# Numeric computation of Black-Scholes equation

**With Anders math**

Sakarias Frette
Anders Vestengen
William Hirst

December 6, 2020

# Contents

# 1 Introduction

"To those who have much, more will be given. To those who have less, more will be taken". This is known as the Matthew principle, and this effect appears no where better than in economics, both at a personal level, and as a group. Companies can choose to go public, sorting ownership in stocks or shares.

This report aims to explore the Black-Scholes equation and to verify its numerical solutions with a comparison to the analytical solution for European options. We will also try the model on a company currently on the Oslo Stack Exchange.

# 2 Theory

## 2.1 Options

Options are type of derivatives for financial instruments. They are called derivatives because they derive their value from an underlying asset. Options are a contract between two parties agreeing to either sell (put) or buy (call) something else for a predetermined price at a point in the future. For example, you recognize that during the pandemic more people are shopping online. You buy calls on a major online retailer which expire later in the year. When the retailer publishes its earnings the stock price immediately goes up, but you already have calls for this retailer and can buy the stocks at a price you set previously when the stock was low. You can now sell those stock for a profit. The Black-Scholes equation aims to predict when you should buy and sell an option, based on the price of the underlying asset.

## 2.2 Black-Scholes equation

The Black-Scholes equation is given as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

where $\sigma$ is the volatility of the stock, S is the stock price, r is the risk-free interest rate, D is the dividend paying rate and V is the value of the option as a function of stock price and time. In order to solve this equation, we can try to model the value of the options after the diffusion equation. In order to that we first have to do a variable change, from S to x. We set this relation to be $x = \ln \frac{S}{E}$. If we differentiate this equation, we get the following relation:

$$\partial x = \frac{\partial S}{S} \tag{2}$$

which gives us the derivation operators

$$\frac{\partial}{\partial x} = S\frac{\partial}{\partial S} \tag{3}$$

Now we also need the second derivative, so we use equation 3 on itself and get

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial x^2} = S\frac{\partial}{\partial S}\left(S\frac{\partial}{\partial S}\right) = S\frac{\partial}{\partial S} + S^2\frac{\partial^2}{\partial S^2} \tag{4}$$

From this equation and equation 3 we have that

$$S^2\frac{\partial^2}{\partial S^2} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \tag{5}$$

We can now rewrite equation 1 in x coordinate:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + (r - D - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} - rV = 0 \tag{6}$$

We then transform the time variable to $\tau = T - t$, where T is the expiration time for the option. This gives us

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} - (r - D - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} + rV = 0 \tag{7}$$

The last thing we need to do is to is to do a substitution for the value of the option:

$$u(x,\tau) = e^{\alpha x + \beta\tau}V(S,t) \tag{8}$$

We then get

$$\begin{aligned}\frac{\partial}{\partial \tau}(u(x,\tau)e^{-\alpha x - \beta\tau}) = \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}(u(x,\tau)e^{-\alpha x - \beta\tau}) \\ + (r - D - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x}(u(x,\tau)e^{-\alpha x - \beta\tau}) \\ - ru(x,\tau)e^{-\alpha x - \beta\tau}\end{aligned} \tag{9}$$

which gives us

$$\begin{aligned}\left[\frac{\partial u}{\partial \tau} - \beta u\right]e^{-\alpha x - \beta\tau} = \frac{\sigma^2}{2}\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}e^{-\alpha x - \beta\tau} - \alpha u e^{-\alpha x - \beta\tau}\right] \\ + (r - D - \frac{\sigma^2}{2})\left[\frac{\partial u}{\partial x}e^{-\alpha x - \beta\tau} - \alpha u e^{-\alpha x - \beta\tau}\right] - rue^{-\alpha x - \beta\tau}\end{aligned} \tag{10}$$

$$\frac{\partial u}{\partial \tau} - \beta u = \frac{\sigma^2}{2}\left[\frac{\partial^2 u}{\partial x^2} - 2\alpha\frac{\partial u}{\partial x} + \alpha^2 u\right] + (r - D - \frac{\sigma^2}{2})\left[\frac{\partial u}{\partial x} - \alpha u\right] - ru \tag{11}$$

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\left[r - D - \frac{1}{2}\sigma^2 - \alpha\sigma^2\right] + u\left[\alpha^2\sigma^2 - \alpha(r - D - \frac{1}{2}\sigma^2) - r + \beta\right] \tag{12}$$

Now, in order to get the diffusion equation we have to set the sum of the coefficients for both first order derivative of u and u to zero. From this we can find $alpha$ and $\beta$ as

$$\alpha = \frac{r-D}{\sigma^2} - \frac{1}{2} \tag{13}$$

and

$$\beta = r + \alpha \left[ r - D - \frac{1}{2}\sigma^2 \right] - \alpha^2\sigma^2 \tag{14}$$

Substituting back we get that the values of the option, given as $V(S,t)$ is given as

$$V(S,t) = u(x,\tau) * e^{-\left[\alpha x + \left(r + \alpha\left[r - D - \frac{1}{2}\sigma^2\right] - \alpha^2\sigma^2\right)\tau\right]} \tag{15}$$

## 2.3 Crank-Nicolson

When learning to solve a Partial Differential Equation(PDE), one often starts with the Forward Euler Method (FEM). FEM is a very intuitive method and can be written as the following:

$$\frac{du}{dx} \approx \frac{u(x_i) - u(x_i - \Delta x)}{\Delta x} \tag{16}$$

, where u is a function of a variable x, and $\Delta x$ is the predetermined step from one value to the next. When using such a definition to solve a PDE, one often calls the method an explicit scheme. It is characterised by the fact that you start the solution in a point $x_0$ and end in another point $x_N$, "moving forward" in the function. We can use (16) on our diffusion equation (INSERT LABEL TO DIFFUSION EQ) for a function u(x,t), deriving twice for x and once for time and we get the following equation:

$$\frac{u(x_i,t_j) - u(x_i,t_j - \Delta t)}{\Delta t} = \frac{u(x_i + \Delta x, t_j) - u(x_i,t_j) + u(x_i - \Delta x, t_j)}{\Delta x} \tag{17}$$

Alternatively you can use an implicit scheme. When using an implicit scheme you begin in a point $x_N$ and integrate "backwards" to a point $x_0$. A common way of doing so is called the Backward Euler Method (BEM). BEM can be written as:

$$\frac{du}{dx} \approx \frac{u(x_i - \Delta x) - u(x_i)}{\Delta x} \tag{18}$$

, where the values are the same as described above. Using (18) on our (INSERT EQ FOR DE) and doing as we did above, we find the following equation:

$$\frac{u(x_i,t_j - \Delta t) - u(x_i,t_j)}{\Delta t} = \frac{u(x_i + \Delta x, t_j) - u(x_i,t_j) + u(x_i - \Delta x, t_j)}{\Delta x} \tag{19}$$

The Cranck-Nicolsen method can be described as a mixture between the two, FEM and BEM. If we introduce a parameter $\theta$ together with our to equations, (17) and (19) the Cranck-Nicolsen method is derived from the following equation:

$$
\begin{aligned}
&\frac{\theta}{\Delta x^2}(u(x_i+\Delta x, t_j) - 2u(x_i, t_j) + u(x_i - \Delta x, t_j)) \\
+&\frac{1-\theta}{\Delta x^2}(u(x_i+\Delta x, t_j - \Delta t) - 2u(x_i, t_j - \Delta t) + u(x_i - \Delta x, t_j - \Delta t)) \\
&\frac{1}{\Delta t}(u(x_i, t_j) - u(x_i, t_j - \Delta t)
\end{aligned}
\tag{20}
$$

When $\theta = 0$, (20) becomes the explicit method, and when $\theta = 1$, (20) becomes the implicit method. When $\theta = 1/2$, 20 becomes the Cranck-Nicolsen method and can be written as:

# 3 Implementation

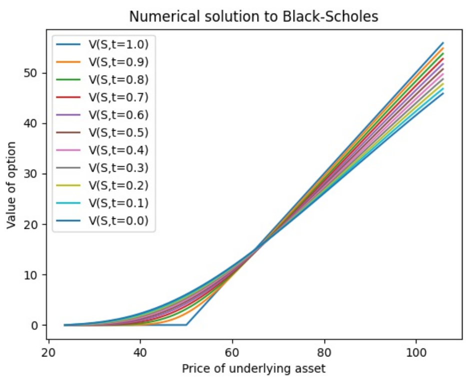
The programs used in this research article can be found on this Github address.

# 4 Results

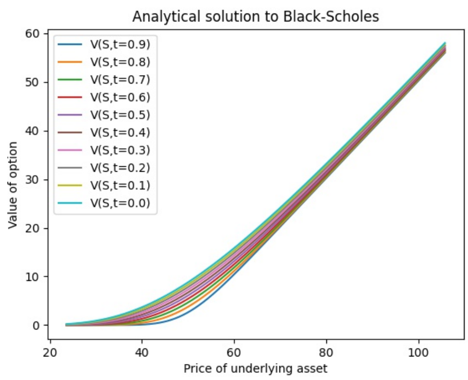
Here are the numerical and analytical solutions to equation 1:

# 5 Discussion and analysis

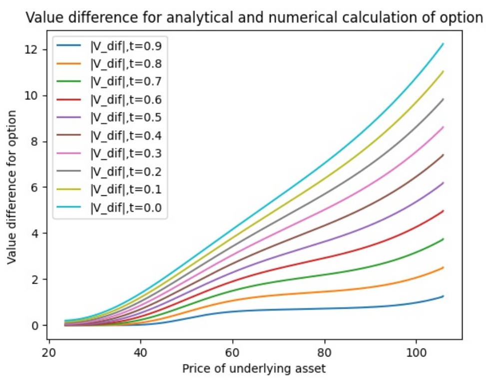
# 6 Conclusion




(a) Numerical solution to the Black-Scholes equation




(b) Analytical solution to the Black-Scholes equation




(c) Absolute difference between analytical and numerical solution

Figure 1: Solutions to Black-Scholes equation as a function of stock price S and time t.