Things to include in the report

1. Create SGD regression
   - Test for different learning rates, mini batches, epochs, adaptive learning rate?
   - Compare with OLS and Ridge on Franke function
   - For ridge study results as function of the hyper-parameters λ and learning rate (heat map)
2. Create NN
   - Discuss intializing weights and biases, cost functions, activation functions and all that stuff. What to do on the output layer regarding activation function.
   - Compare NN regression on Franke function vs. OLS and Ridge (λ and η heat map)
   - Compare to SKLearn/Tensorflow
3. Try out different activation functions (Sigmoid, Relu and Leaky relu
4. Classification (costfunction = logistic)
   - Test logic gates classification
   - Breast cancer data: Important to scale here. Discuss also methods/whether to reduce dimensions. Measure performance with accuracy. Again: optimize for hyper-parameters and various activation functions perhaps.
5. Write logistic regression code using SGD
   - Compare NN to logistic regression on MNIST data
6. Summarize everything

Classification and Regression, from linear and logistic regression to neural networks

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I. INTRODUCTION

The use of Neural Networks have exploded in academia the last 10 years, and continue to impress with its capabilities in complex problem solving. DeepMind made its AlphaGo, which beat the reigning champion 4/5 times in the game Go, facial recognition in China is so good now that it works as ID on the subway, and cars can now drive them selves on the road with higher than 99% success rate [4] .

Neural Networks and regression have somewhat overlapping area of use, especially in with regards to classification and regression. In fact, logistic regression and a Neural Network with no hidden layers are practically the same thing. Neural Networks can however scale much better in dimensionality, such that it can solve complex problems such as quantum many body problems, image recognition and complex differential equations.

II. THEORY

A. Gradient descent

The essence of Machine learning is in a sense just a cost minimization problem with respect to some parameters. For Neural Networks these parameters is the weights and the biases, but for now we just consider a general parameter $\boldsymbol{\theta} = \{\theta_1, \theta_2, \dots \theta_n\}$ for n dimensions. The goal is to choose $\boldsymbol{\theta}$ such that we minimize a given cost function $C(\boldsymbol{\theta})$ from a set of data points in the design matrix $\mathbf{X}$ and corresponding target values $\mathbf{t}$. One common approach to such a problem is the Ordinary Least Squares method (OLS) for which we find the optimal $\boldsymbol{\theta}_{opt}$ that minimize the cost function as

$$\boldsymbol{\theta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{t}$$

This is the main workhorse behind many regression problems, but as the number of data points increases the matrix inversion becomes computationally expensive. Furthermore, the matrix product $\mathbf{X}^T\mathbf{X}$ might be singular and thus the above approach needs additional computations to work out. The concept of gradient descent provides a computationally efficient alternative to this problem.

1. Regular Gradient descent

For a given cost function $C(\boldsymbol{\theta})$ we can approach the minimum by calculating the gradient $\nabla_\theta C$ with respect to the unknown parameters $\boldsymbol{\theta}$. When evaluating the gradient in a specific point $\boldsymbol{\theta}_i$ in the parameter space, the negative of the gradient will correspond the direction for which a small change $d\boldsymbol{\theta}$ in the parameter space will result in the biggest decrease in the cost function. By choosing a step size η, better known as the learning rate, we can iterate forward in the parameter space as

$$\boldsymbol{\theta}_{i+1} = \boldsymbol{\theta}_i - \eta\nabla_\theta C(\boldsymbol{\theta}_i). \tag{1}$$

In order to get convergence towards a minimum we must choose the learning rate η to be sufficiently small such that it does not "step over" the minimum point. The usual approach to choosing η is simply to define it as a hyperparameter and run the gradient descent scheme for different η's and choose the one that yields the best result.

For some simpler cases, where we are able to calculate the so-called Hessian matrix, we can compute an approximation for the optimal learning rate. For a convex problem the gradient is only equal to zero at the global minimum for which the optimal parameter must satisfy

$$g(\boldsymbol{\theta}) = \nabla_\theta C(\boldsymbol{\theta}) = 0.$$

By using Newton-Raphson's method we can iterate towards the root for the gradient by following the tangent line (1. order approximation) as

$$\boldsymbol{\theta}_{i+1} = \boldsymbol{\theta}_i - \frac{g(\boldsymbol{\theta}_i)}{g'(\boldsymbol{\theta}_i)},$$

where

$$g'(\boldsymbol{\theta}_i) = \frac{\partial g(\boldsymbol{\theta}_i)}{\partial \boldsymbol{\theta}_i} = \nabla^2_\theta C(\boldsymbol{\theta}_i) = \mathbf{H}$$

The final iteration scheme reads

$$\boldsymbol{\theta}_{i+1} = \boldsymbol{\theta}_i - \mathbf{H}^{-1}\nabla_\theta C(\boldsymbol{\theta}_i)$$

The above scheme is a quite effective strategy for gradient descent as the matrix multiplication of the hessian inverse will change the gradient according to the second order approximation of the cost function. However

2. Stochastic gradient descent

A variation of the regular gradient descent is the Stochastic gradient Descent (SGD) for which we introduce the concept of mini batches. For each step we divide the data randomly into mini batches of size m. For each batch we compute the gradient and update the unknown parameter using equation 1. Thus we will update the $\boldsymbol{\theta_{i+1}}$ a multiple times for each step corresponding to the number of mini batches. The main benefit of SGD over regular SG is that

1. we might reduce computation time.
2. we might reduce the risk of getting stuck in local local. minima.

the first benefit comes into play when we have big enough data sets. Since the computation of the gradient involves matrix operations, the calculation of the full gradient can be more computational expensive than the calculating the gradient for many smaller batches. The second benefit is due to the fact that SGD introduces some random noise to the movement through parameter space. It is as Grant Sanderson from 3blue1brown put it, it may look as a drunk man stumbling towards the minimum[1]. Thus we for small batch sizes we increase the likelihood of escaping local minima that governs small domains of the parameter space.

3. *Gradient descent with momentum*

As a last alternative to optimize the gradient descent, we introduce momentum also known as inertia. The key idea is that we keep track of the previous gradient step, such that we keep a memory of the direction we are moving in parameter space. When increasing the momentum parameter γ the movement through parameter space becomes more steady and less effected by small fluctuations in the cost function. We can think of this as a heavy ball rolling down a rugged landscape. the heavier the ball the less prone to sudden changes in direction it will be. The main idea is that the momentum concept can reduce the change of getting stuck in local minimums when travelling through parameters space.

Formally we define this as:

$$\mathbf{v}_t = \gamma \mathbf{v}_{t-1} + \eta_t \nabla_\theta E(\boldsymbol{\theta}_t) \tag{2}$$

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \mathbf{v}_t \tag{3}$$

where $\mathbf{v}_t$ is the velocity term for timestep (epoch) t. Notice that whebn choosing $\gamma = 0$ we have zero momentum and equation 3 becomes equivilant with the regular gradient descent method described in equation 1.
Notice that there exist a whole variety of different momentum-based methods, for which some of well known might be the "RMS prop" or "ADAM optimizer".

B. Neural Networks

We now introduce the basic concepts of the Neural Network (NN). Specifically we are going to address a dense feedforward neural network, meaning the information is only moving forward trough the network and each node in a given layer is connected to each and every node following layer. The general structure of the network is sketched in figure 1. In the beginning we have an input layer with one node for each feature of the data. Following we have a series of so-called hidden layers which can have a custom chosen number of nodes for each layer. Finally we have the output layers with one node per target category. For a simple True and False type of problem a single output node which takes values between 0 and 1 is sufficient for the prediction. For classification between multiple classes we would require an output node for each class.

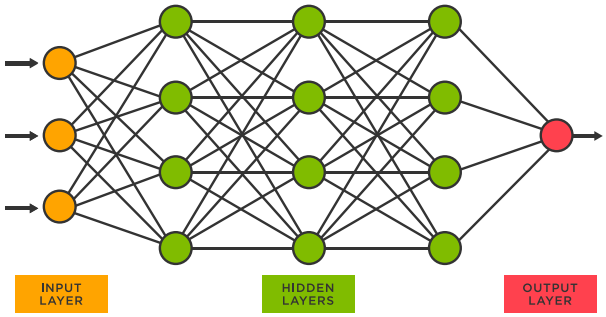


Figure 1: Conceptual illustration of Neural Network. https://www.tibco.com/reference-center/what-is-a-neural-network

The connection between the nodes is controlled by the introduction of the weights w and biases b. The weights and biases serves as the unknown parameter which we need to choose such that the network predictions minimizes a given cost function. For an already trained network we can collect the predictions by performing the so-called feed forward mechanism which is explained in details in the following section.
Howebver, before diving into the details behind the neural network and the key algorithms for performing training and prediction, we want to put down some notation. We aim to use some of the most common notation, but is easy to get lost in the following sections, and thus we provide a table (I) for which you can always go back to clarify the meaning of variables and indexes.

Table I: Notation

Matrices		
Notation	Description	Type
X	Design Matrix (input data).	$\mathbb{R}^{N\times\text{features}}$
t	Target values.	$\mathbb{R}^{N\times\text{categories}}$
y	Model output, the prediction from our network.	$\mathbb{R}^{N\times\text{features}}$
W^l	The weight matrix associated with layer l which handles the connections between layer $l-1$ and l .	$\mathbb{R}^{n_{l-1}\times n_l}$
w^l_{ik}	The weight connecting node i in layer $l-1$ to node k in layer l.	$\mathbb{R}$
B^l	The bias vector associated with layer l which handles the biases for all nodes in layer l.	$\mathbb{R}^{n_l\times 1}$
b^l_j	Bias acting on node j in layer l.	R
z	Node output before activation.	
a	Activated node output.	R
Functions		
C	Cost function	
σ^l	Activation function associated with layer l.	
More...		
n_l	The number of nodes in layer l.	
L	Number of layers in total with $L-2$ hidden layers.	
N	Total number of datapoints.	

1. *Feeding Forward*

The forward mechanism is the main mechanism behind the usage of the neural network. For a single data point the data flow becomes as outlined in the following.

1. The input nodes receive the input data for each feature.

2. Each input node then send the data value into each node of the following hidden layer with a scaling according to the associated weight of every single connection.

3. Each node in the hidden layer sums up the weighted contributions from the input layer an add a bias value associated to that given node. We denote this raw node output by z.

4. The unactivated value z is immediately send through an activation function σ associated with the layer to produce the activated value $a = \sigma(z)$.

5. Each node in the hidden layer then forward the activated value into the next layer using the same procedure. Notice that the number of nodes in the hidden layers no longer corresponds to any given features and one can only speculate on how the complex network creates its own sub-features and interprets the underlying correlations in the data.

6. Finally the stream of forwarded data enters the output layer where the activation values on each node corresponds to the network predictions for each category.

The feed forward mechanism is for a simple four layer network is shown in figure. 2.

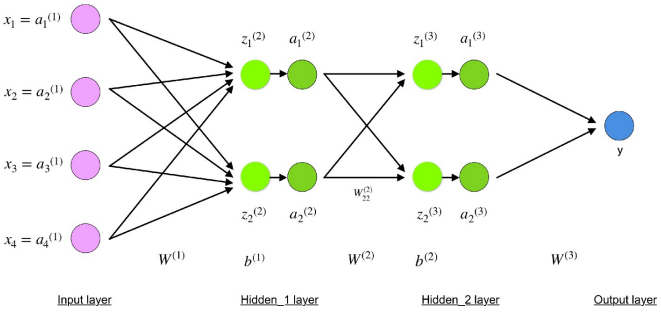


Figure 2: Illustration of simple 4-layer neural network. Source: towardsdatascience.com.

The feed forward step to obtain the activation value on a given layer $l \neq 1$ (not the input layer) is given as

$$z^l_j = \sum_{i=1}^{n_{l-1}} w^l_{ij} a^{l-1}_i + b^l_j, \quad a^l_j = \sigma^l(z^l_j) \tag{4}$$

where z^l_j is the input value to node j in layer l, w^l_{ij} is the weight connecting node i in layer $l-1$ to node j in layer l, a^{l-1}_i is the activation value from node i in layer $l-1$ and b^l_j is the bias associated with node j in layer l. We

can obtain a matrix-variant of equation 4 by defining

$$W^l = \begin{bmatrix} w^l_{11} & w^l_{21} & \dots & w^l_{n_{l-1}1} \\ w^l_{12} & w^l_{22} & \dots & w^l_{n_{l-1}2} \\ \vdots & \vdots & & \vdots \\ w^l_{1n_l} & w^l_{2n_l} & \dots & w^l_{n_l n_{l-1}} \end{bmatrix},$$

$$(W^l)^T = \begin{bmatrix} w^l_{11} & w^l_{12} & \dots & w^l_{1n_l} \\ w^l_{21} & w^l_{22} & \dots & w^l_{2n_l} \\ \vdots & \vdots & & \vdots \\ w^l_{n_{l-1}1} & w^l_{n_{l-1}2} & \dots & w^l_{n_{l-1}n_l} \end{bmatrix},$$

$$A^l = \begin{bmatrix} a^l_1 \\ a^l_2 \\ \vdots \\ a^l_{n_l} \end{bmatrix}, \qquad B^l = \begin{bmatrix} b^l_1 \\ b^l_2 \\ \vdots \\ b^l_{n_l} \end{bmatrix}, \qquad Z^l = \begin{bmatrix} z^l_1 \\ z^l_2 \\ \vdots \\ z^l_{n_l} \end{bmatrix},$$

such that we get

$$Z^l = (W^l)^T A^{l-1} + B^l, \qquad A^l = \sigma(Z^l). \tag{5}$$

The complete feed forward algorithm is simply done by using equation 5 for layer $l = 2, \dots, L$, for which produce the neural network prediction $\mathbf{y}$.

2. *Back Propagation*

The back propagation algorithm is the main workhorse behind the training of the neural network. This serves the purpose of tuning the weights and biases such that the prediction from our neural network (after running the feed forward algorithm) becomes better and better. That is we want to minimize a cost function C, which will evaluate how close the prediction $\mathbf{y}$ comes to the target values $\mathbf{t}$. A common choice for regression type problems is the mean squared error as

$$C(\mathbf{y}) = ||\mathbf{y} - \mathbf{t}||_2^2 = \frac{1}{N}\sum_{i=1}^{N}(y_i - t_i)^2 \tag{6}$$

but one can choose from a wide selection of cost function in general.

By calculating the gradient gradient $\nabla_{w,b}C$ with respect the weights and biases, we can use gradient descent (see section II A) to minimize the cost function. Thus need to evaluate $\partial C/\partial w^l_{ij}$ and $\partial C/\partial b^l_j$. We begin by looking at the last layer $l = L$. By using the chain rule we get

$$\frac{\partial C}{\partial w^L_{ij}} = \frac{\partial C}{\partial a^L_j}\frac{\partial a^L_j}{\partial z^L_j}\frac{\partial z^L_j}{\partial w^L_{ij}}.$$

We remember that

$$a^L_j = \sigma(z^L_j), \qquad z^L_j = \sum_{i=1}^{n_{L-1}} w^L_{ij} a^{L-1}_i + b^L_j,$$

such that we get

$$\frac{\partial C}{\partial w^L_{ij}} = \frac{\partial C}{\partial a^L_j}\sigma'(z^L_j)a^{L-1}_i,$$

Notice that the remaining derivatives can easily be calculated when deciding on a specific cost and activation function. For the bias we simply get

$$\frac{\partial C}{\partial b^L_j} = \frac{\partial C}{\partial a^L_j}\frac{\partial a^L_j}{\partial z^L_j}\frac{\partial z^L_j}{\partial b^L_j} = \frac{\partial C}{\partial a^L_j}\sigma'(z^L_j)\cdot 1$$

We use this to motivate the definition of the local gradient

$$\delta^l_j \equiv \frac{\partial C}{\partial z^l_j},$$

such that

$$\delta^L_j = \frac{\partial C}{\partial z^L_j} = \frac{\partial C}{\partial a^L_j}\frac{\partial a^L_j}{\partial z^L_j} = \frac{\partial C}{\partial a^L_j}\sigma'(z^L_j).$$

This yields the more compact expressions

$$\frac{\partial C}{\partial w^L_{ij}} = \delta^L_j a^{L-1}_i \qquad \frac{\partial C}{\partial b^L_j} = \delta^L_j$$

The local gradient δ^l_j is also commonly called the "error" since it says something about how big an influence the j^{th} node in layer l have on the change of the cost function. We let $\boldsymbol{\delta}^l$ denote the vector containing all the local gradients associated with layer l and we can write it out as a matrix equation for the last layer

$$\boldsymbol{\delta}^L = \nabla_a C \odot \frac{\partial \sigma}{\partial z^L}, \quad \nabla_a C = \left[\frac{\partial C}{\partial a^L_1}, \frac{\partial C}{\partial a^L_2}, \dots, \frac{\partial C}{\partial a^L_{n_L}}\right]^T$$

where $\odot$ is the Hadamard product (element-wise product). We can then define the gradient δ^l_j for the j^{th} node on a general layer l in terms of δ^{l+1}_k for the k^{th} node on the following layer $l+1$ by using the chain rule

$$\begin{aligned} \delta^l_j &\equiv \frac{\partial C}{\partial z^l_j} \\ &= \sum_k \frac{\partial C}{\partial z^{l+1}_k}\frac{\partial z^{l+1}_k}{\partial z^l_j} \\ &= \sum_k \frac{\partial z^{l+1}_k}{\partial z^l_j}\delta^{l+1}_k \end{aligned} \tag{7}$$

We remember

$$z_k^{l+1} = \sum_{j=1}^{n_l} w_{jk}^{l+1} a_j^l + b_k^{l+1} = \sum_{j=1}^{n_l} w_{jk}^{l+1} \sigma(z_j^l) + b_k^{l+1},$$

ann by taking the derivative we obtain

$$\frac{\partial z_k^{l+1}}{\partial z_j^l} = w_{jk}^{l+1} \sigma'(z_j^l). \tag{8}$$

We substitute back 8 into 7 to find

$$\delta_j^l = \sum_k \delta_k^{l+1} w_{ij}^{l+1} \sigma'(z_j^l)$$

The complete back propagation algorithm then becomes

- Compute

$$\delta_j^L = \frac{\partial C}{\partial a_j^L} \sigma'(z_j^L).$$

- For $l = L-1, L-2, \ldots, 1$ compute:

$$\delta_j^l = \sum_k \delta_k^{l+1} w_{jk}^{l+1} \sigma'(z_j^l)$$

- For all layers l update weights and biases:

$$w_{ij}^l \leftarrow w_{ij}^l - \eta \delta_j^l a_i^{l-1},$$
$$b_j^l \leftarrow b_j^l - \eta \delta_j^l,$$

where η is the learning rate.

In order to minimize the risk of over fitting it is common to include a L2 regularization by adding the L2 norm to the cost function as $\lambda ||w||_2^2$ and $\lambda ||b||_2^2$. This result in the modified expressions for the update

$$w_{ij}^l \leftarrow w_{ij}^l - \eta(\delta_j^l a_i^{l-1} + \lambda w_{ij}^l)$$
$$b_j^l \leftarrow b_j^l - \eta(\delta_j^l + \lambda b_j^l),$$

C. Logistic regression

Logistic function definition

$$p(t) = \frac{1}{1+e^{-t}} = \frac{e^t}{1+e^t}$$

We look at the binary case where $y_i = 0$ or $y_i = 1$. We define polynomial model on order n as

$$\hat{y}_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2, \ \ldots \ , \beta_n x_i^n$$

We define the probabilities of getting $y_i = 0 \vee 1$ given a input point x_i and beta values $\boldsymbol{\beta} = (\beta_0, \beta_1, \ \ldots \ , \beta_n)$

$$p(y_i = 1 | x_i, \boldsymbol{\beta}) = \frac{e^{\hat{y}_i}}{1+e^{\hat{y}_i}}$$
$$p(y_i = 0 | x_i, \boldsymbol{\beta}) = 1 - p(y_i = 1 | x_i, \boldsymbol{\beta})$$

We want to choose the parameters $\boldsymbol{\beta}$ such that we maximum the likelihood of saying the observed data $\mathcal{D} \in \{x_i, y_i\}$, that is the Maximum Likelihood Principle (MLE) principle. The likelihood of having the observed data $\mathcal{D}$ given $\boldsymbol{\beta}$

$$P(\mathcal{D}|\boldsymbol{\beta}) = \prod_{i=1}^{n} \Big(p(y_i = 1 | x_i \boldsymbol{\beta}) \Big)^{y_i} \Big(1 - p(y_i = 1 | x_i, \boldsymbol{\beta}) \Big)^{1-y_i}$$

In order to minimze computations we can define the log-likelihood as

$$\log\left(P(\mathcal{D}|\boldsymbol{\beta})\right) = \sum_{i=1}^{n} y_i \log\Big(p(y_i = 1 | x_i \boldsymbol{\beta}) \Big) + (1 - y_i) \log\Big(1 - p(y_i = 1 | x_i, \boldsymbol{\beta}) \Big),$$

and in order to get the cost function on the more conventional form as minimization problem we define the cost function as the negative log-likelihood:

$$C(\boldsymbol{\beta}) = -\log P(\mathcal{D}|\boldsymbol{\beta}). \tag{9}$$

We find the gradients

$$\frac{\partial C(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = -\sum_{i=1}^{n} \begin{pmatrix} y_i - p(\hat{y}_i) \\ x_i y_i - x_i p(\hat{y}_i) \\ \vdots \\ x_i^n y_i - x_i^n p(\hat{y}_i) \end{pmatrix} = -X^T \Big(\mathbf{y} - P(X\boldsymbol{\beta}) \Big)$$

For the L2 regularization we add the L2-norm to the costfunction as

$$C(\boldsymbol{\beta}) = -\log P(\mathcal{D}|\boldsymbol{\beta}) + \lambda ||\boldsymbol{\beta}||_2^2, \qquad \lambda > 0$$

where

$$\lambda ||\boldsymbol{\beta}||_2^2 = \lambda \sum_{i=1}^{n} \beta_i^2 \quad \Longrightarrow \quad \frac{\partial \lambda ||\boldsymbol{\beta}||_2^2}{\partial \boldsymbol{\beta}} = 2\lambda \boldsymbol{\beta}$$

This gives the full gradient expression with L2 regularization as

$$\frac{\partial C(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = -X^T \Big(\mathbf{y} - P(X\boldsymbol{\beta}) \Big) + 2\lambda \boldsymbol{\beta}$$

III. IMPLEMENTATION

The code can be found on this Github address.

A. SGD algorithm

We created a class called SGD, which does Stochastic gradient descent on a given dataset, and returns Θ. A SGD object requires the data to train on, hyperparameters η, λ, γ, and choice of gradient and loss function.

Result: Stochastic gradient descent algorithm
Initialize layers, weights, biases;
while *epoch* < *epochs* **do**
  Create batches;
  while *batch* < *batches* **do**
    Fetch X, y data in batch;
    Find gradient;
    Update velocity;
    Update Θ;
  end
end
Return Θ

B. Neural network

As done for SGD, we created a class to run all Neural network calculations. Each instance of the class must specify the structure of the network such as number of hidden layers, number of nodes per hidden layer and batch size as well as defining all functions such as the cost function and activation function.
Based on the chosen parameters for the network, a function is called to build all matrices needed for forward and backward propagation (see section II B). The function is design such that all hidden layers have the same number of nodes as well as the same activation function. Although this means a more rigid code, the additional flexibility is not needed in this report.
When all matrices and functions are set, the network can be trained by the call of an additional function. The function requires a choice of number of maximum epochs. When this is done it moves into a while loop which will run through all epochs as long as the stopping criteria is not met (more on this in III E 2). As done for SGD, the data is split into batches, and one epoch will conclude when all batches are used. For each batch the code will execute the feed forward calculations followed by the backward propagation, updating all weights and biases as described in section II B.

Result: Neural Net training
Initialize layers, weights, biases;
while *epoch* < *epochs* ***and*** *stopping criteria not met* **do**
  while *batch* < *batches* **do**
    Set first layer to batch;
    Feed Forward;
    Back propagate;
    Update Weights and Biases;
  end
end
The algorithm and structure of the code is described in the algorithm above.

Result: Neural Net prediction
Send in data;
Set data as first layer ;
Feed Forward;
Use activation on last layer;
Return prediction;

C. Activation functions

We defined 4 different cost functions to use in hour network, namely the Sigmoid, RELU, leaky RELU and SOFTMAX. Sigmoid is defined as

$$f(x) = \frac{1}{1+e^{-x}}$$

RELU is defined as

$$f(x) = x^+ = max(0, x)$$

Leaky RELU is defined as

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ 0.01x & \text{if otherwise} \end{cases}$$

SOFTMAX is defined as

$$f(x) = \frac{e^x}{\sum_j e^{x_j}}$$

D. Layers, weights and biases

TensorFlow allows for manually adding a layer, defining how many nodes, and the dimensions of the input data, as well as the activation function for that layer. We chose however to set up our hidden layers as exact copies, meaning that you define a number of layers and nodes, and then all the hidden layers have the same number of nodes, and in fact the same

activation. Lastly one can choose a different activation for the last layer, if needed. The weights and biases follow the same structure, to ensure that the dimensions of the tensor elements, i.e the weights for a given layer or bias for a given layer, are equal to said layer.

Activation functions like the Sigmoid function "squeeze" the data between 0 and 1, which is well behaved under matrix multiplication, but rectifier functions like RELU and Leaky RELU have a linear dependence on the data, and thus can lead to very large values in the hidden layers during matrix multiplication. To mitigate this we added a scaling of the weights by scaling only in the initialization of the network. This is sufficient enough so that RELU or Leaky RELU does not explode.

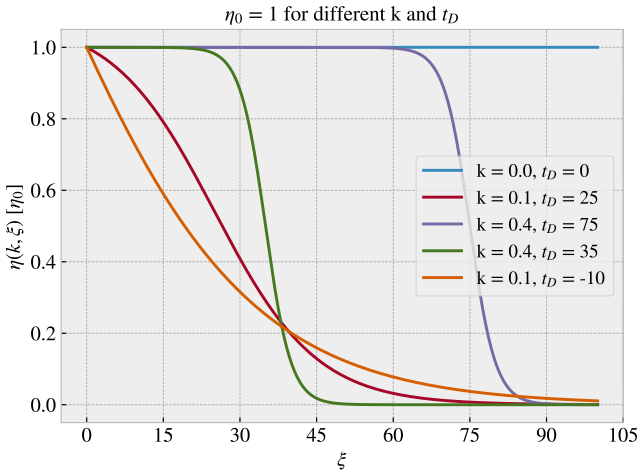


Figure 3: η decay as function of steepness k, drop time t_D, amplitude A and epoch ξ. Drop time increase by three for each iteration of k value.

By fine tuning the steepness and the drop time, these logistic functions can take shape of exponential decay or constant functions.

The learning rate can also be extended by means of being scaled by the batch size, i.e $\eta \rightarrow \eta/n$. This makes the learning rate much smaller, but it allows us to avoid derivations with very small numbers, and thus helps avoiding numerical errors better.

E. Practical implementation

1. *Learning rate*

Using a constant learning rate η can sometimes lead to either too long computation, or overstepping along a slope. A way to correct this is to adjust the learning rate η by how many epochs have been run. We created our own η decay function and it is given as :

$$\eta = \eta_0 A \cdot \sigma_s(k \cdot (t_D - \xi))$$

Here we have an initial η_0 value, an amplitude A, a drop time t_D dictating the half time of decay, epoch ξ and a steepness parameter k. σ_s is the Sigmoid activation function, as defined in the section above. The amplitude is a correction parameter to ensure that the learning rate at epoch 0 is the initial learning rate η_0. Thus we define a piece wise function A given as

$$A = \begin{cases} 2 & \text{if } \xi = 0 \\ 1 + e^{-kt_D} & \text{if } \xi > 0 \end{cases}$$

2. *Handling vanishing and exploding gradients*

Another way to speed up calculations is to ensure that the gradient does not disappear. Vanishing gradients can in extreme cases lead to no training, i.e no updating of weights and biases, and so we added two criteria to mitigate this. First we require the gradient to always be larger than a given tolerance, default to 10^{-8}. Second, we require the gradient to be finite. Sometimes the gradient explodes, leading to extreme changes in the weights and biases, and so we implemented a check to ensure that the gradient is always finite. That way we do not get NaN or inf in our calculations. It is worth mentioning that we just as well could have implemented the so called gradient clipping method, or other methods to control exploding gradients.

3. *Cost functions*

We used two cost functions for the purpose of the datasets we wanted to look at, Mean squared error and Cross entropy. Note from equation 6 that the actual definition of the mean squared error has a factor $1/N$ where m is the number of elements in the dataset y. As mentioned in the section about learning rate, this factor is

implemented in the learning rate, which amongst other things allows AutoGrad, a numerical derivation package for Python, to handle the derivation easier. In other words, the mean squared error cost function in our code is not scaled directly by the factor $1/N$.

4. Scaling

The need to scale the data and the type of scaling varies from data set to data set. For the data set created to predict the Franke's function, we used uniformly distributed values between 0 and 1, and so here there is no need to scale. However, the breast cancer data set has entries spanning from 0 to 4000, so standard scaling is necessary here. Standard scaling subtracts the mean from the data set and divides on the standard deviation.

F. Data sets

Now that we have built a functioning Neural Network, we want to run it on several different data sets and analyse the networks ability to predict values given a set of inputs.

1. Franke's function

In the rapport we will be doing both regression and classification analysis. For the regression case we will be using Franke's function. The Franke's function, f(x,y), is a analytical function often used in machine learning. The function is a weighted sum of four exponentials which we will study in the domain $x, y \in [0,1]$. The function reads as follows

$$\begin{aligned} f(x,y) = &\frac{3}{4}\exp\left(-\frac{(9x-2)^2}{4} - \frac{(9y-2)^2}{4}\right) \\ &+ \frac{3}{4}\exp\left(-\frac{(9x+1)^2}{49} - \frac{(9y+1)}{10}\right) \\ &+ \frac{1}{2}\exp\left(-\frac{(9x-7)^2}{4} - \frac{(9y-3)^2}{4}\right) \\ &- \frac{1}{5}\exp\left(-(9x-4)^2 - (9y-7)^2\right). \end{aligned} \tag{10}$$

For more on the Franke's function, see [3].

2. Breast cancer

For classification we will first be using scikit learns breast cancer data. The data is structured such that each point of the input variables are different attributes (30 in total) of a tumor and the output data is a binary, malignant or benign. Given the number of attributes and the binary nature of the data, the network must contain 30 features and 1 category. The data set includes, in total 569 datapoints.

3. MNIST

The MNIST data set contains 1797 images of handwritten number between 0-9. Each image has a resolution of 8×8 pixels. AN example of the data is shown in figure 4.

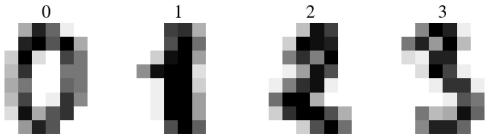


Figure 4: A set of four datapoints from sckit learns breast canser data. Each point with the corresponding value written above.

The MNIST output data, or target is original structured as a list with 1797 elements, where each element is a number between 0-9. This means that the Network still would have 9 categories. As we want the output to be a binary classification case, we restructure the data such that each number is replaced with a 10 dim array. Each array is filled with binary values, where the index equal to the original number is set to 1. For example:

$$3 \rightarrow \begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix} \quad 7 \rightarrow \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix} \quad or \;\; 5 \rightarrow \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix} \tag{11}$$

IV. RESULTS AND DISCUSSION

A. Validation

1. SGD

Note: Say that we choose same seed for repeated experiments for a fair comparison...

We begin by validating and testing the implementation of our SGD method. For this we use Franke's function, which we analyzed in project 1[2]. We use a default 100×100 meshgrid (10^4 data points), a complexity of $n = 5$, referring to the highest polynomial order, and add normal distributed noise to the output with standard deviaion $\sigma = 0.2$. When comparing train and test results we use a default split of 80% training data and 20% test data.

First, we take on analyzing non-momentum OLS-gradient SGD. We investigate the convergence of the model as a function of epochs regarding the hyperparameters: Batch size m and learning rate η. We do this by keeping one parameter constant and changing the other respectively. For this we use the whole data set, and thus the MSE is for the training data. The result for varying batch size and fixed learning rate is shown in figure 5 and the results for varying learning rate and constant batch size is shown in figure 6

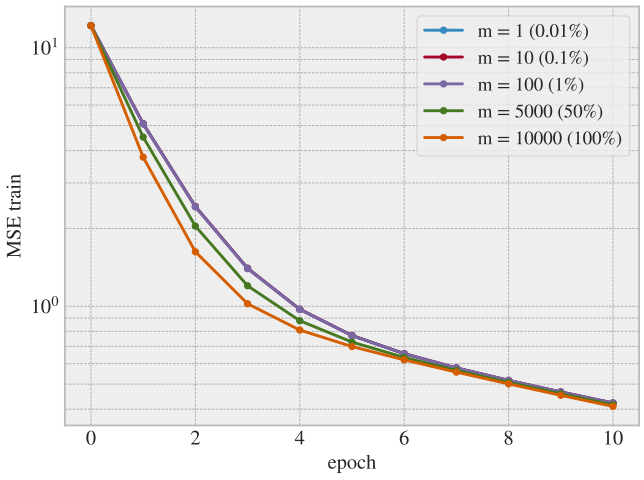


Figure 5: MSE on train data for SGD regression on Franke's function (default settings) with fixed learning rate $\eta = 0.1$ and varying batch size m. The percentage denotes the batch size relative to the number total number of data points.

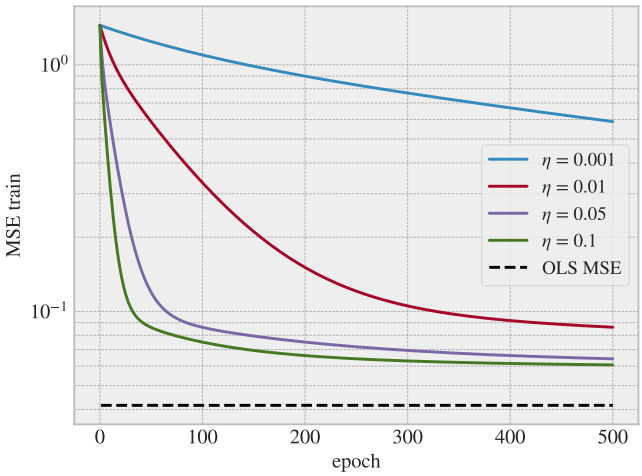


Figure 6: MSE on train data for SGD regression on Franke's function (default settings) with full gradient descent ($m = 10^4$) and varying learning rate η. The black-dotted line indicates the MSE for OLS regression.

From figure 5 we see that the convergence rate is only slightly affected by the batch size, with a small benifit of having the biggest batch size (this was even less clear with other seeds). But the difference becomes negligible already at the 10^{th} epoch onwards. However, from figure 6 we see that the learning rate has a much more distinct impact on the convergence rate. To get a better understanding of the influence from the learning rate we compute the final MSE after 10^3 epochs for both training and test data (default split: 0.2) as a function of the learning rate with full gradient descent (see figure 7).

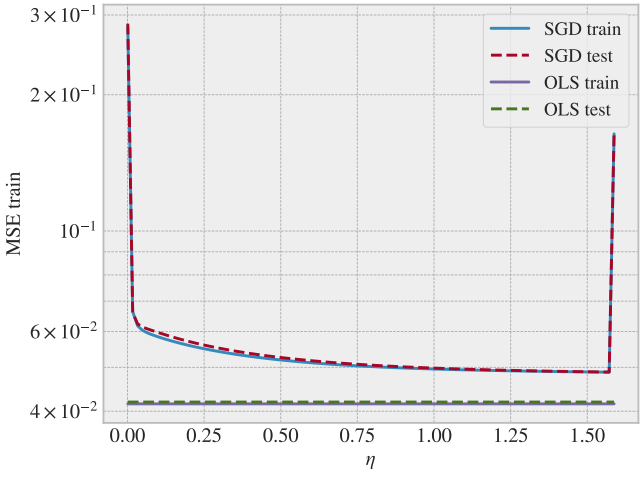


Figure 7: MSE on train and test data for SGD regression on Franke's function (default settings) with full gradient descent ($m = 10^4$) for 10^3 epochs as a function of learning rate η. This is compared to OLS regression for train and test data seen in the bottom of the plot.

From figure 7 we see that the best performance from

the SGD is found with a quite high learning rate. We compute the minimum to be

$$\arg\min_{\eta}(\text{MSE test}) = 1.57 \pm 0.02$$

With a resolution of 0.02 in the leaning rates, the minimum point occurs on the second last index of the tested learning rates, while the last index correspond to the peak on the right-hand side of the plot. Thus we see an apparently non-continuously behaviour (considering our resolution) where the best learning rate comes just before a critical limit where the learning rate will yield poor convergence. However, when running similar test on other data sets we see that this limit changes quite a lot, and thus one should be careful of using a to high learning rate. For the following results we use $\eta = 0.1$ as a safe bet.

We now compare OLS and SGD for different model complexities. Notice that we used $n = 5$ up to this point but know we take on a smaller data set of 20×20 (400 data points) and $n = [1, 20]$. For SGD we use full gradient descent and 10^4 epochs. We continue to use a split of 0.2 and noise $\sigma = 0.2$. The results are shown in figure 8

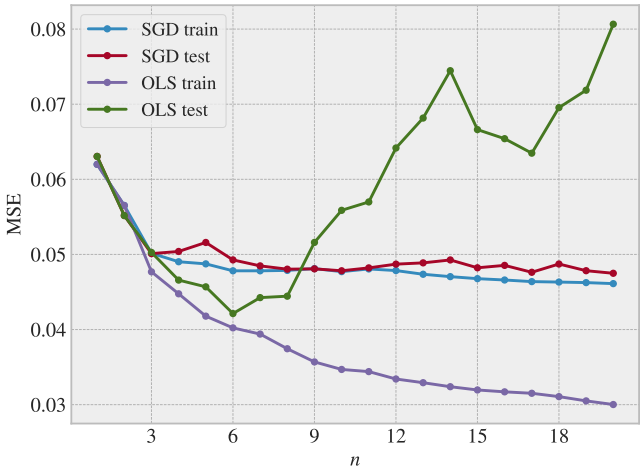


Figure 8: Comparison of SGD and OLS on Franke's function for varying model complexity n. SGD uses full gradient descent with 10^4 epochs and learning rate $\eta = 0.1$. The data consists of 400 data points.

As expected from earlier results, we see that OLS outperformes SGD on the low model complexity which we have already seen for $n = 5$ in figure 7. However, as the model complexity increases the OLS regression results in overfitting and thus the test MSE increases considerable. SGD does not really prone to overfitting in this particular case, and we see that the MSE is steady, actually slightly decreasing, for increasing model complexity.

Until now we have mostly been using full gradient descent as we saw (from figure 5) that this gives a slightly better convergence on a short scale of epochs. When introducing the Neural Network we expect to deal with far bigger gradients (for weights and biases) which will result in heavier matrix operations. For smaller batch sizes we reduce the size of these matrix operations with a cost of having to run through a for-loop for more iterations. In order to get an idea of this computational balance of for-loop iteration and matrix operations we time the algorithm for increasing sizes of the design matrix. We fix $n = 10$, resulting in 66 features and increase the size of the meshgrid from 100×100 (10^4 data points) to 1000×1000 (10^6 data points) while measuring the time pr. epoch. averaged over 10 epochs. We use $\eta = 0.1$ and initialize from same random seed each time. The results are shown in figure 9

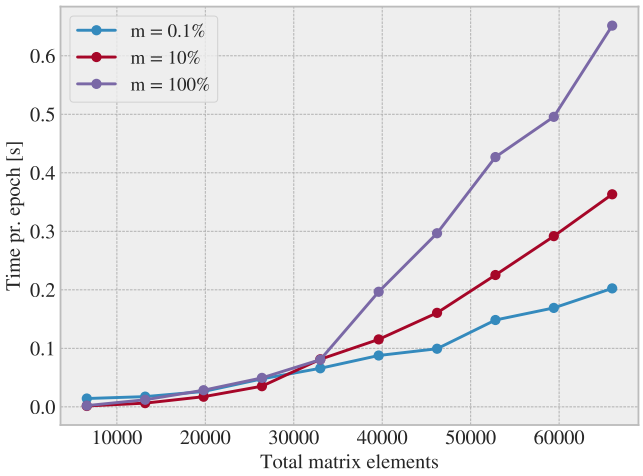


Figure 9: Timing of SGD as a function of the total number of elements in the design matrix for different batch size (given as a percentage of the total number of data points). While keeping $n = 10$, corresponding to 66 features, we varied number of data points in the interval $N \in N = [10^4, 10^6]$ and measured the computation time pr. epoch. We ran each SGD-regression for 10 epochs staring from the same random seed. We see that small mini batches gets computational beneficial for big matrices.

From figure 9 we see that for big matrix sizes, the smallest batch size yields the lowest computation time. For smaller matrices, the small batches actually gives a slightly longer computation time, and the intersection between these domains is visually determined to be around 30,000 matrix elements.

We also want to compare Ridge regression and SGD with Ridge gradient computation. In order to do this we begin by studying the optimal hyperparameters for SGD with with Ridge gradient. When fixing the batch size to full gradient descent we are left with learning rate η and the regularization parameter λ as the main hyperparameters. We go back to using the default data set, as stated in the beginning of this section, and compute the MSE on the test data for different combinations of η and λ. The result is shown in figure 10

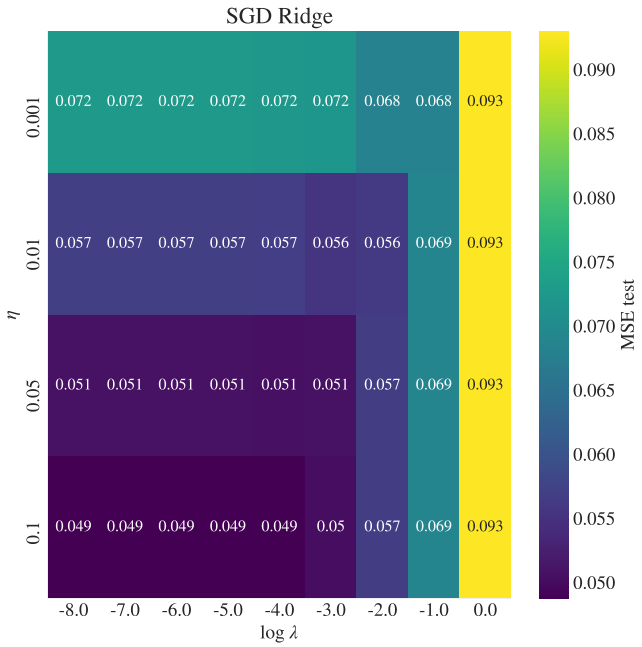


Figure 10: MSE test for SGD with Ridge gradient for different combinations of learing rates η and Ridge regulazation parameter λ on Franke's function. We use the default data set with full gradient descent.

From figure 10 we see that the best MSE test is found at the highest tested learning rate $\eta = 0.1$ and the lowest regularization values $\lambda \leq 10^{-4}$. Looking back to figure 7 one could theorize that we have a similar dependence on the learning rate as we had with the OLS gradient.

This we fix the learning rate to $\eta = 0.1$ and compute the MSE on test data (default data) for different λ-values using 10^4 epochs for the SGD. The result is shown in figure 11.

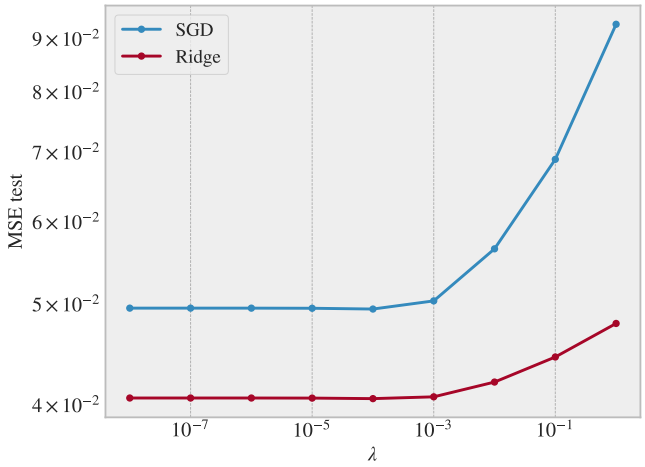


Figure 11: MSE test for SGD (Ridge gradient) and Ridge regression as a function of λ on Franke's function. We used 10^4 epochs and $\eta = 0.1$ with full gradient descent on the default data set.

From figure 11 we see that the lowest MSE is achieved for decreasingly values of λ this reducing the comparison to the one of OLS.

Finally we introduce the SGD with momentum. We use full gradient descent on the default data and investigate the convergence rate for different learning rates and momentum paramter γ. The result is shown in figure 12.

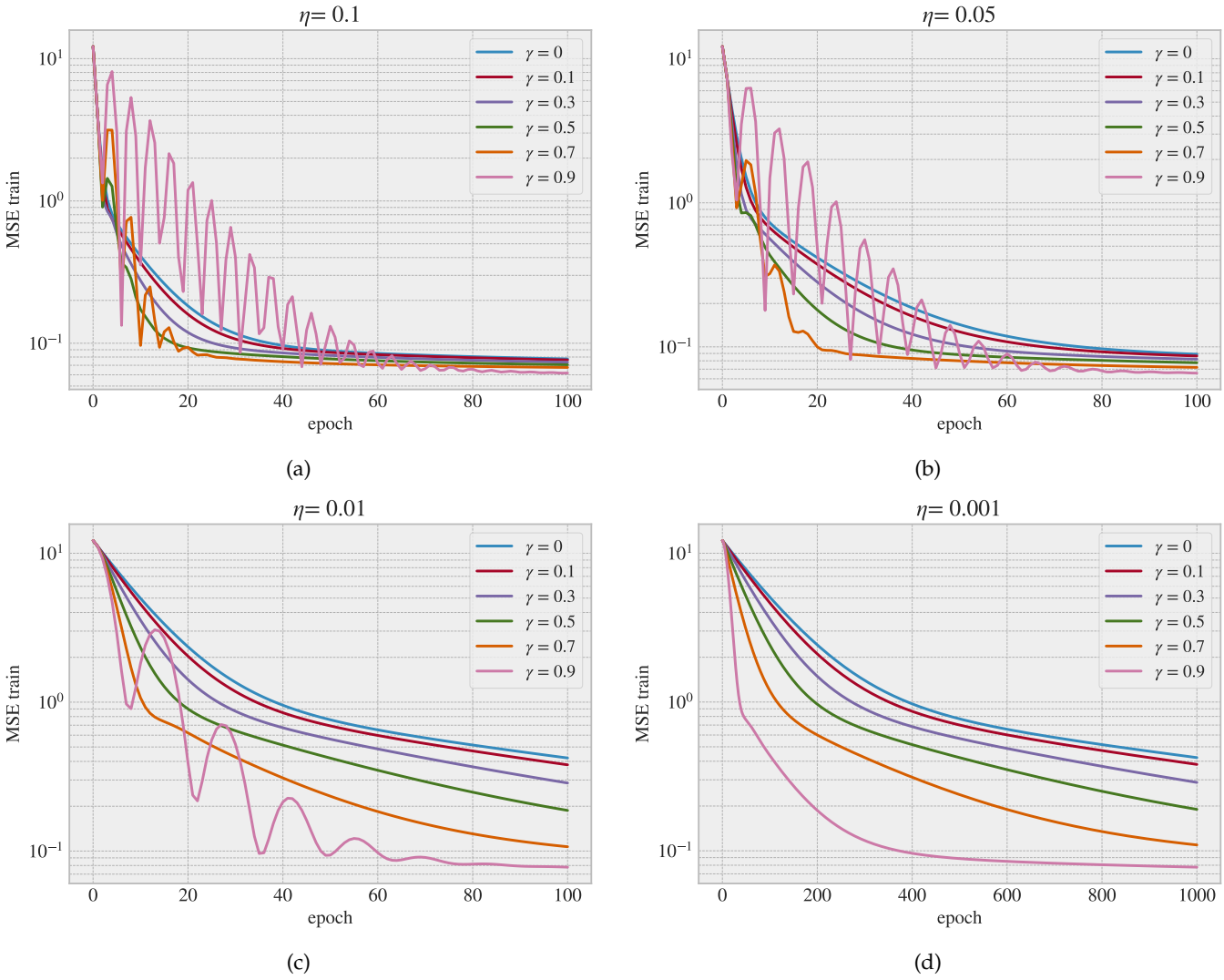


Figure 12: MSE train for different momentum parameters γ and learning rates η as a function of epochs. We used full gradient descent on on the default Franke's function data set.

From figure 12 we see that momentum parameter has a quite significant influence on the convergence rate. For the small learning rate $\eta = 0.001$ (see 12d the highest tested $\gamma = 0.9$ showcased a clear advantage. However, for bigger learning rates, $eta = 0.1$ (figure 12a the higher γ's gave an oscillation MSE and $\gamma = 0.9$ converged the slowest, meaning that it decreases more slowly. But as the oscillations settles down we actually get the lowest MSE after 100 epochs with $\gamma = 0.9$ for all learning rates tested. When increasing γ to 0.99 we see the same effect but on a much bigger scale of epochs.

2. *Neural Network*

In this part we study our implementation of the Neural network. As a starting point we choose three logic gates: AND, OR, XOR (exclusive or) which is represented as the data points in table II

Table II: Logic gates: AND, OR and XOR

Input		Output		
x_1	x_2	AND	OR	XOR
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	1	0

When performing standard OLS we get the predictions shown in table III, where we get 100% accuracy on the AND and OR gate but only 75% on the XOR gate. When displaying the data points graphically it is clear to see that the data points of the XOR-gate cannot be separated by a single straight line (see figure IV A 2. Hence it is clear that OLS will never give a satisfactory prediction on this nonlinear problem.

Table III: OLS predicitons on the logic gates: AND, OR and XOR

Gates	Predictions	Accuracy
AND	$y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	100%
OR	$y = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	100%
XOR	$y = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	75%

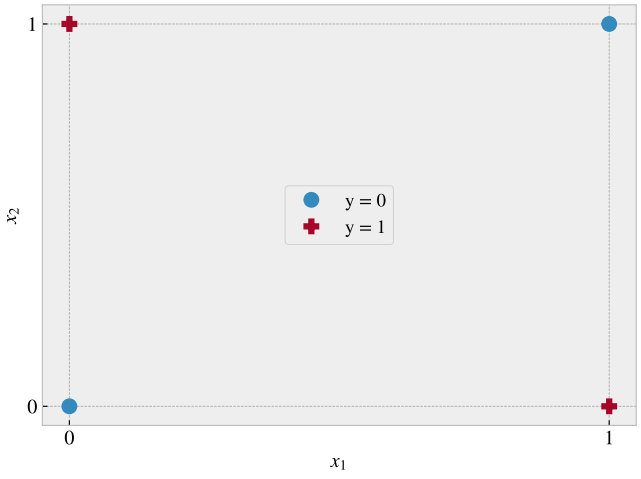


Figure 13: Data points for the XOR-gate. Showcases graphically how we cannot separate the 0's (blue) and 1's (red) by a single straight line

Hence we now use the XOR gate as benchmark for our neural network. We choose a simple architecture with 1 hidden layer containing 4 hidden nodes. Notice that it is possible to solve the XOR-problem with only 2 hidden nodes, but we found that having 4 nodes made the training simpler. We use full gradient descent, $\eta = 1$ and $\lambda = \gamma = 0$ with sigmoid as activation function and cross entropy as the cost function. We manage to get a perfect fit, 100 % accuracy, on the training data after roughly 90 epochs. The learning accuracy development is shown in figure IV A 2.

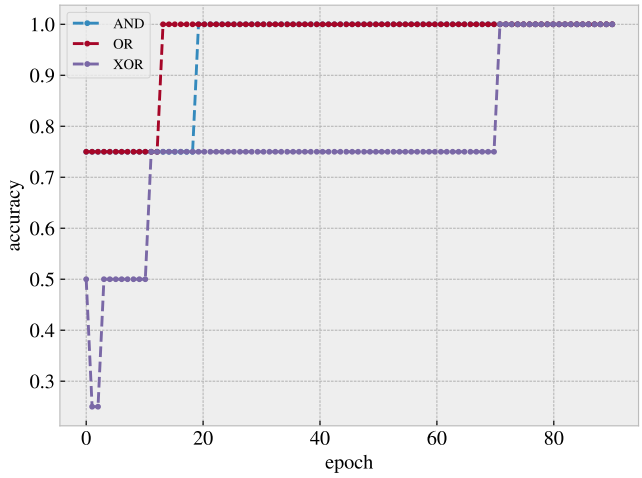


Figure 14: Accuracy as function of epochs for AND, OR and XOR gates.

Here we observe that the network gets 100% accuracy given enough epochs on the XOR gate, thereby outperforming linear regression.

Next we study regression on the Franke's Function from section III F 1 data using our Neural Network. We used the same initial conditions for the Neural Network as with the linear regression models[2], and got the following results:

Table IV: Table with R2 and MSE score for our Neural Net, OLS regression and Ridge regression with $N = 30$, $\sigma = 0.2$, $\lambda = 10^{-4}$, $\gamma = 0$, $\eta = 10^{-1}$, 3 hidden layers, 42 nodes, batch size $= 25$ and at the most 400 epochs.

Score	**NN**	**OLS**	**Ridge**	**TensorFlow**
MSE	0.032	0.034	0.034	0.062
R2	0.844	0.838	0.841	0.489

Here we see that our Neural Network performs as well as OLS and Ridge regression on the Franke's function data, and even Tensorflow. This is however the optimized version, and needed two hyperparameter grid searches, as shown in the images below.

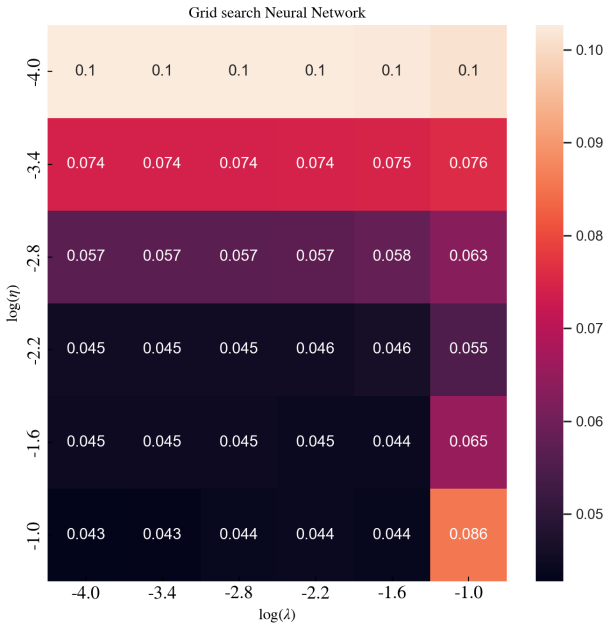


Figure 15: Heatmap showing MSE score for different hyperparameters $\lambda \in [10^{-4}, 10^{1}]$ and $\eta \in [10^{-4}, 10^{1}]$. Max number of epochs is 400, batch size 25, 2 hidden layers and 70 hidden nodes

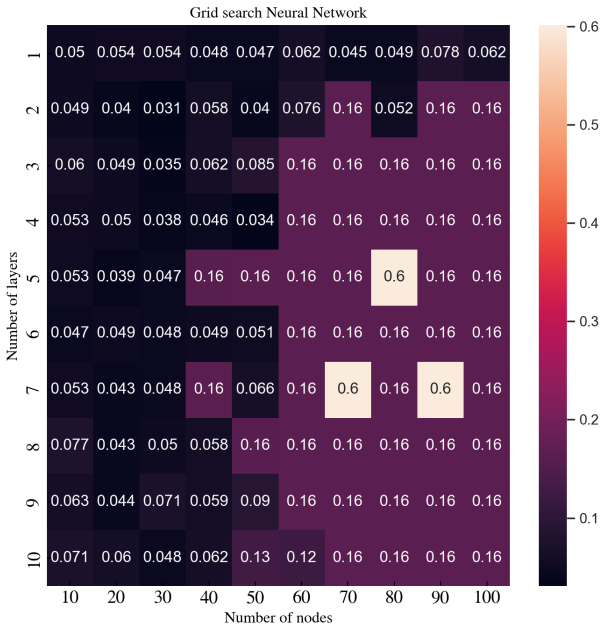


Figure 16: Heatmap showing MSE score for different hyperparameters $\lambda = 10^{-4}$ and $\eta = 10^{-1}$. Max number of epochs is 400, batch size 25, hidden layers $\in [1, 10]$ and hidden nodes $\in [10, 100]$

Here we find the optimal hyper parameters $\lambda = 10^{-4}$ and $\eta = 10^{-1}$. The regularization constant λ acts in the same way as it does for Ridge regression, which further validates our findings of the MSE, given that λ is so small. The next hyper parameter grid search is for number of hidden layers and number of hidden nodes per hidden layer.

From this image we observe that 4 hidden layers and 50 nodes per layer is the optimal choice for the Franke's function data. The grid is spaced with a 10 node and 1 layer increase for each iteration, and this could mean that the actual lowest combination is somewhere in between say 40 and 50, or 50 and 60 nodes. The following image show that there are combinations with lower MSE:

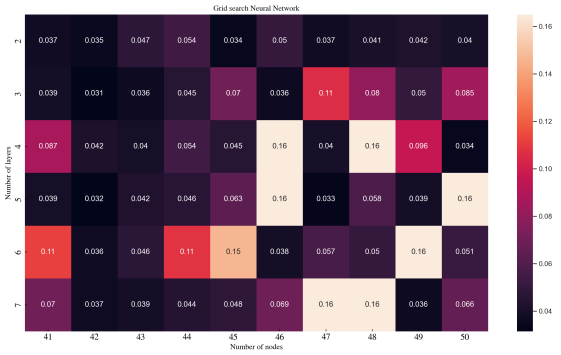
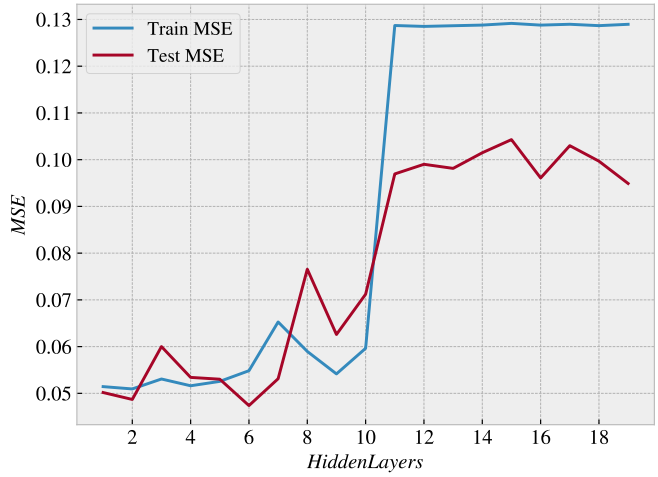


Figure 17: Heatmap showing MSE score for different hyperparameters $\lambda = 10^{-4}$ and $\eta = 10^{-1}$. Number of epochs 400, batch size 25, hidden layers $\in [2, 7]$ and hidden nodes $\in [41, 50]$

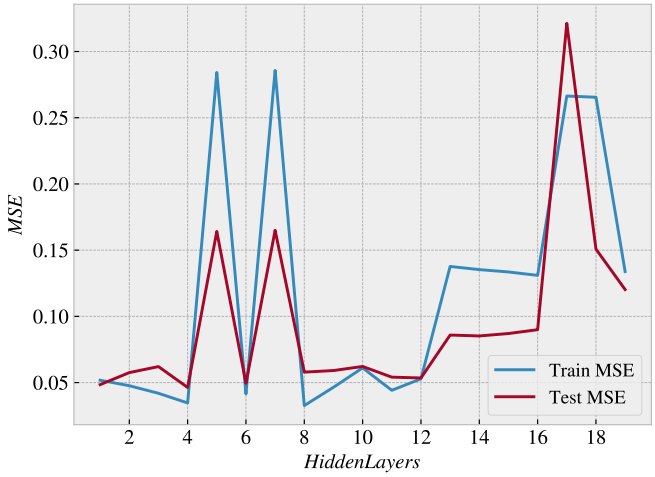
Here we observe that upon further inspection with a much smaller and specified set of nodes, we find the ideal combination of nodes and layers is in fact 3 layers

and 42 nodes.

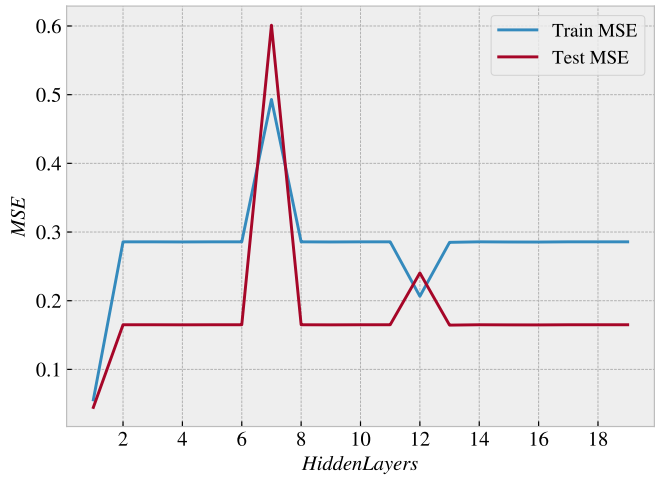
There are certain combinations of the two hyper parameter grid searches that gives a "much" larger MSE score than the optimal parameter combination, and so the images below thus checks if the network is overfitting. If there was any overfitting we would expect the test MSE to diverge from the train MSE for an increase in layers. But as the images show, we observe little to no signs of overfitting. Given these results we feel comfident that our network is good enough to solve more complex problems, such as breast cancer classification and number recognition.



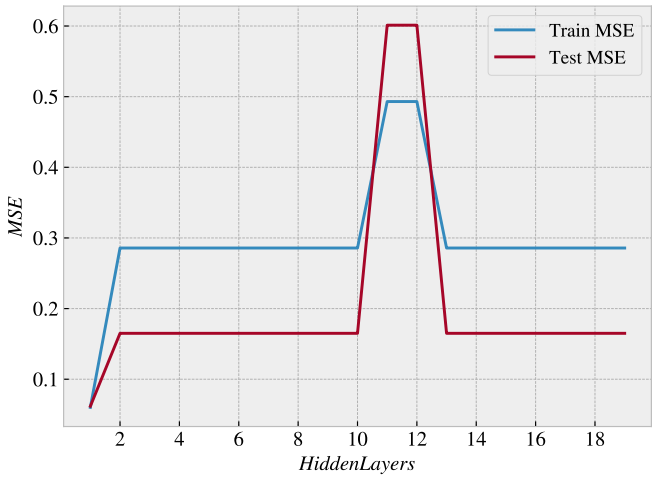
(a) 10 nodes



(b) 40 nodes



(c) 70 nodes



(d) 100 nodes

Figure 18: MSE as function of hidden layers for both training and test data. Top left corner has 10 nodes, top right has 40 nodes, bottom left has 70 nodes and bottom right has 100 nodes. We also have $\eta = 0.1$, $\lambda = 10^{-4}$, batch size $= 25$ and at most 400 epochs.

With these hyperparameters fixed, we plotted our prediction on a new dataset, with the same seed, and got the following results

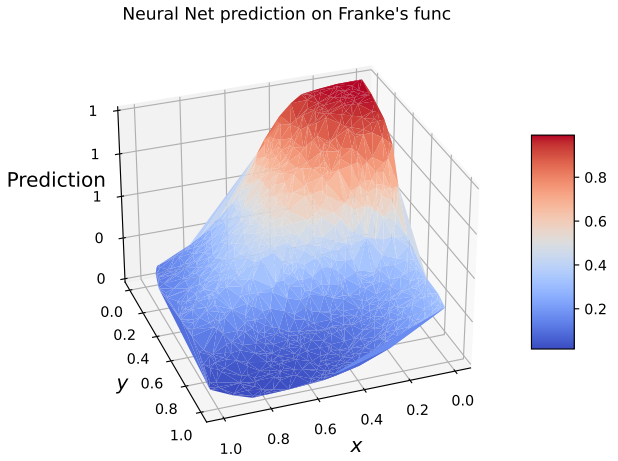


Figure 19: Heatmap for hyperparameters $\lambda = 10^{-4}$ and $\eta = 10^{-1}$. Max number of epochs 200, batch size 25, 2 hidden layers and 30 hidden nodes.

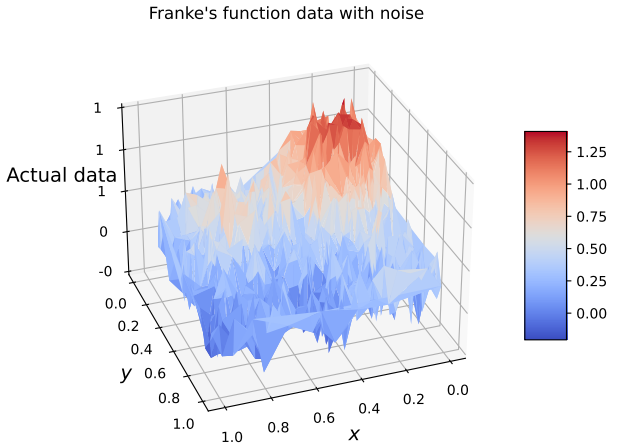


Figure 20: Heatmap for hyperparameters $\lambda = 10^{-4}$ and $\eta = 10^{-1}$. Max number of epochs 200, batch size 25, 2 hidden layers and 30 hidden nodes.

Here we observe that our network makes a very good fit to the noisy data. This is further evidence of good validation of our network, and expected, given the MSE and R2 score from table IV A 2.

B. Franke function

For the Franke data we want to test three different activation functions, Sigmoid, Relu and Leaky relu. We do so by choosing all other parameters to be constant, and observe how the MSE varies as a function of number of epochs, The results are shown in figure 21.

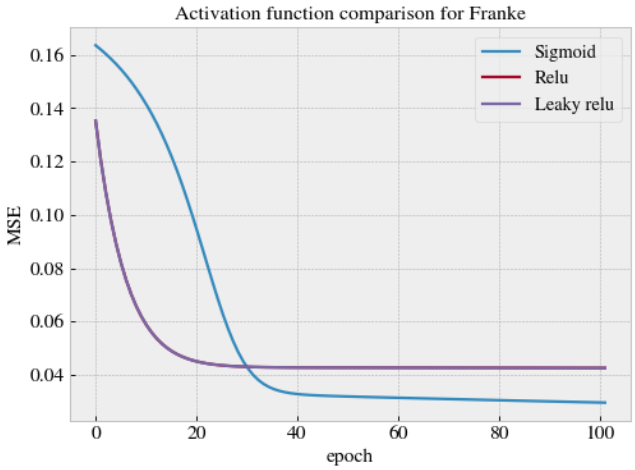


Figure 21: Comparison between sigmoid(blue), relu(red) and leaky relu(purple) as activation functions for calculations on the Franke data. The calculations were done with 2 hidden layers with 30 nodes each. γ and λ values were both set to 0 and η was set to 0.01.

From figure 21 we observe that relu and leaky relu produce the same results. This is an indication that all values are larger than 0. In addition we observe that sigmoid seems to produce larger MSE-values in the first 30 epochs, but stablizes at a smaller MSE value. This means that for the regression case sigmoid produces better predictions than relu and leaky relu, given enough epochs.

C. Breast cancer classification

As done in section IV B, we want to compare different activation functions, but this time for the Breast cancer data. Given the binary nature of the target values, this becomes a classification problem. Therefore the comparison is done with the accuracy score.

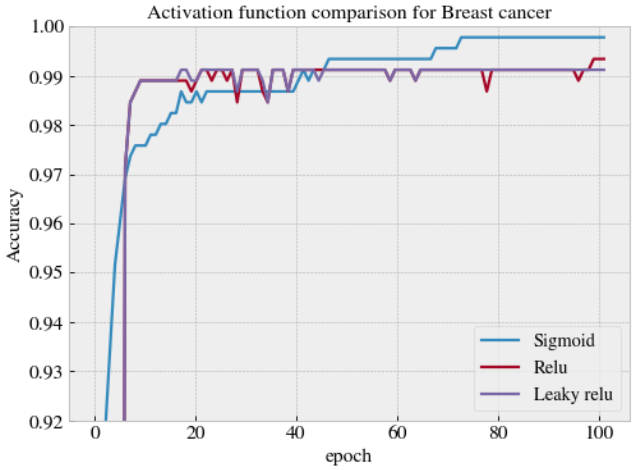


Figure 22: Comparison between sigmoid(blue), relu(red) and leaky relu(purple) as activation functions. The calculations were done with 2 hidden layers with 30 nodes each. γ and λ values were both set to 0 and η was set to 0.0.

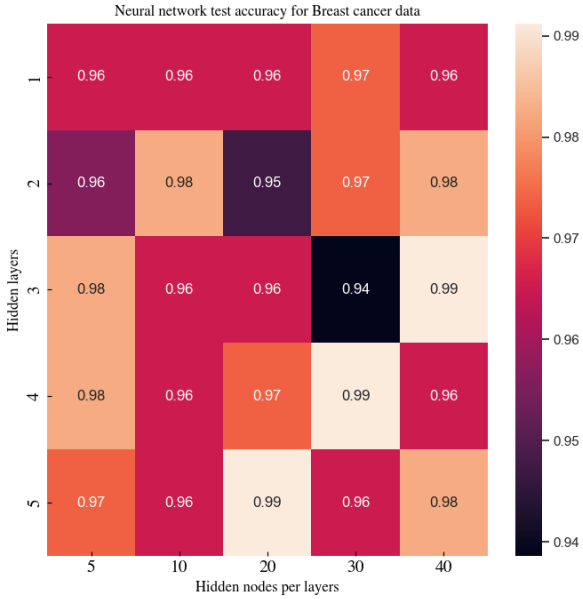


Figure 23: Heatmap for different values of hidden layers and nodes per layer. The calcluation were done with $\eta = 0.01$, $\gamma = 0$, batch size 25 and run for a maximum of 200 epochs.

From the figure 22, we observe that sigmoid produced the highest accuracy, although only slightly. In addition we observe that relu and leaky relu produce rather similar accuracy. Just as in section IV B, sigmoid seems to preform worse for the first 30 epochs.
All though relu and leaky relu are similar, they are not completely equal as in the regression case. We observe that for the classification case the general trend of the accuracy produced from relu is sligthly more stable than that produced from leaky relu.
As well as studying the accuracy as functions of the hyper-parameters and the activation functions, we also want to study for varying number of hidden layers and nodes. The resulting heatmap is shown in figure 23.

From figure **??** we observe a small trend for the values with largest accuracy. The three best values create a diagonal strip in the bottom right corner of the heatmap. As the total number of weights and biases can be viewed as a measure of the complexity in the network, figure **??** shows that the network prefers a certain level of complexity.

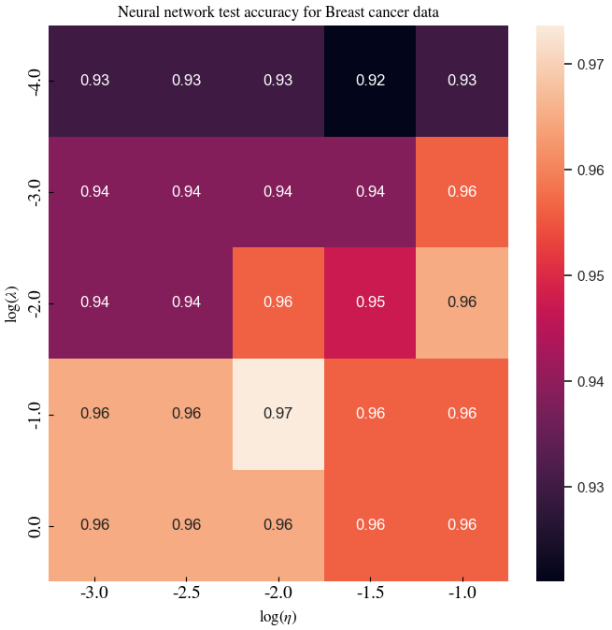


Figure 24: Heatmap for hyperparameters $\lambda \in [10^{-4}, 1]$ and $\eta \in [10^{-4}, 10^{-1}]$. Number of epochs 200, batch size 25, 2 hidden layers and 30 hidden nodes.

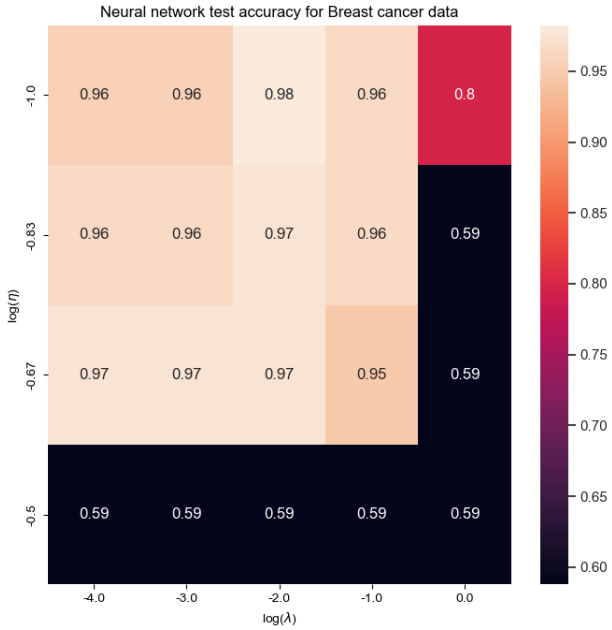


Figure 25: Heatmap for hyperparameters $\lambda \in [10^{-4}, 1]$ and constant $\eta \in [10^{-1}, 10^{-0.5}]$. Number of epochs 200, batch size 25, 2 hidden layers and 30 hidden nodes.

	Neural network	**Scikit learn**
Accuracy	97 %	96 %

Table V: Comparison between the Neural network and Scikit learn. Both with 2 hidden layers with 30 nodes each, batch size equal to 50 and with a maximum of 100 epochs.

1. *Adaptive learning rate*

In all of our calculations on the breast cancer data, we used a constant learning rate. But, as mention in section III E 1 we could also include an adaptive learning rate. For the cases so far we have chosen relatively small learning rates and large number of epochs. But, for larger sets of data a large number of epochs could not be an option. Therefore one must chose a larger learning rate. In this case an adaptive learning rate could be beneficial.
To test our sigmoid function we ran the same grid searches as above, but for only 50 epochs and $\eta \in [10^{-1}, 10^{-0.5}]$. With a constant learning rate the result were

Then by repeating with an adaptive learning rate with k = 0.2 and dropp time equal 10 epochs we found the results

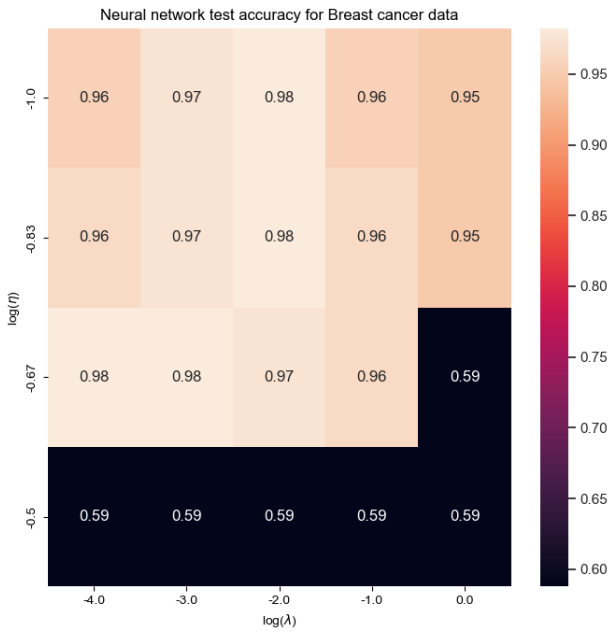


Figure 26: Heatmap for hyperparameters $\lambda \in [10^{-4}, 1]$ and adaptive $\eta \in [10^{-1}, 10^{-0.5}]$. Number of epochs 200, batch size 25, 2 hidden layers and 30 hidden nodes.

By comparing figures 25 and 26 we observe that by

including an adaptive learning rate, we were able to obtain a far better result for some of the combinations. In the cases where both η and λ are large, the adaptive learning rate exceeds the constant learning rate. In addition almost half of the values were imporoved by 1%. All though it is a slight improvement, the size of the data we will be studying in this report are small enough for us to be able to preform large amount of epochs. Therefore there is no need for adaptive learning rates in the later calcualtions.

D. MNIST classification

As stated in section (INSERT SECTION) we will be comparing our Neural Network to our logistic regression solver. All calculations and comparisons will be done using the MNIST data set.
By running a grid search for the hyper-parameters η and λ we find the grid shown in figure (28). The training of the network was done on training data and the accuracy test was done on test data.

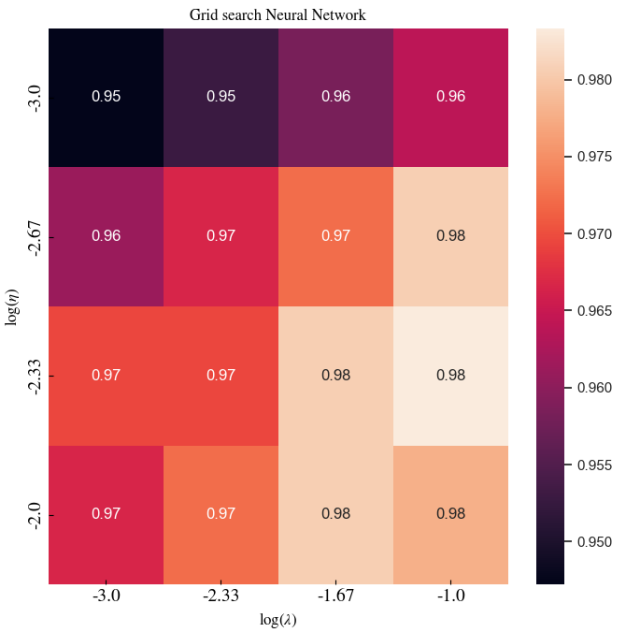


Figure 27: Neural Network heatmap for hyperparameters $\lambda \in [10^{-3}, 10^{-1}]$ and $\eta \in [10^{-4}, 10^{-2}]$. Maximum number of epochs 200, batch size 35, 2 hidden layers and 20 hidden nodes.

From (28) we observe that the optimal values for our Neural Network are $\lambda = 10^{-2.33}$ and $\eta = 10^{-2}$.
By doing the same grid search for our logistic regression solver, we find

Table VI: Accuracy score for the neural network, scikit learn and logistic regression on the MNIST data set

	Neural network	**Logistic regression**	**Scikit**
Accuracy	98%	93 %	95%

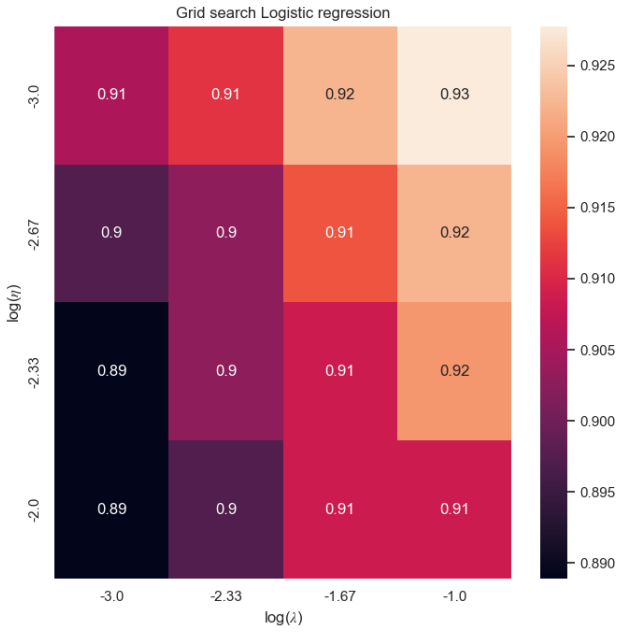


Figure 28: Logistic regression heatmap for hyperparameters $\lambda \in [10^{-3}, 10^{-1}]$ and $\eta \in [10^{-4}, 10^{-3}]$. Number of epochs 200, batch size 35, 2 hidden layers and 20 hidden nodes.

Comparing our Neural Network with our logistic regression solver and Scikit learns logistic regression we find the following results

Figure 29: Digits in test data with the Neural Networks predicted values. Green digits indicate correct prediction, red indicate incorrect. The Neural Network was trained with 2 hidden layers with 35 nodes each, batch size 50, $\eta =$, $\lambda =$ and after 50 epochs.

Figure 30: 64 digits from the test data with the Neural Networks predicted values. Green digits indicate correct prediction, red indicate incorrect. The Neural Network was trained with 2 hidden layers with 35 nodes each, batch size 50 and after a maximum 200 epochs. The hyper-parameters were chosen from a grid search.

V. CONCLUSION

When preforming regression on the Franke function we found that the Neural network outperformed OLS, Ridge and Tensorflow in predicting the surface.
In sections IV B and IV C we studied the MSE and accuracy of our neural network with varying activation functions. We found that for both the regression case and the classification case the sigmoid function gave the best results, allthough Relu and leaky relu were better in the first 30 epochs. In addition we found that in the classification case the leaky relu function produced slightly more stable results than with the relu function.
We tested our network on classification of breast cancer data and digit image data, and found that in the breast cancer our network barely outperforms Scikit learn, whilst with the MNIST digit image case our network outperforms by 3 percent above Scikit and 5 percent above logistic regression.

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