromise between the solvability of the pressure subproblem by AMG and the needs of the outer CPR process. The robustness of this new preconditioning strategy will be demonstrated for several industrial test cases, some of which are very ill-conditioned. Furthermore, we will demonstrate that CPR-AMG can be interpreted in a natural way as a special AMG process applied directly to the coupled Jacobian systems.

Introduction

Reservoir simulators dealing with fully implicit or adaptive implicit models (FIMs, AIMs) require the repeated solution of large linear systems of equations with Jacobian matrices resulting from the linearization (e.g., by Newton's method) of the physical mass-balance equations. These systems are associated with various different types of unknowns such as pressure, saturations, or concentrations. Temperature or mechanical properties could be part of the system as well, depending on the type of application (Aziz and Settari 1979; Cao 2002; Lacroix et al. 2003; Scheichl et al. 2003). Even though the focus in this paper will be on black-oil models, the essential ideas carry over to the flow-related part of more-complex models.

Numerically solving the Jacobian systems in a way that is both fast and robust is still a challenge in reservoir simulation. Standard solvers used today are based either on classical incomplete lower/upper (ILU) factorization techniques directly applied to the full coupled system (Collins et al. 2003) or on the CPR method (Wallis 1983; Wallis et al. 1985). The idea behind the latter is to exploit the fact that ILU techniques are very well-suited for the hyperbolic part, and multigrid methods (in particular, AMG) are known to be very efficient for the elliptic part (Ruge and Stüben 1986; Cleary et al. 1998; Stüben 2001a). CPR attempts to combine both methods in a two-stage preconditioning approach. To further improve this, one usually approximately "decouples" hyperbolic and elliptic parts of the Jacobian system (corresponding to saturation and pressure, respectively).

The combination of ILU and AMG in a CPR approach, usually referred to as CPR-AMG, is generally believed to be much more efficient than plain ILU to solve black-oil models; one reason is the scalability properties of AMG. Indeed, there have been many results presented in the literature demonstrating this (Klie et al. 1996; Lacroix et al. 2003; Masson et al. 2004; Cao et al. 2005; Stüben et al. 2007). Because AMG methods are able to efficiently deal with complex geometries, with discontinuous and varying coefficients, and with anisotropies, they are naturally very attractive for porous-media-flow simulation with its varying permeability fields and resulting heterogeneities. Finally, because AMG in its classical form does not require geometric information, it is also easily integrated into existing reservoir simulators (Stüben et al. 2007; Batycky et al. 2009). Hence, in recent years, AMG has become the most popular choice for the pressure-preconditioning step in reservoir-simulation applications.

Unfortunately, although CPR-AMG with classical decoupling often performs very efficiently, there are applications, especially among the industrial cases considered in this paper, in which AMG faces serious convergence problems and may even not converge. Clearly, convergence problems for the pressure-preconditioning step tend to cause the entire CPR-AMG process to converge more slowly or even not at all. Hence, depending on the concrete situation, the advantage of CPR-AMG over plain ILU is lost. There are essentially two sources for this lack of robustness observed in practice:

- AMG was originally designed to solve matrix problems that correspond to discretized elliptic partial-differential equations (PDEs). However, the pressure matrices considered in reservoir simulations are extracted from Jacobian matrices, and their properties are influenced by the complex physics of porous-media flow. Source terms corresponding to wells and/or fractures especially may locally destroy the "elliptic properties" as required by AMG to a significant extent, causing dramatically reduced AMG convergence or even divergence. Such cases have been observed by users in practice and will be described later in this paper.
- In setting up a CPR process, it is common to decrease the coupling strength between pressure and saturation by applying some "decoupling process" to the Jacobian matrix before starting the CPR process (usually by preconditioning from the left). The primary goal of classical-decoupling processes [such as ABF (alternate block factorization) (Bank et al. 1989) or quasi IMPES (implicit pressure, explicit saturation) (Lacroix et al. 2000; Cao 2002; Stüben et al. 2007)] is to ensure fast "outer" CPR convergence. Unfortunately, as a negative side effect, the use of left preconditioners will algebraically transform the original pressure matrix (in particular, by mixing pressure and saturation quantities) at the expense of an unpredictable influence on AMG's performance. In practice, AMG may no longer converge at all

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despite the fact that it was very efficient at solving the original pressure system (i.e., before the decoupling).

The potential negative impact of the preceding two sources on the CPR-AMG performance is known and has been mentioned previously in the literature (Stüben et al. 2007; Clees and Ganzer 2010). The classical CPR-AMG decoupling processes tend to cause a conflict between fast AMG convergence and fast convergence of the outer CPR iteration.

The main concern of this paper is to design a robust preconditioning strategy that takes both sources of difficulties for CPR-AMG into account. As a result, we will propose a method (based on a suitable combination of left and right preconditioning of the Jacobian matrix) that aims at a compromise between the solvability of the pressure subproblem by AMG and the needs of the outer CPR process. We will demonstrate the robustness of this new strategy for some industrial cases that could not be treated efficiently previously, and we will show that the superiority of CPR-AMG compared with a plain ILU solver is maintained for all considered cases. All AMG-related numerical results have been achieved with Fraunhofer's solver library SAMG (Stüben 2012).

Finally, we will explain how CPR-AMG can be interpreted in a natural way as a special AMG process applied directly to the coupled Jacobian systems. This interpretation is useful in simplifying the process for solving the Jacobian systems (at least from a user's point of view) because all preconditioning steps now naturally can be hidden; they actually can become part of the solver itself. We demonstrate this by applying the solver library SAMG.

The paper is organized as follows. First, a brief introduction to the black-oil model under consideration is given along with a short description of the industrial test cases investigated in this paper. The next two sections briefly summarize the CPR and AMG techniques, respectively. Timing results for full-simulation runs are reported that demonstrate the significant performance benefits of CPR-AMG in those test cases in which AMG works well with simple preconditioners. The remaining sections will concentrate on critical applications. First, the dynamic row sum (DRS) method (a left preconditioner) is described that explicitly focuses on the solvability of a pressure matrix by AMG. Next, a decoupling process is discussed that preserves the pressure matrix and its suitability for AMG (a right preconditioner). Finally, we describe how CPR-AMG can be interpreted as an AMG process for a fully coupled Jacobian system. Conclusions and a discussion of ongoing work are given at the end.

Description of the Equations

This study focuses on black-oil reservoir simulation (Aziz and Settari 1979); however, the principles are applicable for compositional simulations and the flow-related part of more-complex simulations as well. In the black-oil case, the following conservation equations for three components (oil, gas, and water) need to be fulfilled:

$$\frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right)+q_o-\nabla\left[\frac{\lambda_o k}{B_o}(\nabla p_o-\rho_o\overline{g})\right]=0,$$
$$\frac{\partial}{\partial t}\left[\phi\left(\frac{R_s S_o}{B_o}+E_g S_g\right)\right]+R_s q_o+q_g-\nabla\left[\frac{R_s\lambda_o k}{B_o}(\nabla p_o-\rho_o\overline{g})+E_g\lambda_g k(\nabla p_g-\rho_g\overline{g})\right]=0,$$
$$\text{and }\frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right)+q_w-\nabla\left[\frac{\lambda_w k}{B_w}(\nabla p_w-\rho_w\overline{g})\right]=0 \quad \cdots\cdots (1)$$

where B_l = formation volume factor for phase l, E_g = gas-expansion factor, $\overline{g}$ = gravity vector, k = absolute permeability, p_l = pressure in phase l, q_l = well term for phase l, R_s = solution gas/oil ratio, S_l = saturation of phase l, t = time, λ_l = mobility of phase l, ρ_l = density of phase l, and ϕ = porosity.

These equations are discretized into finite-difference form by use of a control-volume approach with central differences in space and upstream mobilities. Three primary variables are chosen for each grid cell. If there is free gas present, the primary variables are oil pressure, water saturation, and oil saturation (p_o, S_w, S_o). When no free gas is present, the bubblepoint pressure replaces the oil saturation (p_o, S_w, p_b). In general, in compositional simulations, the number of saturations and concentration unknowns will be larger, depending on the number of components. Further details of the model can be found in the literature (Aziz and Settari 1979; Forsyth and Sammon 1986).

Because of pressure- and saturation-dependent properties, the system (Eq. 1) is highly nonlinear. It is important to note that, by treating saturations and formation volume factors implicitly within accumulation terms (time derivatives), it is not simple to construct a "true IMPES" equation (Lacroix et al. 2000; Cao 2002) for this formulation. Newton iteration is used to linearize the problem that requires the solution of an associated Jacobian system. Distinguishing between pressure- and saturation-type unknowns, p and S, respectively, the resulting linear system, ordered by physical unknowns, is given by

$$A_{\text{unknown-wise}}\begin{pmatrix}p\\s\end{pmatrix}=\begin{pmatrix}A_{pp}&A_{ps}\\A_{sp}&A_{ss}\end{pmatrix}\begin{pmatrix}p\\s\end{pmatrix}=\begin{pmatrix}f_p\\f_s\end{pmatrix}. \quad \cdots\cdots (2)$$

In Eq. 2, A_{pp} is the submatrix aligned with the pressure unknown, often referred to as the pressure matrix. A_{ss} contains the submatrices aligned with the saturation unknowns. The couplings between the different physical unknowns are given in the submatrices A_{ps} and A_{sp}.

However, we will use a point-wise notation (Eq. 3) to describe the preconditioning methods below. With n_p as the number of gridpoints and n_u as the number of physical unknowns per point, the Jacobian matrix is ordered into different blocks $[A]^{i,j}$. The previous unknown-wise structure now holds within each of these blocks:

$$A_{\text{pointwise}}=\begin{pmatrix}[A]^{1,1}&\cdots&[A]^{1,n_p}\\\vdots&\ddots&\vdots\\[A]^{n_p,1}&\cdots&[A]^{n_p,n_p}\end{pmatrix}$$
$$=\begin{pmatrix}[A_{pp}]^{1,1}&[A_{ps}]^{1,1}&&[A_{pp}]^{1,n_p}&[A_{ps}]^{1,n_p}\\[A_{sp}]^{1,1}&[A_{ss}]^{1,1}&\cdots&[A_{sp}]^{1,n_p}&[A_{ss}]^{1,n_p}\\\vdots&&\ddots&\vdots&\\[A_{pp}]^{n_p,1}&[A_{ps}]^{n_p,1}&&[A_{pp}]^{n_p,n_p}&[A_{ps}]^{n_p,n_p}\\[A_{sp}]^{n_p,1}&[A_{ss}]^{n_p,1}&\cdots&[A_{sp}]^{n_p,n_p}&[A_{ss}]^{n_p,n_p}\end{pmatrix} \quad \cdots\cdots (3)$$

Finally, each single matrix entry can be addressed as $[a_{x,y}]^{i,j}$ with x and y defining the position within the point-block i,j:

$$[A_{p,p}]^{i,j}=[a_{1,1}]^{i,j} \qquad [A_{ps}]^{i,j}=\begin{pmatrix}[a_{1,2}]^{i,j}&\cdots&[a_{1,n_u}]^{i,j}\end{pmatrix}$$
$$[A_{sp}]^{i,j}=\begin{pmatrix}[a_{2,1}]^{i,j}\\\vdots\\[a_{n_u,1}]^{i,j}\end{pmatrix} \qquad [A_{ss}]^{i,j}=\begin{pmatrix}[a_{2,2}]^{i,j}&\cdots&[a_{2,n_u}]^{i,j}\\\vdots&\ddots&\vdots\\[a_{n_u,2}]^{i,j}&\cdots&[a_{n_u,n_u}]^{i,j}\end{pmatrix}. \quad \cdots\cdots (4)$$

In the following, brackets [ ] will refer to this point-based ordering. In situations in which both orderings are used, an implicit reordering is implied.

The methods to be described are tested with the following exemplary test cases supplied by Computer Modelling Group (CMG).

- **SPE10.** This model is based on the fine grid case for Model 2 from the Tenth SPE Comparative Solution Project (Christie and Blunt 2001). It has $60\times220\times85=1.12$ million cells, with 1.09 million active. It consists of a single five-spot pattern, with a geostatistically generated permeability distribution. It is a two-phase (oil/water) water-injection problem. It is run for 2,000 days simulated time. A plot of the vertical permeability, showing its heterogeneity, is given in **Fig. 1a.**

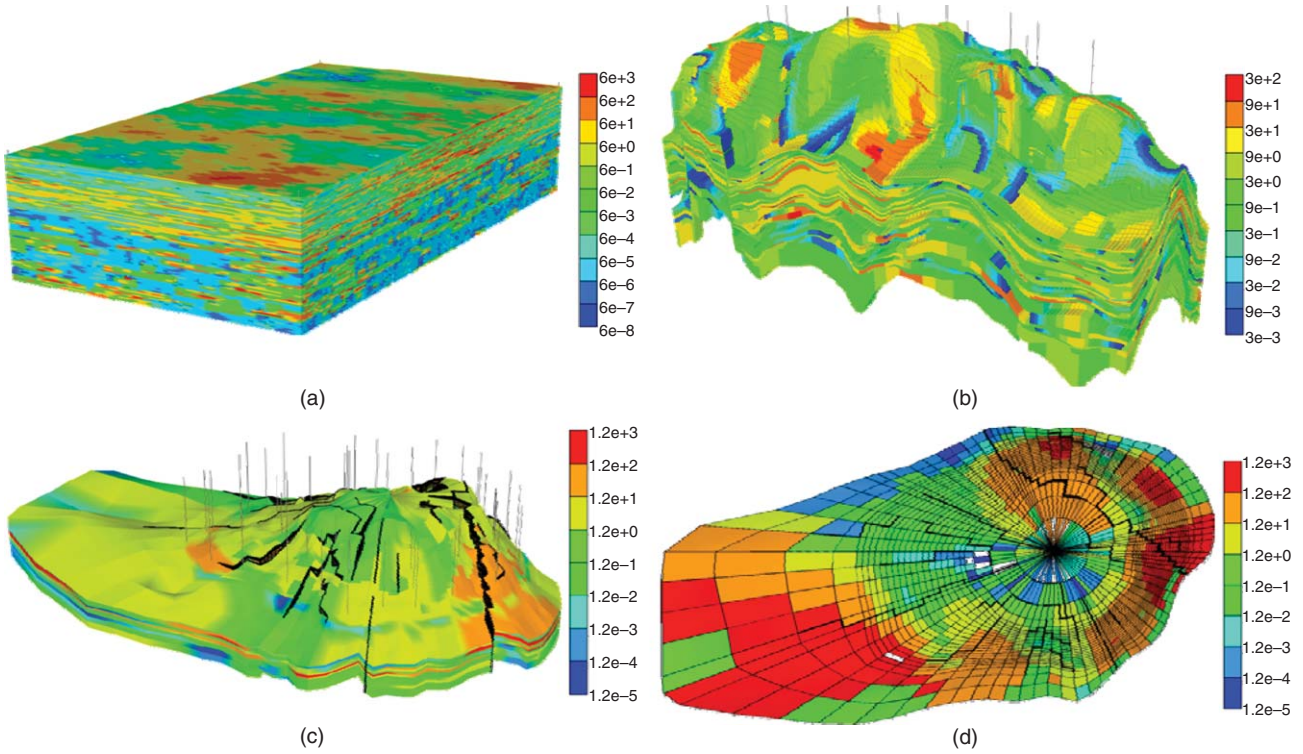

Fig. 1—Vertical-permeability plots of (a) SPE10, (b) TM2T, (c) CPUTEST in three dimensions, and (d) Vertical Layer 5 of CPUTEST.

• **MX1041.** This model consists of 40 five-spot patterns in a 20×2 areal arrangement divided into 10 vertical layers. It has $1041 \times 105 \times 10 = 1.09$ million cells aligned on a Cartesian grid, all active. It is a three-phase (oil/water/gas) water-injection problem and is simulated for 15 years. It has 103 wells. The permeabilities and porosities are constant. This simulation is designed so that free gas appears and then increases throughout the simulation.

• **MX521.** This model consists of 90 five-spot patterns in a 10×9 areal arrangement divided into 20 vertical layers. It has $521 \times 469 \times 20 = 4.88$ million cells aligned on a Cartesian grid, all active. It is a three-phase (oil/water/gas) water-injection problem and is simulated for 15 years. It has 200 wells. The five-spot patterns are identical to those of MX1041, with twice the grid refinement in the vertical direction.

• **PCTST:** This model is based on client data characterized by an irregular region. It is discretized with $123 \times 40 \times 44 = 216.5$ thousand grid cells with 195.1 thousand active. It is a three-phase (oil/water/gas) primary-production problem. It is run for 3,440 days simulated time and contains eleven production wells.

• **TM2T:** This model is a refined grid version of PCTST. Each cell from the original client data set was split equally, areally into $3 \times 3 \times 1$ cells. The model has $369 \times 1200 \times 44 = 1.95$ million cells with 1.64 million active. It is a three-phase (oil/water/gas) primary-production problem. It is run for 3,440 days simulated time and contains eleven production wells. A plot of the vertical permeability is presented in Fig. 1b.

• **CPUTEST:** This model is based on a client data set. It has $26 \times 81 \times 16 = 33.7$ thousand grid cells with 29.5 thousand active. It is a corner-point approximation to a radial-like grid. It is a three-phase (oil/water/gas) water-and-gas-injection problem. It is run for approximately 32 years simulated time. It has 47 wells. Plots of the vertical permeability in three dimensions and areally for the fifth layer in Figs. 1c and 1d give an impression of this test case's heterogeneity and variation of cell sizes. Although this simulation does not result in the largest matrix, AMG still shows advantages for CPUTEST after proper preconditioning. The smaller size of the matrix allows for more-detailed investigation, described later.

CPR Method

The CPR (Wallis 1983; Wallis et al. 1985) method is a common strategy for iteratively solving the described linear systems of equations. This method, also known as two-stage preconditioning, exploits the different properties of the different parts of the matrix. The basic strategy for CPR's kth iteration is as follows (Lacroix et al. 2003; Cao et al. 2005):

1. Roughly approximate the pressure:

$$A_{pp}p' = f_p - A_{ps}S^{k-1}. \quad \ldots\ldots\ldots\ldots\ldots (5)$$

2. Apply an incomplete factorization iteration; starting with $\begin{pmatrix} p' \\ S^{k-1} \end{pmatrix}$, to compute the kth solution $\begin{pmatrix} p^k \\ S^k \end{pmatrix}$.

The pressure is known to drive fluid flows (Axelsson 1994; Lacroix et al. 2000), and thus, investing additional effort in its approximation can be expected to accelerate the total-solution process. Because the pressure-related part A_{pp} in the Jacobian is expected to have properties close to the ones of a matrix derived from the discretization of an elliptic PDE, the AMG methods are a common choice for this purpose. These methods are known to be, in general, very efficient for such kinds of matrices (Ruge and Stüben 1986; Cleary et al. 1998; Stüben 2001b). Fraunhofer's SAMG software library (Stüben 2012) will be used in this study. For the saturation-related part A_{ss}, single-level iterative methods, such as incomplete factorizations, usually are efficient.

Before applying CPR, it is common to left-precondition the Jacobian by a matrix L:

$$L\,A = L\begin{pmatrix} A_{pp} & A_{ps} \\ A_{sp} & A_{ss} \end{pmatrix} =: \begin{pmatrix} \tilde{A}_{pp} & \tilde{A}_{ps} \\ \tilde{A}_{sp} & \tilde{A}_{ss} \end{pmatrix} =: \tilde{A}. \quad \ldots\ldots\ldots (6)$$

The operator L usually aims at approximately decoupling pressure and saturation (i.e., minimizing $||\tilde{A}_{ps}||$, to some extent). To construct this decoupling, an important observation is exploited: The pressure/saturation couplings $[A_{ps}]^{i,i}$ within the diagonal point-blocks of A are expected to dominate those in off-diagonal blocks (Lacroix et al. 2000; Scheichl et al. 2003; Jiang 2007). Thus, the decoupling operator L usually aims at decoupling the

diagonal blocks $[\tilde{A}]^{i,i}$, only. The resulting operator applied to the point-wise ordered matrix will therefore have a block-diagonal structure:

$$L_{\text{point-wise}} := \begin{pmatrix} [L]^1 & & 0 \\ & \ddots & \\ 0 & & [L]^{n_p} \end{pmatrix} \text{ with } [L]^i \in \mathbb{R}^{n_u \times n_u}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Applying any matrix operations on the small $n_u \times n_u$ blocks $[L]^i$ can be expected to be inexpensive. This gives a great flexibility in how to construct L, and various approaches are reported in the literature. ABF (Bank et al. 1989), developed to decouple general discretized systems of PDEs by a block-diagonal scaling, is one option. A more conservative method has been developed on the basis of ABF (Stüben et al. 2007). Other approaches use Schur complements on the diagonal blocks (Cao 2002; Jiang 2007) or try to reduce the impact on the pressure matrix with local QR decompositions (Cao 2002).

A common goal of all developments is to apply as few changes as possible to the linear system. Especially, conserving the "elliptic-like" properties of the pressure matrix (that may be influenced by the application of a left-preconditioning operator) is important for the efficient application of multigrid methods in CPR's first stage. We will describe further approaches focusing on this aspect in later sections.

AMG

AMG methods have been developed to efficiently solve linear systems typically arising in the discretization of scalar elliptic PDEs. Particular properties of such matrices are exploited by a hierarchy of grid levels. In contrast to one-level methods, this approach aims at reducing high- and low-frequency error components to the same extent within a single iteration, thereby making multigrid methods more efficient for such kinds of matrices.

In contrast to geometric multigrid, AMG methods do not explicitly rely on any geometric information. The hierarchy of grids is computed purely algebraically in a so-called "setup" phase before starting the iterative-solution process. Therefore, the user is not concerned with providing a reasonable hierarchy, and the computed coarse levels are moreover adapted to the particular matrix. AMG, in practice, is known to be able to deal with nonsymmetric matrices as well as with heterogeneities and anisotropies. These properties make AMG a preferred choice, especially for the application in reservoir simulation. In particular, AMG methods are a popular method for the pressure-preconditioning step in CPR approaches (Eq. 5).

As new models continue to increase in size, a good scalability of the solver is required. Moreover, the linear problems in this context are often challenging. This is because of the complex simulated physics, including heterogeneous permeability fields, varying external influences, and complex grids. Because of its scalability and robustness for linear systems resulting from elliptic PDEs, AMG, in many such cases, is superior to classical one-level solvers (Stüben 2001b).

Finally, because there are different AMG strategies available, each of which allows for various components, numerous variations of algorithms (e.g., different setup or smoothing schemes) are possible. Thus, AMG offers a great flexibility. Nevertheless, because in concrete applications it can be used as a "black-box" solver with a robust default strategy, in general, the integration of AMG to a simulation code is rather easy. Detailed information on the AMG technique can be found in the literature (Ruge and Stüben 1986; Stüben 2001a, b). Here, for completeness, only some basic remarks shall be given.

In computing its hierarchy during the setup phase, AMG mimics geometric grids by interpreting the matrix as a graph and distinguishing between strong and weak couplings. Here, strong means that the smoother method of the AMG cycle is expected to sufficiently reduce high-frequency error components. On the basis of this graph, a maximally independent set of nodes is computed that defines the next coarser level. The transfer operators between different levels are defined by computing interpolation weights on the basis of the matrix entries. Finally, the coarse-level operator is computed, on the basis of these transfer operators, with the Galerkin principle (Ruge and Stüben 1986; Stüben 2001a).

In this way, the setup adapts AMG to the individual matrix structure so that simple relaxation schemes, such as Gauss-Seidel, usually provide sufficient smoothing. In the solution phase, the efficient interplay of smoothing by simple-relaxation schemes and the coarse-grid correction is exploited, recursively on the basis of the hierarchy of levels.

However, we have to keep in mind that AMG originally was developed for matrices of certain classes, in particular for weakly diagonally dominant M-matrices as they arise in the discretization of scalar elliptic PDEs. In practical applications, especially in industrial contexts, such ideal properties are rarely satisfied strictly.

In the past, AMG approaches have been substantially generalized. However, basic properties of the matrices to solve (such as ellipticity) still have to be satisfied. Otherwise, AMG's performance may degrade. Hence, the purpose of this paper is to describe preconditioning methods that especially focus on the problems' solvability by AMG.

CPR-AMG: Potential and Limitations

To demonstrate the potential acceleration and current limitations when using AMG in the CPR context, benchmarks with IMEX (CMG 2013), a black-oil reservoir simulator with adaptive implicit Jacobian construction, have been made. As a reference method to compare with, CMG's ILU-based sparse linear solver, PARASOL (Collins et al. 2003), is used with OpenMP parallelization. This is a generalized minimum residual (GMRes) solver, preconditioned by a variable-degree ILU. Parallelization is achieved through the partitioning of the reservoir into disjoint sets of grid-cell subdomains (classes). The classes are further organized into levels in which there must be no flow between grid cells of different classes at the same level (Hendrikson and Rothberg 1999; Collins et al. 2003).

The SAMG software package (Stüben 2012) was coupled to PARASOL by means of a CPR method, (Eq. 5), resulting in a reservoir simulator referred to here as IMEX-SAMG. The initial version of the resulting CPR-AMG solver uses the ABF decoupling method (Bank et al. 1989) as a preconditioner. Applying ABF by itself can lead to a less-favorable matrix for an efficient AMG treatment. As a first improvement, an additional preconditioning step is used. Each row of the extracted-pressure matrix is multiplied by the scalar value of the mass or surface-volume weighted sum of the original matrix entries associated with the pressure component of the corresponding grid cell $\left(\text{i.e., } \left|\sum_x^{n_u} [a_{x,1}]^{i,i}\right|\right)$. Indeed, this row-scaling preconditioner can be thought of as a simple approximation to the DRS preconditioner described in later sections.

The number of AMG cycles and the number of ILU steps performed in the first and second preconditioning steps of CPR, respectively, is crucial for the overall efficiency. A loose relative-residual tolerance for the pressure solution from the inner-CPR preconditioning step was chosen rather than the use of a fixed number of AMG cycles. This choice still leads to a small number of AMG cycles per outer iteration, while being more robust. For the considered simulation cases, a desired residual reduction between 0.7 and 0.8 gave the best results.

The following hardware and choice of parameters have been used:

- The simulations were run under Red Hat Enterprise Linux 6.3 on a Dell PowerEdge M820 with four 2.7 GHz Xeon E5-4650 eight-core processors and 256 GB of 1600 MHz DDR2 RAM. Intel's designations of four eight-core processors, for a total of 32 cores, are designated as 32 CPUs.

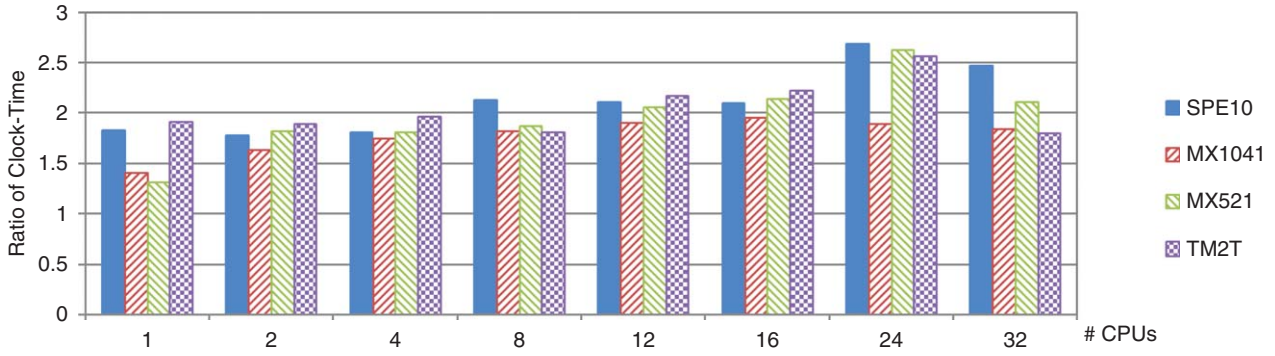

Fig. 2—Improvement ratio in clock-time for the full simulation with IMEX-SAMG compared with IMEX.

• For PARASOL's parallel ILU, the number of subdomains, and therefore first-level solver classes, equals the number of CPUs.

• For IMEX-SAMG, SAMG as a preconditioner for flexible GMRes (Saad 1996) was used, essentially with its default parameters. Some minor-performance improvement was achieved by a modified smoother. One sweep of full Gauss-Seidel and Jacobi was used for pre- and post-smoothing, respectively. In both cases, a Gauss-Seidel F-relaxation sweep was added.

For the selected cases, **Fig. 2** shows a significant improvement up to 2.5 in the ratio of clock-time, defined as the total elapsed time of simulation for IMEX-SAMG divided by that for IMEX, for runs using 1 to 32 CPUs. The solver portion of the simulation takes 60 to 90% of the total run time for IMEX-only simulations. The main reason for the better performance is a reduction in the number of outer CPR-AMG iterations to 5 to 15% of the number of ILU iterations when not using CPR-AMG, as shown in **Table 1** for the example case of 32 CPUs. This more than compensates for the additional cost of AMG's setup phase and, on average, one to three AMG cycles per outer iteration.

Other data sets have been tested that did not perform as well with IMEX-SAMG. For example, the CPUTEST case took 2.7 times longer with IMEX-SAMG compared with IMEX. This was predominantly because of convergence issues for AMG in CPR's inner stage; AMG did not work properly with the constructed pressure submatrix. For these cases, the CPR method, decoupled and preconditioned as stated previously, did not improve the efficiency. In summary, only the ILU-based solver (PARASOL) was able to solve all these test cases. In the next section, a new preconditioning method that allows the efficient solution of all considered test cases with IMEX-SAMG will be described.

In addition to analyzing performance, the accuracy of results must be confirmed. The accuracy of the results computed by IMEX-SAMG is checked by comparing them to IMEX results. Plots of oil-production rate, water-production rate, average reservoir pressure, and average gas saturation are shown for the TM2T case run with 32 CPUs in **Fig. 3.** The production rates are perfectly aligned, and there is very little difference in average pressure and gas saturation. This is expected because the outer control of the solver in each simulation is identical, leaving very little margin for the "drift" of results. Average reservoir pressures and cumulative-production values of oil, water, and gas at the end of simulation for each of the test cases are listed in **Table 2.**

Remarks:

• Results are shown for parallel runs, not to examine the parallel scalability of this method, which is beyond the scope of this paper, but rather to show that the use of IMEX-SAMG is an improvement over IMEX in parallel as well as in series. This is important because the eventual expected use of this solver will be for large data sets when it is necessary to run in parallel.

• For the simulations presented in Fig. 2, generalized red/black ordering was used within each cell subdomain (class). Note that the boundaries of each class were set to all-black. The level of fill for the outer preconditioner was chosen to be ILU(0) for intraclass terms (submatrices relating cells entirely within a subdomain) and ILU(1) for interclass terms (matrix connections between cells in different subdomains).

Natural ordering was also tested for these cases, but for IMEX, as well as IMEX-SAMG, it resulted in increased elapsed times. This was because approximately twice as many variables were solved for by the CPR-AMG method and, in terms of robustness, ILU(1) and ILU(2) were chosen inside each cell subdomain and for interclass terms, respectively.

• Two-stage preconditioners are based on the idea that the coupled-system solutions are mainly determined by the solution of their nearly elliptic portion (i.e., pressure) for which AMG is very efficient. However, within a simulation run, matrices arise in which the hyperbolic portion (nonpressure variables that have a directional dependence) dominates. In these cases, a single-level technique, such as that of standard PARASOL, may be more effective to determine a solution. To further optimize the performance, a threshold-checking mechanism is under development for IMEX-SAMG to control when AMG preconditioning shall be used.

A simple test has been constructed to determine whether AMG should be used for a particular Newton iteration. For each point-block row i, the elements in $[A_{ps}]^{i,j}$ $(j=1,\ldots,n_p)$ are compared with the diagonal element of the pressure matrix. If a certain fraction of the matrix is deemed to have "large-off-diagonal" rows, the two-stage preconditioner is not used for that matrix. This

TABLE 1—DETAILS FOR THE COMPARISON OF IMEX AND IMEX-SAMG SOLVER PERFORMANCE IN THE CASE OF 32 CPUs

Case	Total Elapsed Clock-Time (seconds)		Improvement Ratio in Clock-Time	No. of FGMRes Iterations		Reduction Ratio in Iterations
	IMEX	IMEX-SAMG		IMEX	IMEX-SAMG	
SPE 10	2164.27	877.76	2.466	62875	7132	0.113
MX 1041	316.71	172.27	1.838	12266	1618	0.132
MX 521	2157.69	1023.6	2.108	16556	2110	0.127
TM2T	1366.26	758.99	1.800	44369	2809	0.063

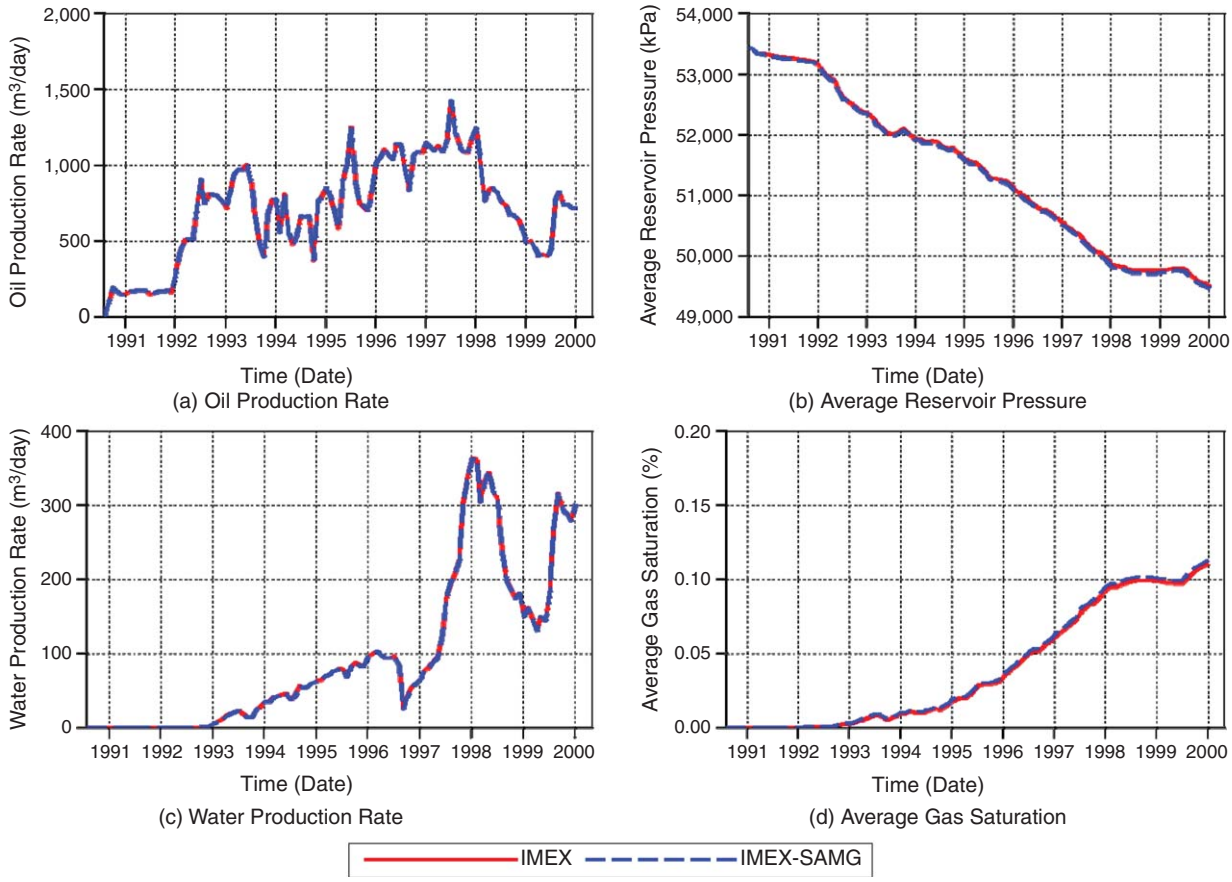

Fig. 3—Simulation results of TM2T case with 32 CPUs.

method is still in development, and default threshold levels for each variable type and the total fraction are being determined. Early tests that use the same thresholds for all test problems show that the two-stage preconditioner is turned off for matrices in which IMEX-SAMG performs less efficiently and remains on for matrices in which IMEX-SAMG outperforms IMEX. However, because it is not finalized, this threshold-checking mechanism was not used for the simulations reported in this paper.

DRS Preconditioner

In the following, a new preconditioner, the DRS method, is developed that aims at constructing pressure matrices that are suitable for AMG and accelerate the CPR process to a significant extent. The preconditioner is based on the approach described by Scheichl et al. 2003 in which, keeping in mind that the pressure is the first unknown, the blocks $[L]^i$ in the preconditioning operator L (Eq. 7) are defined as

$$[L]^i = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

The resulting pressure matrix $\tilde{A}_{pp}$ is the sum of all pressure-aligned parts in the Jacobian system. Consequently, a pressure equation close to the total-pressure equation used in IMPES schemes (Coats 2000) is generated, and thus the method can be considered to give a quasi-IMPES pressure equation.

Unless particular difficulties are present in the original Jacobian system, AMG is an established, highly efficient strategy for solving IMPES-like total-pressure equations (Batycky et al. 2009; Geiger et al. 2009); thus, AMG should be an efficient choice for the quasi-IMPES pressure equation as well. In fact, because the Jacobian's "elliptic parts" are concentrated in $\tilde{A}_{pp}$ (Masson et al. 2004), this resulting pressure equation can be expected to have comparable properties. Thus, no new difficulties from AMG's perspective are introduced by the preconditioner. This differs from preconditioning methods primarily focused on decoupling pressure and saturation, which results in a purely algebraic scaling for the pressure submatrix leading to possible convergence problems for AMG and the total CPR-AMG, as seen later in **Figs. 4 through 7.**

However, difficulties might already have been present in the original Jacobian system. This especially holds for the matrices under consideration that do not directly result from an elliptic PDE but are rather extracted parts of Jacobian matrices. If the

TABLE 2—COMPARISON OF IMEX AND IMEX-SAMG SIMULATION RESULTS IN THE CASE OF 32 CPUs (EACH TABLE ENTRY READS AS: VALUE COMPUTED BY IMEX | VALUE COMPUTED BY IMEX-SAMG | % DIFFERENCE)

Case	Cumulative Oil Production (10^3 m^3)			Cumulative Water Production (10^3 m^3)			Cumulative Gas Production (10^6 m^3)			Average Reservoir Pressure [kPa]		
SPE10	474.7	471.0	−0.775	1133.5	1119.3	−1.247				30908	30894	*-0.046*
MX1041	16778	16777	−0.006	0.1	0.1	+0.011	12557	12558	+0.008	5157	5156	*-0.009*
MX521	36271	36268	−0.008	0.2	0.2	+0.005	28300	28299	−0.004	5173	5170	*-0.068*
TM2T	2478.5	2479.4	+0.036	322.5	321.5	−0.307	575.3	574.8	−0.078	49682	49640	*-0.084*

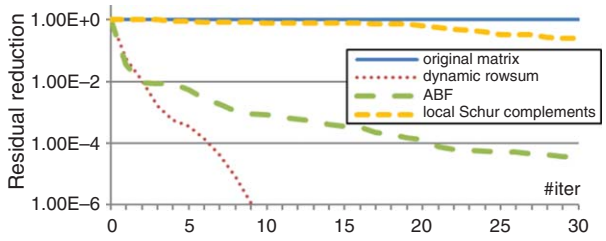

Fig. 4—AMG convergence for the pressure matrix, case CPUTEST.

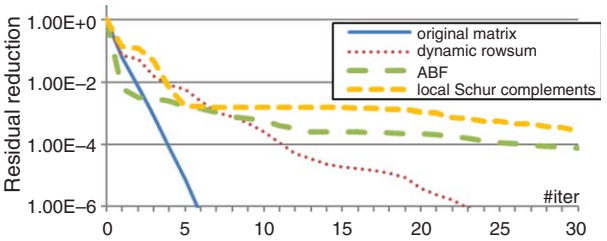

Fig. 5—AMG convergence for the pressure matrix, case PCTST.

properties expected by AMG are violated too strongly, the interplay between smoothing and coarse-grid correction does not work properly, which may result in AMG convergence troubles.

In practice, the extracted original pressure matrices A_{pp} may even be indefinite. **Fig. 8** shows the plot of eigenvalues in the complex plane for A_{pp} from a single Newton iteration of the test case CPUTEST before applying any preconditioning. To focus on eigenvalues with negative real parts (here, approximately 2% of all eigenvalues), only near-zero eigenvalues are shown. In fact, 11% of the diagonal entries are negative. Their appearance seems to be directly related to the relatively high number of active wells at the particular timestep.

Applying AMG directly to this matrix is a challenge. However, even for such ill-conditioned matrices, there are fallback strategies to make AMG converge. For this particular matrix, a combination of a Schwarz iterative method (Schwarz 1870; Lions 1988), defect correction (Boehmer et al. 1984; Naumovich et al. 2010), and a coarsening strategy that actively prevents negative diagonals at coarser levels allows the solution of this matrix by the use of AMG.

Clearly, it is not desirable to investigate such fallback strategies individually per matrix. Instead, it is preferable to avoid such difficulties to begin with to allow standard-AMG strategies to work. For this purpose, the block diagonals of the preconditioner (Eq. 8) are modified to be dynamic:

$$[L]^i = \begin{pmatrix} \delta_1^i & \delta_2^i & \cdots & \delta_{n_u}^i \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

The entries δ_x^i are computed individually for each point-block i because the crucial physical forces of the system, which give rise to the difficult matrix properties, do not necessarily appear in all phases to the same extent. The idea of the dynamic preconditioner is to exploit the different properties of the phases to avoid unwanted matrix properties with respect to the applicability of AMG. We select the dynamic entries as follows:

$$\delta_x^i = \begin{cases} 0 & \text{if} \quad \dfrac{[a_{x,1}]^{i,i}}{\sum_{j=1,j\neq i}^{n_{\text{points}}} |[a_{x,1}]^{i,j}|} < \varepsilon_{dd} \\ 1 & \text{else} \end{cases}$$

$$\text{where } 0 \leq \varepsilon_{dd} \leq 1. \quad \ldots\ldots\ldots\ldots\ldots (10)$$

Only those parts of A are considered for the construction of $\tilde{A}_{pp}$ that are not expected to introduce significant difficulties for the efficient application of AMG. In particular, the appearance of negative diagonal entries is prevented. Furthermore, the parameter ε_{dd} controls the severity of the violation of diagonal dominance. In the limiting case $\varepsilon_{dd} = 1$, DRS will attempt to construct a (weakly) diagonally dominant $\tilde{A}_{pp}$. However, in this case, the outer CPR convergence may unnecessarily slow down. Hence, a compromise has to be made between the solvability of the matrix by AMG and the necessity to exclude as few rows as possible to obtain a pressure equation close to the one in IMPES schemes. For the matrices under consideration, a value of $\varepsilon_{dd} = 0.2$ was found to give the best compromise.

By use of the DRS method described thus far, it is possible to shield AMG from the "most destructive" physical influences and to ensure robust AMG convergence for all the considered test cases. In **Fig. 9,** the plot of eigenvalues for the pressure matrix from the test case CPUTEST, after applying the DRS preconditioner, shows its success. In this particular case, negative eigenvalues no longer occur in $\tilde{A}_{pp}$.

Remarks:

- To prevent ε_{dd} from being chosen too small, a further check is introduced. After all δ_x^i are defined, the diagonal dominance in the resulting ith row of $\tilde{A}_{pp}$ is checked. If it should not be sufficient, the δ_x^i are recomputed with a larger ε_{dd}. Because of the small size of the block $[L]^i$, this check can be expected to be inexpensive.
- If $\delta_1^i = 0$, then $[L]^i$ – and thus L – will become singular. This is corrected by the following procedure:
  - If $\delta_x^i \neq 0$ for some $x > 1$, in the xth row of $[L]^I$, the diagonal has to be moved to the first column.
  - If $\delta_x^i = 0$ for all x, set $\delta_x^i = 1$, where x corresponds to the row that least violates diagonal dominance. To ensure a regular matrix L, some unwanted physical influences on the pressure matrix need to be accepted in this case. In the test cases under consideration, this problem occurs at well equations only, and did not cause further problems. A well-decoupling was not necessary.

With those two conditions imposed, the DRS preconditioner can be guaranteed to be regular. Consequently, solving the modified system of equations $\tilde{A}=LA$ is equivalent to solving the original system.

- For all considered cases, the convergence of CPR-AMG was robust after applying the DRS preconditioner. However, there is not yet a rigorous proof of convergence. As an emergency strategy, it is always possible to implement an automatic strategy that, on the fly, switches off the pressure preconditioning, if ever necessary.

In addition to avoiding unwanted properties in $\tilde{A}_{pp}$, a second dynamic mechanism is introduced. This aims at further improving CPR's overall convergence by monitoring the pressure/saturation couplings. If, for a block-row i, there exists an $x > 1$, such that

$$\sum_{j=1}^{n_p} |[a_{1,x}]^{i,j}| < \varepsilon_{ps} |[a_{1,1}]^{i,i}|, \quad \ldots\ldots\ldots\ldots\ldots (11)$$

then, because the corresponding saturation can be assumed to have a negligible influence on the pressure, this particular pressure/saturation coupling is considered to be weak. Accounting for it when computing the total-pressure equation in gridblock i does not contribute to accelerating the total convergence. Instead, "noise" would be added to the resulting pressure equation. Moreover, one would risk strengthening a weak pressure/saturation coupling. Therefore, for a block-row i in which Eq. 11 holds for some $x>1$, the corresponding δ_x^i is set to zero (unless this would make L singular).

Because excluding a row because of a weak pressure/saturation coupling will, in a sense, neglect the corresponding pressure-related information, the value of ε_{ps} should be chosen carefully. For the considered matrices, a value of $\varepsilon_{ps} = 0.02$ gave the best results.

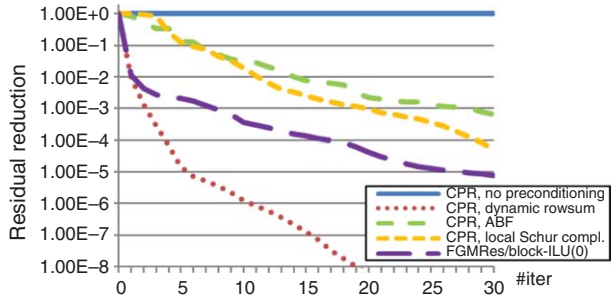

Fig. 6—Results for the full-system matrix, case CPUTEST.

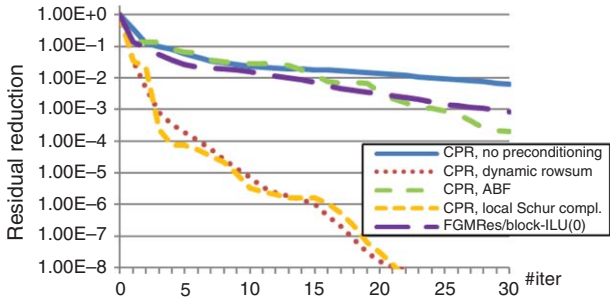

Fig. 7—Results for the full-system matrix, case PCTST.

As a first step, we focus on the solvability of the pressure subproblem by AMG. To check whether AMG gives a stable convergence, these tests are solved to a higher accuracy than usually needed in CPR applications. The pressure submatrix computed by the DRS method is compared with the original pressure problem and the ones after the application of two further popular preconditioning methods. These methods, the ABF (Bank et al. 1989) and the application of Schur complements to each diagonal block of the Jacobian (Lacroix et al. 2000; Jiang 2007), primarily aim at decoupling pressure and saturation. The results in Figs. 4 and 5 show that AMG is able to continuously reduce the residual in both test cases after the application of DRS. The residual reduction is especially continuous and does not stagnate—which would indicate some remaining problematic eigenvalues. This, in addition to the plots of eigenvalues (Figs. 7 and 8), indicates there are no problematic properties remaining in the pressure matrix.

There is one thing to observe. The convergence speed for solving the pressure subproblem of PCTST has decreased after applying DRS compared with the original pressure matrix. However, AMG still converges robustly and continuously. Moreover, the reason for the slower convergence is the introduction of matrix elements from the submatrix A_{sp}. These might have a weaker diagonal dominance than the original pressure matrix in some rows, leading to the slower AMG convergence after DRS. But they do contain further pressure-related information that is beneficial for the total-CPR convergence in Fig. 7.

The results also show that AMG can still accelerate CPR even if it does not converge robustly (e.g., if its residual reduction stagnates). This might happen if occasionally ILU is able to reduce those error frequencies efficiently that AMG has difficulties with. However, this is an application-dependent effect. In general, a robust convergence of AMG is desirable for the given pressure matrix to accelerate the global CPR process. By use of the DRS preconditioner, convergence was obtained for all pressure matrices considered in this study, whereas for other methods, as well as for the original pressure matrix, sometimes a stagnation of the residual reduction was observed for the full linear system, as shown in Figs. 6 and 7.

Remarks:

- Because we are currently focusing on the numerical effect of the new preconditioning method, parallelization has not yet been considered.
- To focus on the effect of the pressure preconditioner on the convergence, a default AMG strategy without any tuning of parameters is used. Convergence histories are given for representative matrices. Flexible GMRes (FGMRes) was always used as an accelerator, and an ILU method was applied in CPR's second stage. Because a block structure was available and performance was not of primary interest, we have used standard block-ILU(0) (Chow and Saad 1997) instead of PARASOL for these convergence comparisons.
- Clearly, CPUTEST and PCTST are too small to fully benefit from AMG's major strength—its scalability. However, difficulties, as observed for these test cases, also occur for bigger problems, and establishing robust AMG processes, which is the primary interest of this paper, is a prerequisite for considering such bigger cases.

Decoupling Operator

When constructing the left-preconditioned matrix $\tilde{A}$, the algebraic decoupling of pressure and saturation (i.e., reducing $||\tilde{A}_{ps}||$) has not yet been considered to this point. To perform such a decoupling, we present two right-preconditioning methods. We do it in a way such that we can ensure that the pressure matrix constructed by the DRS method is not influenced by the decoupling—which was the problem with left-decoupling methods. The right preconditioner, R, analogously to the left one (Eq. 7) is constructed as a block-diagonal matrix:

$$\hat{A} := \tilde{A}R \text{ with } R_{\text{point-wise}} = \begin{pmatrix} [R]^1 & & 0 \\ & \ddots & \\ 0 & & [R]^{n_p} \end{pmatrix}$$
$$\text{and } [R]^i \in \mathbb{R}^{n_u \times n_u} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

The *first approach,* we call it quasi-ABF (qABF), constructs $[R]^i$ for block-row i (comparable to the ABF) by inverting small matrices V^i on the basis of the diagonal blocks $[\tilde{A}]^{i,i}$:

$$V^i = \begin{pmatrix} [\tilde{A}_{pp}]^{i,i} & [\tilde{A}_{ps}]^{i,i} \\ 0 & [\tilde{A}_{ss}]^{i,i} \end{pmatrix} \text{ and } (V^i)^{-1} = \begin{pmatrix} S^i & T^i \\ 0 & U^i \end{pmatrix},$$
$$\text{where } S^i \in \mathbb{R},\ T^i \in \mathbb{R}^{1 \times n_u},\ U^i \in \mathbb{R}^{n_u \times n_u}. \quad \ldots\ldots\ldots (13)$$

Now, $[R]^i$ is defined as

$$[R]^i = \begin{pmatrix} 1 & T^i \\ 0 & U^i \end{pmatrix}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

It can easily be shown that the resulting operator gives a pressure/saturation decoupling in the diagonal blocks. However, there

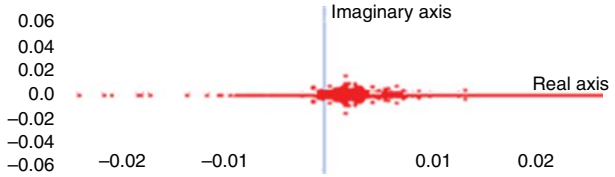

Fig. 8—Excerpt of eigenvalues in the complex plane of the pressure matrix from the original Jacobian of test case CPUTEST.

Fig. 9—Excerpt of eigenvalues in the complex plane of the pressure matrix for CPUTEST after DRS.

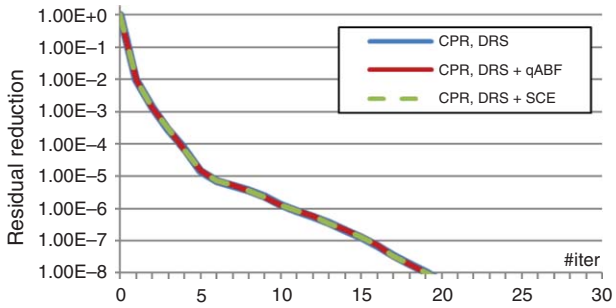

Fig. 10—Results for CPR-AMG, considering decoupling with right-preconditioning, case CPUTEST.

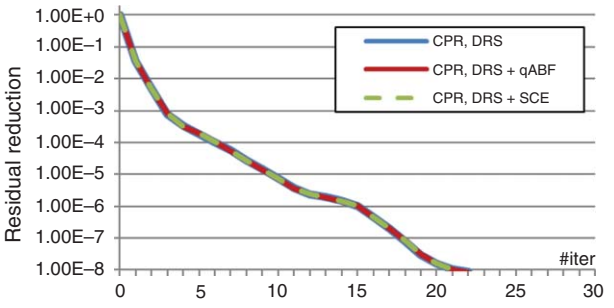

Fig. 11—Results for CPR-AMG, considering decoupling with right-preconditioning, case PCTST.

is one disadvantage; after the application of the right qABF, new off-diagonal entries might have been introduced in $\hat{A}_{ss}$. It is guaranteed that no new nonzero blocks are introduced, but new entries might occur in those already-existing blocks i,j in which $[\tilde{A}_{sp}]^{i,j} \neq 0$. Therefore, this is not a problem for block-ILU.

We present a *second approach* that reduces the risk of additional fill-in. Because one saturation is used to eliminate those in the other columns, we call it the saturation-column-eliminating (SCE) method. This method will not perfectly decouple pressure and saturation, not even in the diagonal block, but it is guaranteed not to mix pressure-and-saturation-related columns, thus reducing the risk of additional fill-in for the matrix.

First, the residual vector $r = Au - f$ is computed. Then, the saturation unknown z with the smallest residual is chosen. The construction of $[R]^i$ again is based on the diagonal block $[\tilde{A}]^{i,i}$:

$$R^i = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \\ & & 1 & & \\ 0 & -\frac{\tilde{a}^{i,i}_{1,1}}{\tilde{a}^{i,i}_{1,z}} & \cdots -\frac{\tilde{a}^{i,i}_{1,z-1}}{\tilde{a}^{i,i}_{1,z}} & 1 & -\frac{\tilde{a}^{i,i}_{1,z+1}}{\tilde{a}^{i,i}_{1,z}} \cdots -\frac{\tilde{a}^{i,i}_{1,n_B}}{\tilde{a}^{i,i}_{1,z}} \\ \vdots & & & 1 & \\ & & & & \ddots \\ 0 & & & & 1 \end{pmatrix} \leftarrow z\text{th row} \quad \cdots\cdots\cdots\cdots\cdots\cdots (15)$$

After the application of the SCE operator, the pressure in the diagonal is decoupled from all saturations except for the zth one. But this is the one with smallest residual. Because of the off-diagonal entries in A_{ps}, a perfect decoupling (meaning $\hat{A}_{ps} = 0$) will usually not be possible for other decoupling methods as well.

The results in **Figs. 10 and 11** indicate that, for the given test matrices, generating quasi-IMPES pressure equations is more important than an algebraic decoupling of pressure and saturation.

The new decoupling methods were also tested without the addition of DRS, but the results were not as good. For CPUTEST, this was expected because of the problematic properties of the original pressure matrix, which are not shielded from AMG in the case in which only qABF or SCE decoupling is applied. For the PCTST case, the use of one of the new decoupling operators alone gives nearly no advantage compared with CPR without any preconditioning.

When using the decoupling methods in addition to the DRS preconditioner, the convergence behavior is nearly identical compared with using DRS only. For the given matrices, this seems to be a result of relatively large off-diagonal pressure/saturation couplings. Consequently, decoupling only the diagonal blocks does not sufficiently weaken the total pressure/saturation couplings $||\hat{A}_{ps}||$. However, the decoupling will pay off for matrices with smaller pressure/saturation couplings in the off-diagonal blocks. In such cases, where the diagonal blocks' couplings dominate, $||\hat{A}_{ps}||$ can be successfully decreased. In particular, this holds for those cells of adaptive implicit discretizations in which the saturation is treated explicitly. In such cases, the decoupling allows the elimination of these saturations from the system.

Solving the Full Jacobian System by AMG

AMG, although originally developed for matrices obtained by the discretization of scalar elliptic PDEs, in principle, can also deal with matrices resulting from coupled systems of PDEs (Clees 2005; Stüben et al. 2007). In such systems, different physical unknowns (e.g., pressure and saturation) are coupled to one another. When solving full Jacobian systems entirely within an AMG context, referred to as a system-AMG approach, it is possible to consider couplings between different physical unknowns on coarser levels as well. This allows (in contrast to operator-splitting approaches) the updating of all unknowns simultaneously.

To construct a hierarchy of levels for different physical unknowns, the user has to provide some additional information in addition to the matrix. Each entry of the solution vector needs to be assigned to one of the physical unknowns. Furthermore, the knowledge of which variables are located at the same point may be exploited by system-AMG.

For systems of mixed type, the unknown that drives the solution often corresponds to an elliptic component that might serve as a so-called "primary unknown." That is, the hierarchy of grids is based on this unknown, and coarsening can be performed similar to a matrix resulting from a scalar elliptic PDE. For Jacobian matrices in fully implicit or adaptive implicit reservoir simulations, the pressure unknown would be a natural primary unknown. The fact that the pressure drives the fluid flow (Axelsson 1994; Lacroix et al. 2000) also is the motivation for the pressure-preconditioning step of CPR.

If there exists a smoothing scheme that is not only able to smooth but also to solve some particular unknowns sufficiently well, then there may be no need to coarsen these unknowns at all. In the FIM/AIM context, this holds for the saturation and concentration unknowns. For such unknowns, the ILU(0) method not only provides a sufficient smoothing, but also is even able to serve as a solver.

From the point of view of system-AMG, one iteration of CPR can consequently be interpreted as a multigrid cycle. In fact, the system-AMG and CPR show a comparable convergence behavior, as seen in **Figs. 12 and 13.** However, even for the black-oil simulations considered here, there are advantages for the user:

- The total number of operations decreases by merging fine-level post-smoothing and CPR's second stage. More precisely, Gauss-Seidel post-smoothing is no longer needed at the finest level.
- High-error frequencies of pressure and saturation at the finest level are reduced simultaneously. Hence, local saturation effects are already taken care of by the pressure's post-smoothing.
- The entire linear-solution process reduces to a single linear solver. This, for instance, allows the incorporation of preconditioners such as DRS into the solution process, and monitoring the

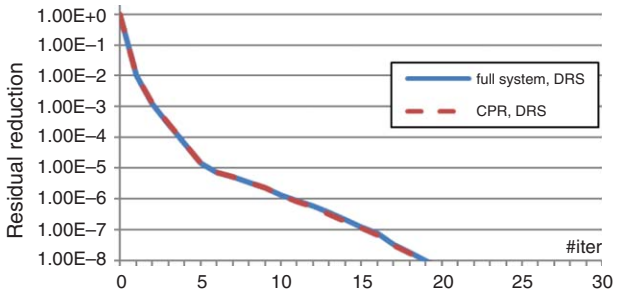

Fig. 12—System-AMG and CPR, both with DRS, case CPUTEST.

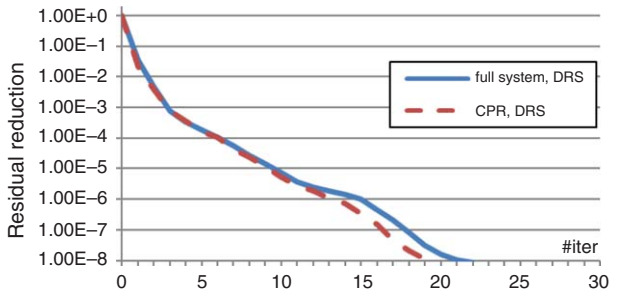

Fig. 13—System-AMG and CPR, both with DRS, case PCTST.

residual reduction in terms of the original system. Moreover, parallelism can be easily exploited for the full system.

Because coarsening still is based on the pressure-related submatrix, it remains important to ensure properties in this submatrix that AMG is able to exploit. This makes preconditioners such as DRS also necessary for the system-AMG approach. Ignoring suitable preconditioning, system-AMG applied to extremely ill-conditioned matrices such as those of CPUTEST will end up with convergence troubles, as in the CPR case.

However, by passing the original Jacobian system to system-AMG, it can be ensured inside the solver that no artificial influences, such as those from decoupling, are introduced. In fact, any preconditioning would take place within the solver, and care could be taken to mitigate destructive influences.

Finally, combining the entire linear solution within an AMG context gives a higher flexibility in selecting details of the AMG process. The full set of existing AMG strategies for systems is available. This, of course, also includes the exploitation of information obtained during the preconditioning stage by the AMG method. If necessary, for instance, even more robustness can be achieved by on-the-fly application of special strategies, as described by Clees and Ganzer (2010), on the basis of information obtained during the application of DRS.

The method also is expandable. More-complex reservoir simulations might deal with additional unknowns. For those corresponding to elliptic components, AMG is an efficient option for the resulting subsystems in the Jacobian. However, instead of solving for these unknowns in a segregated approach, applying AMG to the full system combines the linear solution in a single iterative scheme.

Summary and Conclusions

Classical approaches used to precondition the Jacobian matrix primarily aim at (approximately) decoupling pressure and saturation to accelerate the CPR process. In cases when AMG converges robustly for the considered pressure subblock, such decoupling indeed often helps to substantially reduce the total number of linear-solver iterations. This was demonstrated for some selected applications by applying ABF factorization, combined with a simple row-scaling method, to precondition the Jacobian matrices.

In practice, however, the applicability of AMG sometimes seriously suffers from these types of preconditioning approaches. Moreover, if the pressure matrix is not selected properly to begin with, AMG may not even be robustly applicable at all. The results presented within this study show not only the importance of constructing a pressure matrix that is well-suited for AMG but also the importance of avoiding the potentially destructive influence of standard-decoupling processes.

The so-called DRS method has been introduced as a new possibility to construct reasonable pressure problems. This method basically constructs quasi-IMPES pressure matrices, with additional focus on the suitability for AMG methods. The latter aims at ensuring matrix properties that AMG is able to exploit for achieving its scalability, even in ill-conditioned cases. The effectiveness and robustness of the DRS approach have been demonstrated for some selected cases in which standard approaches failed.

As an alternative to CPR, the possibility of using system-AMG approaches to solve the full Jacobian systems has been examined. Although being algorithmically close to CPR, system-AMG provides a simpler solution technique—in particular, in more-challenging simulations. In fact, the integration of all linear-solver components into a single multigrid process allows an easier application for the user.

Further research will focus on generalizing system-AMG. First, efficient preconditioning methods will systematically be integrated into AMG's setup so that it can benefit from the information obtained while preconditioning. Second, the AMG-cycle can be tuned to the matrix properties as well. Smoothers can be varied, depending on the interplay between pressure and satura-tion.

The DRS preconditioning method will be incorporated in IMEX-SAMG to investigate its performance in case of full-simulation runs. Modifications to DRS may be necessary to treat IMPES and fully implicit grid cells differently. Additional scaling techniques will also be explored.

Finally, our approaches will be extended to Jacobian matrices arising in reservoir simulations that take further physical effects into account. For such more-complex simulations, system-AMG approaches may give even more advantages by allowing for a coupled solution.

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