№ 1

$F \sim R(0, \theta)$ $\quad \tilde{\theta}_3 = \max x_i$

Состоятельность:

$\forall \theta \in \Theta \leftrightarrow \tilde{\theta} \xrightarrow{P} \theta$

$P(|\max x_i - \theta| \geq \varepsilon) = P(\max x_i \geq \theta + \varepsilon) +$

$+ P(\max x_i \leq \theta - \varepsilon) =$

$= 1 - \underbrace{P(\max x_i < \theta + \varepsilon)}_{(F(\theta+\varepsilon))^n = 1} + \underbrace{P(\max x_i < \theta - \varepsilon)}_{(F(\theta-\varepsilon))^n = (1-\frac{\varepsilon}{\theta})^n \xrightarrow[n\to\infty]{} 0} + \underbrace{P(\max x_i = \theta - \varepsilon)}_{0}$

$\Rightarrow \tilde{\theta}_3 \xrightarrow{P} \theta$ состоятельна

№ 3

$$p = \begin{cases} e^{-\frac{x}{\theta}} \cdot \frac{1}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$n = 3$ $\quad \tilde{\theta}_1 = \overline{x}$, $\tilde{\theta}_2 = x_{(2)}$

$M[\xi] = \int_{-\infty}^{\infty} e^{-\frac{x}{\theta}} \frac{x}{\theta} dx = \theta \int_0^{\infty} e^{-t} t \, dt =$

$= [e^{-t} t |_0^{\infty} + \int_0^{\infty} e^{-t} dt] \theta = \theta$

$M[\xi^2] = \int_0^{\infty} e^{-\frac{x}{\theta}} \frac{x^2}{\theta} dx = \theta^2 \int_0^{\infty} e^{-t} t^2 dt =$

$= \theta^2 [-e^{-t} t^2 |_0^{\infty} + \int_0^{\infty} 2e^{-t} t \, dt] = 2\theta^2$

a) $M[\overline{x}] = \frac{1}{n} \cdot n M[\xi] = \theta$

$\tilde{\theta}_1$ несмещенная

$F(t) = \int_0^{x} p(x)dx =$

$F(x) = \int_0^x p(t)dt = 1 - e^{-\frac{x}{\theta}}$

$\varkappa = n\, p(x)\, C_{n-1}^{k-1} (1-F)^{n-k} F^{k-1} =$

$= 3\, p(x)\, C_2^1 \left(e^{-\frac{x}{\theta}}\right)' \left(1 - e^{-\frac{x}{\theta}}\right) =$

$= \frac{6}{\theta}\left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right)$

$M[\tilde{\theta}_2] = 6\int_0^\infty \frac{x}{\theta}\left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right)dx$

$\int_0^\infty x\frac{e^{-\frac{2x}{\theta}}}{\theta}dx = -\frac{1}{2}\int_0^\infty x\, de^{-\frac{2x}{\theta}} =$

$= -\frac{1}{2}\left[xe^{-\frac{2x}{\theta}} - \int_0^\infty e^{-\frac{2x}{\theta}}dx\right] = \frac{\theta}{4}$

$\int_0^\infty x\frac{e^{-\frac{3x}{\theta}}}{\theta}dx = -\frac{1}{3}\left[xe^{-\frac{3x}{\theta}}\Big|_0^\infty - \int_0^\infty e^{-\frac{3x}{\theta}}dx\right] = \frac{\theta}{9}$

$M[\tilde{\theta}_2] = 6\left[\frac{\theta}{4} - \frac{\theta}{9}\right] = \frac{5}{6}\theta$

$\tilde{\theta}_2$ – смещ. $\tilde{\theta}_2' = \frac{6}{5}x_{(2)}$

в) $D[\tilde{\theta}_1] = D[\bar{X}] = \frac{1}{n^2}\cdot n\, D[\xi] = \frac{\theta^2}{n} = \frac{\theta^2}{3}$

$M[\tilde{\theta}_2'^2] = \frac{36}{25}\cdot 6 \cdot \int_0^\infty \frac{x^2}{\theta}\left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right)dx$

$\int_0^\infty \frac{x^2}{\theta}e^{-\frac{2x}{\theta}}dx = -\frac{1}{2}\int_0^\infty x^2 de^{-\frac{2x}{\theta}} =$

$= -\frac{1}{2}\left[x^2 e^{-\frac{2x}{\theta}}\Big|_0^\infty - \int_0^\infty 2xe^{-\frac{2x}{\theta}}dx\right] = \frac{\theta^2}{4}$

$\int_0^\infty \frac{x^2}{\theta} e^{-\frac{3x}{\theta}} dx = -\frac{1}{3}\int_0^\infty x^2 d e^{-\frac{3x}{\theta}} =$

$= -\frac{1}{3}\left[x^2 e^{-\frac{3x}{\theta}}\Big|_0^\infty - 2\int_0^\infty x e^{-\frac{3x}{\theta}} dx\right] = \frac{2\theta^2}{27}$

$M[\tilde{\theta}_2'^2] = \frac{36}{25}\cdot 6\left(\frac{\theta^2}{4} - \frac{2\theta^2}{27}\right) = \frac{38}{25}\theta^2$

$D[\theta_2'] = \frac{13}{25}\theta^2$

$D[\tilde{\theta}_1] < D[\tilde{\theta}_2'] \Rightarrow \tilde{\theta}_1$ более эффект.

c) регулярность модели:

1) $p(x,\theta)$ - непр. дифф. при $\theta > 0$ ✓

2) $\frac{\partial}{\partial\theta}\int_0^\infty p(x,\theta)dx = 0$

$\int_0^\infty \frac{\partial}{\partial\theta} p(x,\theta)dx = \int_0^\infty \frac{\frac{x}{\theta^2}\cdot e^{-\frac{x}{\theta}}\cdot\theta - e^{-\frac{x}{\theta}}}{\theta^2}dx =$

$= \int_0^\infty \frac{x}{\theta^3} e^{-\frac{x}{\theta}} dx - \int_0^\infty \frac{1}{\theta^2} e^{-\frac{x}{\theta}} dx = 0$

$\Rightarrow \frac{\partial}{\partial\theta}\int_0^\infty p\,dx = \int_0^\infty \frac{\partial}{\partial\theta} p\,dx$ ✓

3) $I(\theta) = M\left[\left(\frac{\partial \ln p(x,\theta)}{\partial\theta}\right)^2\right] =$

$= M\left[\left(\frac{\partial}{\partial\theta}\ln\frac{e^{-\frac{x}{\theta}}}{\theta}\right)^2\right] = M\left[\left(+\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2\right] =$

$= M\left[\frac{x^2}{\theta^4} - 2\frac{x}{\theta^3} + \frac{1}{\theta^2}\right] =$

$= \int_0^\infty \left(\frac{x^2}{\theta^4} - 2\frac{x}{\theta^3} + \frac{1}{\theta^2}\right)\frac{e^{-\frac{x}{\theta}}}{\theta}dx = \frac{1}{\theta^2}$

$I(\theta)$ непр. и >0 при $\theta > 0$ ✓

модель регулярна

регулярность оценок:
оценка несмещ и D огран
на любом компакте из Θ
выполнено и для $\tilde{\theta}_1$ и для $\tilde{\theta}_2'$
оценки регулярны

$$D[\tilde{g}] \geq \frac{g'^2(\theta)}{nI(\theta)}$$

$\tilde{\theta}_1 : \frac{\theta^2}{3} \geq \frac{\theta^2}{3}$ $\quad$ $\tilde{\theta}_1$ - эффективная

$\tilde{\theta}_2' : \frac{13}{25}\theta^2 \geq \frac{\theta^2}{3}$ $\quad$ $\tilde{\theta}_2'$ - дост. усл. эф. не вып.

№1

$$\tilde{\theta}_3' = \frac{n+1}{n} X_{max}$$

$$P\left(\left|\frac{n+1}{n} X_{max} - \theta\right| \geq \varepsilon\right) = P\left(X_{max} \geq \frac{(\theta+\varepsilon)n}{n+1}\right) + P\left(X_{max} < \frac{(\theta-\varepsilon)n}{n+1}\right) + \underbrace{P\left(X_{max} = \frac{(\theta-\varepsilon)n}{n+1}\right)}_{=0} =$$

$$= 1 - P\left(X_{max} < \frac{(\theta+\varepsilon)n}{n+1}\right) + P\left(X_{max} < \frac{(\theta-\varepsilon)n}{n+1}\right) =$$

$$\cancel{\xrightarrow[n\to\infty]{} 1 - P(X_{max} < (\theta+\varepsilon)) + P(X_{max} < (\theta-\varepsilon)) =}$$

$$\cancel{= 1 - (F(\theta+\varepsilon))'}$$

$$= 1 - \left(F\left(\frac{(\theta+\varepsilon)n}{n+1}\right)\right)^n + \overbrace{\left(F\left(\frac{n}{n+1}\left(1-\frac{\theta}{\varepsilon}\right)\right)\right)^n}^{<1} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow \tilde{\theta}_3'$ - состоятельна