№4

$p(x) = \frac{\theta}{2}\{(-1,1)/\{0\}\} + \frac{1-\theta}{2}\{0\} + \frac{1-\theta}{2}\{2\}$

$\theta \in (0,1)$

<u>ОММ</u>

a) $\alpha_1 = M[\xi] = \int_{-1}^{1} x\frac{\theta}{2}dx + 2\frac{1-\theta}{2} =$

$= \frac{x^2}{2}\Big|_{-1}^{1} \cdot \frac{\theta}{2} + 1 - \theta = 1-\theta$

$\alpha_2 = \int_{-1}^{1} x^2 \cdot \frac{\theta}{2}dx + 4 \cdot \frac{1-\theta}{2} =$

$= \frac{x^3}{3}\Big|_{-1}^{1} \cdot \frac{\theta}{2} + 2 - 2\theta = 2 - \frac{5}{3}\theta$

$D\xi = 2 - \frac{5}{3}\theta - 1 + 2\theta - \theta^2 = 1 + \frac{\theta}{3} - \theta^2$

$\alpha_1 = \tilde{\alpha}_1 = \bar{X} \qquad \tilde{\theta} = 1 - \bar{X}$

~~a)~~ b) несмещ.:

$M[\tilde{\theta}] = M[1-\bar{X}] = 1 - M\xi = \theta$

<u>несмещенная</u>

состоятельность:

$D[\tilde{\theta}] = D\bar{X} = \frac{1}{n}D\xi = \frac{1}{n}(1+\frac{\theta}{3}-\theta^2) \xrightarrow[n\to\infty]{} 0$

<u>сост.</u> по дост. усл.

c) регулярность модели

$\int_{-1}^{1} \frac{\partial}{\partial\theta}\left(\frac{\theta}{2}\right)dx + \frac{\partial}{\partial\theta}(1-\theta) =$

$= 1 - 1 = 0$

$$I(\theta)=\int_{-1}^{1}\left(\frac{\partial \ln(\frac{\theta}{2})}{\partial\theta}\right)^2\cdot\frac{\theta}{2}\,dx+2\cdot\left(\frac{\partial \ln\frac{1-\theta}{2}}{\partial\theta}\right)^2\left(\frac{1-\theta}{2}\right)=$$

$$=\int_{-1}^{1}\left(\frac{2}{\theta}\cdot\frac{1}{2}\right)^2\cdot\frac{\theta}{2}\,dx+2\left(\frac{2}{1-\theta}\cdot(-1)\right)^2\left(\frac{1-\theta}{2}\right)=$$

$$=\int_{-1}^{1}\frac{1}{2\theta}\,dx+\frac{8}{(1-\theta)^2}\cdot\left(\frac{1-\theta}{2}\right)=$$

$$=\frac{1}{\theta}+\frac{4}{1-\theta}>0\quad\forall\theta\in\Theta$$

$$=\frac{1-\theta+4\theta}{\theta-\theta^2}=\frac{1+3\theta}{\theta-\theta^2}$$

регулярность оценки:

~~дисп. огран. на ∀ компакте~~

дисп. огран. на ∀ компакте из Θ => по дост. усл. неравенство Крамера-Рао:

$$D[\tilde{\theta}]\geqslant\frac{1}{nI(\theta)}=\frac{1}{n}\cdot\frac{\theta-\theta^2}{1+3\theta}$$

$$\frac{3+\theta-3\theta^2}{3}\geqslant\frac{\theta-\theta^2}{1+3\theta}$$

не вып. дост. усл. эф.

<u>ОМП:</u>

m - кол-во нулей или двоек

$$L(\theta)=\left(\frac{\theta}{2}\right)^{n-m}\left(\frac{1-\theta}{2}\right)^m$$

$$\ln L=(n-m)\ln\frac{\theta}{2}+m\ln\frac{1-\theta}{2}$$

$(\ln L)' = (n-m)\frac{2}{\theta} + m\frac{2}{1-\theta}\cdot(-1) = 0$

$\cancel{\frac{n-m-n\theta+m\theta-m\theta}{\theta(1-\theta)}} = 0$ ~~fгх~~

$n\theta = n-m \quad \tilde{\theta} = 1-\nu$

$M\tilde{\theta} = M[1-\nu] = 1 - M\nu = 1-1+\theta = \theta$

несм.

$D\tilde{\theta} = D[1-\nu] = D\nu = \frac{(1-\theta)\cdot\theta}{n} \xrightarrow[n\to\infty]{} 0$

сост

регулярная оценка не дост. усл.

$\frac{(1-\theta)\cdot\theta}{n} \geq \frac{\theta-\theta^2}{1+3\theta}\cdot\frac{1}{n}$

не вып. дост. усл. эф.

№5

$p(x) = \frac{1}{\theta}\{[\theta, 2\theta]\}$

ОММ: $\alpha_1 = M[\xi] = 1,5\theta$

$\alpha_2 = M[\xi^2] = \int_0^\theta \frac{1}{\theta}x^2dx = \frac{7\theta^2}{3}$

$D[\xi] = \alpha_2 - \alpha_1^2 =$

$= \frac{7\theta^2}{3} - \frac{9\theta^2}{4} = \frac{\theta^2}{12}$

$\alpha_1 = \tilde{\alpha}_1 = \bar{X} \Rightarrow \tilde{\theta}_1 = \frac{2}{3}\bar{X}$

$M[\tilde{\theta}_1] = M[\frac{2}{3}\bar{X}] = \frac{2}{3} M\xi = \theta$

<u>несмещ</u>

$D[\tilde{\theta}_1] = \frac{4}{9}\frac{1}{n} D\xi = \frac{4\theta^2}{9 \cdot 12} \cdot \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

<u>сост</u> по дост. усл.

ОМП:

$L(\theta) = (\frac{1}{\theta})^n \{\forall i, \theta < x_i < 2\theta\} =$

$= \frac{1}{\theta^n}\{\theta < \min x_i \leq \max x_i < 2\theta\}$

$\frac{\max x_i}{2} \leq \min x_i \quad x_i \in (\theta, 2\theta)$

$\tilde{\theta}_2 = \frac{x_{max}}{2}$

$M[\tilde{\theta}_2] = M[\frac{1}{2} x_{max}] = \frac{1}{2}\int_\theta^{2\theta} x \cdot n(\frac{x}{\theta} - 1)^{n-1}\frac{dx}{\theta}$

$= \frac{1}{2}\int_0^1 (t+1)\theta n(t)^{n-1} dt = \frac{\theta}{2}(\frac{n}{n+1} + 1) =$

$= \frac{\theta}{2}\left(\frac{2n+1}{n+1}\right)$ <u>смещ</u>

$\tilde{\theta}_2' = \frac{2n+2}{2n+1} \cdot \frac{x_{max}}{2} = \frac{n+1}{2n+1} x_{max}$

$$D[\tilde{\theta}_2'] = \left(\frac{n+1}{2n+1}\right)^2$$

$$M[X_{max}] = \theta\,\frac{2n+1}{n+1}$$

$$M[X_{max}^2] = \int_\theta^{2\theta} x^2 n\left(\frac{x}{\theta}-1\right)^{n-1}\frac{1}{\theta}\,dx =$$

$$= \int_0^1 (t+1)^2\theta^2 n t^{n-1}\,dt =$$

$$= n\theta^2\left(\frac{t^{n+2}}{n+2} + 2\frac{t^{n+1}}{n+1} + \frac{t^n}{n}\right)\Bigg|_0^1 =$$

$$= n\theta^2\,\frac{n^2+n+2n^2+4n+n^2+3n+2}{(n+2)(n+1)n} =$$

$$= \theta^2\,\frac{4n^2+8n+2}{(n+2)(n+1)}$$

$$\cancel{D[X_{max}] = \theta^2\left[\frac{4n^3+4n^2+6n+4n^2+4n+6-4n^3-4n^2-n-8n^2-2}{(n+2)(n+1)^2}\right] =}$$

$$\cancel{= \theta^2}$$

$$D[X_{max}] = \theta^2\left[\frac{(4n^2+8n+2)(n+1)-(4n^2+4n+1)(n+2)}{(n+2)(n+1)^2}\right] =$$

$$= \theta^2\left[\frac{4n^3+8n^2+2n+4n^2+8n+2-4n^3-4n^2-n-8n^2-8n-2}{(n+2)(n+1)^2}\right] =$$

$$= \theta^2\,\frac{n}{(n+2)(n+1)^2}$$

$$D[\tilde{\theta}_2'] = \left(\frac{n+1}{2n+1}\right)^2\cdot\theta^2\,\frac{n}{(n+2)(n+1)^2}\xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ сост. по дост. усл.

$$D[\tilde{\theta}_1] = \frac{\theta^2}{27n} \geqslant D[\tilde{\theta}_2] = \frac{n\theta^2}{(2n+1)^2(n+2)} \quad \forall n \geq 3$$

Точный довер. инт.

$$f = \frac{X_{max}}{\theta}$$

$$f \sim \varphi(y) = P(X_{max} < \theta y) =$$

$$= (F(\theta y))^n = \left(\frac{y}{\theta} - 1\right)^n = (y-1)^n \quad y \in [1, 2]$$

$$g(y) = \varphi'(y) = n(y-1)^{n-1}$$

$$t_1 = g_{\frac{1-\beta}{2}} \quad t_2 = g_{\frac{1+\beta}{2}} \quad \beta = 0,95$$

$$(t_1 - 1)^n = \frac{1-\beta}{2} \quad t_1 = 1 + \sqrt[n]{\frac{1-\beta}{2}}$$

$$P\left(t_1 < \frac{X_{max}}{\theta} < t_2\right) = \beta$$

$$\frac{X_{max}}{t_2} < \theta < \frac{X_{max}}{t_1} \quad \frac{X_{max}}{1+\sqrt[n]{0,975}} < \theta < \frac{X_{max}}{1+\sqrt[n]{0,025}}$$

Асимпт. дов. инт.

$$\tilde{\theta} = g(\alpha_1) = \frac{2}{3}\tilde{\alpha}_1 \quad \beta = 0,95 \quad g(\alpha_1) = \theta$$

$$\frac{\sqrt{n}\,(\tilde{\theta} - \theta) \cdot \frac{2}{3}}{\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}\,\sqrt{\frac{4}{9}}} \rightarrow N(0, 1)$$

$$\nabla g(\tilde{\alpha}_1) = \frac{2}{3} \quad K_{11} = \alpha_2 - \alpha_1^2$$

$$t_1 = u_{0,025} = -1,96 \quad t_2 = u_{0,975} = 1,96$$

$$t_1 < \frac{\sqrt{n}(\tilde{\theta}-\theta)}{\sqrt{\frac{4}{9}(\overline{\alpha}_2-\overline{\alpha}_1^2)}} < t_2$$

$$\frac{-1{,}96\sqrt{\frac{4}{9}(\overline{\alpha}_2-\overline{\alpha}_1^2)}}{\sqrt{n}} + \frac{2}{3}\alpha_1 < \theta < \frac{1{,}96\sqrt{\frac{4}{9}(\overline{\alpha}_2-\overline{\alpha}_1^2)}}{\sqrt{n}} + \frac{2}{3}\alpha_1$$

№6

$$p(x) = \frac{\theta-1}{x^\theta}\{x \geq 1\} \quad \theta > 1$$

$$\ln L(\theta) = \ln\prod_{i=1}^{n}\left(\frac{\theta-1}{x_i^\theta}\right) = n\ln(\theta-1) - \theta\sum_{i=1}^{n}\ln x_i$$

$$\ln L\Big|'_\theta = \frac{n}{\theta-1} - \sum_{i=1}^{n}\ln x_i = 0$$

$$\tilde{\theta} = \frac{n}{\sum_{i=1}^{n}\ln x_i} + 1$$

Довер. инт. мед:

$$\int_{-\infty}^{x_{med}} p(x)\,dx = \frac{1}{2}$$

$$\int_1^{x_{med}} \frac{\theta-1}{x^\theta}\,dx = -\frac{1}{x^{\theta-1}}\Big|_1^{x_{med}} = -x_{med}^{1-\theta} + 1 = \frac{1}{2}$$

$$x_{med}^{1-\theta} = \frac{1}{2} \qquad \theta_{med} \quad x_{med} = 2^{\frac{1}{\theta-1}} = f(\theta)$$

Асимп ОМП:

$$\nabla f = 2^{\frac{1}{\theta-1}}\ln(2)\cdot\left(-\frac{1}{(\theta-1)^2}\right)$$

$$\sqrt{n}\,\frac{f(\tilde{\theta})-f(\theta)}{\sqrt{\nabla^T f I^{-1}(\theta)\nabla f}} \rightrightarrows N(0,1)$$

$$I(\theta)=\int_1^\infty \left(\frac{\partial \ln p}{\partial \theta}\right)^2 \cdot \frac{\theta-1}{x^\theta}\,dx=\int_1^\infty \left(\frac{1}{\theta-1}-\ln x\right)^2\frac{\theta-1}{x^\theta}dx=$$

$$=\int_1^\infty \frac{1}{(\theta-1)x^\theta}\,dx-2\int_1^\infty \frac{\ln x}{x^\theta}dx+\int_1^\infty \frac{(\theta-1)\ln^2 x}{x^\theta}dx$$

$$-2\int_1^\infty \frac{\ln x}{x^\theta}dx=\frac{2}{\theta+1}\int_1^\infty \ln x\, dx^{-\theta+1}=$$

$$=\frac{2}{\theta-1}\ln x\, x^{-\theta+1}\Big|_1^\infty-\frac{2}{\theta-1}\int_1^\infty x^{-\theta}dx=\frac{-2}{(\theta-1)^2}$$

$$\int_1^\infty \frac{(\theta-1)\ln^2 x}{x^\theta}dx=-\ln^2 x\, x^{-\theta+1}\Big|_1^\infty+2\int_1^\infty \frac{\ln x}{x^\theta}dx$$

$$I(\theta)=\frac{1}{(\theta-1)^2}$$

$$\sqrt{\nabla^T f I^{-1}\nabla f}=\sqrt{\left(2^{\frac{1}{\theta-1}}\ln 2\,\frac{1}{(\theta-1)^2}\right)^2(\theta-1)^2}=$$

$$=\frac{2^{\frac{1}{\theta-1}}\ln 2}{\theta-1}$$

$$-1{,}96<\frac{\sqrt{n}(\theta-1)(\tilde{X}_{med}-X_{med})}{2^{\frac{1}{\theta-1}}\ln 2}<1{,}96$$

$$\left(\frac{1{,}96\ln 2}{\sqrt{n}(\theta-1)}+1\right)\tilde{X}_{med}<X_{med}<\left(-\frac{1{,}96\ln 2}{\sqrt{n}(\theta-1)}+1\right)\tilde{X}_{med}$$

Асимп. довер. инт. θ (ОМП):

$$\sqrt{n}\,\frac{\hat{\theta}-\theta}{\hat{\theta}-1} \rightsquigarrow N(0,1)$$

$$\tilde{\theta}-\frac{1{,}96(\hat{\theta}-1)}{\sqrt{n}} < \theta < \hat{\theta}+\frac{1{,}96(\hat{\theta}-1)}{\sqrt{n}}$$