A numerical approach to estimating implied risk in financial instruments using Black-Scholes

Anders Vestengen
Sakarias Frette
William Hirst

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In this report we solve the Black-Scholes equation by modeling it as a diffusion equation, applying the Cranck-Nicholsen scheme and solving it numerically using a tridiagonal matrix algorithm. We analyse our results by calculating the Greeks and build credibility to our code by comparing our solutions to the Black-Scholes formula. Lastly we apply our code to a real-life problem by finding the implied volatility to the option-values of Aker BP. We found that parameters like the volatility of the option values and the interest rate greatly impacts our numerical option values. This is notably the case when the interest rate approaches zero. For this occurrence we find that the Black-Scholes solver breaks down, and therefore works poorly for today's stock market. Finally we estimated the implied volatility of Aker Bp to be 30.4% (annualized).

Contents

1 Introduction

"To those who have much, more will be given. To those who have less, more will be taken". This is known as the Matthew principle, and this effect appears no where better than in economics, both at a personal level, and as a group. For ever since man became civil, economy has been a cornerstone in our societies. As early as ancient Greece (ca 9th century BC) there are records of people buying and selling contracts speculating on future marked prices of crops from olive harvests. Therefore, being able to analyse the value of options is not only crucial now, but has been for thousands of years.

In this report we explore the Black-Scholes equation to further understand options and to verify its numerical solutions with a comparison to the analytical solution for European options. These options have an expiration date for which they can be exercised, and are locked until that time, differing them from American style options which can be exercised at any point. We will also calculate the Greeks[4], to understand risk assessment for portfolios. To understand how Black-Scholes is used in real life, we will look at publicly traded options and try to find its implied volatility. In section (2) we explore the theoretical background for the solver, in addition we break down our methodology in section (3). We display our results in section (4), and discuss the implications as well as the limitations of the model in section (5). Finally section (6) encompasses our conclusions and parting thoughts.

2 Theory

2.1 Options

Options are type of derivatives for financial instruments. They are called derivatives because they derive their value from an underlying asset. Options are a contract between two parties agreeing to either sell (put) or buy (call) a financial asset for a predetermined price at a point in the future. For example, a trader recognizes that during the pandemic more people are shopping online. The trader buys calls on a major online retailer which expire later in the year. When the retailer publishes its earnings the stock price immediately goes up, but the trader already have calls for this retailer and can buy the stocks at a price you set previously when the stock was low. The trader can now sell those stocks for a profit.

2.2 The Black-Scholes model

To predict the price of an option, Myron Scholes and Fischer Black derived a partial differential equation for European calls, known as the Black-Scholes equation. The derivation makes a lot of assumptions, both regarding the market and the assets[10].

- Assume lognormal stock price, based on principle that asset prices cannot have negative value.
- Assumes that stocks move according to a Brownian motion [2], as they are hard to predict and highly volatile.
- Assumes no transaction costs, including commission and brokerage.
- Assumes constant interest rate.
- Assumes no arbitrage, avoiding riskless profit.
- Assumes the rate of return on riskless assets (cash, bonds) is constant and therefore risk-free.
- Assumes European style options; it is assumed that the option can only be exercised on the expiration date.

2.2.1 The Black-Scholes equation

The Black-Scholes equation is given as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

where σ is the volatility of the stock, S is the stock price, r is the risk-free interest rate, D is the dividend paying rate and V is the value of the option as a function of stock price and time. In order to solve this equation, we can try to model the value of the options after the diffusion equation. In order to that we first have to do a variable change, from S to x. We set this relation to be $x = \ln \frac{S}{E}$. If we differentiate this equation, we get the following relation:

$$\partial x = \frac{\partial S}{S} \tag{2}$$

which gives us the derivation operators

$$\frac{\partial}{\partial x} = S\frac{\partial}{\partial S} \tag{3}$$

Now we also need the second derivative, so we use equation (3) on itself and get

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial x^2} = S\frac{\partial}{\partial S}\left(S\frac{\partial}{\partial S}\right) = S\frac{\partial}{\partial S} + S^2\frac{\partial^2}{\partial S^2} \tag{4}$$

From this equation and equation (3) we have that

$$S^2\frac{\partial^2}{\partial S^2} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \tag{5}$$

We can now rewrite equation (1) in x coordinate:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}+(r-D-\frac{1}{2}\sigma^2)\frac{\partial V}{\partial x}-rV=0 \tag{6}$$

We then transform the time variable to $\tau = T-t$, where T is the expiration time for the option. This gives us

$$\frac{\partial V}{\partial \tau}-\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}-(r-D-\frac{1}{2}\sigma^2)\frac{\partial V}{\partial x}+rV=0 \tag{7}$$

The last thing we need to do is to is to do a substitution for the value of the option:

$$u(x,\tau)=e^{\alpha x+\beta\tau}V(S,t) \tag{8}$$

We then get

$$\begin{aligned}\frac{\partial}{\partial \tau}(u(x,\tau)e^{-\alpha x-\beta\tau}) &= \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}(u(x,\tau)e^{-\alpha x-\beta\tau}) \\ &\quad +(r-D-\frac{1}{2}\sigma^2)\frac{\partial V}{\partial x}(u(x,\tau)e^{-\alpha x-\beta\tau})-ru(x,\tau)e^{-\alpha x-\beta\tau}\end{aligned} \tag{9}$$

which gives us

$$\begin{aligned}\left[\frac{\partial u}{\partial \tau}-\beta u\right]e^{-\alpha x-\beta\tau} &= \frac{\sigma^2}{2}\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}e^{-\alpha x-\beta\tau}-\alpha u e^{-\alpha x-\beta\tau}\right] \\ &\quad +(r-D-\frac{\sigma^2}{2})\left[\frac{\partial u}{\partial x}e^{-\alpha x-\beta\tau}-\alpha u e^{-\alpha x-\beta\tau}\right]-rue^{-\alpha x-\beta\tau}\end{aligned} \tag{10}$$

$$\frac{\partial u}{\partial \tau}-\beta u=\frac{\sigma^2}{2}\left[\frac{\partial^2 u}{\partial x^2}-2\alpha\frac{\partial u}{\partial x}+\alpha^2 u\right]+(r-D-\frac{\sigma^2}{2})\left[\frac{\partial u}{\partial x}-\alpha u\right]-ru \tag{11}$$

$$\frac{\partial u}{\partial \tau}=\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\left[r-D-\frac{1}{2}\sigma^2-\alpha\sigma^2\right]+u\left[\alpha^2\sigma^2-\alpha(r-D-\frac{1}{2}\sigma^2)-r+\beta\right] \tag{12}$$

Now, in order to get the diffusion equation we have to set the sum of the coefficients for both first order derivative of u and u to zero. From this we can find α and β as

$$\alpha=\frac{r-D}{\sigma^2}-\frac{1}{2} \tag{13}$$

and

$$\beta=r+\alpha\left[r-D-\frac{1}{2}\sigma^2\right]-\alpha^2\sigma^2 \tag{14}$$

If we use these two values for α and β, we get a modified diffusion equation

$$\frac{\partial u}{\partial \tau}=\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} \tag{15}$$

Substituting back we get that the values of the option, given as $V(S,t)$ is given as

$$V(S,t)=u(x,\tau)*e^{-\left[\alpha x+\left(r+\alpha\left[r-D-\frac{1}{2}\sigma^2\right]-\alpha^2\sigma^2\right)\tau\right]} \tag{16}$$

2.2.2 The Analytical solution

The Black-Scholes equation has an analytical solution as well. It is given as

$$V(S_t, t) = N(d_1)S_t - N(d_2)PV(E) \tag{17}$$

where V is the value of the option, S_t is a stock price at a given time t, PV the present value of the exercise price $PV(E) = Ee^{-r(T-t)}$, and N is the cumulative distribution function [1]. Here, d_1 and d_2 are given as

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\frac{S_t}{T} + \left((r+\frac{\sigma^2}{2})(T-t)\right)\right] \tag{18}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{19}$$

Note that the analytical expression is inversely proportional to $\tau = T - t$. Therefore, the solution breaks down for $\tau = 0$.

2.3 Crank-Nicolson and its application to the Black-Scholes equation

When learning to solve a Partial Differential Equation(PDE), one often starts with the Forward Euler Method (FEM). FEM is a very intuitive method and can be written as the following:

$$\frac{du}{dx} \approx \frac{u(x_i) - u(x_{i-1})}{\Delta x} \tag{20}$$

, where u is a function of a variable x_i ($i \in [0, N]$), and Δx is the predetermined step from one value to the next ($\rightarrow x_{i+1} - x_i$). When using such a definition to solve a PDE, one often calls the method an explicit scheme. It is characterised by the fact that you start the solution in a point x_0 and end in another point x_N, "moving forward" in the function. We can use (20) on a diffusion equation (15 without the factor $\frac{\sigma^2}{2}$) for a function u(x,t), deriving twice for x and once for time and we get the following equation:

$$\frac{u(x_i, t_j) - u(x_i, t_{j-1})}{\Delta t} = \frac{u(x_{i+1}, t_j) - u(x_i, t_j) + u(x_{i-1}x, t_j)}{\Delta x} \tag{21}$$

Alternatively you can use an implicit scheme. When using an implicit scheme you begin in a point x_N and integrate "backwards" to a point x_0. A common way of doing so is called the Backward Euler Method (BEM). BEM can be written as:

$$\frac{du}{dx} \approx \frac{u(x_{i-1}) - u(x_i)}{\Delta x} \tag{22}$$

, where the values are the same as described above. Using (22) on a diffusion equation and doing as we did above, we find the following equation:

$$\frac{u(x_i,t_{j-1})-u(x_i,t_j)}{\Delta t}=\frac{u(x_{i+1},t_j)-u(x_i,t_j)+u(x_{i-1},t_j)}{\Delta x} \tag{23}$$

The Cranck-Nicolsen method can be described as a mixture between the two, FEM and BEM. One of the benefits of the Cranck-Nicolsen is that it is stable for all sizes of $\alpha=\frac{dt}{h^2}$ [1]. If we introduce a parameter θ together with our to equations, (21) and (23) the Cranck-Nicolsen method is derived from the following equation:

$$\begin{aligned} &\frac{\theta}{\Delta x^2}(u(x_{i+1},t_j)-2u(x_i,t_j)+u(x_{i-1},t_j))\\ +&\frac{1-\theta}{\Delta x^2}(u(x_{i+1},t_{j-1})-2u(x_i,t_{j-1})+u(x_{i-1},t_{j-1}))\\ &=\frac{1}{\Delta t}(u(x_i,t_j)-u(x_i,t_{j-1})) \end{aligned} \tag{24}$$

When $\theta=0$, (24) becomes the explicit method, and when $\theta=1$, (24) becomes the implicit method. When $\theta=1/2$, (24) becomes the Cranck-Nicolsen method and can be written as:

$$\begin{aligned} &\frac{1}{2\Delta x^2}(u(x_{i+1},t_j)-2u(x_i,t_j)+u(x_{i-1},t_j))\\ +&u(x_{i+1},t_{j-1})-2u(x_i,t_{j-1})+u(x_{i-1},t_{j-1}))\\ &=\frac{1}{\Delta t}(u(x_i,t_j)-u(x_i,t_{j-1})) \end{aligned} \tag{25}$$

The equation above represents the Crank-Nicholsens approximation to the diffusion equation. But, we are not solving the diffusion equation, because we have an extra factor $\frac{\sigma^2}{2}$ on the left side of the equation and we have substituted t for $\tau=T-t$. With this in mind, we can rearrange this equation putting every element in time t_j on the left side and all the elements in time t_{j-1} on the right:

$$\begin{aligned} &-\frac{\sigma^2}{2}\alpha u(x_{i+1},\tau_j)+(2+\alpha\sigma^2)u(x_i,\tau_j)-\frac{\sigma^2}{2}\alpha u(x_{i-1},\tau_j)\\ =&\frac{\sigma^2}{2}\alpha u(x_{i+1},\tau_{j-1})+(2-\alpha\sigma^2)u(x_i,\tau_{j-1})\frac{\sigma^2}{2}\alpha u(x_{i-1},\tau_{j-1}) \end{aligned} \tag{26}$$

Finally we can write (26) as a matrix equation:

$$(2\hat{I}+\frac{\sigma^2}{2}\alpha\hat{B})V_j=(2\hat{I}-\frac{\sigma^2}{2}\alpha\hat{B})V_{j-1} \tag{27}$$

[1]For more information on why this is the case, see [5]

, where $V_j = \{u(x_0, \tau_j), u(x_1, \tau_j), ..., u(x_N, \tau_j)\}$ is a vector with our function u for a specific time-step τ_j, $\hat{I}$ is the identity matrix and $\hat{B}$ is a tridiagonal matrix withe 2 on the diagonal and -1 on both of the subdiagonals.

To make calculations simpler, we can rewrite the right side of (27) as the following:

$$\tilde{V}_{j-1} = (2\hat{I} - \frac{\sigma^2}{2}\alpha\hat{B})V_{j-1} \tag{28}$$

When we do this, we can find V_j by solving (27) as a systems of linear equations with a tridiagonal matrix , where $\tilde{V}_{j-1}$ and the matrix $(2\hat{I} + \alpha\hat{B})$ is known. We chose to solve this using a tridiagonal matrix algorithm.

2.4 Tridiagonal matrix algorithm

The second row/equation in the matrix equation (index i=1, where i=0 is the first row) with time $\tau_j = \tau_1$ (27) can be written in the following way:

$$au(x_0, \tau_1) + bu(x_1, \tau_1) + cu(x_2, \tau_1) = \tilde{u}(x_1, \tau_0) \tag{29}$$

, where $a = c = -\frac{\sigma^2}{2}\alpha$, $b = 2 + \alpha\sigma^2$ and $\acute{u}(x_i, \tau_{j-1})$ is the second element in $\tilde{V}_{j-1}$.
If we subtract (29) with the first equation/row (multiplying the first with a and the second with b), we find and a new equation:

$$(b^2 - ac)u(x_1, \tau_1) + cbu(x_2, \tau_1) = b\tilde{u}(x_1, \tau_0) - a\tilde{u}(x_0, \tau_0) \tag{30}$$

This is equation does no longer include $u(x_0, \tau_0)$. We can again subtract this equation from the third equation/row, now multiplying the third row with the new constant in front of $u(x_1, \tau_1)$, and multiplying (31) with b. We then find our third equation:

$$(b(b^2 - ac) - cba)u(x_2, \tau_1) + (b^2 - ca)u(x_3, \tau_1) = (b^2 - ca)\tilde{u}(x_2, \tau_0) - (b\tilde{u}(x_1, \tau_0) - a\tilde{u}(x_0, \tau_0))a \tag{31}$$

Now this equation no longer includes $u(x_0, \tau_0)$ or $u(x_1, \tau_0)$. What we have done is started with the first row in (27), updating our constants, and subtracting it from rows further down in our matrix in such a way that they loose dependency of u for a certain index. If we do this process enough times, we end up with an equation only dependent on $u(x_{N-1}, \tau_0)$ and $u(x_N, \tau_0)$. Given boundary conditions for u, we are able to solve for $u(x_{N-1}, \tau_0)$, and then use $u(x_{N-1}, \tau_0)$ to solve for $u(x_{N-2}, \tau_0)$ and so one. In this way we are able to solve for all values of x_i in an efficient way.

2.5 The Greeks

"The Greeks"[4] are a set of varying metrics to try to better understand the aspects of a financial instrument. They tell us about how to the asset evolves in the market in much the same way as acceleration and speed tells us about a ball flying in the air. And from these metrics we can make predictions about the outcome.

2.5.1 Δ

$$\Delta = \frac{\partial V}{\partial S} \tag{32}$$

Is the first derivative of the option price by the underlying asset price. Therefore δ is most often used to asses the price sensitivity of the option. Namely, how much the value of the option varies when the price of the stock varies.

2.5.2 γ

$$\gamma = \frac{\partial^2 V}{\partial S^2} \tag{33}$$

Is the derivative of delta, meaning its a metric of how much the delta will change according to the asset price.

2.5.3 Θ

$$\Theta = -\frac{\partial V}{\partial t} \tag{34}$$

Is the derivative of the option value relative to time, and often used to see how much the option changes in value relative to the decreasing time to expiration.

2.5.4 ρ

$$\rho = \frac{\partial V}{\partial r} \tag{35}$$

Measures the changes of the option value relative to the changes of the interest rate. This is often much more stable given the interest rate rarely changes compared to the movements of the other Greeks. Like θ, the derivative is measured as the change in the option value relative to a percentage change in the derivative.

2.5.5 ν

$$\nu = \frac{\partial V}{\partial \sigma} \tag{36}$$

Is the rate of change in the option value by the change in the implied volatility of the underlying asset, given as σ.

2.5.6 Black-Scholes equation and Greeks

Now that we have written down our greeks, we can rewrite our Black-Scholes equation. Using the equations we introduced above, we can write:

$$\Theta + \frac{1}{2}\sigma^2 S^2 \gamma = rV - rS\Delta \tag{37}$$

, where all of the variables are defined above.

3 Implementation

The programs used in this research article can be found on this address.

3.1 Initial and boundary conditions

As shown below in the Crank-Nicholsen algorithm, we have to set the initial conditions for V_j at $j = 0$ corresponding to $\tau = 0$. In other words, at the expiration time for the option. At the expiration date we then have two options. Either the stock value is below the exercise price or above it. In the first case, we then value the option at zero, as the Black-Scholes solution can not have negative values. In the latter case however, we set the value to the difference between the stock and the exercise price. This means that when the time has expired, the option value is equal to whatever the stock is worth minus what you paid for it. Mathematically we can write this as:

$$V(S, \tau = 0) = \max(0, S - E) \tag{38}$$

Now that we khow initial conditions for V, we can use (8) to substitute to find the initial conditions for u.
Additionally to initial conditions, we need to have boundary conditions ($V(0,\tau)$ and $V(S_{max},\tau)$) for all τ. This follows from our algorithm to solve the traditional matrix algorithm. For very small S values, the option value is naturally very low. Therefore we set $V(0,\tau) = 0$.

Another boundary condition is needed for the spacial part of the solution. We have converted the original Black-Scholes equation (1) to a variant of the diffusion equation (15), and to do that we did a variable change from S to x. Ideally we want the Solver to cover all possible variations of S, and thus the boundary conditions for x are $x \in [-\infty, \infty]$. This boundary is not possible to represent on a computer, as it is infinite, and computers can only represent finite numbers, so we have to choose a sufficiently large value L to act as our large number. At the tail ends of the x values, we imposed special initial conditions for the value of the option. For stock prices close to zero, we expect the option to be worth nothing, as it is both below the strike price and, due to the assumption that stocks do not dramatically change(although it can change a little), tend to remain stable, there is no reason to expect non-zero values there. On the other hand, the boundary condition for the largest stock price is given below.

$$V(x = L, \tau) = Se^{-D\tau} - Ee^{-r\tau}$$

For $\tau = 0$ the boundary condition just yields the initial condition above. The extra terms are added to represent the risks for an option to decrease as we approach the expiration time. Thus, the longer you wait, the safer the value of the option is, as is consistent with real life options.

3.2 The algorithms

The code revolves around two main algorithms, the Crank-Nicholsen implementation, and the Tridiagonal function. The Cranck-Nicholsen implementation is as follows:

Algorithm 1: *Crank − Nicholsen*

```
input: Set time = 0
input: Set max iteration number N
1 Set up file to write;
2 Set initial condition/V_0;
3 for i = 0 to N do
4     Update time;
5     Update boundary condition for V_j;
6     Calculate Ṽ_{j-1}(time);
7     Call Tridiagonal function, get V_j;
8     Transform u(x,τ)-values in V_j to V(S,t) Print V_j;
9     Set V_{j-1} = V_j ;
```

Firstly we set up our initial conditions for V_j using theory discussed in (3.1). In step 6 we use equation (28) to set up our known vector using V for the previous time-step, V_{j-1}. In step 7 we use the method from section (2.4) to calculate V for the next step and finding V_j. After calculating V_j, we then transform our vector to the option-values using (8). In the actual code, we also have an if statement that checks what iteration value we have, and prints the vector V_j for each 10 percent of the expiration time. This is not included in the algorithm to make it tidier. When the values are printed, the method repeats.

3.3 The Greeks

The Greeks when used for educational purposes are usually only found using the analytical expression as they are easy to derive. However, we have found them numerically.

3.3.1 Δ

From section 2.5.1, we have that $\Delta = \frac{\partial V}{\partial S}$. To find this numerically we interpolated the data for the option values. Scipy has an interpolate package[7], and we used spline methods to interpolate the data, and then derived the given polynomial using Scipy's derivation function for spline methods.

3.3.2 γ

From section 2.5.2, we have that $\gamma = \frac{\partial^2 V}{\partial S^2}$. Spline methods work well with zero'th order and first order derivatives, but usually deviates at higher order derivates. This led us to use the numpy.diff function[6]. This uses the forward finite difference method, and allows for recursion, which, makes it easy to find the second derivative.

3.3.3 Θ

From section 2.5.3, we have that $\Theta = -\frac{\partial V}{\partial \tau}$. Here we hold all parameters except for time constant, and differentiate with respect to τ. This is done by picking a specific option value for a given stock, and using the same index for all the other timesteps. We then use Numpy's differentiation method, as explained in the section above.

3.3.4 ρ

From section 2.5.4, we have that $\rho = \frac{\partial V}{\partial r}$. Here we hold all parameters constant except for the risk free interest rate, i.e we loop over multiple r values, and calculate the option value for that specific value. This is then written to a file for analysis. As above, we then use forward finite difference method from Numpy.

3.3.5 ν

From section 2.5.5, we have that $\nu = \frac{\partial V}{\partial \sigma}$. Here we hold all parameters constant except for the volatility, i.e we loop over multiple σ values, and calculate the option value for that specific value. This is then written to a file for analysis. As above, we then use forward finite difference method from Numpy.

3.4 Application to real stock analysis

Up until now, we have explained how we solve the Black-Scholes equation to find option values. In reality the option-values are often known and most of the parameters in the Black-Scholes equation can easily be found/derived. Therefore, the Black-Scholes equation is often used to find the volatility of a stock σ. As will be discussed in (5.1), the volatility will greatly affect the different option values. If the option values are known, we can derive different values for the parameters and vary our σ until our numerical curve matches that of the actual curve. To be able to do this for an actual company, we need to choose a company with public option values and a large turnover of stock.

We chose to find the volatility of AkerBP. To find the information we needed we went on the Oslo Stock Exchange [3]. In reality the underlying stock price is constant for a given time, and the strike-price (E) varies from option to option. Therefore, comparing our data to the real data doesn't make sense in a plot like (1a), where the stock price varies for a given time. Instead we have to find the option price for a given strike price and vary the time to be able to vary the stock price. The resulting comparison will be a plot where the stock price varies as a function of time. This way we can produce plots that help us determine the credibility of the σ we are testing. But, first we need some parameters.

Risk free interest rate or r is usually derived from annual interest rates used by national banks when issuing bonds, as this is considered risk free interest (governments rarely default on their obligations). This has compounded effects for options trading as the interest rates influences all facets of the economy on a macro-scale. For 2020 this rate is 0.11%, and this is also the r value we used in our calculations for Aker BP. The Dividend rate is also something which changes given the times we are in, or the changing philosophy of the company. Dividends are payments to the shareholders of the companies profit. It is given as a currency amount per share. A company does not have to do this and while we initially assumed a dividend of 1.2 percent, it is something which changes from one company to another. When we calculated implied volatility for Aker BP we also found their dividend rate to be 5.14% for the 2020 fiscal year.

3.5 Unit tests

To ensure that the code works for different input, we made three unit tests. Firstly, we check that value of the option for a given time step always is equal to or larger than zero. Options, or any other commodity either has zero value, or a given value, and so we expect this, regardless of time.

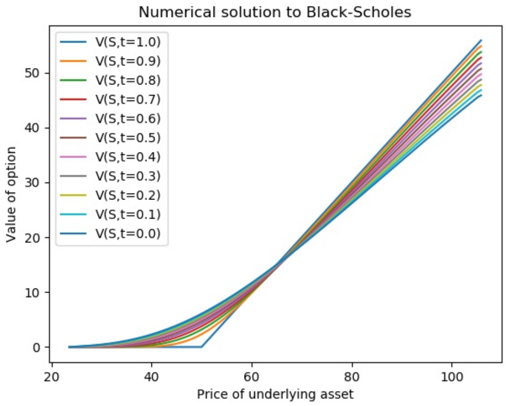
The second one checks that if the volatility σ increases, it will also increase the option value for lower stock prices(we discuss this in further detail in the discussion section of the report (5.1)). We do this by choosing a random $\sigma \in [0, 0.5]$, and then a second which is twice as big. The test is successful if the larger σ produces larger option-values for the first 10% of the integration points. We only chose this amount since the premise is no longer valid after a certain point.

The third test checks that the Cranck-Nicholsen algorithm is somewhat independent of step size. As we discussed in the (2.3), the Cranck-Nicholsen method should be stable for different values of $\alpha = \frac{dt}{h^2}$. In our implementation we choose $dt = T/N$ and $h = 2L/N$, where T is the total time of the calculation and N is the amount of steps. Therefore we vary α by varying N. Our last unit test is therefore to compare the option values for a random time-step, for different N values(N=1e3 and N=1e4). If the difference in the average option value for the two N is smaller than 1% of the average value of $N = 1e3$, we define the method as stable. Since the time-step is random, the credibility of the code increases for every test.

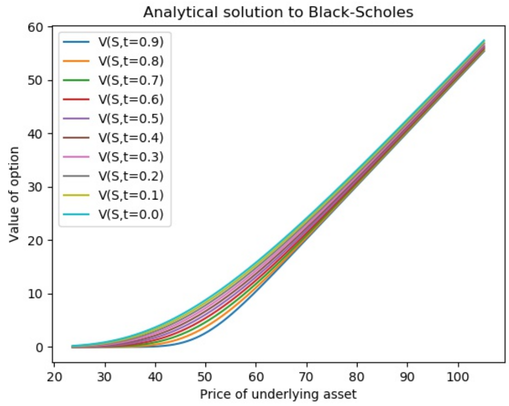
4 Results

4.1 The Black-Scholes solution

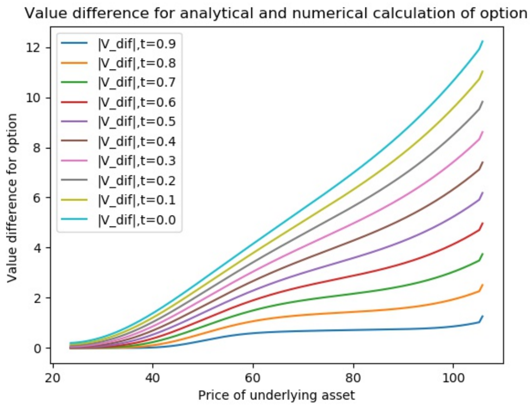
Here are the numerical and analytical solutions to equation (1). All of the figures bellow (1a,1b,1c,1d) were produced with $E = 50$, $r = 0.04$, $D = 0.12$ and $\sigma = 0.04$. In the plots below, t means normal time, from zero to T, the expiration date.



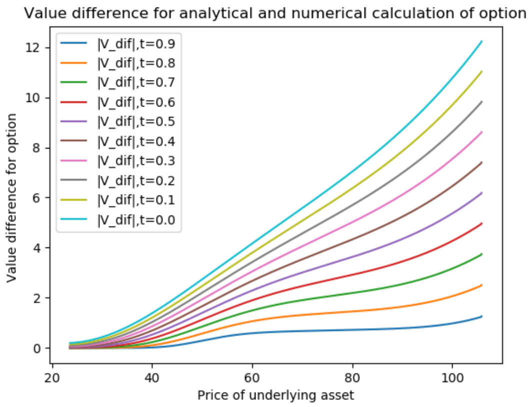
(a) A set of numerical solutions to the Black-Scholes equation with N=1e3 for times ranging from $t \in [0, 1]$.



(b) Analytical solution to the Black-Scholes equation with N=1e3 for times ranging from $t \in [0, 0.9]$.



(c) Absolute difference between analytical and numerical solution with N=1e3 for times ranging from $t \in [0, 0.9]$.



(d) Absolute difference between analytical and numerical solution with N=1e4 for times ranging from $t \in [0, 0.9]$.

Figure 1: Solutions to Black-Scholes equation as a function of stock price S and time t.

Plot (1a) shows the numerical solution with set parameters for the value of the option as a function of the value of the stock (s(x)) ranging from $x \in [-0.75, 0.75]$. Plot (1b) shows the analytical solution to the same problem, using equation (17), and plot (1c) shows the difference between the two. From the analytical solution we can see that for small stock prices (S<E), the smaller the t-value the larger the option price. But when the stock prices reaches a certain point (S≈58), the option value starts to become lower for smaller t-values(curve for t=1 overlaps other curves). And finally when the stock prices are large enough (≈S >70) the value of the options are rearranged proportionally to the time.

To highlight the effect of the volatility, σ we compute the same plot as above, but this time with $\sigma = 2$. The result were the following:

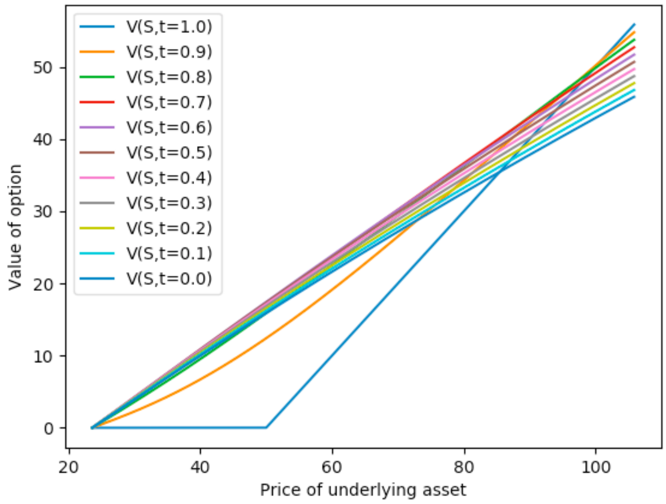


Figure 2: A set of numerical solutions to the Black-Scholes equation with N=1e3 for times ranging from $t \in [0, 1]$, with $\sigma = 2$

With $\sigma = 2$, the time the dependency of the value of the options changes. Now the overlapping of the t=1 curve happens around S=85, not S=58.
Finally we want to highlight the affect of the dividend 0, on the system. Therefore we plot a similar plot as (1c) with the same parameters, but for D=0. This yields the following result.

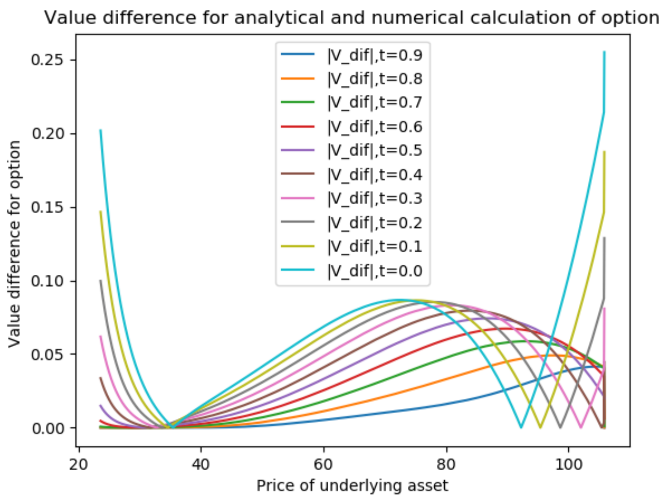
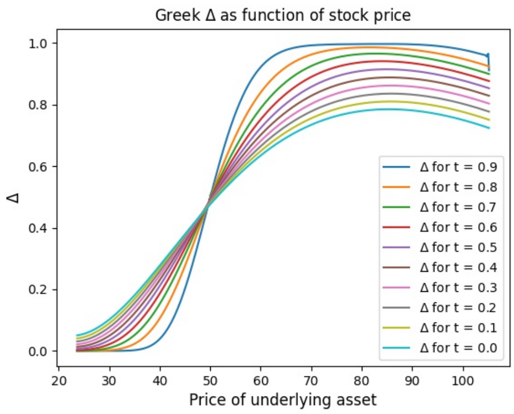


Figure 3: Absolute difference between analytical and numerical solution with N=1e3 and D=0

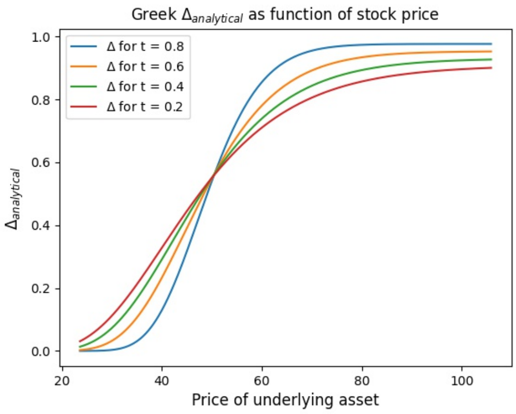
When D=0 the difference in the two methods is greatly decreased. Although the difference seems more chaotic, the size of the difference is decreased by a factor of 60. This implies that our numerical solver works very well.

4.2 The Greeks

Here we will study the Greeks, mentioned in section 2.5. We will compare the numerical results to the analytical ones[2]. Note that some of the analytical results for the greeks differ slightly in size to the numerical, or have fewer time values. This is because we are focusing mainly on the shape of the greeks, since we are mostly interested in how they behave and we know (as will be discussed in section (5.1)) that the analytical solutions tend to differ slightly from our numerical.
Firstly we see Δ as a function of stock price S in figure (4a). We see here that the function starts to stabilize for $S \approx 60$. The figure shows that our numerical graph 4a shows great resemblance to the analytical (4b).



(a) Numerical solution to Δ as a function of stock price

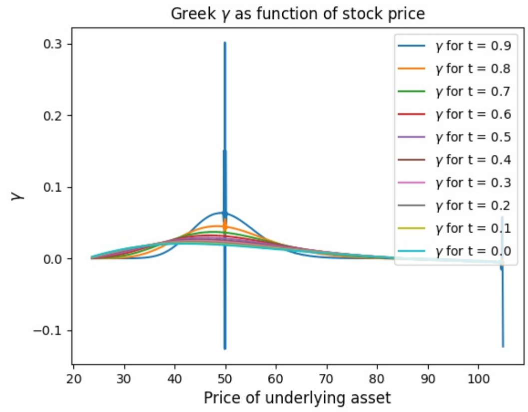


(b) Analytical solution of Δ as a function of stock price

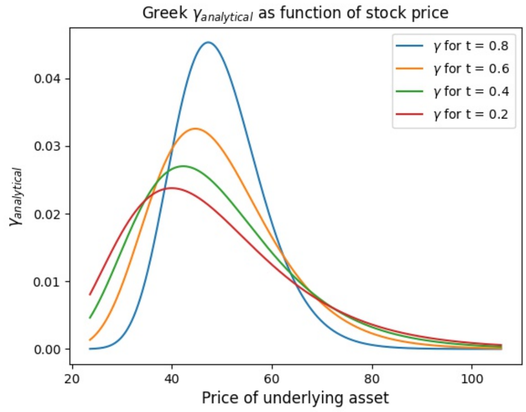
Figure 4: Δ solved both numerically and analytically for $\sigma = 0.4$, $r = 0.04$, D=0.12 and E=50 with N=1e3

Secondly we see γ as a function of stock price S in figure (5). Because the derivative for the expiration time is so much larger, we also have a zoomed in photo to see more clearly the behavior of the other timesteps.

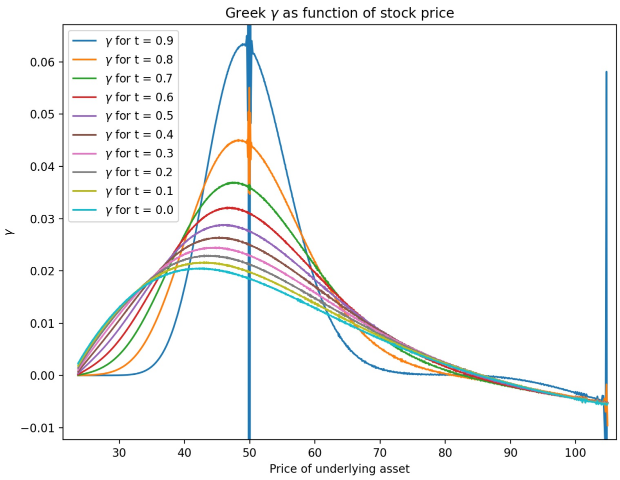
[2]The analytical expressions of the Greeks are symbolic derivations of the analytical solution to the Black-Scholes equation, and thus we chose not to derive them in this report, but rather to link to the expression[9].



(a) Numerical solution to γ as a function of stock price



(b) Analytical solution to γ as a function of stock price

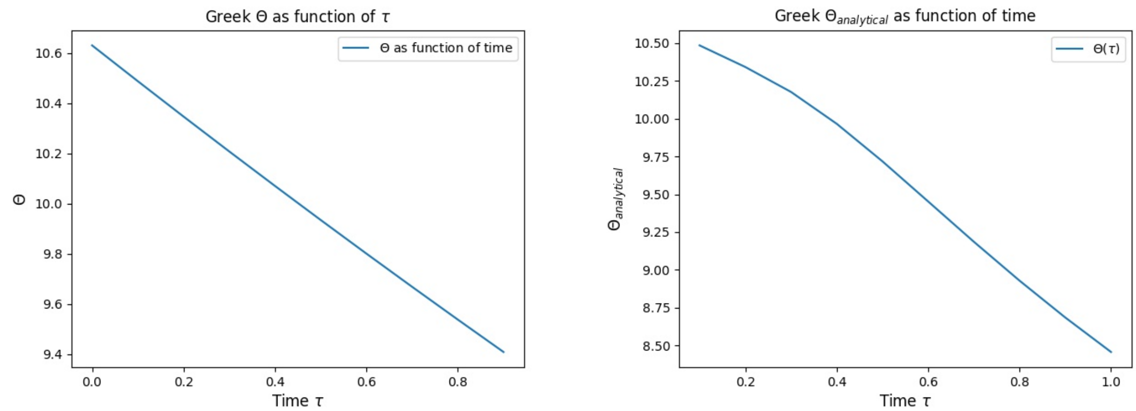


(c) Zoomed in figure of the numerical solution to γ

Figure 5: γ solved both numerically and analytically for $\sigma = 0.4$, $r = 0.04$, D=0.12 and E=50 with N=1e3.

Apart from the great spike in the numerical result, which is a result of the initial boundary-condition, the numerical graph (5a) seems to compare very well to the analytical (5b).

Thirdly we see Θ as a function of time τ in figure (6), with a constant stock price of 105.8. Θ describes the change of the value as function of time. Remember that $\tau = T - t$, so Θ increases over time.

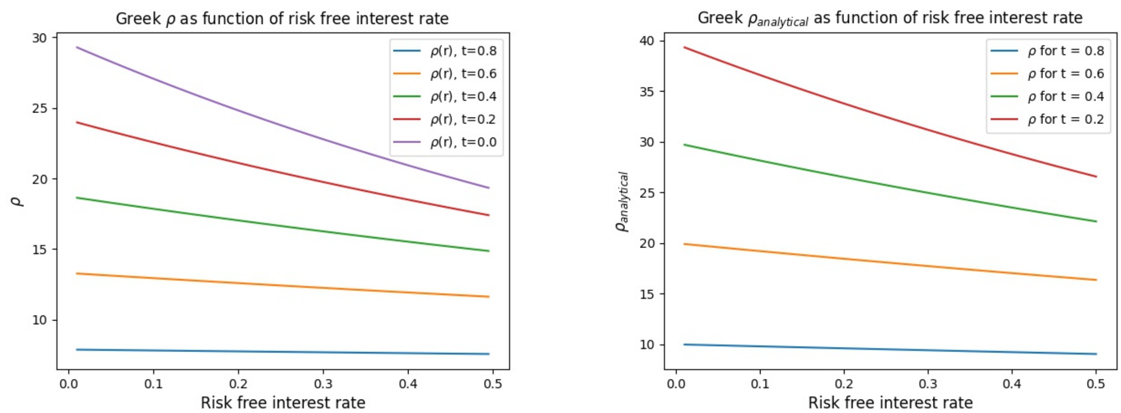


(a) Numerical solution to Θ as a function of time

(b) Analytical solution of Θ as a function of time

Figure 6: Θ solved both numerically and analytically for $\sigma = 0.4$, $r = 0.04$, D=0.12, E=50, with N=1e3 and a constant stock price ≈ 105.8.

It is clear from the plot that both the numerical (6a) and the analytical result (6b) show great resemblance. Note that in the other plots we had multiple curves adhering to their timestep. Here we have plottet one option value for a given timestep, the option value corresponding to the largest stock price.
Fourthly we see ρ as a function of risk free interest rate r for multiple timesteps with fixed stock price of 105.8 in figure (7). We see here that the analytical and numerical calculations for rho differ with a value of about 10 for $t < T/2$.



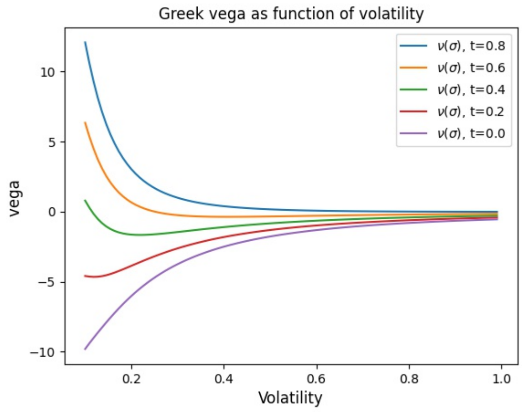
(a) Numerical solution to ρ as a function of risk free interest rate

(b) Analytical solution of ρ as a function of risk free interest rate

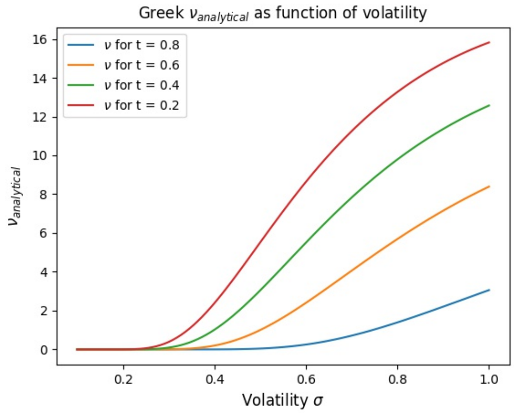
Figure 7: ρ solved both numerically and analytically for $\sigma = 0.4$, D=0.12, E=50, N=1e3 and $r \in [0.01, 0.5]$.

Lastly we see ν as a function of the volatility σ, with a large fixed stock price of

S=105.8 in figure (8). We see here that the closer you get to the expiration time for the option, the change in option value goes to zero.



(a) Numerical solution to ν as a function of volatility



(b) Analytical solution of ν as a function of volatility

Figure 8: ν solved both numerically and analytically

This time the numerical (8a) does not seem to match the analytical (8b). They seem to converge in different ends of the σ-values, as well as the numerical solutions have negative values but the analytical do not.

4.3 Akerp BP σ

As discussed in section (3.4), we are interested in recreating the option-values for a given day/stock price, by trying different values of σ. In our discussion (5.2) we cover how we found a realistic value for r. Using r=0.0011, D=0.05 and N=1e4 we found the following result:

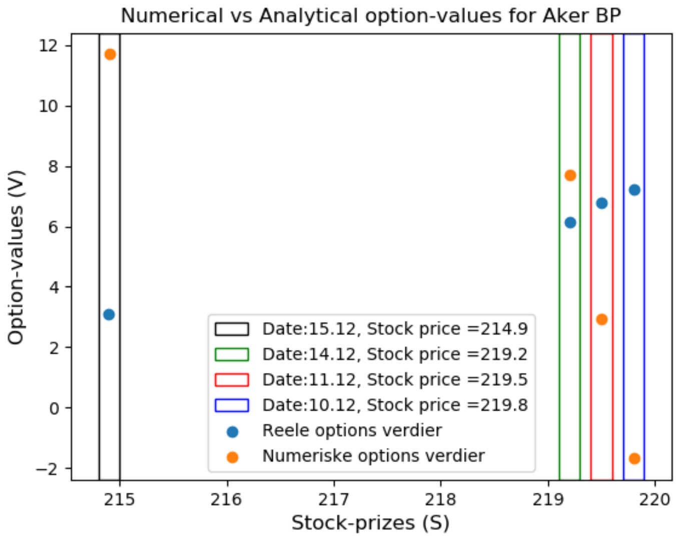


Figure 9: Numerical vs Analytical comparison for option values of Aker BP, where numerical values were calculated for r=0.0011, D = 0.05, N=1e4 and $\sigma = 0.3$

It is clear from the plot that $\sigma = 0.3$ does not create a good approximation to the actual values, as well as creates negative option values. But, through further trial and errors, we found that this was the case for every σ we tried. As we discuss in section (5.2) we found this to be the fault of our r-value. The r-value was too low. We then tried our original r of 4%. Using the parameters r=0.04, D = 0.05, N=8e4, we found the following result:

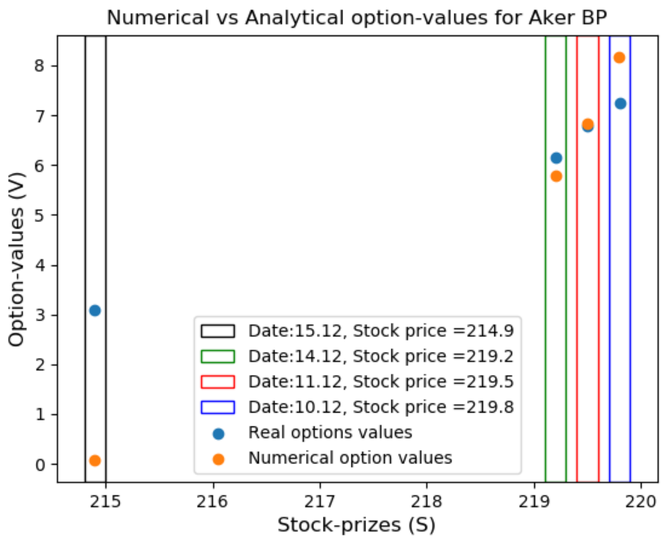


Figure 10: Numerical vs Analytical comparison for option values of Aker BP, where numerical values were calculated for r=0.04, D = 0.05, N=8e4 and $\sigma = 0.007$

Figure (10) was created using $\sigma = 0.07$, for a total of 6 days. This is the equivalent of a $\sigma = 0.07 * 100 * \frac{6}{252} = 30.4\%$ annualized (where 252 is about the amount of trading days in a year, and 6 is the amount of time we calculated over). We can see that with our new values for r and σ our numerical result are a fare better approximation. Only the option value for the first date (15.12) has a deviation of any significance.

5 Discussion and analysis

5.1 Analytical vs Numerical

In (1c) we plot the numerical results and subtracted them from the analytical, so that we could visualize the difference. From the plot it is obvious that they compare better for lower S-values. One of the reasons why the difference varies for S-values could be because of the distribution in the vector of S-values. When calculating the different option-values, we did so with a uniformly distributed vector for our x-values. From (2.2.1), we can see that our definition of x means that S is defined as $S = Ee^x$). This means that the larger the x-values get, the quicker S grows. In other words, the distribution of steps will be larger for lower S-values and smaller for larger S-values. But, if this was the cause of the difference, the difference should decrease for a larger amount of steps. In (1d), we increased N, but the difference remained the same.

An other cause of the difference could be the analytical expression. The expression (17) is independent of D. This means that the analytical solution is for a non-dividend-paying underlying stock. This is why we are interested in the case where we set D=0 for our numerical solver. From figure (3), we can see that the difference in the two curves drastically decreases when D is set to 0. This means that the difference can be contributed to our value of D, not the credibility of the code.

5.2 Black-Scholes parameters

In section (4.1) when presenting our results, we initialized Black-Scholes for a number of parameters. These were useful springboards for testing the implementation, but they can be far from realistic. We also saw in figure (9) that the realistic value r=0.0011 made it very hard for our solver to create realistic results. Therefore, we will discuss why the parameter r is so crucial and what the implications of a low value for r is.
In the Black-Scholes equation r functions as an approximation of the effect interest rates will have on the option value. A valuable metric for this is the ρ function which we discuss in theory (2.5.4) and in effect (7a). Furthermore, the interest rate today is historically low. We initially set the rate at 4% which is quite bullish. The rate today is closer to zero[3]. This has major implications for our own model.

[3]Norges Bank [8] currently issues rates of 0.11 percent, 40 times less than our initial guess.

If we look at equation (37), the right side approaches 0 as r approaches 0. This means that the left side must do so as well. This implies that the effect of γ neutralizes the effect of Θ. By studying the Greeks, this also implies that the option-value is no longer effected by a change in time or stock-price. This means as r approaches zero the trade becomes risk-free. The Black-Scholes equation (1) therefore implies that we can hedge our option perfectly. In the real world this cannot be true, and this is supported by the assumption of the Black-Scholes model that there must always be risk free rent attached (2.2). It would then seem like our current market is outside the scope of the model, and anyone hedging investments through it may expose themselves to significant financial losses.

σ being implied volatility is the statistical deviation of an asset price. Assets with higher volatility can therefore give both higher returns and losses. Initially we assumed a volatility of 40 percent. For a better approximation see section (4.3), where we estimate the implied volatility of a traded company.

5.3 The unit tests

We chose three tests to verify that the code produced reasonable data for analysis. All three were designed to check that the Crank-Nicholsen algorithm worked, but they did not touch other aspects of the code, such as the computation of the Greeks. We made a choice to not create more unit tests, as there were little time left to design them. Also, the Greeks function were designed after the Crank-Nicholsen algorithm, and so we assume that if the code pass the unit tests designed for that algorithm, then we will not have any problem with the Greek one either. Regardless, we feel confident that they produce correct values. An idea for a test however could be to ensure that the computed value for the option behaves somewhat as expected when varying either σ or ρ. Another point could be to check that the choice of static values makes sense. In other words, compute the Greeks for the chosen values to evaluate how realistic they are, to ensure more realistic option values.

5.4 The effect of volatility on the value of options as a function of time

When presenting figure (1a), we discussed how the value of the option varies as a function of time in different intervals of stock value. It is interesting to note that for stock values smaller than the exercise price, the value of the option is larger than 0 for all t except t=(terminal time). The larger than 0 option value and the different effect of time for certain intervals of S, can be explained by our volatility. Volatility gives us a sense of how much the value of the stock might fluctuate. Meaning that even though the stock might be worth less than the exercise price, if there is time left, there is still a chance that the stock market might fluctuate in you're favour and increasing the value of the stock above that of the exercise price. This is also true the other way around. If you're stock is worth more than the exercise price in a time t<(terminal time), there is still a chance that the stock market fluctuates so that the stock looses value.

The effect of the volatility becomes apparent in figure (2). We see that when we increase the fluctuations in the stock market, we also increase the value of the options for stock-values smaller than E. This is because we are increasing the fluctuations in the stock market. This is also the reason why our curve for t=1 does not start overlapping other curves until $S \approx 85$.

5.5 The Greeks

As shown in section 4.2, the numerical and analytical solutions vary both in shape and accuracy, and there are multiple reasons for that. Overall it seems like our numerical results match those of the analytical very well, and this brings credibility to our code.
Lets start with Δ and γ. For Δ we see that they both show an extreme value around $S = 60$, and then stabilizes for higher stock prices. This is what we would expect since this is the area where the stock prices rises above the strike price. As for γ, the analytical and the numerical behave rather similarly, which we would expect since they do so for Δ as well. This behavior is probably due to the fact that we used the same stock range, volatility, strike price and risk free interest rate. Again it is no shock that γ peaks for stock-prices close to the strike price.
We also see a nice fit for the numerical and analytical solution to Θ. Since we choose a stock price twice the size of the strike price, it makes sense for the value to increase as we approach the terminal time $\tau = 0$. This is because the lack of time decreases the possibility of a drop in stock price.
Similarly to the greeks before it, ρ has a nice fit between analytical and numerical, differing only in the size of the value. This could either be due to numerical error or due to the fact that we for the numerical solution pick the option value for the largest stock price. This choice is not rooted in anything other than a random choice. Alternatively this could be because for some off the analytical expressions, a non-dividend-paying stock is assumed. But, although the size differs, the shape does not. It is clear from both plots that with an increasing interest rate, the effect it has on the option- value decreases.

The last one is Vega. Numerically, we expect that for large stock values, the increase of volatility will only make the value of the option decrease. This is (like we have discussed previously) because with an increase of σ comes an increase of risk of the stock-value loosing value, for stock prices much larger than the strike price. Therefore it does not make sense for the effect of the volatility on the option-value to be strictly positive. Also, we expect the effect of increasing the volatility has on a large stock price to be very large for low values of σ. This is because when increasing the volatility from nothing to something, you are introducing a possibility of the stock loosing great amount of value. Based on this we believe that out numerical curve better explains our system and its deviation from the analytical result could be because of certain assumptions made when deriving the analytical expression.

5.6 The approximation of the volatility of Aker BP and the deviation for 15.12

In section (5.2) we discussed why our model poorly simulates reality for low values of r, as it implies the possibility of complete elimination of risk. Therefore we choose r to be larger then it actually is, to better approximate the option values of Aker BP. When doing so we found that for $\sigma = 30.4\%$, our numerical model approximates the reel values to a sufficient degree. Note that 30.4% is well in the ball park of what we would expect. But, we can see that our approximation of the option value at 15.02, is quite poor.

The reason why this is so poor can be many things. We believe that the main the reason is our boundary conditions. As we discussed in (3.1), we are trying to approximate $x \in [-\infty, \infty]$. So our boundary condition for $V(S = S_0, \tau) = 0$ based on the fact that S_0 is very small. But, to make results where we would find an option-value for a stock-price close enough to the stock-prices of Aker BP, we had to limit ourselves to a rather small interval for $x \in [0.25, 0.25]$, otherwise N is too large. Therefore the residue of this boundary condition effects stock-prices close to the boundary, like the option value at 15.12.

Alternatively it could also be because of our change in r-value. To make (10), we increased r by a factor of 40. This could effect our value of σ. As we previously discussed, a low value for σ will decrease the value of the option for low values of S. This could therefore be an implication that the σ we landed on is to small.

A last remark is that we only use 4 different values for stock prices for 4 different days. Originally we wanted to use at least a month worth of data, but the data was not available to us. This could be the cause of some of our complications. The Black-Scholes model is an approximation of reality. The real market is not nearly so predictable and would include many deviation from the trend. But, given enough data, models like Black-Scholes work well due to the fact that deviations will even out. Therefore, with so little data it is impossible to tell if our approximation of the option-value at 15.12 is due to an error in our model, or because the data in it self is a deviation from the trend our model is based on.

6 Conclusion

We found that our numerical solver was able to solve the Black-Scholes equation by modeling it as a diffusion equation, and by doing so creating solutions that show resemblance to the Black-Scholes formula. Through further analysis of our results we found that the Black-Scholes formula assumes a non-dividend-paying stock, and when we set D=0 we reduce the deviation from the formula by a factor of ≈ 60. Furthermore we found that like the interest rate, the option price greatly impacted our volatility. We found that when a stock was priced under the strike price, the volatility increases the value by introducing a chance of the stock rising in value. When the stock is priced above the strike price, often the volatility introduced a risk of the stock losing money, thereby decreasing the value of the stock.

With regards to the solution of the equation, we found that the analytical solution is restricted as τ approaches zero, so when looking at the expiration time for the option, we have only the numerical solution to rely on. We found that for the Greeks, the numerical solutions made more sense for our model than the analytical ones, which is expected due to the restrictiveness of the analytical expression for the Black-Scholes equation. And finally we found that when using our initial value for r=0.04, we were able to approximate the value of the volatility in the option value of Aker Bp to be 30.4%. Though this number shows great promise, a larger number of real data would greatly improve the credibility of our result. As well as a larger amount of data, we could have explored a larger interval for S values to better approximate conditions that inspired our boundary conditions.

Brokers on the financial markets rely heavily on managing their exposure to risk. For a long time the Black-Scholes equation has been used to predict this by giving an estimate for implied volatility. We set out to implement this model numerically so we could try and predict the volatility of a real option trade on the market. During our research we found that the model may be out of bound in the current financial climate. The model assumes there to always be some interest rate attached to safe assets like cash or bonds, but when this interest approaches zero the predictions from the model become inaccurate. Our findings in figure (9) shows that the model predicts entirely different outcomes than what the market trends look like. Therefore, brokers hedging risk with Black-Scholes as a baseline model may find that the number of assumptions (2.2) needed severely restricts the domain of real world applications. As a result we believe that unchanged, the Black-Scholes model is today better suited towards historical data.

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