# Thermal Radiation - part 1

Octavian Dănilă

## 1 Brief historical background

By the end of the 19th century, it seemed like physics had achieved its main goal - to describe every observed phenomenon. Mechanics had its two cornerstones set by Newton and Lagrange, electromagnetic phenomena could completely be described by Maxwell's equations, optics could be attributed to a special field of electromagnetics, thermodynamics and statistical physics had laws derived directly from applying mechanics principles to large-number systems of non-interacting particles. All was seemingly fine in the physics "kingdom", and all its major builders were satisfied and could finally rest. That was, until someone rushed through the front gate with a new phenomenon, and tried to test all known physics to explain it. Shock! - it could not! The phenomenon ultimately came to be known as the external photoelectric effect, which needs a lot of new physics to explain (and that someone was H. Hertz). Starting from that, others started building a new wing of the physics kingdom. This is an account of their efforts.

## 2 Radiometric properties of an incandescent source

One of the unexplained phenomena was *the emission of electromagnetic radiation by any body having some temperature $T$*. This experimental truth was unable to be explained by means of classical physics. And indeed, it was weird to even try: nothing in thermodynamics had mentioned that the atoms and/or molecules of either ideal or real gases generate electromagnetic waves, because there was no way to link electromagnetic to thermal properties theoretically. Some of the properties of the emitted radiation are:

1. The electromagnetic energy density is orders of magnitude smaller than the thermal agitation energy density;
2. The emitted radiation is homogeneous, isotropic and unpolarized;
3. The emission of radiation is temperature-dependent, rising with the temperature of the source;
4. The emission spectrum is continuous. The spectral range is determined by the temperature of the source;
5. The electromagnetic energy density is dependent on the spectrum components, not every spectral component carries the same energy density (non-uniform spectral distribution of energy);
6. Every body emits electromagnetic radiation with a specific energy - two bodies heated at the same temperature may emit electromagnetic radiation with different energy depending on their internal properties.

7. A body that emits electromagnetic radiation can also absorb the same amount of radiation by means of process reversal.

Based on the observations above, we can make the following definitions:

1. *The volumetric electromagnetic energy density* is defined as the distribution of electromagnetic energy across the volume of the system:

$$w = \frac{dW}{dV} \tag{1}$$

2. *The integral emission power* is the electromagnetic energy that is emitted across unit surface, in unit time, across all the spectrum, at source temperature $T$:

$$P_T = \frac{d}{dA}\left(\frac{dw}{dt}\right) \tag{2}$$

   Here, instead of total electromagnetic energy, we take the electromagnetic energy density, as this is the quantity linked to power flow through a surface (see the Poynting theorem in electromagnetic optics).

3. *The spectral emission power* is the distributed emission power across the spectrum. This can be expressed in terms of frequency $\nu$ or wavelength $\lambda$:

$$P_{\nu,T} = \frac{dP}{d\nu}; \quad P_{\lambda,T} = \frac{dP}{d\lambda} \tag{3}$$

   The two quantities defined above have to obey the relationship:

$$P_{\nu,T}\,d\nu = -P_{\lambda,T}\,d\lambda \tag{4}$$

   where the minus sign accounts for the fact that when $d\nu$ increases, $d\lambda$ decreases, based on the relation between the two.

4. *The absorption coefficient* is the ratio between the absorbed electromagnetic energy $dW_{abs}$ with respect to the incident energy $dW_{inc}$ transmitted through unit surface, in unit time, by incoming waves belonging to the spectral window $(\nu, \nu + d\nu)$:

$$A_{\nu,T} = \left.\frac{dW_{abs}}{dW_{inc}}\right|_{\nu,T} \tag{5}$$

   This ratio takes values between 0 and 1. Typical bodies have an $A_{\nu,T}$ that takes different values at different frequencies $\nu$.

   We define *a black body* a body that absorbs all the incoming electromagnetic radiation for any temperature $T$ and for all spectral components. For such bodies, the absorption coefficient reads:

$$A_{\nu,T} = 1 \tag{6}$$

   We define a *perfectly reflecting body* a body that reflects all the incoming electromagnetic radiation for any temperature $T$ and for all spectral components. For such bodies, the absorption coefficient reads:

$$A_{\nu,T} = 0 \tag{7}$$

We define a *a gray body* a body that has the same value of the absorption coefficient across the whole spectrum. The absorption coefficient value depends on the temperature and the nature of the material. For gray bodies, the absorption coefficient reads:

$$A_{\nu,T} = A_T \tag{8}$$

5. *The volumetric spectral energy density* $w_{\nu,T}$ is the parameter equal to the electromagnetic energy emitted per unit volume and per unit frequency:

$$w_{\nu,T} = \frac{d}{d\nu}\left(\frac{dW}{dV}\right) = \frac{dw}{d\nu} \tag{9}$$

All of the above parameters can be redefined equivalently in terms of wavelength. The monotony of the respective functions in terms of spectral components has to be reversed when switching between variables.

# 3 Creating a black body

Typically, the absorption coefficient for a random radiating body is a function of both frequency and temperature. There are some naturally-occuring gray bodies that have an absorption coefficient close to unity, such as coal smoke, platinum smoke, some metallic oxides. However, none of these systems are at equilibrium, since they are byproducts of chemical reactions. A sufficiently-good approximation of a black body at equilibrium is a black box with a small hole, that lets in some light rays. The light is reflected on the internal walls and with each reflection, part of the energy carried by the rays is absorbed inside the walls, leading rapidly to a total absorption of the incident rays. This is easy to test experimentally - if you try to look through the hole from a distance you can see that the inside of the box is completely dark if the walls are absorbing the radiation, and becomes visible as the walls are made to reflect more (for example if the walls have mirrors on them).

# 4 The laws of thermal radiation

## Kirchhoff's radiation law

According to the law of Prevost, *if two bodies absorb different quantities of energy, their emission has to be different.* This law, however is only qualitative in nature. Based on this law and the definition of radiometric quantities, Kirchhoff stated that:

*The ratio between the spectral emission power of a radiating source and its absorption coefficient is independent on the nature of the body, and is only a universal function dependent on frequency and temperature $f(\nu, T)$.*

$$\frac{P_{\nu,T}}{A_{\nu,T}} = f(\nu, T) \tag{10}$$

Since the function is universal, this means the same rules apply for the black body. However, for the black body, we have $A^{bb}_{\nu,T} = 1$, where the superscript "bb" means black body.

This means that *for any body*, the ratio between the spectral emission power and the absorption coefficient is equal to the spectral emission power of the black body (which is in fact the universal function):

$$\frac{P_{\nu,T}}{A_{\nu,T}} = \frac{P_{\nu,T}^{bb}}{A_{\nu,T}^{bb}} = P_{\nu,T}^{bb} = f(\nu, T) \tag{11}$$

## Representing the spectral energy distribution

When plotting the experimental data representing the dependency of the spectral emission power for a black body, a gray body and a typical body, we can see that:

1. The integral emission power - the area of the surface under the graph - is finite, as is expected;
2. The maximum integral emission power is the one of the black body;
3. The shape of the spectral emission power function of the gray body is the same as the one of the black body, only with lesser values. To deduce the function of the gray body, it is sufficient to multiply the black body function with the absorption coefficient of the gray body;
4. The shape of the spectral emission power function of the typical body is the result of the modulation between the underlying gray body function and various absorption functions. The integral emission power of a typical body is less than that of the black body, and typically less than that of a gray body (with some exceptions).

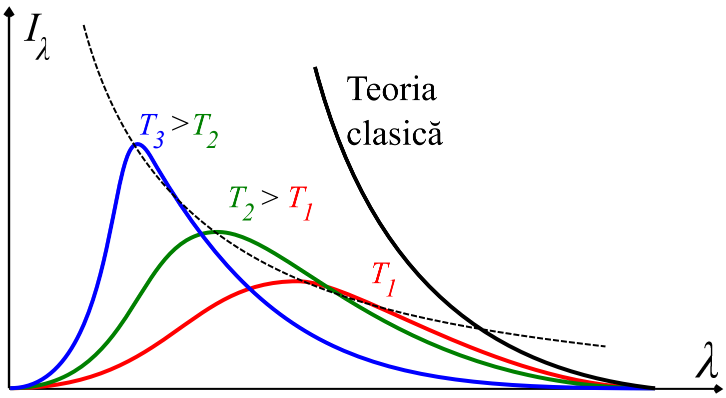


Figure 1: The experimental curve of the spectral emission power of a black body at different temperatures, as well as the classical thermodynamic approximation.

## The Stefan-Boltzmann law

The law states that:

*The integral emission power of a black body is proportional to the fourth power of temperature.*

$$P_T^{bb} = \sigma T^4 \tag{12}$$

Here, the constant $\sigma = 5.6687 \cdot 10^{-8}$ W m$^{-2}$ s$^{-1}$ grad $^{-4}$, is known as *the Stefan-Boltzmann constant*. This law only applies to black bodies, for any other radiating bodies, the law becomes:

$$P_T = \sigma(T)\, T^{\gamma(T)} \tag{13}$$

which means that both $\sigma$ and the exponent $\gamma$ become temperature-dependent. For gray bodies, the approximation:

$$P_T^{gb} = A_T \sigma T^4 \tag{14}$$

is still valid, due to the fact that $\sigma(\nu, T) \simeq \sigma$, and $\gamma \simeq 4$.

## Approximating the emission power function

Here, we arrive at the main issue regarding black body radiation: *The spectral emission power function could not be described by means of classical physics.* However, some approximations could be made: For low frequencies (i.e. $\nu \to 0$), the function can be approximated by the function $\nu^2 T$, while for high frequencies (i.e. $f_1(\nu, T) = \nu \to \infty$), the function could be described by the function $f_2(\nu, T) = \nu^3 \exp\left(-\frac{a\nu}{T}\right)$, where $a$ is a constant. However, these functions are not the result of any physical considerations, rather some mathematical fitting functions. Moreover, $f_1$ and $f_2$ both tend to infinity for the frequency corresponding to the peak of the experimental function, which means that "the middle" part of the function still could not be explained.

## Wien's displacement law

The law states that:

> *For a black body, the frequency corresponding to the maximum of the spectral emission power function is directly proportional to the temperature by means of a constant.*

Conversely, when considering the wavelength, the law states:

> *For a black body, the wavelength corresponding to the maximum of the spectral emission power function is inversely proportional to the temperature by means of a constant.*

Mathematically, we write:

$$\nu_m = b_1 T \quad \text{or} \quad \lambda T = b_2 \tag{15}$$

where $b_1$ and $b_2$ are the Wien constants. The constant $b_2 = 2.898 \cdot 10^{-3}$ m$\cdot$ K was experimentally determined.

## The Wien function for the spectral emission power function

By means of the Doppler effect (measuring the deviation in frequency of a radiation after being reflected by a mobile mirror), Wien concluded that, for a black body, the spectral emission power function has the form:

$$f_{\nu,T} = c \cdot \nu^3 f\left(\frac{\nu}{T}\right) \tag{16}$$

where $c$ is the speed of light in vacuum. Notice that the function $f\left(\frac{\nu}{T}\right)$ is still not determined. However, this form does lead to the law of Stefan-Boltzmann. By integrating across all spectrum components, we find:

$$P_T^{bb} = \int_0^\infty f_{\nu,T} d\nu \tag{17}$$

We introduce the variable $\eta = \frac{\nu}{T}$, and therefore $\nu = T\eta$, and $d\nu = Td\eta$. Returning to the integral, we get:

$$P_T^{bb} = \int_0^\infty cT^4\eta^3 f(\eta)\, d\eta = T^4 c \int_0^\infty \eta^3 f(\eta)\, d\eta \tag{18}$$

The integral cannot be computed analytically. Numerical results however, yield that:

$$c\int_0^\infty \eta^3 f(\eta) = \sigma \tag{19}$$

and therefore:

$$P_T^{bb} = \sigma T^4 \tag{20}$$

which is none other than the law of Stefan Boltzmann.

The function is also able to obtain Wien's displacement law. When converting from frequency to wavelength (i.e. $\lambda = \frac{c}{\nu}$ and $d\lambda = \frac{d\lambda}{d\nu} d\nu = -\frac{c}{\nu^2} d\nu$), the spectral emission power function becomes:

$$f_{\lambda,T} = \frac{c^4}{\lambda^5} f\left(\frac{c}{\lambda T}\right) \tag{21}$$

where $f\left(\frac{c}{\lambda T}\right)$ is again some analytically unknown function. We change variables by introducing $\eta = \frac{c}{\lambda T}$, and we get:

$$f_{\lambda,T} = \frac{\eta^5 T^5}{c} f(\eta) \tag{22}$$

To analyze the maximum, we impose the first-derivative condition $\left.\frac{df_{\lambda,T}}{d\lambda}\right|_{\lambda_m} = 0$, and get:

$$\frac{df_{\lambda,T}}{d\lambda} = \frac{1}{c}\left(5\eta^4 T^5 f(\eta)\frac{d\eta}{d\lambda} + \eta^5 T^5 f(\eta)\frac{d\eta}{d\lambda}\right) = \frac{\eta^4 T^5}{c}\frac{d\eta}{d\lambda}\left(5f(\eta) + \eta\frac{df(\eta)}{d\eta}\right) \tag{23}$$

This derivative is zero for:

$$5f(\eta_m) + \eta_m \left.\frac{df(\eta)}{d\eta}\right|_{\eta_m} = 0 \tag{24}$$

Just as before, due to the fact that $f(\eta)$ is unknown, this means that the above relation cannot be solved analytically. There is, however, a numerically-available value $\eta_m$ that satisfies the above equation, and which has its corresponding $\lambda_m$. We have $\eta_m = \frac{c}{\lambda_m T}$, from which we get:

$$\lambda_m T = \frac{c}{\eta_m} = b \tag{25}$$

# 5 The Rayleigh-Jeans model

The Rayleigh-Jeans model was developed as part of an effort to attribute physical significance to Wien's function and to the fitting functions. The model assumes as a black body a three-dimensional closed cavity in thermodynamic equilibrium at temperature $T$, in which we place some electromagnetic energy. This energy will generate stationary electromagnetic waves, which will form across a discrete number of modes $N$. The system thus formed is isolated from the exterior, and therefore, any energy exchange is only allowed between the stationary waves and the cavity walls. The statistical distribution describing the system is the canonical distribution:

$$\rho(E) = \frac{1}{Z}\exp\left(-\frac{E}{kT}\right) \tag{26}$$

where $Z = \int\limits_0^\infty \exp\left(-\frac{E}{kT}\right)$ is the partition function. For such systems, each mode receives the average energy:

$$\langle E\rangle = \int\limits_0^\infty E\rho(E)\,dE = kT \tag{27}$$

where $k = 1.38 \cdot 10^{23}$ J/K is the Boltzmann constant.

For a cubic closed cavity, the allowed oscillation modes of the standing waves are given by the wavenumbers:

$$k_{m,x} = \pm\frac{m\pi}{L};\quad k_{n,y} = \pm\frac{n\pi}{L};\quad k_{q,z} = \pm\frac{q\pi}{L} \tag{28}$$

where $m, n$ and $q$ are natural numbers, and the "$\pm$" sign refers to both progressive and regressive waves. For $m = n = q = a$, the effective k-space representation is:

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\pi^2}{L^2}a^2 \tag{29}$$

which describes a sphere. The symmetry planes are given by $k_x$, $k_y$ and $k_z$. The idea here is to represent the allowed number of modes as a function of wavenumber $k$. Between two allowed values of $k$ we can find a smaller, unit-cell cube with size $\pi/L$. The number of modes in this configuration is:

$$N = 2 \times \frac{\text{Volume of a spherical quadrant}}{\text{Volume of a unit cell cube}} \tag{30}$$

where the $2\times$ factor is taken because the oscillations can occur on two polarization modes, either TE or TM. The total number of allowed modes is:

$$N(k) = 2\frac{1}{8}\frac{4\pi k^3}{3}\left(\frac{L}{\pi}\right)^3 = \frac{k^3L^3}{3\pi^2} \tag{31}$$

To determine the distribution of those modes as a function of $k$, we choose a sheet having radius $k$ and thickness $dk$, and we scan across the whole solid quadrant (see Figure 2). The number of allowed modes inside the sheet is:

$$dN = \frac{dN}{dk}dk = \frac{k^2L^3}{\pi^2}dk \tag{32}$$

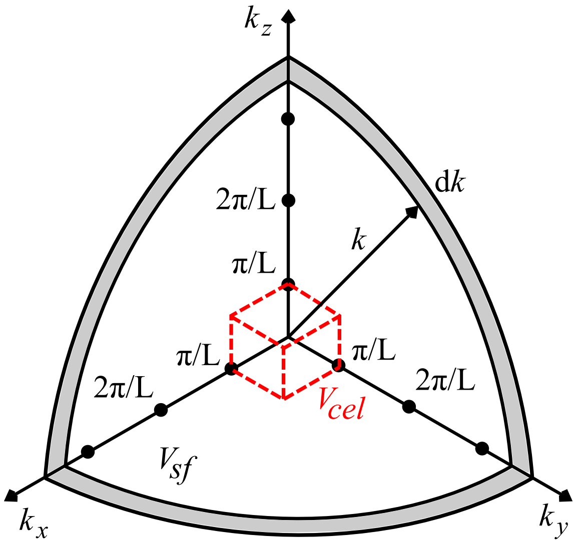

Figure 2: The equivalent volume of a unit cell and a spherical quadrant in $k$-space.

and the distribution is:

$$\frac{dN}{dk} = \frac{k^2 L^3}{\pi^2} \tag{33}$$

To switch to frequencies spectrum, we write $k = \frac{2\pi}{c}\nu$, and we obtain the total number of modes in the cavity:

$$N(\nu) = \frac{8\pi}{3}\frac{L^3}{c^3}\nu^3 \tag{34}$$

with the associated frequency distribution:

$$\frac{dN}{d\nu} = 8\pi\frac{L^3}{c^3}\nu^2 d\nu \tag{35}$$

According to the classical theory, the energy deposited inside the allowed oscillation modes between $\nu$ și $\nu + d\nu$ is:

$$d\mathcal{W}(\nu) = \langle\mathcal{W}\rangle dN = \langle\mathcal{W}\rangle\frac{dN}{d\nu}d\nu = 8\pi\frac{n^3 L^3}{c^3}k_B T\nu^2 d\nu \tag{36}$$

or as a function of wavelength:

$$\frac{dN}{d\lambda} = \frac{dN}{dk}\frac{dk}{d\lambda} \tag{37}$$

and $k = 2n\pi/\lambda$. We further obtain:

$$dN(\lambda) = \frac{4\pi^2 n^2 L^3}{\pi^2\lambda^2}\frac{2n\pi}{\lambda^2}d\lambda = \frac{8\pi n^3 L^3}{\lambda^4}d\lambda \tag{38}$$

with the associated deposited energy:

$$d\mathcal{W}(\lambda) = \frac{8\pi n^3 L^3}{\lambda^4}k_B T d\lambda \tag{39}$$

Given the fact that this energy can be expressed as a function of the spectral emission power and the volumetric energy density integrated in unit time:

$$dW(\lambda) = (f_{\lambda,T} d\lambda) dV \Delta t \tag{40}$$

and setting $\Delta t = 1$ and knowing that $V = L^3$ and the energy distribution is uniform, we get the expression of the spectral emission density:

$$f_{\lambda,T} = 8\pi \frac{k_B T}{\lambda^4} \tag{41}$$

and in frequency:

$$f_{\nu,T} = f_{\lambda,T} \frac{d\lambda}{d\nu} = 8\pi \frac{\nu^2}{c^3} k_B T \tag{42}$$

This distribution is a candidate of the Wien function, due to the fact that it fits the description. However, at a given temperature $T$, when integrating over all spectrum, we get:

$$P_T = \int_0^\infty f_{\nu,T} d\nu = \infty \tag{43}$$

This is impossible, due to the fact that the radiated power has to be finite! The relation was dubbed "the ultraviolet catastrophe", and leaves the model at the status of only an approximation of the experimental data.