Chapter 2

Single-Phase Fluid Flow Equations in Multidimensional Domain

2.1 Introduction

The development of flow equations requires an understanding of the physics of the flow of fluids in porous media; the knowledge of fluid properties, rock properties, fluid-rock properties, and reservoir discretization into blocks; and the use of basic engineering concepts. This chapter discusses the items related to single-phase flow. Discussions of fluid-rock properties are postponed until Chapter 10, which deals with the simulation of multiphase flow. The engineering approach is used to derive a fluid flow equation. This approach involves three consecutive steps: 1) discretization of the reservoir into blocks, 2) derivation of the algebraic flow equation for a general block in the reservoir using basic engineering concepts such as material balance, formation volume factor (FVF), and Darcy's Law, and 3) approximation of time integrals in the algebraic flow equation derived in the second step. Even though petroleum reservoirs are geometrically three dimensional, fluids may flow in one direction (1D flow), two directions (2D flow), or three directions (3D flow). This chapter presents the flow equation for single-phase in 1D reservoir. Then it extends the formulation to 2D and 3D in Cartesian coordinates. In addition, this chapter presents the derivation of the single-phase flow equation in 3D radial-cylindrical coordinates for single-well simulation.

2.2 Properties of Single-Phase Fluid

Fluid properties that are needed to model single-phase fluid flow include those that appear in the flow equations, namely, density (ρ), formation volume factor (B), and viscosity (μ). Fluid density is needed for the estimation of fluid gravity (γ) using

$$\gamma = \gamma_c \rho g \tag{2.1}$$

where γ_c = the gravity conversion factor and g = acceleration due to gravity. In general, fluid properties are a function of pressure. Mathematically, the pressure dependence of fluid properties is expressed as:

$$\rho = \mathrm{f}(p) \tag{2.2}$$

$$B = \mathrm{f}(p) \tag{2.3}$$

$$\mu = \mathrm{f}(p). \tag{2.4}$$

The derivation of the general flow equation in this chapter does not require more than the general dependence of fluid properties on pressure as expressed by Eqs. 2.2 through 2.4. In Chapter 7, the specific pressure dependence of fluid properties is required for the derivation of the flow equation for each type of fluid.

2.3 Properties of Porous Media

Modeling single-phase fluid flow in reservoirs requires the knowledge of basic rock properties such as porosity and permeability, or more precisely, effective porosity and absolute permeability. Other rock properties include reservoir thickness and elevation below sea level. *Effective porosity* is the ratio of interconnected pore spaces to bulk volume of a rock sample. Petroleum reservoirs usually have heterogeneous porosity distribution; i.e., porosity changes with location. A reservoir is homogeneous if porosity is constant independent of location. Porosity depends on reservoir pressure because of solid and pore compressibilities. It increases as reservoir pressure (pressure of the fluid contained in the pores) increases and vice versa. This relationship can be expressed as

$$\phi = \phi^{\circ}[1 + c_{\phi}(p - p^{\circ})] \tag{2.5}$$

where ϕ° = porosity at reference pressure (p°) and c_{ϕ} = porosity compressibility. *Permeability* is the capacity of the rock to transmit fluid through its connected pores when the same fluid fills all the interconnected pores. Permeability is a directional rock property. If the reservoir coordinates coincide with the principal directions of permeability, then permeability can be represented by k_x, k_y, and k_z. The reservoir is described as having isotropic permeability distribution if $k_x = k_y = k_z$; otherwise, the reservoir is anisotropic if permeability shows directional bias. Usually $k_x = k_y = k_H$ and $k_z = k_V$ with $k_V < k_H$ because of depositional environments.

2.4 Reservoir Discretization

Reservoir discretization means that the reservoir is described by a set of gridblocks (or gridpoints) whose properties, dimensions, boundaries, and locations in the reservoir are well defined. Chapter 4 deals with reservoirs discretized using a block-centered grid and Chapter 5 discusses reservoirs discretized using a point-distributed grid. Figure 2–1 shows reservoir discretization in the x direction as the focus is on block i.

The figure shows how the blocks are related to each other—block i and its neighboring blocks $i-1$ and $i+1$—block dimensions (Δx_i , Δx_{i-1} , Δx_{i+1}), block boundaries ($x_{i-1/2}$, $x_{i+1/2}$), distances between the point that represents the block and block boundaries (δx_{i^-} , δx_{i^+}), and distances between the points representing the blocks ($\Delta x_{i-1/2}$, $\Delta x_{i+1/2}$). The terminology presented in Figure 2–1 is applicable to both block-centered and point-distributed grid systems in 1D flow in the direction of the x axis. Reservoir discretization in the y and z directions uses similar terminology. In addition, each gridblock (or gridpoint) is assigned elevation and rock properties such as porosity and permeabilities in the x, y, and z directions. The transfer of fluids from one block to the rest of reservoir takes place through

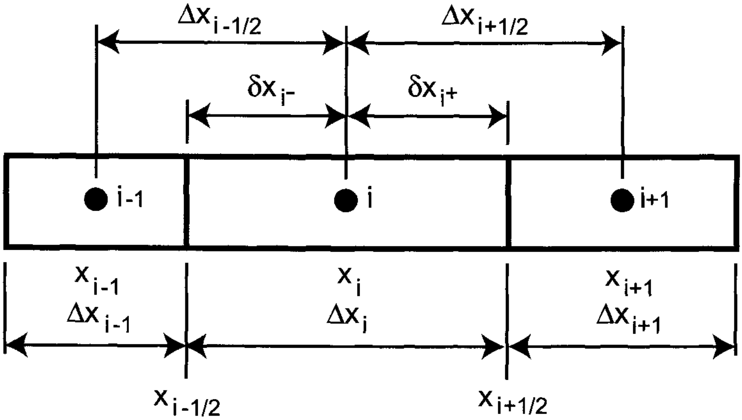
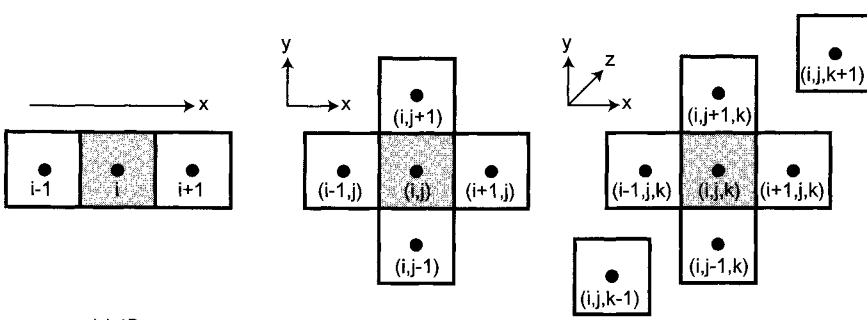


Figure 2–1 Relationships between block *i* and its neighboring blocks in 1D flow.

the immediate neighboring blocks. When the whole reservoir is discretized, each block is surrounded by a set (group) of neighboring blocks. Figure 2–2a shows that there are two neighboring blocks in 1D flow along the x axis, Figure 2–2b shows that there are four neighboring blocks in 2D flow in the x-y plane, and Figure 2–2c shows that there are six neighboring blocks in 3D flow in x-y-z space.

It must be made clear that once the reservoir is discretized and rock properties are assigned to gridblocks (or gridpoints), space is no longer a variable and functions that depend on space, such as interblock properties, become well-defined. In other words, reservoir discretization removes space from being a variable in the formulation of the problem. More elaboration follows in Section 2.6.2.



(a) 1D. (b) 2D. (c) 3D.

Figure 2–2 A block and its neighboring blocks in 1D, 2D, and 3D flow using engineering notation.

2.5 Basic Engineering Concepts

The basic engineering concepts include mass conservation, equation of state, and constitutive equation. The principle of *mass conservation* states that the total mass of fluid entering minus the fluid leaving a volume element of the reservoir, shown in Figure 2–3 as block i, must equal the net increase in the mass of the fluid in the reservoir volume element; i.e.,

$$m_i - m_o + m_s = m_a \tag{2.6}$$

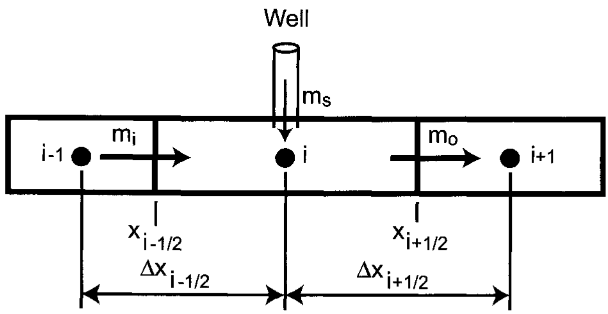


Figure 2–3 Block *i* as a reservoir volume element in 1D flow.

where $m_i =$ the mass of fluid entering the reservoir volume element from other parts of the reservoir, $m_o =$ the mass of fluid leaving the reservoir volume element to other parts of the reservoir, $m_s =$ the mass of fluid entering or leaving the reservoir volume element externally through wells, and $m_a =$ the mass of excess fluid stored in or depleted from the reservoir volume element over a time interval.

An *equation of state* describes the density of fluid as a function of pressure and temperature. For single-phase fluid,

$$B = \rho_{sc} / \rho \tag{2.7a}$$

for oil or water, and

$$B_g = \frac{\rho_{gsc}}{\alpha_c \rho_g} \tag{2.7b}$$

for gas, where ρ and $\rho_g =$ fluid densities at reservoir conditions, ρ_{sc} and $\rho_{gsc} =$ fluid densities at standard conditions, and $\alpha_c =$ the volume conversion factor.

A *constitutive equation* describes the rate of fluid movement into (or out of) the reservoir volume element. In reservoir simulation, Darcy's Law is used to relate fluid flow rate to potential gradient. The differential form of Darcy's Law in a 1D inclined reservoir is

$$u_x = q_x / A_x = -\beta_c \frac{k_x}{\mu} \frac{\partial \Phi}{\partial x} \tag{2.8}$$

where $\beta_c =$ the transmissibility conversion factor, $k_x =$ absolute permeability of rock in the direction of the x axis, $\mu =$ fluid viscosity, $\Phi =$ potential, and $u_x =$ volumetric (or superficial) velocity of fluid defined as fluid flow rate (q_x) per unit cross-sectional area (A_x) normal to flow direction x. The potential is related to pressure through the following relationship:

$$\Phi - \Phi_{ref} = (p - p_{ref}) - \gamma(Z - Z_{ref}) \tag{2.9}$$

where Z = elevation from datum, with positive values downward.

Therefore,

$$\frac{\partial \Phi}{\partial x} = (\frac{\partial p}{\partial x} - \gamma \frac{\partial Z}{\partial x}) \tag{2.10}$$

and the potential differences between block i and its neighbors, block $i-1$ and block $i+1$, are

$$\Phi_{i-1} - \Phi_i = (p_{i-1} - p_i) - \gamma_{i-1/2}(Z_{i-1} - Z_i) \tag{2.11a}$$

$$\Phi_{i+1} - \Phi_i = (p_{i+1} - p_i) - \gamma_{i+1/2}(Z_{i+1} - Z_i) \tag{2.11b}$$

2.6 Multidimensional Flow in Cartesian Coordinates

2.6.1 Block Identification and Block Ordering

Before writing the flow equation for a 1D, 2D, or 3D reservoir, the blocks in the discretized reservoir must be identified and ordered. Any block in the reservoir can be identified either by engineering notation or by the number the block holds in a given ordering scheme. Engineering notation uses the order of the block in the x, y, and z directions; i.e., it identifies a block as (i, j, k), where i, j, and k are the orders of the block in the three directions x, y, and z, respectively. The engineering notation for block identification is the most convenient for entering reservoir description (input) and for printing simulation results (output). Figure 2–4 shows the engineering notation for block identification in a 2D reservoir consisting of 4×5 blocks. Block ordering not only serves to identify blocks in the reservoir but also minimizes matrix computations in obtaining the solution of linear equations.

(1,5)	(2,5)	(3,5)	(4,5)
(1,4)	(2,4)	(3,4)	(4,4)
(1,3)	(2,3)	(3,3)	(4,3)
(1,2)	(2,2)	(3,2)	(4,2)
(1,1)	(2,1)	(3,1)	(4,1)

Figure 2–4 Engineering notation for block identification.

There are many block ordering schemes, including natural ordering, zebra ordering, diagonal (D2) ordering, alternating diagonal (D4) ordering, cyclic ordering, and cyclic-2 ordering. If the reservoir has inactive blocks within its external boundaries, such blocks

17	18	19	20
13	14	15	16
9	10	11	12
5	6	7	8
1	2	3	4

(a) Natural ordering.

9	10	11	12
17	18	19	20
5	6	7	8
13	14	15	16
1	2	3	4

(b) Zebra ordering.

14	17	19	20
10	13	16	18
6	9	12	15
3	5	8	11
1	2	4	7

(c) Diagonal (D2) ordering.

8	19	10	20
16	7	18	9
4	15	6	17
12	3	14	5
1	11	2	13

(d) Alternating diagonal (D4) ordering.

11	10	9	8
12	19	18	7
13	20	17	6
14	15	16	5
1	2	3	4

(e) Cyclic ordering.

9	19	10	20
17	7	18	8
5	15	6	16
13	3	14	4
1	11	2	12

(f) Cyclic-2 ordering.

Figure 2–5 Block ordering schemes used in reservoir simulation.

will be skipped and ordering of active blocks will continue (Abou-Kassem and Ertekin 1992). For multidimensional reservoirs, natural ordering is the simplest to program but is the least efficient in solving linear equations, whereas D4 ordering requires complicated programming but is the most efficient in obtaining the solution when the number of blocks is large. If the number of blocks is very large, however, the zebra ordering scheme becomes twice as efficient as D4 ordering in obtaining the solution (McDonald and Trimble 1977). Figure 2–5 shows the various block ordering schemes for the 2D reservoir shown in Figure 2–4. Given the engineering notation for block identification, block ordering is generated internally in a simulator. Any ordering scheme becomes even more efficient computationally if the ordering is performed along the shortest direction, followed by the intermediate direction, and finally the longest direction (Abou-Kassem and Ertekin 1992). Details related to various ordering schemes and computational efficiency in solving linear equations are not discussed further in this book but can be found elsewhere (Woo, Roberts, and Gustavson 1973; Price and Coats 1974; McDonald and Trimble

1977). The natural ordering scheme is used throughout this book because it produces equations that are readily solvable with handheld calculators and easily programmable for computer usage. The following three examples demonstrate the use of engineering notation and natural ordering to identify blocks in multi dimensions.

Example 2.1 Consider the 1D reservoir shown in Figure 2–6a. This reservoir is discretized using four blocks in the x direction as shown in the figure. Order the blocks in this reservoir using natural ordering.

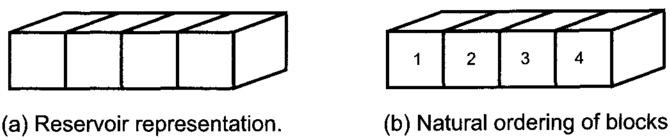


Figure 2–6 1D reservoir representation in Example 2.1.

Solution

We first choose one of the corner blocks (say the left-corner block), identify it as block 1, and then move along a given direction to the other blocks, one block at a time. The order of the next block is obtained by incrementing the order of the previous block by one. The process of block ordering (or numbering) continues until the last block in that direction is numbered. The final ordering of blocks in this reservoir is shown in Figure 2–6b.

Example 2.2 Consider the 2D reservoir shown in Figure 2–7a. This reservoir is discretized using 4×3 blocks as shown in the figure. Identify the blocks in this reservoir using engineering notation and natural ordering.

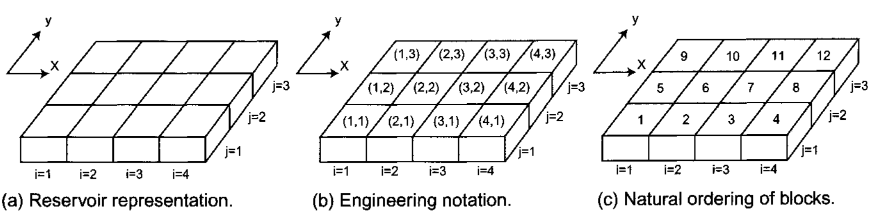


Figure 2–7 2D reservoir representation in Example 2.2.

Solution

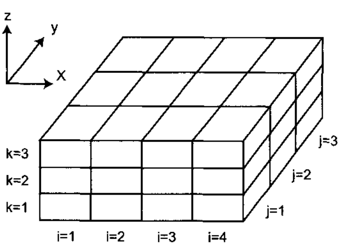
(a) The engineering notation for block identification is shown in Figure 2–7b.

(b) We start by choosing one of the corner blocks in the reservoir. In this example, we arbitrarily choose the lower-left corner block, block (1,1), and identify it as block 1. In addition, we choose to order blocks along rows. The rest of the blocks in the first row ($j=1$) are numbered as explained in Example 2.1. Block (1,2) in the first column ($i=1$) and second row ($j=2$) is numbered next as block 5, and block numbering along this row continues as in Example 2.1. Block numbering continues row by row until all the blocks are numbered. The final ordering of blocks in this reservoir is shown in Figure 2–7c.

Example 2.3 Consider the 3D reservoir shown in Figure 2–8a. This reservoir is discretized into 4×3×3 blocks as shown in the figure. Identify the blocks in this reservoir using:

(a) Engineering notation.

(b) Natural ordering.



(a) Reservoir representation.

	i=1	i=2	i=3	i=4	
k=3	(1,3,3)	(2,3,3)	(3,3,3)	(4,3,3)	j=3
	(1,2,3)	(2,2,3)	(3,2,3)	(4,2,3)	j=2
	(1,1,3)	(2,1,3)	(3,1,3)	(4,1,3)	j=1
k=2	(1,3,2)	(2,3,2)	(3,3,2)	(4,3,2)	j=3
	(1,2,2)	(2,2,2)	(3,2,2)	(4,2,2)	j=2
	(1,1,2)	(2,1,2)	(3,1,2)	(4,1,2)	j=1
k=1	(1,3,1)	(2,3,1)	(3,3,1)	(4,3,1)	j=3
	(1,2,1)	(2,2,1)	(3,2,1)	(4,2,1)	j=2
	(1,1,1)	(2,1,1)	(3,1,1)	(4,1,1)	j=1

(b) Engineering notation.

	i=1	i=2	i=3	i=4	
k=3	33	34	35	36	j=3
	29	30	31	32	j=2
	25	26	27	28	j=1
k=2	21	22	23	24	j=3
	17	18	19	20	j=2
	13	14	15	16	j=1
k=1	9	10	11	12	j=3
	5	6	7	8	j=2
	1	2	3	4	j=1

(c) Natural ordering of blocks.

Figure 2–8 3D reservoir representation in Example 2.3.

Solution

(a) The engineering notation for block identification in this reservoir is shown in Figure 2–8b.

(b) We arbitrarily choose the bottom-lower-left corner block, block (1,1,1), and identify it as block 1. In addition, we choose to order blocks layer by layer and along rows. The blocks in the first (bottom) layer ($k = 1$) are ordered as shown in Example 2.2. Next, block (1,1,2) is numbered as block 13 and the ordering of blocks in this second layer is carried out as in the first layer. Finally, block (1,1,3) is numbered as block 25 and the ordering of blocks in this third layer ($k = 3$) is carried out as before. Figure 2–8c shows the resulting natural ordering of blocks in this reservoir.

2.6.2 Derivation of the One-Dimensional Flow Equation in Cartesian Coordinates

Figure 2–3 shows block i and its neighboring blocks $i-1$ and $i+1$ in the x direction. At any instant in time, fluid enters block i, coming from block $i-1$ across its $x_{i-1/2}$ face at a mass rate of $w_x|_{x_{i-1/2}}$ and leaves to block $i+1$ across its $x_{i+1/2}$ face at a mass rate of $w_x|_{x_{i+1/2}}$. The fluid also enters block i through a well at a mass rate of q_{m_i}. The mass of fluid contained in a unit volume of rock in block i is m_{v_i}. Therefore, the material balance equation for block i written over a time step $\Delta t = t^{n+1} - t^n$ can be rewritten as

$$m_i\big|_{x_{i-1/2}} - m_o\big|_{x_{i+1/2}} + m_{s_i} = m_{a_i} \tag{2.12}$$

Terms like $w_x|_{x_{i-1/2}}$, $w_x|_{x_{i+1/2}}$, and q_{m_i} are functions of time only because space is not a variable for an already discretized reservoir as discussed in Section 2.4. Further justification is presented later in this section. Therefore,

$$m_i\big|_{x_{i-1/2}} = \int_{t^n}^{t^{n+1}} w_x\big|_{x_{i-1/2}}\,\mathrm{d}t \tag{2.13}$$

$$m_o\big|_{x_{i+1/2}} = \int_{t^n}^{t^{n+1}} w_x\big|_{x_{i+1/2}}\,\mathrm{d}t \tag{2.14}$$

$$m_{s_i} = \int_{t^n}^{t^{n+1}} q_{m_i}\,\mathrm{d}t \tag{2.15}$$

Using Eqs. 2.13 through 2.15, Eq. 2.12 can be rewritten as

$$\int_{t^n}^{t^{n+1}} w_x\big|_{x_{i-1/2}}\,\mathrm{d}t - \int_{t^n}^{t^{n+1}} w_x\big|_{x_{i+1/2}}\,\mathrm{d}t + \int_{t^n}^{t^{n+1}} q_{m_i}\,\mathrm{d}t = m_{a_i}. \tag{2.16}$$

The mass accumulation is defined as

$$m_{a_i} = \Delta_t (V_b m_v)_i = V_{b_i}(m_{v_i}^{n+1} - m_{v_i}^n). \tag{2.17}$$

Note that mass rate and mass flux are related through

$$w_x = \dot{m}_x A_x \tag{2.18}$$

Mass flux ($\dot{m}_x$) can be expressed in terms of fluid density and volumetric velocity,

$$\dot{m}_x = \alpha_c \rho u_x \tag{2.19}$$

mass of fluid per unit volume or rock (m_v) can be expressed in terms of fluid density and porosity,

$$m_v = \phi\rho \tag{2.20}$$

and mass of injected or produced fluid (q_m) can be expressed in terms of well volumetric rate (q) and fluid density,

$$q_m = \alpha_c \rho q \tag{2.21}$$

Substitution of Eqs. 2.17 and 2.18 into Eq. 2.16 yields

$$\int_{t^n}^{t^{n+1}} (\dot{m}_x A_x)\Big|_{x_{i-1/2}} \mathrm{d}t - \int_{t^n}^{t^{n+1}} (\dot{m}_x A_x)\Big|_{x_{i+1/2}} \mathrm{d}t + \int_{t^n}^{t^{n+1}} q_{m_i} \mathrm{d}t = V_{b_i}(m_{v_i}^{n+1} - m_{v_i}^n). \tag{2.22}$$

Substitution of Eqs. 2.19 through 2.21 into Eq. 2.22 yields

$$\int_{t^n}^{t^{n+1}} (\alpha_c \rho u_x A_x)\Big|_{x_{i-1/2}} \mathrm{d}t - \int_{t^n}^{t^{n+1}} (\alpha_c \rho u_x A_x)\Big|_{x_{i+1/2}} \mathrm{d}t + \int_{t^n}^{t^{n+1}} (\alpha_c \rho q)_i \mathrm{d}t = V_{b_i}[(\phi\rho)_i^{n+1} - (\phi\rho)_i^n]. \tag{2.23}$$

Substitution of Eq. 2.7a into Eq. 2.23, dividing by $\alpha_c \rho_{sc}$, and noting that $q/B = q_{sc}$ yields

$$\int_{t^n}^{t^{n+1}} (\frac{u_x A_x}{B})\Bigg|_{x_{i-1/2}} \mathrm{d}t - \int_{t^n}^{t^{n+1}} (\frac{u_x A_x}{B})\Bigg|_{x_{i+1/2}} \mathrm{d}t + \int_{t^n}^{t^{n+1}} q_{sc_i} \mathrm{d}t = \frac{V_{b_i}}{\alpha_c}[(\frac{\phi}{B})_i^{n+1} - (\frac{\phi}{B})_i^n]. \tag{2.24}$$

Fluid volumetric velocity (flow rate per unit cross-sectional area) from block $i-1$ to block i ($u_x|_{x_{i-1/2}}$) at any time instant t is given by the algebraic analog of Eq. 2.8,

$$u_x\Big|_{x_{i-1/2}} = \beta_c \frac{k_x|_{x_{i-1/2}}}{\mu|_{x_{i-1/2}}}[\frac{(\Phi_{i-1} - \Phi_i)}{\Delta x_{i-1/2}}] \tag{2.25a}$$

where $k_x|_{x_{i-1/2}}$ is rock permeability between blocks $i-1$ and i that are separated by a distance $\Delta x_{i-1/2}$, Φ_{i-1} and Φ_i are the potentials of blocks $i-1$ and i, and $\mu|_{x_{i-1/2}}$ is viscosity of the fluid contained in blocks $i-1$ and i.

Likewise, fluid flow rate per unit cross-sectional area from block i to block $i+1$ is

$$u_x\Big|_{x_{i+1/2}} = \beta_c \frac{k_x|_{x_{i+1/2}}}{\mu|_{x_{i+1/2}}}[\frac{(\Phi_i - \Phi_{i+1})}{\Delta x_{i+1/2}}]. \tag{2.25b}$$

Substitution of Eq. 2.25 into Eq. 2.24 and grouping terms results in

$$\int_{t^n}^{t^{n+1}} [(\beta_c \frac{k_x A_x}{\mu B \Delta x})\bigg|_{x_{i-1/2}} (\Phi_{i-1} - \Phi_i)] dt - \int_{t^n}^{t^{n+1}} [(\beta_c \frac{k_x A_x}{\mu B \Delta x})\bigg|_{x_{i+1/2}} (\Phi_i - \Phi_{i+1})] dt$$
$$+ \int_{t^n}^{t^{n+1}} q_{sc_i} dt = \frac{V_{b_i}}{\alpha_c} [(\frac{\phi}{B})_i^{n+1} - (\frac{\phi}{B})_i^n] \tag{2.26}$$

or

$$\int_{t^n}^{t^{n+1}} [T_{x_{i-1/2}} (\Phi_{i-1} - \Phi_i)] dt + \int_{t^n}^{t^{n+1}} [T_{x_{i+1/2}} (\Phi_{i+1} - \Phi_i)] dt + \int_{t^n}^{t^{n+1}} q_{sc_i} dt = \frac{V_{b_i}}{\alpha_c} [(\frac{\phi}{B})_i^{n+1} - (\frac{\phi}{B})_i^n] \tag{2.27}$$

where

$$T_{x_{i\mp1/2}} = (\beta_c \frac{k_x A_x}{\mu B \Delta x})\bigg|_{x_{i\mp1/2}} \tag{2.28}$$

is transmissibility in the x direction between block i and the neighboring block $i \mp 1$. The derivation of Eq. 2.27 is rigorous and involves no assumptions other than the validity of Darcy's Law (Eq. 2.25) to estimate fluid volumetric velocity between block i and its neighboring block $i \mp 1$. The validity of Darcy's Law is well accepted. Note that similar derivation can be made even if Darcy's Law is replaced by another flow equation, such as Brinkman's equation, etc. (Mustafiz, Biazar, and Islam 2005; Belhaj et al. 2005). For heterogeneous block permeability distribution and irregular grid blocks (neither constant nor equal Δx, Δy, and Δz), the part

$$(\beta_c \frac{k_x A_x}{\Delta x})\bigg|_{x_{i\mp1/2}}$$

of transmissibility $T_{x_{i\mp1/2}}$ is derived in Chapter 4 for a block-centered grid and in Chapter 5 for a point-distributed grid. Note that for a discretized reservoir, blocks have defined dimensions and permeabilities; therefore, interblock geometric factor

$$[\, (\beta_c \frac{k_x A_x}{\Delta x})\bigg|_{x_{i\mp1/2}} \,]$$

is constant, independent of space and time. In addition, the pressure dependent term $(\mu B)|_{x_{i\mp1/2}}$ of transmissibility uses some average viscosity and formation volume factor (FVF) of the fluid contained in block i and the neighboring block $i \mp 1$ or some weight (up-stream weighting, average weighting) at any instant of time t. In other words, the term $(\mu B)|_{x_{i\mp1/2}}$ is not a function of space but a function of time as block pressures change with time. Hence, transmissibility $T_{x_{i\mp1/2}}$ between block i and its neighboring block $i \mp 1$ is a function of time only; it does not depend on space at any instant of time.

Again, the accumulation term in Eq. 2.27 can be expressed in terms of the change in the pressure of block i as shown in Eq. 2.29a;

$$\int_{t^n}^{t^{n+1}} [T_{x_{i-1/2}}(\Phi_{i-1} - \Phi_i)]\mathrm{d}t + \int_{t^n}^{t^{n+1}} [T_{x_{i+1/2}}(\Phi_{i+1} - \Phi_i)]\mathrm{d}t + \int_{t^n}^{t^{n+1}} q_{sc_i}\mathrm{d}t = \frac{V_{b_i}}{\alpha_c}\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\phi}{B}\right)_i [p_i^{n+1} - p_i^n] \tag{2.29a}$$

or after substituting Eq. 2.11 for potential,

$$\int_{t^n}^{t^{n+1}} \{T_{x_{i-1/2}}[(p_{i-1} - p_i) - \gamma_{i-1/2}(Z_{i-1} - Z_i)]\}\mathrm{d}t + \int_{t^n}^{t^{n+1}} \{T_{x_{i+1/2}}[(p_{i+1} - p_i) - \gamma_{i+1/2}(Z_{i+1} - Z_i)]\}\mathrm{d}t$$
$$+ \int_{t^n}^{t^{n+1}} q_{sc_i}\mathrm{d}t = \frac{V_{b_i}}{\alpha_c}\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\phi}{B}\right)_i [p_i^{n+1} - p_i^n], \tag{2.29b}$$

where $\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\phi}{B}\right)_i$ = the chord slope of $\left(\frac{\phi}{B}\right)_i$ between p_i^{n+1} and p_i^n.

2.6.3 Approximation of Time Integrals

If the argument of an integral is an explicit function of time, the integral can be evaluated analytically. This is not the case for the integrals appearing on the Left-Hand Side (LHS) of either Eq. 2.27 or Eq. 2.29. If Eq. 2.29b is written for every block $i = 1, 2, 3 \ldots n_x$, then the solution can be obtained by one of the Ordinary Difference Equations (ODE) methods (Euler's method, the modified Euler method, the explicit Runge-Kutta method, or the implicit Runge-Kutta method) reviewed by Aziz and Settari (1979). ODE methods, however, are not efficient for solving reservoir simulation problems. Therefore, performing these integrations necessitates making certain assumptions.

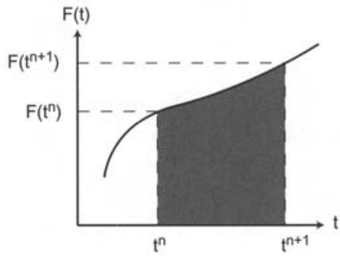
Consider the integral

$$\int_{t^n}^{t^{n+1}} F(t)\mathrm{d}t$$

shown in Figure 2–9. This integral is equal to the area under the curve $F(t)$ in the interval $t^n \le t \le t^{n+1}$. This area is also equal to the area of a rectangle with the dimensions of $F(t^m)$, where F is evaluated at time t^m, where $t^n \le t^m \le t^{n+1}$, and Δt, where $\Delta t = (t^{n+1} - t^n)$, as shown in Figure 2–10. Therefore,

$$\int_{t^n}^{t^{n+1}} F(t)\mathrm{d}t = \int_{t^n}^{t^{n+1}} F(t^m)\mathrm{d}t = \int_{t^n}^{t^{n+1}} F^m\mathrm{d}t = F^m \int_{t^n}^{t^{n+1}} \mathrm{d}t = F^m \times t\Big|_{t^n}^{t^{n+1}} = F^m \times (t^{n+1} - t^n) = F^m \times \Delta t. \tag{2.30}$$

The value of this integral can be calculated using above equation provided that the value of F^m or $F(t^m)$ is known. In reality, however, F^m is not known and, therefore, it needs to be approximated. The area under the curve in Figure 2–9 can be approximated by one of the following four methods:



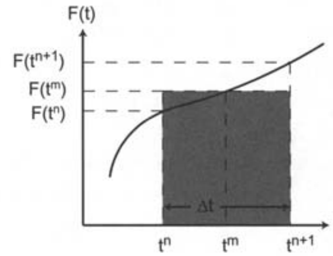


Figure 2–9 Representation of the integral of a function as the area under the curve (left).
Figure 2–10 Representation of the integral of a function as F(t^m)×Δt (right).

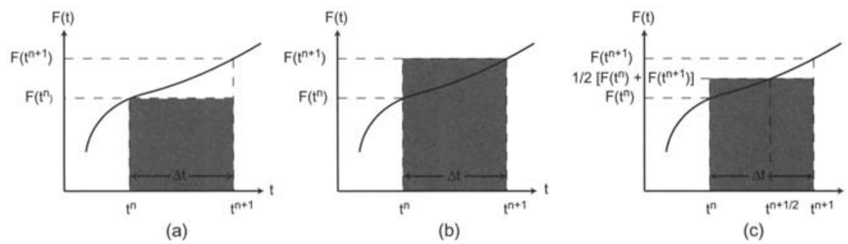


Figure 2–11 Approximations of the integral of function.

$F(t^n)\times\Delta t$ as shown in Figure 2–11a, $F(t^{n+1})\times\Delta t$ as shown in Figure 2–11b, $\frac{1}{2}[F(t^n)+F(t^{n+1})]\times\Delta t$ as shown in Figure 2–11c, or numerical integration.

The argument F in Eq. 2.30 stands for $[T_{x_{i-1/2}}(\Phi_{i-1}-\Phi_i)]$, $[T_{x_{i+1/2}}(\Phi_{i+1}-\Phi_i)]$, or q_{sc_i} that appears on the LHS of Eq. 2.27, and F^m = value of F at time t^m.

Therefore, Eq. 2.27 after this approximation becomes

$$[T^m_{x_{i-1/2}}(\Phi^m_{i-1}-\Phi^m_i)]\Delta t+[T^m_{x_{i+1/2}}(\Phi^m_{i+1}-\Phi^m_i)]\Delta t+q^m_{sc_i}\Delta t=\frac{V_{b_i}}{\alpha_c}[(\frac{\phi}{B})^{n+1}_i-(\frac{\phi}{B})^n_i] \tag{2.31}$$

Dividing the above equation by Δt gives

$$T^m_{x_{i-1/2}}(\Phi^m_{i-1}-\Phi^m_i)+T^m_{x_{i+1/2}}(\Phi^m_{i+1}-\Phi^m_i)+q^m_{sc_i}=\frac{V_{b_i}}{\alpha_c\Delta t}[(\frac{\phi}{B})^{n+1}_i-(\frac{\phi}{B})^n_i] \tag{2.32}$$

Substituting Eq. 2.11 into Eq. 2.32, we obtain the flow equation for block i,

$$\begin{aligned}&T^m_{x_{i-1/2}}[(p^m_{i-1}-p^m_i)-\gamma^m_{i-1/2}(Z_{i-1}-Z_i)]+T^m_{x_{i+1/2}}[(p^m_{i+1}-p^m_i)-\gamma^m_{i+1/2}(Z_{i+1}-Z_i)]\\&+q^m_{sc_i}=\frac{V_{b_i}}{\alpha_c\Delta t}[(\frac{\phi}{B})^{n+1}_i-(\frac{\phi}{B})^n_i].\end{aligned} \tag{2.33}$$

The right-hand side (RHS) of the flow equation expressed as Eq. 2.33, known as the fluid accumulation term, vanishes in problems involving the flow of incompressible fluid ($c=0$) in an incompressible porous medium ($c_\phi=0$). This is the case where both B

and ϕ are constant independent of pressure. Reservoir pressure in this type of flow problem is independent of time. Example 2.4 demonstrates the application of Eq. 2.33 for an interior block in a 1D reservoir using a regular grid. In Chapter 7, the explicit, implicit, and Crank-Nicolson formulations are derived from Eq. 2.33 by specifying the approximation of time t^m as t^n , t^{n+1} , or $t^{n+1/2}$, which are equivalent to using the first, second, and third integral approximation methods mentioned above. The fourth integration method mentioned above leads to the Runge-Kutta solution methods of ordinary differential equations. Table 2–1 presents the units of all the quantities that appear in flow equations.

Example 2.4 Consider single-phase fluid flow in a 1D horizontal reservoir. The reservoir is discretized using four blocks in the x direction, as shown in Figure 2–12. A well located in block 3 produces at a rate of 400 STB/D. All grid blocks have $\Delta x = 250$ ft, $w = 900$ ft, $h = 100$ ft, and $k_x = 270$ md. The FVF and the viscosity of the flowing fluid are 1.0 RB/STB and 2 cp, respectively. Identify the interior and boundary blocks in this reservoir. Write the flow equation for block 3 and give the physical meaning of each term in the equation.

Solution

Blocks 2 and 3 are interior blocks, whereas blocks 1 and 4 are boundary blocks. The flow equation for block 3 can be obtained from Eq. 2.33 for *i* = 3; i.e.,

$$T^m_{x_{3-1/2}}[(p^m_2 - p^m_3) - \gamma^m_{3-1/2}(Z_2 - Z_3)] + T^m_{x_{3+1/2}}[(p^m_4 - p^m_3) - \gamma^m_{3+1/2}(Z_4 - Z_3)]$$
$$+ q^m_{sc_3} = \frac{V_{b_3}}{\alpha_c \Delta t}[(\frac{\phi}{B})^{n+1}_3 - (\frac{\phi}{B})^n_3]. \tag{2.34}$$

For block 3, $Z_2 = Z_3 = Z_4$ for horizontal reservoir and $q^m_{sc_3} = -400$ STB/D.

Because $\Delta x_{3\mp 1/2} = \Delta x$, and because μ and *B* are constant,

$$T^m_{x_{3-1/2}} = T^m_{x_{3+1/2}} = \beta_c \frac{k_x A_x}{\mu B \Delta x} = 0.001127 \times \frac{270 \times (900 \times 100)}{2 \times 1 \times 250} = 54.7722 \text{ STB/D-psi} \tag{2.35}$$

Substitution of Eq. 2.35 into Eq. 2.34 gives

$$(54.7722)(p^m_2 - p^m_3) + (54.7722)(p^m_4 - p^m_3) - 400 = \frac{V_{b_3}}{\alpha_c \Delta t}[(\frac{\phi}{B})^{n+1}_3 - (\frac{\phi}{B})^n_3]. \tag{2.36}$$

The LHS of Eq. 2.36 comprises three terms. The first term represents the rate of fluid flow from block 2 to block 3, the second term represents the rate of fluid flow from block 4 to block 3, and the third term represents the rate of fluid production from the well in block 3. The RHS of Eq. 2.36 represents the rate of fluid accumulation in block 3. All terms have the units of STB/D.

Table 2–1 Quantities Used in Flow Equations in Different Systems of Units

Quantity	Symbol	System of Units		
		Customary Units	SPE Metric Units	Lab Units
Length	x, y, z, r, Z	ft	m	cm
Area	$A, A_x, A_y, A_z, A_r, A_\theta$	ft^2	m^2	cm^2
Permeability	$k, k_x, k_y, k_z, k_r, k_\theta$	md	m^2	darcy
Phase viscosity	μ, μ_o, μ_w, μ_g	cp	mPa.s	cp
Gas FVF	B, B_g	RB/scf	m^3/std m^3	cm^3/std cm^3
Liquid FVF	B, B_o, B_w	RB/STB	m^3/std m^3	cm^3/std cm^3
Solution GOR	R_s	scf/STB	std m^3/std m^3	std cm^3/std cm^3
Phase pressure	p, p_o, p_w, p_g	psia	kPa	atm
Phase potential	$\Phi, \Phi_o, \Phi_w, \Phi_g$	psia	kPa	atm
Phase gravity	$\gamma, \gamma_o, \gamma_w, \gamma_g$	psi/ft	kPa/m	atm/cm
Gas flow rate	q_{sc}, q_{gsc}	scf/D	std m^3/d	std cm^3/sec
Oil flow rate	q_{sc}, q_{osc}	STB/D	std m^3/d	std cm^3/sec
Water flow rate	q_{sc}, q_{wsc}	B/D	std m^3/d	std cm^3/sec
Volumetric velocity	u	RB/D-ft^2	m/d	cm/sec
Phase density	$\rho, \rho_o, \rho_w, \rho_g$	lbm/ft^3	kg/m^3	g/cm^3
Block bulk volume	V_b	ft^3	m^3	cm^3
Compressibility	c, c_o, c_ϕ	psi^{-1}	kPa^{-1}	atm^{-1}
Compressibility factor	z	dimensionless	dimensionless	dimensionless
Temperature	T	°R	K	K
Porosity	ϕ	fraction	fraction	fraction
Phase saturation	S, S_o, S_w, S_g	fraction	fraction	fraction
Relative permeability	k_{ro}, k_{rw}, k_{rg}	fraction	fraction	fraction
Gravitational acceleration	g	32.174 ft/sec^2	9.806635 m/s^2	980.6635 cm/sec^2
Time	$t, \Delta t$	day	day	sec
Angle	θ	rad	rad	rad
Transmissibility conversion factor	β_c	0.001127	0.0864	1
Gravity conversion factor	γ_c	0.21584×10^{-3}	0.001	0.986923×10^{-6}
Volume conversion factor	α_c	5.614583	1	1

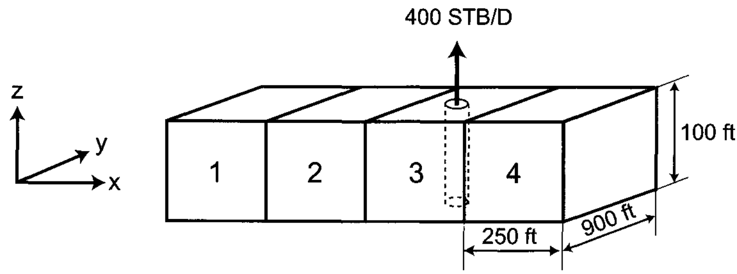


Figure 2–12 1D reservoir representation in Example 2.4.

2.6.4 Flow Equations in Multidimensions Using Engineering Notation

A close inspection of the flow equation expressed as Eq. 2.33 reveals that this equation involves three different groups: the interblock flow terms between block i and its two neighboring blocks in the x direction

$$\{ T^m_{x_{i-1/2}}[(p^m_{i-1} - p^m_i) - \gamma^m_{i-1/2}(Z_{i-1} - Z_i)] \text{ and }$$

$$T^m_{x_{i+1/2}}[(p^m_{i+1} - p^m_i) - \gamma^m_{i+1/2}(Z_{i+1} - Z_i)] \},$$

the source term due to injection or production ($q^m_{sc_i}$), and the accumulation term

$$\{ \frac{V_{b_i}}{\alpha_c \Delta t}[(\frac{\phi}{B})^{n+1}_i - (\frac{\phi}{B})^n_i] \}$$

Any block in the reservoir has one source term and one accumulation term, but the number of interblock flow terms equals the number of its neighboring blocks. Specifically, any block has a maximum of two neighboring blocks in 1D flow (Figure 2–2a), four neighboring blocks in 2D flow (Figure 2–2b), and six neighboring blocks in 3D flow (Figure 2–2c). Therefore, for 2D flow, the flow equation for block (i, j) in the x-y plane is

$$\begin{aligned}
&T^m_{y_{i,j-1/2}}[(p^m_{i,j-1} - p^m_{i,j}) - \gamma^m_{i,j-1/2}(Z_{i,j-1} - Z_{i,j})] \\
&+T^m_{x_{i-1/2,j}}[(p^m_{i-1,j} - p^m_{i,j}) - \gamma^m_{i-1/2,j}(Z_{i-1,j} - Z_{i,j})] \\
&+T^m_{x_{i+1/2,j}}[(p^m_{i+1,j} - p^m_{i,j}) - \gamma^m_{i+1/2,j}(Z_{i+1,j} - Z_{i,j})] \\
&+T^m_{y_{i,j+1/2}}[(p^m_{i,j+1} - p^m_{i,j}) - \gamma^m_{i,j+1/2}(Z_{i,j+1} - Z_{i,j})] + q^m_{sc_{i,j}} = \frac{V_{b_{i,j}}}{\alpha_c \Delta t}[(\frac{\phi}{B})^{n+1}_{i,j} - (\frac{\phi}{B})^n_{i,j}].
\end{aligned} \tag{2.37}$$

For 3D flow, the flow equation for block (i, j, k) in the x-y-z space is

$$
\begin{aligned}
&T^{m}_{z_{i,j,k-1/2}}[(p^{m}_{i,j,k-1}-p^{m}_{i,j,k})-\gamma^{m}_{i,j,k-1/2}(Z_{i,j,k-1}-Z_{i,j,k})]\\
&+T^{m}_{y_{i,j-1/2,k}}[(p^{m}_{i,j-1,k}-p^{m}_{i,j,k})-\gamma^{m}_{i,j-1/2,k}(Z_{i,j-1,k}-Z_{i,j,k})]\\
&+T^{m}_{x_{i-1/2,j,k}}[(p^{m}_{i-1,j,k}-p^{m}_{i,j,k})-\gamma^{m}_{i-1/2,j,k}(Z_{i-1,j,k}-Z_{i,j,k})]\\
&+T^{m}_{x_{i+1/2,j,k}}[(p^{m}_{i+1,j,k}-p^{m}_{i,j,k})-\gamma^{m}_{i+1/2,j,k}(Z_{i+1,j,k}-Z_{i,j,k})]\\
&+T^{m}_{y_{i,j+1/2,k}}[(p^{m}_{i,j+1,k}-p^{m}_{i,j,k})-\gamma^{m}_{i,j+1/2,k}(Z_{i,j+1,k}-Z_{i,j,k})]\\
&+T^{m}_{z_{i,j,k+1/2}}[(p^{m}_{i,j,k+1}-p^{m}_{i,j,k})-\gamma^{m}_{i,j,k+1/2}(Z_{i,j,k+1}-Z_{i,j,k})]\\
&+q^{m}_{sc_{i,j,k}}=\frac{V_{b_{i,j,k}}}{\alpha_c\Delta t}[(\frac{\phi}{B})^{n+1}_{i,j,k}-(\frac{\phi}{B})^{n}_{i,j,k}],
\end{aligned}
\tag{2.38}
$$

where

$$T_{x_{i\mp1/2,j,k}}=(\beta_c\frac{k_xA_x}{\mu B\Delta x})\bigg|_{x_{i\mp1/2,j,k}}=(\beta_c\frac{k_xA_x}{\Delta x})_{x_{i\mp1/2,j,k}}(\frac{1}{\mu B})_{x_{i\mp1/2,j,k}}=G_{x_{i\mp1/2,j,k}}(\frac{1}{\mu B})_{x_{i\mp1/2,j,k}} \tag{2.39a}$$

$$T_{y_{i,j\mp1/2,k}}=(\beta_c\frac{k_yA_y}{\mu B\Delta y})\bigg|_{y_{i,j\mp1/2,k}}=(\beta_c\frac{k_yA_y}{\Delta y})_{y_{i,j\mp1/2,k}}(\frac{1}{\mu B})_{y_{i,j\mp1/2,k}}=G_{y_{i,j\mp1/2,k}}(\frac{1}{\mu B})_{y_{i,j\mp1/2,k}} \tag{2.39b}$$

$$T_{z_{i,j,k\mp1/2}}=(\beta_c\frac{k_zA_z}{\mu B\Delta z})\bigg|_{z_{i,j,k\mp1/2}}=(\beta_c\frac{k_zA_z}{\Delta z})_{z_{i,j,k\mp1/2}}(\frac{1}{\mu B})_{z_{i,j,k\mp1/2}}=G_{z_{i,j,k\mp1/2}}(\frac{1}{\mu B})_{z_{i,j,k\mp1/2}} \tag{2.39c}$$

Expressions for the geometric factors G for irregular grids in heterogeneous reservoirs are presented in Chapters 4 and 5. It should be mentioned that the interblock flow terms in the flow equations for 1D (Eq. 2.33), 2D (Eq. 2.37), or 3D (Eq. 2.38) problems appear in the sequence shown in Figure 2–13 for neighboring blocks. As will be shown in Chapter 9, the sequencing of neighboring blocks as in Figure 2–13 produces flow equations with unknowns already ordered as they appear in the vector of unknowns for the whole reservoir.

The following two examples demonstrate the application of Eqs. 2.37 and 2.38 for interior blocks in multidimensional anisotropic reservoirs using regular grids.

Example 2.5 Consider single-phase fluid flow in a 2D horizontal reservoir. The reservoir is discretized using 4×3 blocks as shown in Figure 2–14. A well that is located in block (3,2) produces at a rate of 400 STB/D. All gridblocks have

$\Delta x = 250$ ft, $\Delta y = 300$ ft, h = 100 ft, $k_x = 270$ md, and $k_y = 220$ md. The FVF and the viscosity of the flowing fluid are 1.0 RB/STB and 2 cp, respectively. Identify the interior and boundary blocks in this reservoir. Write the flow equation for block (3,2) and give the physical meaning of each term in the flow equation. Write the flow equation for block (2,2).

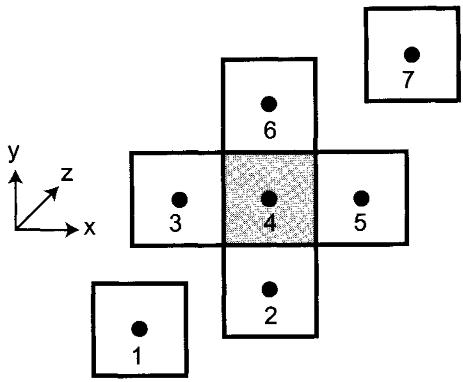


Figure 2–13 The sequence of neighboring blocks in the set $\psi_{i,j,k}$ or ψ_n.

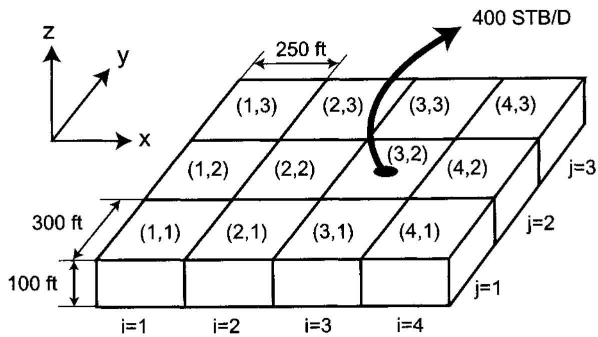


Figure 2–14 2D reservoir representation in Example 2.5.

Solution

Interior blocks in this reservoir include reservoir blocks that are located in the second and third columns in the second row. Other reservoir blocks are boundary blocks. In explicit terms, blocks (2,2) and (3,2) are interior blocks, whereas Blocks (1,1), (2,1), (3,1), (4,1), (1,2), (4,2), (1,3), (2,3), (3,3), and (4,3) are boundary blocks.

The flow equation for block (3,2) can be obtained from Eq. 2.37 for $i = 3$ and $j = 2$; i.e.,

$$\begin{aligned}
&T^m_{y_{3,2-1/2}}[(p^m_{3,1} - p^m_{3,2}) - \gamma^m_{3,2-1/2}(Z_{3,1} - Z_{3,2})] \\
&+T^m_{x_{3-1/2,2}}[(p^m_{2,2} - p^m_{3,2}) - \gamma^m_{3-1/2,2}(Z_{2,2} - Z_{3,2})] \\
&+T^m_{x_{3+1/2,2}}[(p^m_{4,2} - p^m_{3,2}) - \gamma^m_{3+1/2,2}(Z_{4,2} - Z_{3,2})] \\
&+T^m_{y_{3,2+1/2}}[(p^m_{3,3} - p^m_{3,2}) - \gamma^m_{3,2+1/2}(Z_{3,3} - Z_{3,2})] + q^m_{sc_{3,2}} = \frac{V_{b_{3,2}}}{\alpha_c \Delta t}[(\frac{\phi}{B})^{n+1}_{3,2} - (\frac{\phi}{B})^n_{3,2}].
\end{aligned} \tag{2.40}$$

For block (3,2), $Z_{3,1} = Z_{2,2} = Z_{3,2} = Z_{4,2} = Z_{3,3}$ for a horizontal reservoir and $q^m_{sc_{3,2}} = -400$ STB/D. Because $\Delta x_{3\mp 1/2,2} = \Delta x = 250$ ft, $\Delta y_{3,2\mp 1/2} = \Delta y = 300$ ft, and μ and B are constant,

$$T^m_{x_{3-1/2,2}} = T^m_{x_{3+1/2,2}} = \beta_c \frac{k_x A_x}{\mu B \Delta x} = 0.001127 \times \frac{270 \times (300 \times 100)}{2 \times 1 \times 250} = 18.2574 \tag{2.41a}$$

$$T^m_{y_{3,2-1/2}} = T^m_{y_{3,2+1/2}} = \beta_c \frac{k_y A_y}{\mu B \Delta y} = 0.001127 \times \frac{220 \times (250 \times 100)}{2 \times 1 \times 300} = 10.3308 \tag{2.41b}$$

Substitution into Eq. 2.40 gives

$$\begin{aligned}&(10.3308)(p^m_{3,1} - p^m_{3,2}) + (18.2574)(p^m_{2,2} - p^m_{3,2}) + (18.2574)(p^m_{4,2} - p^m_{3,2}) \\ &+(10.3308)(p^m_{3,3} - p^m_{3,2}) - 400 = \frac{V_{b_{3,2}}}{\alpha_c \Delta t}[(\frac{\phi}{B})^{n+1}_{3,2} - (\frac{\phi}{B})^{n}_{3,2}].\end{aligned} \tag{2.42}$$

The LHS of Eq. 2.42 comprises five terms. The first term represents the rate of fluid flow from block (3,1) to block (3,2), the second term from block (2,2) to block (3,2), the third from block (4,2) to block (3,2), and the fourth from block (3,3) to block (3,2). Finally, the fifth term represents the rate of fluid production from the well in block (3,2). The RHS of Eq. 2.42 represents the rate of fluid accumulation in block (3,2). All terms have the units STB/D.

The flow equation for block (2,2) can be obtained from Eq. 2.37 for $i = 2$ and $j = 2$; i.e.,

$$\begin{aligned}&T^m_{y_{2,2-1/2}}[(p^m_{2,1} - p^m_{2,2}) - \gamma^m_{2,2-1/2}(Z_{2,1} - Z_{2,2})] \\ &+T^m_{x_{2-1/2,2}}[(p^m_{1,2} - p^m_{2,2}) - \gamma^m_{2-1/2,2}(Z_{1,2} - Z_{2,2})] \\ &+T^m_{x_{2+1/2,2}}[(p^m_{3,2} - p^m_{2,2}) - \gamma^m_{2+1/2,2}(Z_{3,2} - Z_{2,2})] \\ &+T^m_{y_{2,2+1/2}}[(p^m_{2,3} - p^m_{2,2}) - \gamma^m_{2,2+1/2}(Z_{2,3} - Z_{2,2})] + q^m_{sc_{2,2}} = \frac{V_{b_{2,2}}}{\alpha_c \Delta t}[(\frac{\phi}{B})^{n+1}_{2,2} - (\frac{\phi}{B})^{n}_{2,2}].\end{aligned} \tag{2.43}$$

For block (2,2), $Z_{2,1} = Z_{1,2} = Z_{2,2} = Z_{3,2} = Z_{2,3}$ for a horizontal reservoir,

$q^m_{sc_{2,2}} = 0$ STB/D because block (2,2) does not host a well,

$T^m_{x_{2-1/2,2}} = T^m_{x_{2+1/2,2}} = 18.2574$ STB/D-psi, and $T^m_{y_{2,2-1/2}} = T^m_{y_{2,2+1/2}} = 10.3308$ STB/D-psi.

Substitution into Eq. 2.43 gives

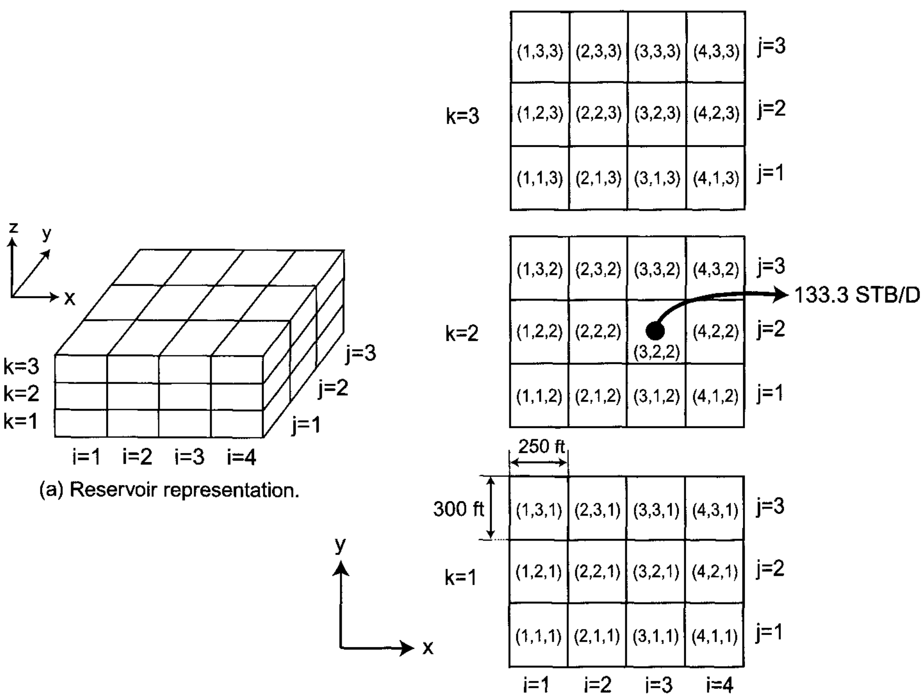
$$(10.3308)(p^m_{2,1} - p^m_{2,2}) + (18.2574)(p^m_{1,2} - p^m_{2,2}) + (18.2574)(p^m_{3,2} - p^m_{2,2})$$

$$+(10.3308)(p_{2,3}^{m}-p_{2,2}^{m})=\frac{V_{b_{2,2}}}{\alpha_c \Delta t}[(\frac{\phi}{B})_{2,2}^{n+1}-(\frac{\phi}{B})_{2,2}^{n}]. \tag{2.44}$$

Example 2.6 Consider single-phase fluid flow in a 3D horizontal reservoir. The reservoir is discretized using 4×3×3 blocks as shown in Figure 2–15a. A well that is located in block (3,2,2) produces at a rate of 133.3 STB/D. All grid blocks have $\Delta x = 250$ ft, $\Delta y = 300$ft, $\Delta z = 33.333$ ft, $k_x = 270$md, $k_y = 220$ md, and $k_z = 50$ md. The FVF, density, and viscosity of the flowing fluid are 1.0 RB/STB, 55 lbm/ft^3, and 2 cp, respectively. Identify the interior and boundary blocks in this reservoir. Write the flow equation for block (3,2,2).

Solution

As can be seen in Figure 2–15b, interior blocks include reservoir blocks that are located in the second and third columns in the second row in the second layer, i.e., blocks (2,2,2) and (3,2,2). All other reservoir blocks are boundary blocks.



(a) Reservoir representation.

(b) Engineering notation.

Figure 2–15 3D reservoir representation in Example 2.6.

The flow equation for block (3,2,2) can be obtained from Eq. 2.38 for $i = 3, j = 2$, and $k = 2$; i.e.,

$$
\begin{aligned}
& T^m_{z_{3,2,2-1/2}}[(p^m_{3,2,1} - p^m_{3,2,2}) - \gamma^m_{3,2,2-1/2}(Z_{3,2,1} - Z_{3,2,2})] \\
& + T^m_{y_{3,2-1/2,2}}[(p^m_{3,1,2} - p^m_{3,2,2}) - \gamma^m_{3,2-1/2,2}(Z_{3,1,2} - Z_{3,2,2})] \\
& + T^m_{x_{3-1/2,2,2}}[(p^m_{2,2,2} - p^m_{3,2,2}) - \gamma^m_{3-1/2,2,2}(Z_{2,2,2} - Z_{3,2,2})] \\
& + T^m_{x_{3+1/2,2,2}}[(p^m_{4,2,2} - p^m_{3,2,2}) - \gamma^m_{3+1/2,2,2}(Z_{4,2,2} - Z_{3,2,2})] \\
& + T^m_{y_{3,2+1/2,2}}[(p^m_{3,3,2} - p^m_{3,2,2}) - \gamma^m_{3,2+1/2,2}(Z_{3,3,2} - Z_{3,2,2})] \\
& + T^m_{z_{3,2,2+1/2}}[(p^m_{3,2,3} - p^m_{3,2,2}) - \gamma^m_{3,2,2+1/2}(Z_{3,2,3} - Z_{3,2,2})] \\
& + q^m_{sc_{3,2,2}} = \frac{V_{b_{3,2,2}}}{\alpha_c \Delta t}[(\frac{\phi}{B})^{n+1}_{3,2,2} - (\frac{\phi}{B})^n_{3,2,2}].
\end{aligned}
\tag{2.45}
$$

For block (3,2,2), $Z_{3,1,2} = Z_{2,2,2} = Z_{3,2,2} = Z_{4,2,2} = Z_{3,3,2}$, $Z_{3,2,1} - Z_{3,2,2} = 33.333$ ft, $Z_{3,2,3} - Z_{3,2,2} = -33.333$ ft, and $q^m_{sc_{3,2,2}} = -133.3$ STB/D. Because $\Delta x_{3\mp 1/2,2,2} = \Delta x = 250$ ft, $\Delta y_{3,2\mp 1/2,2} = \Delta y = 300$ ft, $\Delta z_{3,2,2\mp 1/2} = \Delta z = 33.333$ ft, and μ , ρ , and B are constant, then

$$\gamma^m_{3,2,2-1/2} = \gamma^m_{3,2,2+1/2} = \gamma_c \rho g = 0.21584 \times 10^{-3} \times 55 \times 32.174 = 0.3819 \text{ psi/ft,}$$

$$T^m_{x_{3\mp 1/2,2,2}} = \beta_c \frac{k_x A_x}{\mu B \Delta x} = 0.001127 \times \frac{270 \times (300 \times 33.333)}{2 \times 1 \times 250} = 6.0857 \text{ STB/D-psi} \tag{2.46a}$$

$$T^m_{y_{3,2\mp 1/2,2}} = \beta_c \frac{k_y A_y}{\mu B \Delta y} = 0.001127 \times \frac{220 \times (250 \times 33.333)}{2 \times 1 \times 300} = 3.4436 \text{ STB/D-psi} \tag{2.46b}$$

$$T^m_{z_{3,2,2\mp 1/2}} = \beta_c \frac{k_z A_z}{\mu B \Delta z} = 0.001127 \times \frac{50 \times (250 \times 300)}{2 \times 1 \times 33.333} = 63.3944 \text{ STB/D-psi} \tag{2.46c}$$

Substitution into Eq. 2.45 gives

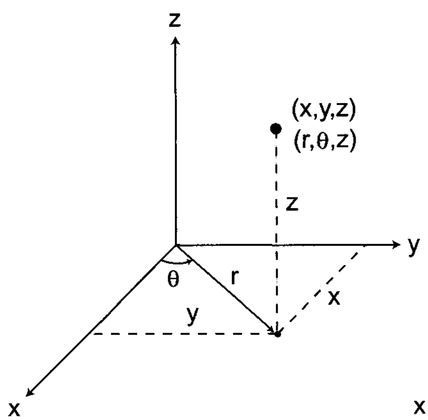
$$
\begin{aligned}
& (63.3944)[(p^m_{3,2,1} - p^m_{3,2,2}) - 12.7287] + (3.4436)(p^m_{3,1,2} - p^m_{3,2,2}) \\
& + (6.0857)(p^m_{2,2,2} - p^m_{3,2,2}) + (6.0857)(p^m_{4,2,2} - p^m_{3,2,2}) + (3.4436)(p^m_{3,3,2} - p^m_{3,2,2}) \\
& + (63.3944)[(p^m_{3,2,3} - p^m_{3,2,2}) + 12.7287] - 133.3 = \frac{V_{b_{3,2,2}}}{\alpha_c \Delta t}[(\frac{\phi}{B})^{n+1}_{3,2,2} - (\frac{\phi}{B})^n_{3,2,2}].
\end{aligned}
\tag{2.47}
$$

2.7 Multidimensional Flow in Radial-Cylindrical Coordinates

2.7.1 Reservoir Discretization for Single-Well Simulation

Single-well simulation uses radial-cylindrical coordinates. A point in space in radial-cylindrical coordinates is identified as point (r, θ, z) as shown in Figure 2–16. A cylinder with the well coinciding with its longitudinal axis represents the reservoir in single-well simulation. Reservoir discretization involves dividing the cylinder into n_r concentric radial segments with the well passing through the center. Rays from the center divide the radial segments into n_θ cake-like slices. Planes normal to the longitudinal axis divide the cake-like slices into n_z segments.

A reservoir block in a discretized reservoir is identified as block (i,j,k) , where i, j, and k are respectively the orders of the block in r, θ, and z directions with $1 \le i \le n_r$, $1 \le j \le n_\theta$, $1 \le k \le n_z$. This block has the shape shown in Figure 2–17.



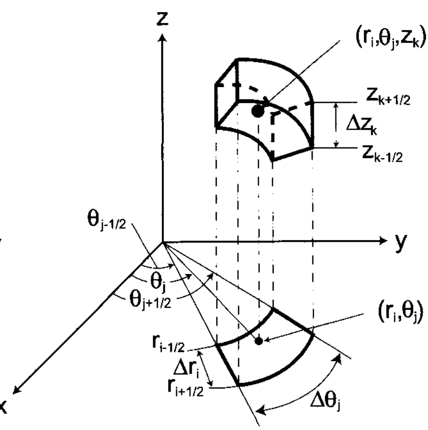
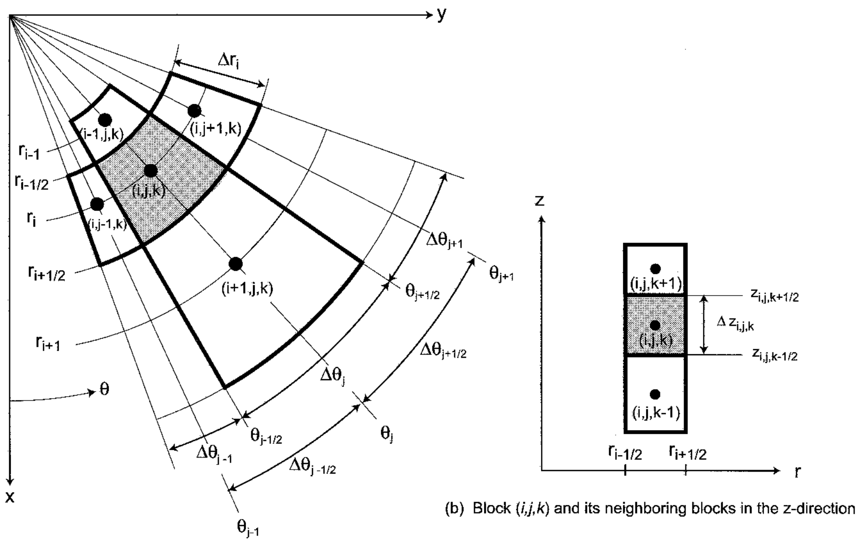


Figure 2–16 Graphing a point in Cartesian and radial coordinates (left).
Figure 2–17 Block (i,j,k) in single-well simulation (right).

Figure 2–18a shows that block (i,j,k) is surrounded by blocks $(i-1,j,k)$ and $(i+1,j,k)$ in the r direction, and by blocks $(i,j-1,k)$ and $(i,j+1,k)$ in the θ direction. In addition, the figure shows the boundaries between block (i,j,k) and its neighboring blocks: block boundaries $(i-\frac{1}{2},j,k)$, $(i+\frac{1}{2},j,k)$, $(i,j-\frac{1}{2},k)$, and $(i,j+\frac{1}{2},k)$. Figure 2–18b shows that block (i,j,k) is surrounded by blocks $(i,j,k-1)$ and $(i,j,k+1)$ in the z direction. The figure also shows block boundaries $(i,j,k-\frac{1}{2})$ and $(i,j,k+\frac{1}{2})$. We will demonstrate block identification and ordering in a single-well simulation in the following two examples. In the absence of fluid flow in the θ direction, block ordering and identification in radial and rectangular coordinates are identical.



(a) Block (i,j,k) and its neighboring blocks in horizontal plane.

(b) Block (i,j,k) and its neighboring blocks in the z-direction.

Figure 2–18 Block (i, j, k) and its neighboring blocks in single-well simulation.

Example 2.7 In single-well simulation, a reservoir is discretized in the r direction into four concentric cylindrical blocks as shown in Figure 2–19a. Order blocks in this reservoir using natural ordering.

Solution

We identify the innermost block enclosing the well as block 1. Then we move to other blocks, one block at a time, in the direction of increasing radius. The order of the next block is obtained by incrementing the order of the previous block by one. We continue the process of block ordering (or numbering) until the outermost block is numbered. The final ordering of blocks in this reservoir is shown in Figure 2–19b.

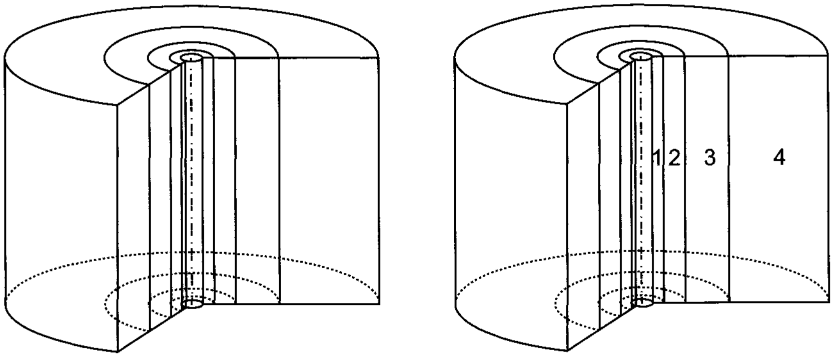
Example 2.8 Let the reservoir in Example 2.7 consists of three layers as shown in Figure 2–20a.

Identify the blocks in this reservoir using

(a) Engineering notation.
(b) Natural ordering.

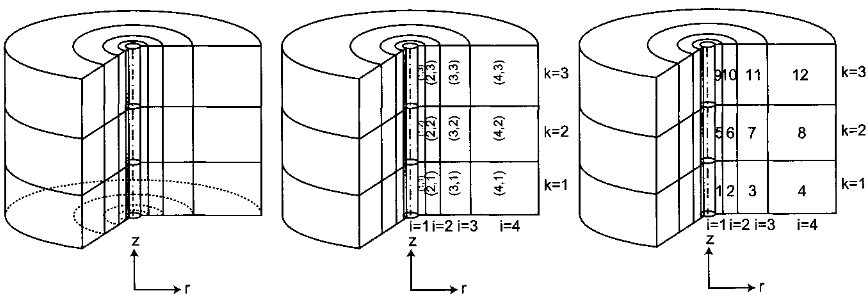
Solution

(a) The engineering notation for block identification in this reservoir is shown in Figure 2–20b.
(b) We arbitrarily choose to order blocks in each layer along rows. Blocks in the first layer ($k = 1$) are numbered as explained in Example 2.7. Block (1,2) in



(a) Reservoir representation. (b) Natural ordering of blocks.

Figure 2–19 1D radial-cylindrical reservoir representation in Example 2.7.



(a) Reservoir representation. (b) Engineering notation. (c) Natural ordering of blocks.

Figure 2–20 2D radial-cylindrical reservoir representation in Example 2.8.

the first column ($i = 1$) and second plane ($k = 2$) is numbered next as block 5, and block numbering continues as in Example 2.7. Block numbering continues (layer by layer) until all blocks are numbered. The final ordering of blocks in this reservoir is shown in Figure 2–20c.

2.7.2 Derivation of the Multidimensional Flow Equation in Radial-cylindrical Coordinates

To write the material balance for block (i,j,k) in Figure 2–18 over a time step $\Delta t = t^{n+1} - t^n$, we assume that the fluid coming from neighboring blocks enters block (i,j,k) through block boundaries $(i-\tfrac{1}{2},j,k)$, $(i,j-\tfrac{1}{2},k)$, and $(i,j,k-\tfrac{1}{2})$ and leaves through block boundaries $(i+\tfrac{1}{2},j,k)$, $(i,j+\tfrac{1}{2},k)$, and $(i,j,k+\tfrac{1}{2})$. The application of Eq. 2.6 results in

$$(m_i|_{r_{i-1/2,j,k}} - m_o|_{r_{i+1/2,j,k}}) + (m_i|_{\theta_{i,j-1/2,k}} - m_o|_{\theta_{i,j+1/2,k}}) + (m_i|_{z_{i,j,k-1/2}} - m_o|_{z_{i,j,k+1/2}}) + m_{s_{i,j,k}} = m_{a_{i,j,k}}. \quad \textbf{(2.48)}$$

Terms like mass rates,

$$w_r\Big|_{r_{i-1/2,j,k}}, w_\theta\Big|_{\theta_{i,j-1/2,k}}, w_z\Big|_{z_{i,j,k-1/2}}, w_r\Big|_{r_{i+1/2,j,k}}, w_\theta\Big|_{\theta_{i,j+1/2,k}}, w_z\Big|_{z_{i,j,k+1/2}}$$

and well mass rate, $q_{m_{i,j,k}}$, are functions of time only (see justification in Sec. 2.6.2); therefore,

$$m_i\Big|_{r_{i-1/2,j,k}} = \int_{t^n}^{t^{n+1}} w_r\Big|_{r_{i-1/2,j,k}} \mathrm{d}t \tag{2.49a}$$

$$m_i\Big|_{\theta_{i,j-1/2,k}} = \int_{t^n}^{t^{n+1}} w_\theta\Big|_{\theta_{i,j-1/2,k}} \mathrm{d}t \tag{2.49b}$$

$$m_i\Big|_{z_{i,j,k-1/2}} = \int_{t^n}^{t^{n+1}} w_z\Big|_{z_{i,j,k-1/2}} \mathrm{d}t \tag{2.49c}$$

$$m_o\Big|_{r_{i+1/2,j,k}} = \int_{t^n}^{t^{n+1}} w_r\Big|_{r_{i+1/2,j,k}} \mathrm{d}t \tag{2.50a}$$

$$m_o\Big|_{\theta_{i,j+1/2,k}} = \int_{t^n}^{t^{n+1}} w_\theta\Big|_{\theta_{i,j+1/2,k}} \mathrm{d}t \tag{2.50b}$$

$$m_o\Big|_{z_{i,j,k+1/2}} = \int_{t^n}^{t^{n+1}} w_z\Big|_{z_{i,j,k+1/2}} \mathrm{d}t \tag{2.50c}$$

$$m_{s_{i,j,k}} = \int_{t^n}^{t^{n+1}} q_{m_{i,j,k}} \mathrm{d}t \tag{2.51}$$

In addition, mass accumulation is defined as

$$m_{a_{i,j,k}} = \Delta_t (V_b m_v)_{i,j,k} = V_{b_{i,j,k}} (m_{v_{i,j,k}}^{n+1} - m_{v_{i,j,k}}^{n}) \tag{2.52}$$

Mass rates and mass fluxes are related through

$$w_r\Big|_r = \dot{m}_r A_r \tag{2.53a}$$

$$w_\theta\Big|_\theta = \dot{m}_\theta A_\theta \tag{2.53b}$$

$$w_z\Big|_z = \dot{m}_z A_z \tag{2.53c}$$

Mass fluxes can be expressed in terms of fluid density and volumetric velocities,

$$\dot{m}_r = \alpha_c \rho u_r \tag{2.54a}$$

$$\dot{m}_\theta = \alpha_c \rho u_\theta \tag{2.54b}$$

$$\dot{m}_z = \alpha_c \rho u_z \tag{2.54c}$$

and m_v can be expressed in terms of fluid density and porosity,

$$m_{v_{i,j,k}} = (\phi\rho)_{i,j,k} \tag{2.55}$$

Also, the well mass rate can be expressed in terms of well volumetric rate and fluid density,

$$q_{m_{i,j,k}} = (\alpha_c \rho q)_{i,j,k} \tag{2.56}$$

Substitution of Eq. 2.54 into Eq. 2.53 yields

$$w_r\big|_r = \alpha_c \rho u_r A_r \tag{2.57a}$$

$$w_\theta\big|_\theta = \alpha_c \rho u_\theta A_\theta \tag{2.57b}$$

$$w_z\big|_z = \alpha_c \rho u_z A_z \tag{2.57c}$$

Substitution of Eq. 2.57 into Eqs. 2.49 and 2.50 yields

$$m_i\big|_{r_{i-1/2,j,k}} = \int_{t^n}^{t^{n+1}} \alpha_c (\rho u_r A_r)\big|_{r_{i-1/2,j,k}} \mathrm{d}t \tag{2.58a}$$

$$m_i\big|_{\theta_{i,j-1/2,k}} = \int_{t^n}^{t^{n+1}} \alpha_c (\rho u_\theta A_\theta)\big|_{\theta_{i,j-1/2,k}} \mathrm{d}t \tag{2.58b}$$

$$m_i\big|_{z_{i,j,k-1/2}} = \int_{t^n}^{t^{n+1}} \alpha_c (\rho u_z A_z)\big|_{z_{i,j,k-1/2}} \mathrm{d}t \tag{2.58c}$$

$$m_o\big|_{r_{i+1/2,j,k}} = \int_{t^n}^{t^{n+1}} \alpha_c (\rho u_r A_r)\big|_{r_{i+1/2,j,k}} \mathrm{d}t \tag{2.59a}$$

$$m_o\big|_{\theta_{i,j+1/2,k}} = \int_{t^n}^{t^{n+1}} \alpha_c (\rho u_\theta A_\theta)\big|_{\theta_{i,j+1/2,k}} \mathrm{d}t \tag{2.59b}$$

$$m_o\big|_{z_{i,j,k+1/2}} = \int_{t^n}^{t^{n+1}} \alpha_c (\rho u_z A_z)\big|_{z_{i,j,k+1/2}} \mathrm{d}t \tag{2.59c}$$

Substitution of Eq. 2.56 into Eq. 2.51 yields

$$m_{s_{i,j,k}} = \int_{t^n}^{t^{n+1}} (\alpha_c \rho q)_{i,j,k} \mathrm{d}t \tag{2.60}$$

Substitution of Eq. 2.55 into Eq. 2.52 yields

$$m_{a_{i,j,k}} = V_{b_{i,j,k}}[(\phi\rho)^{n+1}_{i,j,k} - (\phi\rho)^{n}_{i,j,k}] \tag{2.61}$$

Substitution of Eqs. 2.58 through 2.61 into Eq. 2.48 results in

$$\begin{aligned}&\int_{t^n}^{t^{n+1}} \alpha_c(\rho u_r A_r)\Big|_{r_{i-1/2,j,k}} \mathrm{d}t - \int_{t^n}^{t^{n+1}} \alpha_c(\rho u_r A_r)\Big|_{r_{i+1/2,j,k}} \mathrm{d}t + \int_{t^n}^{t^{n+1}} \alpha_c(\rho u_\theta A_\theta)\Big|_{\theta_{i,j-1/2,k}} \mathrm{d}t \\ &- \int_{t^n}^{t^{n+1}} \alpha_c(\rho u_\theta A_\theta)\Big|_{\theta_{i,j+1/2,k}} \mathrm{d}t + \int_{t^n}^{t^{n+1}} \alpha_c(\rho u_z A_z)\Big|_{z_{i,j,k-1/2}} \mathrm{d}t - \int_{t^n}^{t^{n+1}} \alpha_c(\rho u_z A_z)\Big|_{z_{i,j,k+1/2}} \mathrm{d}t \\ &+ \int_{t^n}^{t^{n+1}} (\alpha_c \rho q)_{i,j,k} \mathrm{d}t = V_{b_{i,j,k}}[(\phi\rho)^{n+1}_{i,j,k} - (\phi\rho)^{n}_{i,j,k}].\end{aligned} \tag{2.62}$$

Substitution of Eq. 2.7a into Eq. 2.62, dividing by $\alpha_c \rho_{sc}$, and noting that $q_{sc} = q/B$ yields

$$\begin{aligned}&\int_{t^n}^{t^{n+1}} (\frac{u_r A_r}{B})\Big|_{r_{i-1/2,j,k}} \mathrm{d}t - \int_{t^n}^{t^{n+1}} (\frac{u_r A_r}{B})\Big|_{r_{i+1/2,j,k}} \mathrm{d}t + \int_{t^n}^{t^{n+1}} (\frac{u_\theta A_\theta}{B})\Big|_{\theta_{i,j-1/2,k}} \mathrm{d}t \\ &- \int_{t^n}^{t^{n+1}} (\frac{u_\theta A_\theta}{B})\Big|_{\theta_{i,j+1/2,k}} \mathrm{d}t + \int_{t^n}^{t^{n+1}} (\frac{u_z A_z}{B})\Big|_{z_{i,j,k-1/2}} \mathrm{d}t - \int_{t^n}^{t^{n+1}} (\frac{u_z A_z}{B})\Big|_{z_{i,j,k+1/2}} \mathrm{d}t \\ &+ \int_{t^n}^{t^{n+1}} q_{sc_{i,j,k}} \mathrm{d}t = \frac{V_{b_{i,j,k}}}{\alpha_c}[(\frac{\phi}{B})^{n+1}_{i,j,k} - (\frac{\phi}{B})^{n}_{i,j,k}].\end{aligned} \tag{2.63}$$

Fluid volumetric velocities in the r, θ, and z directions are given by the algebraic analogs of Eq. 2.8; i.e.,

$$u_r\Big|_{r_{i-1/2,j,k}} = \beta_c \frac{k_r\big|_{r_{i-1/2,j,k}}}{\mu\big|_{r_{i-1/2,j,k}}}[\frac{(\Phi_{i-1,j,k} - \Phi_{i,j,k})}{\Delta r_{i-1/2,j,k}}] \tag{2.64a}$$

$$u_r\Big|_{r_{i+1/2,j,k}} = \beta_c \frac{k_r\big|_{r_{i+1/2,j,k}}}{\mu\big|_{r_{i+1/2,j,k}}}[\frac{(\Phi_{i,j,k} - \Phi_{i+1,j,k})}{\Delta r_{i+1/2,j,k}}]. \tag{2.64b}$$

Likewise,

$$u_z\Big|_{z_{i,j,k-1/2}} = \beta_c \frac{k_z\big|_{z_{i,j,k-1/2}}}{\mu\big|_{z_{i,j,k-1/2}}}[\frac{(\Phi_{i,j,k-1} - \Phi_{i,j,k})}{\Delta z_{i,j,k-1/2}}] \tag{2.65a}$$

$$u_z\Big|_{z_{i,j,k+1/2}} = \beta_c \frac{k_z\Big|_{z_{i,j,k+1/2}}}{\mu\Big|_{z_{i,j,k+1/2}}}[\frac{(\Phi_{i,j,k} - \Phi_{i,j,k+1})}{\Delta z_{i,j,k+1/2}}]. \tag{2.65b}$$

Similarly,

$$u_\theta\Big|_{\theta_{i,j-1/2,k}} = \beta_c \frac{k_\theta\Big|_{\theta_{i,j-1/2,k}}}{\mu\Big|_{\theta_{i,j-1/2,k}}}[\frac{(\Phi_{i,j-1,k} - \Phi_{i,j,k})}{r_{i,j,k}\Delta\theta_{i,j-1/2,k}}] \tag{2.66a}$$

$$u_\theta\Big|_{\theta_{i,j+1/2,k}} = \beta_c \frac{k_\theta\Big|_{\theta_{i,j+1/2,k}}}{\mu\Big|_{\theta_{i,j+1/2,k}}}[\frac{(\Phi_{i,j,k} - \Phi_{i,j+1,k})}{r_{i,j,k}\Delta\theta_{i,j+1/2,k}}]. \tag{2.66b}$$

Substitution of Eqs. 2.64 through 2.66 into Eq. 2.63 and grouping terms results in

$$\begin{aligned}
&\int_{t^n}^{t^{n+1}} [(\beta_c \frac{k_r A_r}{\mu B \Delta r})\Big|_{r_{i-1/2,j,k}} (\Phi_{i-1,j,k} - \Phi_{i,j,k})]dt + \int_{t^n}^{t^{n+1}} [(\beta_c \frac{k_r A_r}{\mu B \Delta r})\Big|_{r_{i+1/2,j,k}} (\Phi_{i+1,j,k} - \Phi_{i,j,k})]dt \\
&+ \int_{t^n}^{t^{n+1}} [\frac{1}{r_{i,j,k}}(\beta_c \frac{k_\theta A_\theta}{\mu B \Delta\theta})\Big|_{\theta_{i,j-1/2,k}} (\Phi_{i,j-1,k} - \Phi_{i,j,k})]dt \\
&+ \int_{t^n}^{t^{n+1}} [\frac{1}{r_{i,j,k}}(\beta_c \frac{k_\theta A_\theta}{\mu B \Delta\theta})\Big|_{\theta_{i,j+1/2,k}} (\Phi_{i,j+1,k} - \Phi_{i,j,k})]dt \\
&+ \int_{t^n}^{t^{n+1}} [(\beta_c \frac{k_z A_z}{\mu B \Delta z})\Big|_{z_{i,j,k-1/2}} (\Phi_{i,j,k-1} - \Phi_{i,j,k})]dt + \int_{t^n}^{t^{n+1}} [(\beta_c \frac{k_z A_z}{\mu B \Delta z})\Big|_{z_{i,j,k+1/2}} (\Phi_{i,j,k+1} - \Phi_{i,j,k})]dt \\
&+ \int_{t^n}^{t^{n+1}} q_{sc_{i,j,k}} dt = \frac{V_{b_{i,j,k}}}{\alpha_c}[(\frac{\phi}{B})_{i,j,k}^{n+1} - (\frac{\phi}{B})_{i,j,k}^{n}].
\end{aligned} \tag{2.67}$$

Eq. 2.67 can be rewritten as

$$\begin{aligned}
&\int_{t^n}^{t^{n+1}} [T_{z_{i,j,k-1/2}}(\Phi_{i,j,k-1} - \Phi_{i,j,k})]dt + \int_{t^n}^{t^{n+1}} [T_{\theta_{i,j-1/2,k}}(\Phi_{i,j-1,k} - \Phi_{i,j,k})]dt \\
&+ \int_{t^n}^{t^{n+1}} [T_{r_{i-1/2,j,k}}(\Phi_{i-1,j,k} - \Phi_{i,j,k})]dt + \int_{t^n}^{t^{n+1}} [T_{r_{i+1/2,j,k}}(\Phi_{i+1,j,k} - \Phi_{i,j,k})]dt \\
&+ \int_{t^n}^{t^{n+1}} [T_{\theta_{i,j+1/2,k}}(\Phi_{i,j+1,k} - \Phi_{i,j,k})]dt + \int_{t^n}^{t^{n+1}} [T_{z_{i,j,k+1/2}}(\Phi_{i,j,k+1} - \Phi_{i,j,k})]dt \\
&+ \int_{t^n}^{t^{n+1}} q_{sc_{i,j,k}} dt = \frac{V_{b_{i,j,k}}}{\alpha_c}[(\frac{\phi}{B})_{i,j,k}^{n+1} - (\frac{\phi}{B})_{i,j,k}^{n}],
\end{aligned} \tag{2.68}$$

where

$$T_{r_{i\pm 1/2,j,k}} = (\beta_c \frac{k_r A_r}{\mu B \Delta r})\bigg|_{r_{i\mp 1/2,j,k}} = (\beta_c \frac{k_r A_r}{\Delta r})_{r_{i\mp 1/2,j,k}} (\frac{1}{\mu B})_{r_{i\mp 1/2,j,k}} = G_{r_{i\pm 1/2,j,k}} (\frac{1}{\mu B})_{r_{i\mp 1/2,j,k}} \quad \textbf{(2.69a)}$$

$$T_{\theta_{i,j\mp 1/2,k}} = \frac{1}{r_{i,j,k}} (\beta_c \frac{k_\theta A_\theta}{\mu B \Delta \theta})\bigg|_{\theta_{i,j\mp 1/2,k}} = (\beta_c \frac{k_\theta A_\theta}{r_{i,j,k} \Delta \theta})_{\theta_{i,j\mp 1/2,k}} (\frac{1}{\mu B})_{\theta_{i,j\mp 1/2,k}}$$
$$= G_{\theta_{i,j\mp 1/2,k}} (\frac{1}{\mu B})_{\theta_{i,j\mp 1/2,k}}, \quad \textbf{(2.69b)}$$

$$T_{z_{i,j,k\mp 1/2}} = (\beta_c \frac{k_z A_z}{\mu B \Delta z})\bigg|_{z_{i,j,k\mp 1/2}} = (\beta_c \frac{k_z A_z}{\Delta z})_{z_{i,j,k\mp 1/2}} (\frac{1}{\mu B})_{z_{i,j,k\mp 1/2}} = G_{z_{i,j,k\mp 1/2}} (\frac{1}{\mu B})_{z_{i,j,k\mp 1/2}} \quad \textbf{(2.69c)}$$

Expressions for geometric factors G for irregular grids in heterogeneous reservoirs are presented in Chapters 4 and 5.

2.7.3 Approximation of Time Integrals

Using Eq. 2.30 to approximate integrals in Eq. 2.68 and dividing by Δt , the flow equation in radial-cylindrical coordinates becomes

$$\begin{aligned}
&T^m_{z_{i,j,k-1/2}}[(\Phi^m_{i,j,k-1} - \Phi^m_{i,j,k})] + T^m_{\theta_{i,j-1/2,k}}[(\Phi^m_{i,j-1,k} - \Phi^m_{i,j,k})] \\
&+T^m_{r_{i-1/2,j,k}}[(\Phi^m_{i-1,j,k} - \Phi^m_{i,j,k})] + T^m_{r_{i+1/2,j,k}}[(\Phi^m_{i+1,j,k} - \Phi^m_{i,j,k})] \\
&+T^m_{\theta_{i,j+1/2,k}}[(\Phi^m_{i,j+1,k} - \Phi^m_{i,j,k})] + T^m_{z_{i,j,k+1/2}}[(\Phi^m_{i,j,k+1} - \Phi^m_{i,j,k})] \\
&+q^m_{sc_{i,j,k}} = \frac{V_{b_{i,j,k}}}{\alpha_c \Delta t}[(\frac{\phi}{B})^{n+1}_{i,j,k} - (\frac{\phi}{B})^n_{i,j,k}].
\end{aligned} \quad \textbf{(2.70)}$$

Using the definition of potential difference, Eq. 2.70 becomes

$$\begin{aligned}
&T^m_{z_{i,j,k-1/2}}[(p^m_{i,j,k-1} - p^m_{i,j,k}) - \gamma^m_{i,j,k-1/2}(Z_{i,j,k-1} - Z_{i,j,k})] \\
&+T^m_{\theta_{i,j-1/2,k}}[(p^m_{i,j-1,k} - p^m_{i,j,k}) - \gamma^m_{i,j-1/2,k}(Z_{i,j-1,k} - Z_{i,j,k})] \\
&+T^m_{r_{i-1/2,j,k}}[(p^m_{i-1,j,k} - p^m_{i,j,k}) - \gamma^m_{i-1/2,j,k}(Z_{i-1,j,k} - Z_{i,j,k})] \\
&+T^m_{r_{i+1/2,j,k}}[(p^m_{i+1,j,k} - p^m_{i,j,k}) - \gamma^m_{i+1/2,j,k}(Z_{i+1,j,k} - Z_{i,j,k})] \\
&+T^m_{\theta_{i,j+1/2,k}}[(p^m_{i,j+1,k} - p^m_{i,j,k}) - \gamma^m_{i,j+1/2,k}(Z_{i,j+1,k} - Z_{i,j,k})] \\
&+T^m_{z_{i,j,k+1/2}}[(p^m_{i,j,k+1} - p^m_{i,j,k}) - \gamma^m_{i,j,k+1/2}(Z_{i,j,k+1} - Z_{i,j,k})] \\
&+q^m_{sc_{i,j,k}} = \frac{V_{b_{i,j,k}}}{\alpha_c \Delta t}[(\frac{\phi}{B})^{n+1}_{i,j,k} - (\frac{\phi}{B})^n_{i,j,k}].
\end{aligned} \quad \textbf{(2.71)}$$

Eq. 2.38, the flow equation in Cartesian coordinates (x-y-z), is used for field simulation, whereas Eq. 2.71, the flow equation in radial-cylindrical coordinates (r-θ-z), is used for single-well simulation. These two equations are similar in form. The RHS of both equations represents fluid accumulation in block (i,j,k). On the LHS, both equations have a source term represented by well production or injection and six flow terms representing interblock flow between block (i,j,k) and its six neighboring blocks: blocks $(i-1,j,k)$ and $(i+1,j,k)$ in the x direction (or r direction), blocks $(i,j-1,k)$ and $(i,j+1,k)$ in the y direction (or θ direction), and blocks $(i,j,k-1)$ and $(i,j,k+1)$ in the z direction. The coefficients of potential differences are transmissibilities T_x, T_y, and T_z in the x-y-z space and T_r, T_θ, and T_z in the r-θ-z space. Eqs. 2.39 and 2.69 define these transmissibilities. The geometric factors in these equations are presented in Chapter 4 for the block-centered grid and in Chapter 5 for the point-distributed grid.

2.8 Summary

This chapter has reviewed various engineering steps involved in rendering governing equations into algebraic equations. Governing equations, involving both the rock and fluid properties, are discretized without conventional finite-difference or finite element manipulation of PDEs. Fluid properties such as density, FVF, and viscosity are, in general, functions of pressure. Reservoir porosity depends on pressure and has heterogeneous distribution, and reservoir permeability is usually anisotropic. The basic knowledge of material balance, FVF, potential difference, and Darcy's Law are necessary for deriving flow equations in petroleum reservoirs. Rectangular coordinates and radial coordinates are two ways of describing reservoirs in space. Although it is common to study reservoirs using rectangular coordinates, there are a few applications that require using radial-cylindrical coordinates. Using the engineering approach, the single-phase flow equation can be derived in any coordinate system. In this approach, the reservoir first is discretized into blocks, which are identified using the engineering notation or any block ordering scheme. The second step involves writing the fluid material balance for a general reservoir block in a multidimensional reservoir over the time interval $t^n \le t \le t^{n+1}$ and combining it with Darcy's Law and the formation volume factor. The third step provides for an evaluation method of the time integrals in the flow equation that was obtained in the second step. The result is a flow equation in algebraic form with all functions evaluated at time t^m, where $t^n \le t^m \le t^{n+1}$. In Chapter 7, we demonstrate how the choice of time t^m as the old time level t^n, new time level t^{n+1}, or intermediate time level $t^{n+1/2}$ gives rise to the explicit formulation, implicit formulation, or the Crank-Nicolson formulation of the flow equation.

2.9 Exercises

2-1 List the physical properties of rock and fluid necessary for the derivation of a single-phase flow equation.

2-2 Enumerate the three basic engineering concepts or equations used in the derivation of a flow equation.

2-3 Eq. 2.33 has four major terms, three on the LHS and one on the RHS. What is the physical meaning of each major term? What are units of each major term in the three systems of units? Using customary units, state the units of each variable or function that appears in Eq. 2.33.

2-4 Compare Eq. 2.33 with Eq. 2.37; i.e., identify the similar major terms and the extra major terms in Eq. 2.37. What is the physical meaning of each of these extra terms and to which direction do they belong?

2-5 Compare Eq. 2.33 with Eq. 2.38; i.e., identify the similar major terms and the extra major terms in Eq. 2.38. What is the physical meaning of each of these extra terms? Group the extra terms according to the direction they belong.

2-6 Compare the 3D flow equation in rectangular coordinates (x-y-z) in Eq. 2.38, with the 3D flow equation in radial-cylindrical coordinates (r-θ-z) in Eq. 2.71. Elaborate on the similarities and differences in these two equations. Note the differences in the definition of geometric factors.

2-7 Consider the 2D reservoir shown in Figure 2–21. This reservoir is discretized using 5×5 blocks but has irregular boundaries, as shown in the figure.

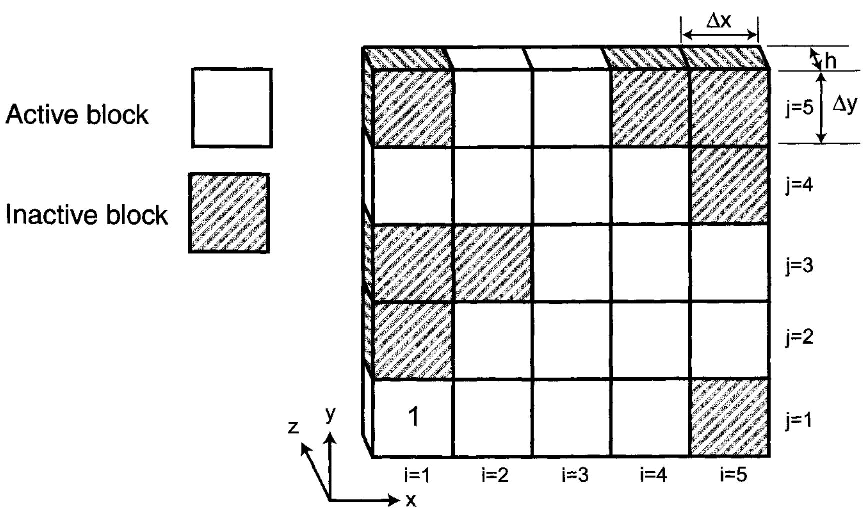


Figure 2–21 2D reservoir representation in Exercise 2-7.

Use the following schemes to identify and order the blocks in this reservoir:

- engineering notation
- natural ordering by rows
- natural ordering by columns
- diagonal (D2) ordering
- alternating diagonal (D4) ordering

- zebra ordering
- cyclic ordering
- cyclic-2 ordering

2-8 Consider single-phase flow in a 1D inclined reservoir. The flow equation for block i in this reservoir is expressed as Eq. 2.33.

(a) Write Eq. 2.33 for block i assuming $t^m = t^n$. The resulting equation is the explicit formulation of the flow equation for block i.
(b) Write Eq. 2.33 for block i assuming $t^m = t^{n+1}$. The resulting equation is the implicit formulation of the flow equation for block i.
(c) Write Eq. 2.33 for block i assuming $t^m = t^{n+1/2}$. The resulting equation is the Crank-Nicolson formulation of the flow equation for block i.

2-9 Consider single-phase flow of oil in a 1D horizontal reservoir. The reservoir is discretized using six blocks as shown in Figure 2–22. A well that is located in block 4 produces at a rate of 600 STB/D. All blocks have $\Delta x = 220$ ft, $\Delta y = 1000$ ft, $h = 90$ ft, and $k_x = 120$ md. The oil FVF, viscosity, and compressibility are 1.0 RB/STB, 3.5 cp, and 1.5×10^{-5} psi^{-1}, respectively.

(a) Identify the interior and boundary blocks in this reservoir.
(b) Write the flow equation for every interior block. Leave the RHS of the flow equation without substitution of values.
(c) Write the flow equation for every interior block assuming incompressible fluid and porous medium.

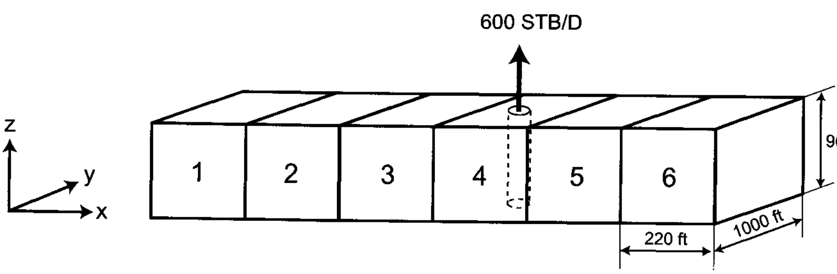


Figure 2–22 1D reservoir representation in Exercise 2-9.

2-10 Consider single-phase flow of water in a 2D horizontal reservoir. The reservoir is discretized using 4×4 blocks as shown in Figure 2–23. Two wells are located in blocks (2,2) and (3,3), and each produces at a rate of 200 STB/D. All blocks have $\Delta x = 200$ ft, $\Delta y = 200$ ft, $h = 50$ ft, and $k_x = k_y = 180$ md. The oil FVF, viscosity, and compressibility are 1.0 RB/STB, 0.5 cp, and 1×10^{-6} psi^{-1}, respectively.

(a) Identify the interior and boundary blocks in this reservoir.
(b) Write the flow equation for every interior block. Leave the RHS of flow equation without substitution of values.

(c) Write the flow equation for every interior block assuming incompressible fluid and porous medium.

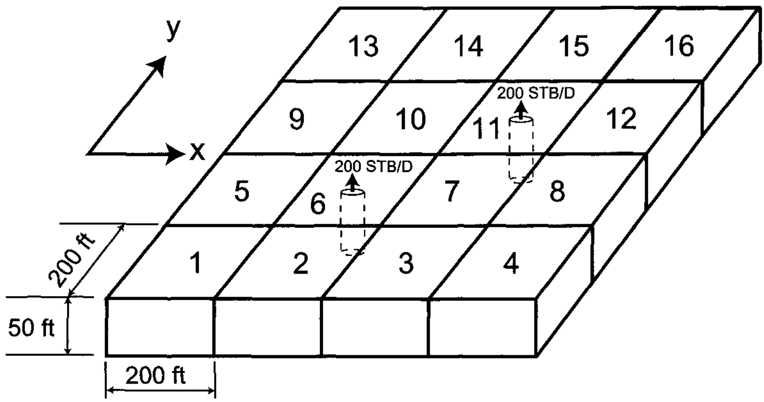


Figure 2–23 2D reservoir representation in Exercise 2-10.

2-11 Consider the 2D horizontal reservoir presented in Figure 2–21. All blocks have same dimensions ($\Delta x = 300$ ft, $\Delta y = 300$ ft, and $h = 20$ ft) and rock properties ($k_x = 140$ md, $k_y = 140$ md, and $\phi = 0.13$). The oil FVF and viscosity are 1.0 RB/STB and 3 cp, respectively. Write the flow equations for the interior blocks in this reservoir assuming incompressible fluid flow in an incompressible porous medium.

2-12 Consider the 1D radial reservoir presented in Figure 2–19. Write the flow equations for the interior blocks in this reservoir. Do not estimate interblock radial transmissibility. Leave the RHS of flow equations without substitution.

2-13 Consider the 2D radial reservoir presented in Figure 2–20b. Write the flow equations for the interior blocks in this reservoir. Do not estimate interblock radial or vertical transmissibilities. Leave the RHS of the flow equations without substitution.

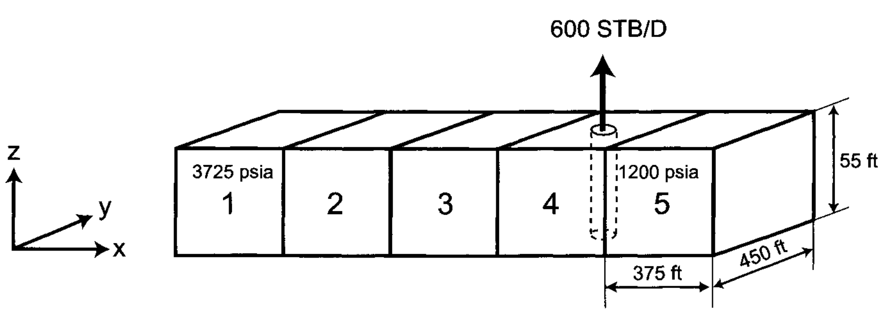


Figure 2–24 1D reservoir representation in Exercise 2-14.

2-14 A single-phase oil reservoir is described by five equal blocks as shown in Figure 2–24. The reservoir is horizontal and has homogeneous and isotropic rock properties, $k = 210$ md and $\phi = 0.21$. Block dimensions are $\Delta x = 375$ ft, $\Delta y = 450$ ft, and $h = 55$ ft. Oil properties are $B = 1$ RB/STB and $\mu = 1.5$ cp. The pressure of blocks 1 and 5 is 3725 psia and 1200 psia, respectively. Block 4 hosts a well that produces oil at a rate of 600 STB/D. Find the pressure distribution in the reservoir assuming that the reservoir rock and oil are incompressible. Estimate the rates of oil loss or gain across the right boundary of block 5 and that across the left boundary of block 1.

2-15 A single-phase water reservoir is described by five equal blocks as shown in Figure 2–25. The reservoir is horizontal and has $k = 178$ md and $\phi = 0.17$. Block dimensions are $\Delta x = 275$ ft, $\Delta y = 650$ ft, and $h = 30$ ft. Water properties are $B = 1$ RB/B and $\mu = 0.7$ cp. The pressure of blocks 1 and 5 is maintained at 3000 psia and 1000 psia, respectively. Block 3 hosts a well that produces water at a rate of 240 B/D. Find the pressure distribution in the reservoir assuming that the reservoir water and rock are incompressible.

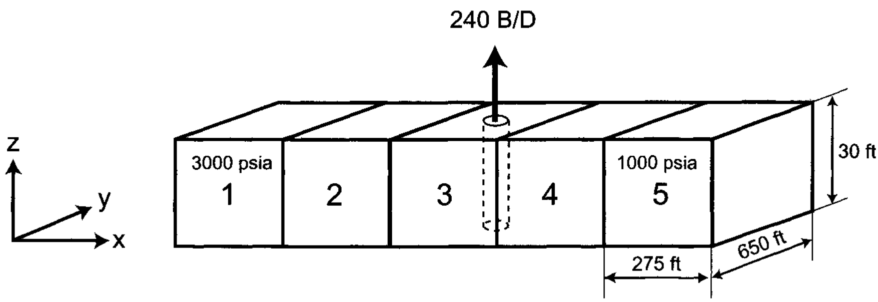


Figure 2–25 1D reservoir representation in Exercise 2-15.

2-16 Consider the reservoir presented in Figure 2–14 and the flow problem described in Example 2.5. Assuming that both the reservoir fluid and rock are incompressible and given that a strong aquifer keeps the pressure of all boundary blocks at 3200 psia, estimate the pressure of blocks (2,2) and (3,2).

2-17 Consider single-phase flow of water in a 2D horizontal reservoir. The reservoir is discretized using 4×4 equal blocks as shown in Figure 2–26. Block 7 hosts a well that produces 500 B/D of water. All blocks have $\Delta x = \Delta y = 230$ ft, $h = 80$ ft, and $k_x = k_y = 65$ md. The water FVF and viscosity are 1.0 RB/B and 0.5 cp, respectively. The pressure of reservoir boundary blocks is specified as $p_2 = p_3 = p_4 = p_8 = p_{12} = 2500$ psia, $p_1 = p_5 = p_9 = p_{13} = 4000$ psia, and $p_{14} = p_{15} = p_{16} = 3500$ psia. Assuming that the reservoir water and rock are incompressible, calculate the pressure of blocks 6, 7, 10, and 11.

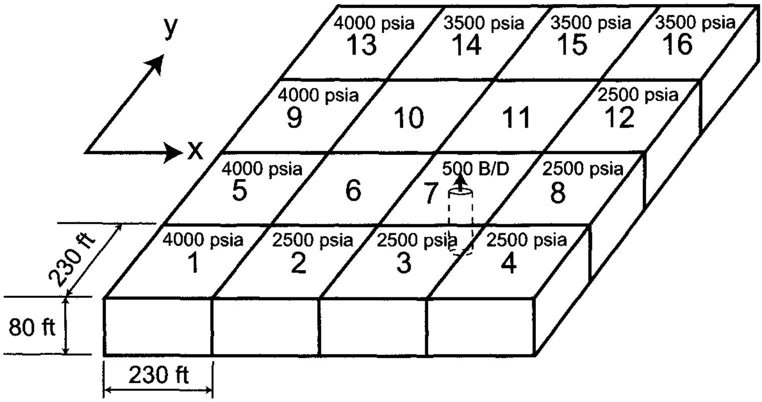


Figure 2–26 2D reservoir representation in Exercise 2-17.